Comp6211e: Optimization for Machine Learning

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Lecture 18: Randomized Coordinate Descent and Acceleration

Regularized Loss Minimization

Last lecture, we consider the composite optimization problem, but with an added finite sum structure as follows,

$$\phi(w) = \frac{1}{n}\sum_{i=1}^{n} f_i(X_i^\top w) + \lambda g(w), \tag{1}$$

where $w \in \mathbb{R}^d$ is the model parameter:

We assume that $g(w)$ is strongly convex.

Decomposable Linear Model

In this lecture, we consider optimization problem with the model parameter $w \in \mathbb{R}^d$. Here w can be decomposed into p components $w = [w_1, \ldots, w_p]$, where each w_j is a d_j dimensional vector, with $\sum_{j=1}^p d_j = d$.

We consider the following form of optimization problem:

$$\phi(w) = f(w) + g(w), \tag{2}$$

where

$$f(w) = \psi\left(\sum_{j=1}^p A_j w_j\right), \quad g(w) = \sum_{j=1}^p g(w_j).$$

We assume that $f(\cdot)$ is L_i-smooth with respect tor w_j, and $g(\cdot)$ is convex but may not be smooth.

Dual Formulation

Example

Consider the dual formulation of the regularized loss minimization problem:

$$\phi_D(\alpha) = \frac{1}{n}\sum_{i=1}^{n} -f_i^*(-\alpha_i) - \lambda g^*\left(\frac{1}{\lambda n}\sum_{i=1}^{n} X_i\alpha_i\right),$$

where each $\alpha_i \in \mathbb{R}^k$. Here $-\phi_D(\alpha)$ can be written as

$$\tilde{\psi}\left(\sum_{j=1}^{p} A_j \tilde{w}_j\right) + \sum_{j=1}^{p} \tilde{g}_j(\tilde{w}_j).$$

Here $\tilde{w}_j = \alpha_j$, $p = n$, $d = nk$, $\tilde{\psi}(u) = \lambda g^*(u)$, $A_j = (\lambda n)^{-1} X_j$, $\tilde{g}_j(\tilde{w}_j) = n^{-1} f_j^*(-\alpha_j)$ for $j = 1, \ldots, p$.

Randomized Coordinate Descent

In randomized coordinate descent algorithm for solving (2), we randomly select a variable i from 1 to p, and minimize the objective with respect to w_i using proximal gradient.

That is, we select i, and optimize with respect $w_i + \Delta w_i$:

$$f\left(\sum_{j=1}^{p} A_j w_j + A_i \Delta w_i\right) + \sum_{j=1}^{p} g_j\left(w_j + \Delta w_i \delta_i^j\right).$$

Proximal Coordinate Optimization

Given $\eta_i \leq 1/L_i$, we use an upper bound of $f(\cdot)$ as follows:

$$\psi\left(\sum_{j=1}^{p} A_j w_j\right) + \nabla\psi\left(\sum_{j=1}^{p} A_j w_j\right)^\top (A_i \Delta w_i) + \frac{1}{2\eta_i}\|\Delta w_i\|_2^2 + g_i(w_i + \Delta w_i).$$

Let

$$u = \sum_{j=1}^{p} A_j w_j,$$

Derivation of Primal CD Method

We can optimize

$$
\begin{aligned}
\Delta w_i =& \arg\min_{\Delta w} \left[(A_i^\top \nabla f(u))^\top \Delta w + \frac{1}{2\eta} \|A_i\|_2^2 \|\Delta w\|_2^2 + g_i(w_i + \Delta w) \right] \\
=& \arg\min_{\Delta w} \left[\frac{1}{2\eta_i} \|\Delta w + \eta_i A_i^\top \nabla f(u)\|_2^2 + g_i(w_i + \Delta w) \right] \\
=& \mathrm{prox}_{\eta_i \cdot g_i}(w_i - \eta_i A_i^\top \nabla f(u)) - w_i,
\end{aligned}
$$

and

$$
\mathrm{prox}_{\eta_i g_i}(w) = \arg\min_{z \in \mathbb{R}^{d_i}} \left[\frac{1}{2} \|z - w\|_2^2 + \eta_i g_i(z) \right].
$$

Primal Coordinate Descent

Algorithm 1: Randomized Proximal Coordinate Descent

Input: $\phi(\cdot)$, $\eta_i \leq 1/L_i (i = 1, \ldots, p)$, $w^{(0)}$
Output: $w^{(T)}$

Let $u^{(0)} = \sum_{j=1}^{p} A_j w_j^{(0)}$
for $t = 1, 2, \ldots, T$ **do**
  Randomly pick $i \sim [1, \ldots, p]$
  Let $w_i^{(t)} = \mathrm{prox}_{\eta_i g_i}(w_i^{(t)} - \eta_i A_i^\top \nabla f(u^{(t-1)}))$
  Let $w_j^{(t)} = w_j^{(t-1)}$ for $j \neq i$
  Let $u^{(t)} = u^{(t-1)} + A_i(w_i^{(t)} - w_i^{(t-1)})$

Return: $w^{(T)}$

Theorem

In Algorithm 1, assume that $\eta \leq 1/L$, then $\forall w = [w_1, \ldots, w_p] \in \mathbb{R}^d$:

$$\frac{p-1}{T}\mathbf{E}\phi(w^{(T)}) + \frac{1}{T}\sum_{t=1}^{T}\mathbf{E}\phi(w^{(t)})$$

$$\leq \frac{p-1}{T}\phi(w^{(0)}) + \phi(w) + \frac{1}{T}\sum_{i=1}^{p}\frac{1}{2\eta_i}\|w_i^{(0)} - w_i\|_2^2.$$

Proof

Let $\sum_j A_j w_j^{(t)} = \sum_j A_j w_j^{(t-1)} + A_i(w_i^{(t)} - w_i^{(t-1)})$. We have for all $w \in \mathbb{R}^d$:

$$
\begin{aligned}
\phi(w^{(t)}) = & \left[\psi\left(u^{(t-1)} + A_i(w_i^{(t)} - w_i^{(t-1)})\right) + g(w^{(t)})\right] \\
\leq & \psi\left(u^{(t-1)}\right) + \left(A_i^\top \nabla\psi\left(u^{(t-1)}\right)\right)^\top (w_i^{(t)} - w_i^{(t-1)}) \\
& + \frac{1}{2\eta_i}\|w_i^{(t)} - w_i^{(t-1)}\|_2^2 + g(w^{(t)}) \\
\leq & \psi\left(u^{(t-1)}\right) + \left(A_i^\top \nabla\psi\left(u^{(t-1)}\right)\right)^\top (w_i - w_i^{(t-1)}) \\
& + \frac{1}{2\eta_i}\|w_i - w_i^{(t-1)}\|_2^2 + g(w_i) \\
& + \sum_{j\neq i} g(w_j^{(t-1)}) - \frac{1}{2\eta_i}\|w_i - w_i^{(t)}\|_2^2.
\end{aligned}
\tag{3}
$$

Proof

Take expectation with respect to i, we obtain

$$\begin{aligned}
\mathbf{E}_i \phi(w^{(t)}) \leq & \psi\left(u^{(t-1)}\right) + \frac{1}{p}\nabla\psi\left(u^{(t-1)}\right)^\top \left(\sum_{i=1}^{p} A_i w_i - u^{(t-1)}\right) \\
& + \frac{1}{p} g(w) + \frac{p-1}{p} g(w^{(t-1)}) \\
& + \frac{1}{p}\sum_{i=1}^{p} \frac{1}{2\eta_i}\|w_i - w_i^{(t-1)}\|_2^2 - \frac{1}{p}\sum_{i=1}^{p} \frac{1}{2\eta_i}\|w_i - w_i^{(t)}\|_2^2 \\
\leq & \frac{p-1}{p}\phi(w^{(t-1)}) + \frac{1}{p}\phi(w) + \frac{1}{p}\sum_{i=1}^{p}\frac{1}{2\eta_i}\|w_i - w_i^{(t-1)}\|_2^2 \\
& - \frac{1}{p}\sum_{i=1}^{p}\frac{1}{2\eta_i}\|w_i - w_i^{(t)}\|_2^2 .
\end{aligned}$$

Acceleration

It is possible to derive accelerated coordinate descent methods.

We present an accelerated method for SDCA in Algorithm 2, which applies to the dual formulation for regularized loss minimization problem:

$$\frac{1}{n}\sum_{i=1}^{n} -f^*(-\alpha_i) - g^*\left(\frac{1}{n}\sum_{i=1}^{n} X_i\alpha_i\right)$$

strongly convex problems.

SPDC

Algorithm 2: Stochastic Primal-Dual Coordinate Method (SPDC)

Input: $\phi(\cdot)$, L, λ, $\alpha^{(0)}$, and R such that $\|X_i\|_2 \leq R$
Output: $\alpha^{(T)}$, $w^{(T)}$

Let $\tau = 1/(2R\sqrt{n\lambda L})$
Let $\sigma = \sqrt{n\lambda L}/(2R)$
Let $\theta = 1 - 1/(n + R\sqrt{nL/\lambda})$
Let $u^{(0)} = n^{-1}\sum_{i=1}^{n} X_i\alpha_i$
Let $w^{(0)} = \nabla g^*(u^{(0)})$
let $\bar{w}^{(0)} = w^{(0)}$
for $t = 1, 2, \ldots, T$ **do**
    Randomly pick i
    Let $\Delta\alpha_i \in \arg\max_{\Delta\alpha_i}\left[-f_i^*(-(\alpha_i^{(t-1)} + \Delta\alpha_i)) - \bar{w}^{(t-1)^\top} X_i\Delta\alpha_i - \frac{1}{2\sigma}\|\Delta\alpha_i\|_2^2\right]$
    Let $\alpha_i^{(t)} = \alpha_i^{(t-1)} + \Delta\alpha_i$ and $\alpha_j^{(t)} = \alpha_j^{(t-1)}$ when $j \neq i$
    Let $w^{(t)} = \mathrm{prox}_{\tau g}(w^{(t-1)} + \tau(u^{(t-1)} + X_i\Delta\alpha_i))$
    Let $u^{(t)} = u^{(t-1)} + n^{-1}X_i\Delta\alpha_i$
    Let $\bar{w}^{(t)} = w^{(t)} + \theta(w^{(t)} - w^{(t-1)})$

Return: $\alpha^{(T)}$, $w^{(T)}$

Convergence

Theorem (Convergence of SPDC)

Assume that $f_i^*(\cdot)$ is $1/L$-strongly convex, and $g(\cdot)$ is λ-strongly convex. Let $R = \max_i \|X_i\|_2$. We have

$$
\begin{aligned}
&\left(\frac{1}{2\tau} + \lambda\right) \mathbf{E}\|w^{(t)} - w_*\|_2^2 + \left(\frac{1}{4\sigma} + \frac{1}{L}\right) \mathbf{E}\|\alpha^{(t)} - \alpha_*\|_2^2 \\
\leq &\theta^t \left(\left(\frac{1}{2\tau} + \lambda\right) \mathbf{E}\|w^{(0)} - w_*\|_2^2 + \left(\frac{1}{4\sigma} + \frac{1}{L}\right) \mathbf{E}\|\alpha^{(0)} - \alpha_*\|_2^2\right).
\end{aligned}
$$

Empirical Studies

We study the smoothed hinge loss function $\phi_\gamma(z)$ with $\gamma = 1$, and solves the following $L_1 - L_2$ regularization problem:

$$\min_w \left[\underbrace{\frac{1}{n} \sum_{i=1}^{n} \phi_\gamma(w^\top x_i y_i)}_{f(w)} + \underbrace{\frac{\lambda}{2} \|w\|_2^2 + \mu \|w\|_1}_{g(w)} \right].$$

We compare different algorithms

Comparisons

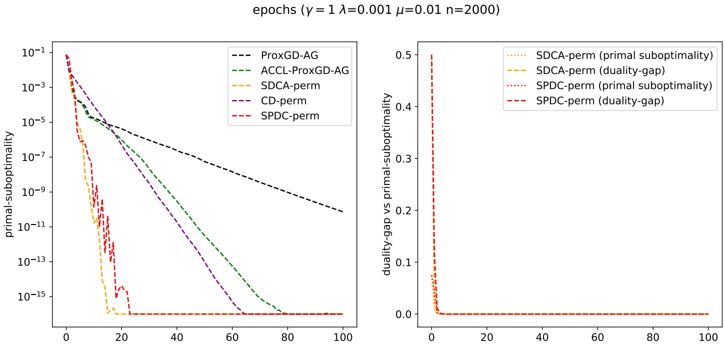

Figure: Comparisons of Proximal Gradient, SDCA and primal CD, SPDC

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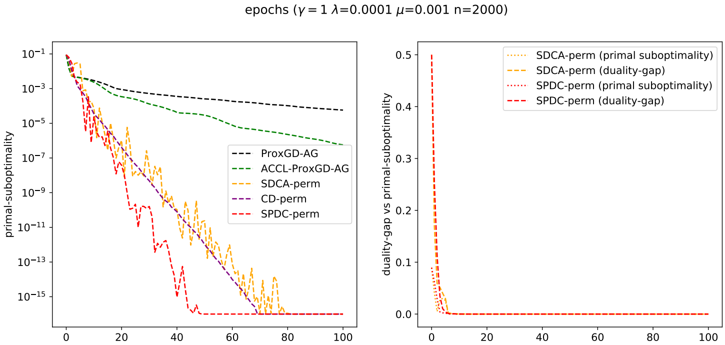

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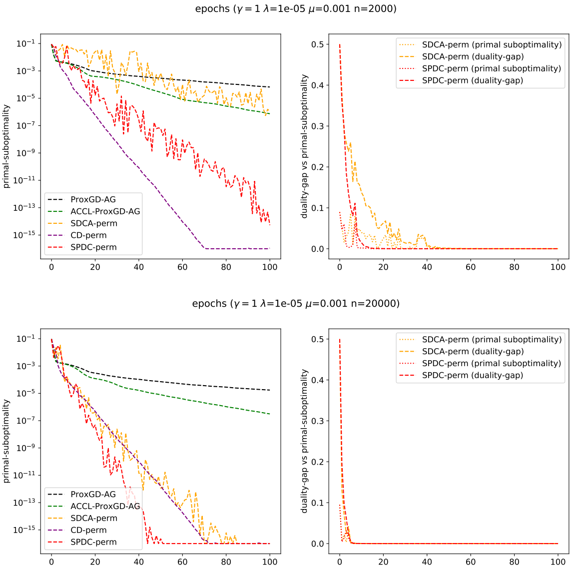

Summary

Regularized Loss Minimization

- finite sum structure
- decomposable linear model

Primal Coordinate Descent

- primal variables
- insensitive to λ and n

Primal-Dual SPDC

- accelerated dual coordinate
- works better when $\lambda n \ll 1$