Semidefinite Relaxations of Truncated Least-Squares in Robust Rotation Search: Tight or Not

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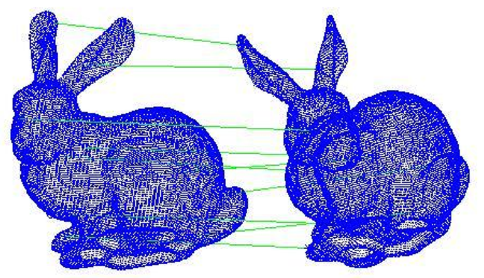
Rotation Search: Problem Setup

Goal:

- Find 3D rotation R_0^* that best aligns 2 point sets

Case without Outliers (Wahba, 1965 [1]):

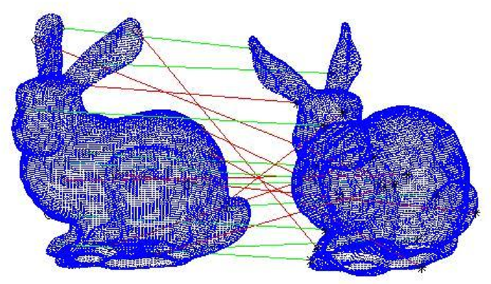
- $y_i \approx R_0^* x_i, \quad i = 1, \dots, l$
- $R_0^* \in SO(3)$
- Optimization:



$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \|y_i - R_0 x_i\|_2^2$$

Case with Outliers:

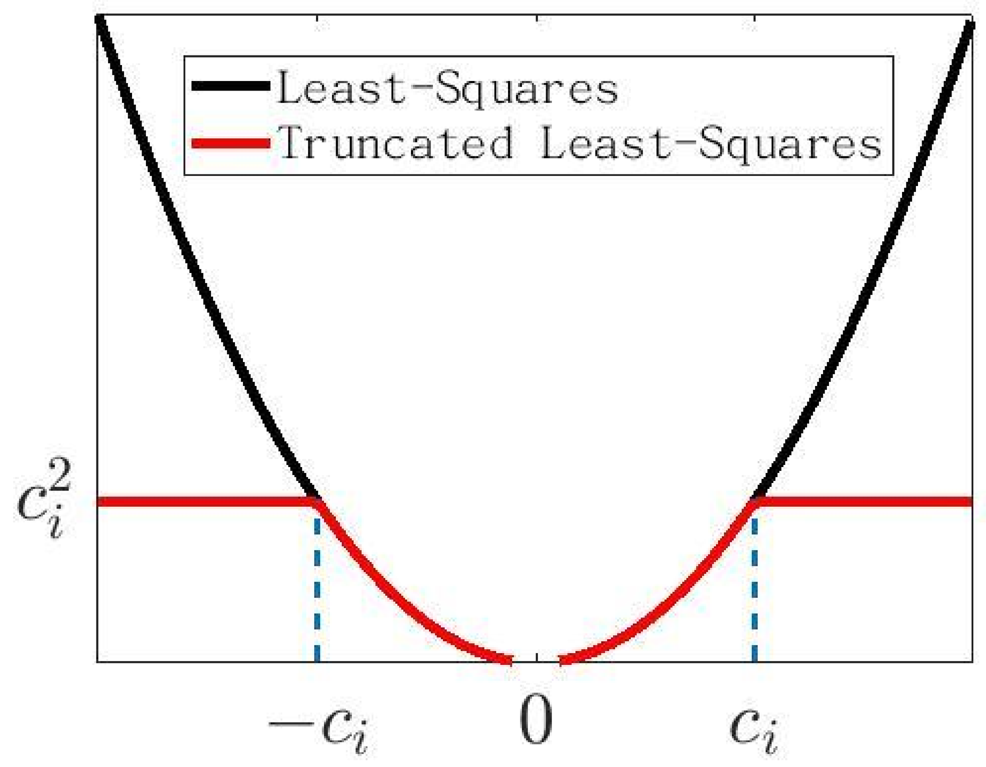
- Inliers: $y_i \approx R_0^* x_i$
- Outliers: (x_i, y_i) arbitrary
- Optimization:



Truncated Least-Squares **(TLS-R)** [1]:

$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \min\left\{ \|y_i - R_0 x_i\|_2^2,\ c_i^2 \right\}$$

(Huber, 1964 [2]) Truncation Parameter



From (TLS-R) to Semidefinite Relaxations (SDR)

$$\text{(TLS-R)} \overset{(1)}{\Longleftrightarrow} \text{(TLS-Q)} \overset{(2),\ (3)}{\Longleftrightarrow} \text{(QCQP)} \overset{(4)\ \text{lifting}}{\Longrightarrow} \text{(SDR)}$$

Step (1): 3D rotations $R_0 \Longleftrightarrow$ unit quaternions w_0

$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \min\left\{ \|y_i - R_0 x_i\|_2^2,\ c_i^2 \right\}$$

$$\min_{w_0 \in \mathbb{S}^3} \sum_{i=1}^{\ell} \min\left\{ w_0^\top Q_i w_0,\ c_i^2 \right\}$$

Step (2): $\min\{a, b\} = \min_{\theta_i \in \{0,1\}} \theta_i a + (1 - \theta_i) b$

$$\min_{\substack{w_0 \in \mathbb{S}^3 \\ \theta_i \in \{0,1\}}} \sum_{i=1}^{\ell} \left(\theta_i w_0^\top Q_i w_0 + (1 - \theta_i)\, c_i^2 \right)$$

Step (3): $w_i := \theta_i w_0$, so $\theta_i = w_i^\top w_0$ [3]

$$\min_{\substack{w_0 \in \mathbb{S}^3 \\ w_i \in \{0, w_0\}}} \sum_{i=1}^{\ell} \left(w_i^\top Q_i w_0 + (1 - w_i^\top w_0)\, c_i^2 \right)$$

$$\begin{aligned} \min_{\boldsymbol{w} \in \mathbb{R}^{4(\ell+1)}} \quad & \mathrm{trace}\left(\boldsymbol{Q} \boldsymbol{w} \boldsymbol{w}^\top \right) \\ \text{s.t.} \quad & [\boldsymbol{w}\boldsymbol{w}^\top]_{0i} = [\boldsymbol{w}\boldsymbol{w}^\top]_{ii}, \quad \forall\, i \in \{1, \dots, \ell\} \\ & \mathrm{trace}\left([\boldsymbol{w}\boldsymbol{w}^\top]_{00} \right) = 1 \end{aligned}$$

Step (4): From (QCQP) to (SDR)

$$\begin{aligned} \min_{\boldsymbol{W} \succeq 0} \quad & \mathrm{trace}\left(\boldsymbol{Q} \boldsymbol{W} \right) \\ \text{s.t.} \quad & [\boldsymbol{W}]_{0i} = [\boldsymbol{W}]_{ii}, \quad \forall\, i \in \{1, \dots, \ell\} \\ & \mathrm{trace}\left([\boldsymbol{W}]_{00} \right) = 1 \end{aligned}$$

$$\begin{array}{c} \text{(TLS-R)} \\ \Updownarrow (1) \\ \text{(TLS-Q)} \\ \Updownarrow (2) \\ (\text{TLS-}\boldsymbol{\theta_i}) \\ \Updownarrow (3) \\ (\text{TLS-}\boldsymbol{w_i}) \\ \Updownarrow \\ \text{(QCQP)} \\ \Downarrow (4) \\ \text{(SDR)} \end{array} \qquad \boldsymbol{w} := \begin{bmatrix} w_0 \\ \vdots \\ w_\ell \end{bmatrix}$$

Tightness of (SDR)

Definition (Tightness):
Let $w \in \mathbb{R}^{4(l+1)}$ be a global minimizer of (QCQP).
We say (SDR) is tight if ww^T.

Main Results:

Positive Result
(SDR) is tight for small noise and random outliers.

Negative Result
(SDR) is not tight for “adversarial” outliers.

Remarks:

- Our theorems assume truncation parameters c_i^2 are chosen properly; see paper for details.
- “Adversarial” outliers can be point pairs that are defined by a rotation far from the ground-truth.

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References

[1] G. Wahba. A least squares estimate of satellite attitude. SIAM Review, 1965.

[2] P. J. Huber. Robust estimation of a location parameter. Ann. Math. Stat., 1964.

[3] H. Yang & L. Carlone. A quaternion-based certifiably optimal solution to the Wahba problem with outliers. ICCV 2019.

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