SI112: Advanced Geometry Spring 2018

Lecture 11 — Apr. 3rd, Tuesday

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1 Lecture 11-12

1.1 Overview of This Lecture

1.2 Proof of Things

Definition 1.2.1 (connected, definition 4.2.1)**.** A topological space X is said to be connected if the only two subsets of X that are simultaneously open and closed are X itself and the empty set $\emptyset$. A topological space which is not connected is said to be disconnected.

Example 1.2.2. Discrete topology is not connected, since every point is open and closed. $[0, 1] \cup [2, 3]$ on the real line is not connected.

Lemma 1.2.3 (lemma 4.2.3)**.** *Let A be a subspace of a topological space X. Then A is disconnected if and only if there exist two open subsets P and Q of X such that*

$$A \subset P \cup Q, P \cap Q \subset A^C, \text{ and } P \cap A \neq \emptyset, Q \cap A \neq \emptyset.$$

Proof. On the one hand, suppose that A is disconnected. Then there is a subset P' of A, different from $\emptyset$ and from A, such that P' is both relatively open and relatively closed in A. This means that P'^C is also different from $\emptyset$ and from A and relatively open. Let P, Q be such that $P' = P \cap A, P'^C = Q \cap A$, where P and Q are open subsets of X. We therefore have that $A = P' \cup P'^C \subset P \cup Q$, for $P' \subset P$ and $P'^C \subset Q$, and also $P \cap Q \cap A = (P \cap A) \cap (Q \cap A) = P' \cap P'^C = \emptyset$ so that $P \cap Q \subset A^C$. Finally, $P' = P \cap A$ and $P'^C = Q \cap A$ are non-empty.

On the other hand, given open sets P and Q satisfying the stated conditions, set $P' = P \cap A$ and $Q' = Q \cap A$. Then $A = A \cap (P \cup Q) = (A \cap P) \cup (A \cap Q) = P' \cup Q'$ and $P' \cap Q' = (A \cap P) \cap (A \cap Q) = \emptyset$. Thus $P' = Q'^C$, and P' is both relatively open and relatively closed in A. Since $P' \neq \emptyset$ and $P' \neq A$, A is disconnected. $\square$

Theorem 1.2.4 (theorem 4.2.5)**.** *Let X and Y be topological spaces, and le $f : X \to Y$ be continuous. If X is connected, then $f(X)$ is connected.*

Proof. Suppose $f(X)$ is disconnected. Use 1.2.3, and after some steps we can derive that X is not connected, a contradiction. Hence $f(X)$ is connected. $\square$

Theorem 1.2.5 (lemma 4.2.8)**.** *Let $Y = \{0, 1\}$ with discrete topology be a topological space. A topological space X is connected if and only if the only continuous mappings $f : X \to Y$ are the constant mappings.*

Proof. Let $f : X \to Y$ be a continuous non-constant mapping. Then $P = f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are both non-empty (why?). Thus $P \neq \emptyset$ and $P \neq X$ (why?). $\{0\}$ and $\{1\}$ are open subsets of Y (why?) and f is continuous, therefore P and Q are open subsets of X. But $P = Q^C$ (why?), so P is both open and closed and consequently X is disconnected. Thus, if X is connected, the only continuous mappings $f : X \to Y$ are constant mappings.

Conversely, suppose X is disconnected. Then there are non-empty open subsets P, Q of X such that $P \cap Q = \emptyset$ and $P \cup Q = X$. Define a mapping $f : X \to Y$ as follows: If $x \in P$, set $f(x) = 0$; if $x \in Q$, set $f(x) = 1$. f is continuous, for there are four open subsets, $\emptyset, \{0\}, \{1\}$, and Y of Y and $f^{-1}(\emptyset) = \emptyset, f^{-1}(\{0\}) = P, f^{-1}(\{1\}) = Q$, and $f^{-1}(Y) = X$, so that the inverse image of an open set is open. □

Theorem 1.2.6 (theorem 4.2.9)**.** *Let X and Y be connected topological spaces. Then $X \times Y$ is connected.*

Proof. It is enough to show that the only continuous mappings $f : X \times Y \to \{0, 1\}$ are constant mappings. Suppose, on the contrary, that there is a continuous mapping $f : X \times Y \to \{0, 1\}$ that is not constant. Then there are points $(x_0, y_0), (x_1, y_1) \in X \times Y$ such that $f(x_0, y_0) = 0, (x_1, y_1) = 1$. If □

Theorem 1.2.7. *The product of connected spaces is connected.*

Theorem 1.2.8 (theorem 4.3.4)**.** *A subset A of the real line that contains at least two distinct points is connected if and only if it is an interval.*

Theorem 1.2.9 (Intermediate Value Theorem, theorem 4.4.1)**.** *$f : [a, b] \to \mathbb{R}$ continuous. $a \neq b$. v is any number between $f(a)$ and $f(b)$, i.e., $f(a) < v < f(b)$. then there is $x \in [a, b]$ such that $f(x) = v$.*

Proof. $[a, b]$ is connected. It follows that $f([a, b])$ is connected and hence is an interval, which means $v \in f([a, b])$. □

Theorem 1.2.10 (theorem 4.5.1)**.** *The component of a is the largest connected set that contains a.*

Lemma 1.2.11 (lemma 4.5.2)**.** *In a topological space X, let $b \in Cmp(a)$. Then $Cmp(b) = Cmp(a)$.*

Theorem 1.2.12 (corollary 4.5.3)**.** *In a topological space X, define a b if $b \in Cmp(a)$. Then is an equivalence relation.*

Theorem 1.2.13 (path connectedness, 4.6.2)**.**

homotopy equivalent.

Remark 1.2.14*.* disconnected, jump

define topology for graph.

1.3 Further Reading

5.1-5.4 in Mendelson.