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Homomorphic Sensing: Sparsity and Noise

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Homomorphic Sensing

$$A \in \mathbb{R}^{m\times n}$$

Unlabeled Sensing

$$y = \Pi A x, \quad \Pi \in \mathcal{S}_m, \quad x \in \mathbb{R}^n \qquad (1)$$

$\mathcal{S}_m$: the set of $m \times m$ permutation matrices
Assume the existence of a solution x^* to (1):

$$y = \Pi^* A x^*, \text{ for some } \Pi^* \in \mathcal{S}_m$$

Question. When is x^* unique? [1, 2, 3]

Unlabeled Sensing with Missing Entries

$$y = SAx, \quad S \in \mathcal{S}_{r,m}, \quad x \in \mathbb{R}^n \qquad (2)$$

$\mathcal{S}_{r,m}$: the set of $r \times m$ selection matrice
Assume the existence of a solution x^* to (2):

$$y = S^* A x^*, \text{ for some } S^* \in \mathcal{S}_{r,m}$$

Question. When is x^* unique? [1, 4]

Real Phase Retrieval

$$y = BAx, \quad B \in \mathcal{B}_m, \quad x \in \mathbb{R}^n \qquad (3)$$

$\mathcal{B}_m$: the set of $m \times m$ sign matrices
Assume the existence of a solution x^* to (3):

$$y = B^* A x^*, \text{ for some } B^* \in \mathcal{B}_m$$

Question. When is x^* unique up to sign? [5]

Generalization: Homomorphic Sensing

$$y = TAx, \quad T \in \mathcal{T}, \quad x \in \mathbb{R}^n \qquad (4)$$

$\mathcal{T}$: a finite set of $r \times m$ arbitrary matrices
Assume the existence of a solution x^* to (4):

$$y = T^* A x^*, \text{ for some } T^* \in \mathcal{T}$$

Question. When is x^* unique? [6, 7, 8]

Sparse Homomorphic Sensing

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad y = TAx, \ \ T \in \mathcal{T} \qquad (5)$$

x^*: a k-sparse solution to (5).

Theory: Uniqueness Guarantee

Rank Constraint.

$$\operatorname{rank}(T) \le 2k, \quad \forall\, T \in \mathcal{T}$$

Quasi-variety Constraint.

$$\dim(\mathcal{Y}_{T_1,T_2} \backslash \mathcal{Z}_{T_1,T_2}) \le m - k, \quad \forall\, T_1, T_2 \in \mathcal{T},$$

where $\mathcal{Y}_{T_1,T_2}$ and $\mathcal{Z}_{T_1,T_2}$ are defined as

$$\mathcal{Y}_{T_1,T_2} = \{u \in \mathbb{C}^m : \operatorname{rank}[T_1 u, \ T_2 u] \le 1\}$$
$$\mathcal{Z}_{T_1,T_2} = \{u \in \mathbb{C}^m : T_1 u = T_2 u\}$$

Theorem 1. *Sparse homomorphic sensing (5) admits x^* as the unique solution for $A \in \mathbb{R}^{m\times n}$ generic, as long as the rank constraint and quasi-variety constraint are satisfied.*

Example: Unlabeled Compressed Sensing

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad y = \Pi A x, \ \ \Pi \in \mathcal{S}_m \qquad (6)$$

Corollary 1. *Unlabeled compressed sensing (6) admits x^* as the unique solution for $A \in \mathbb{R}^{m\times n}$ generic, as long as $m \ge 2k$.*

Example: Sparse Real Phase Retrieval

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad y = BAx, \ \ B \in \mathcal{B}_m \qquad (7)$$

Corollary 2 ([9, 10])**.** *Sparse real phase retrieval (7) admits $\pm x^*$ as the unique solution for $A \in \mathbb{R}^{m\times n}$ generic, as long as $m \ge 2k$.*

Noisy Homomorphic Sensing

$$(\hat{x}, \hat{T}) \in \operatorname*{arg\,min}_{x\in\mathbb{R}^n, T\in\mathcal{T}} \|\overline{y} - TAx\|_2 \qquad (8)$$

where $\overline{y} = y + \epsilon = T^* A x^* + \epsilon$

$$\mathcal{T}_1 = \{T \in \mathcal{T} : y \in R(TA)\}$$

where $R(TA)$ is the range space of TA

Theorem 2. *Suppose i) homomorphic sensing (4) admits a unique solution x^*, ii) $\mathcal{T}_1 = \mathcal{T}$ or*

$$\|\epsilon\|_2 < \|y\|_2 \Big(1 - \max_{T\in\mathcal{T}\backslash\mathcal{T}_1, x\in\mathbb{R}^n} \frac{y^\top TAx}{\|y\|_2 \|TAx\|_2}\Big),$$

then $\hat{x} - x^* = (\hat{T}A)^\dagger \epsilon$, and in particular

$$\|\hat{x} - x^*\|_2 \le \sigma((\hat{T}A)^\dagger)\|\epsilon\|_2.$$

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Algorithm & Experiments

$$\min_{x\in\mathbb{R}^n} \|y - Ax\|_1 \quad \text{s.t.} \quad \|x\|_0 \le k. \qquad (9)$$

Unlabeled Compressed Sensing Algorithm

Initialization. $x^{(0)} = 0$

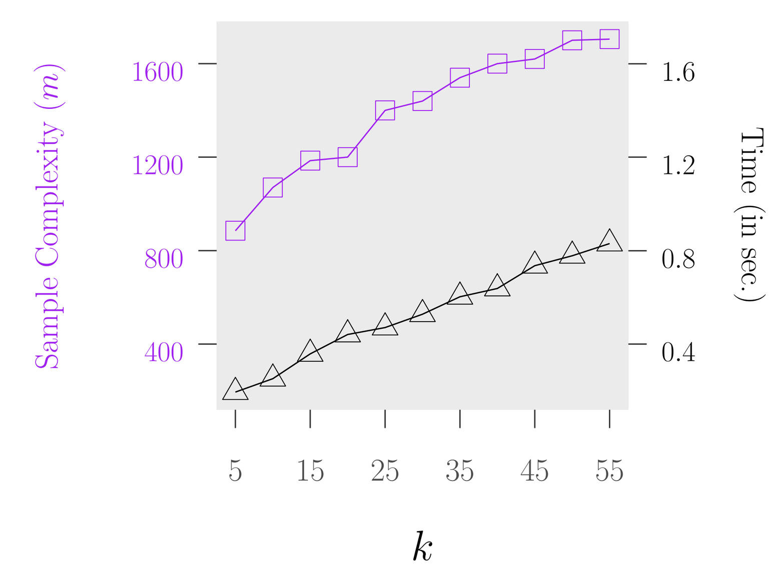
Iterative Step.

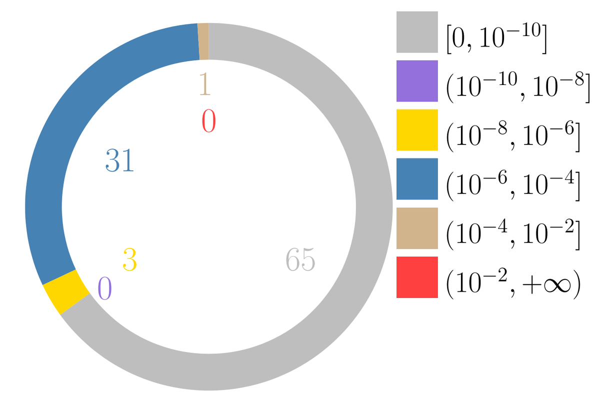
$$x^{(t+1)} \leftarrow \operatorname{Proj}_{\mathcal{K}}(x^{(t)} - \mu \cdot \operatorname{subg}(x^{(t)}))$$
$$J \leftarrow \text{the support } \{i : x_i^{(t+1)} \ne 0\} \text{ of } x^{(t+1)}$$
$$x_J^{(t+1)} \leftarrow \operatorname*{arg\,min}_{x\in\mathbb{R}^n} \|y - A_J x\|_1$$

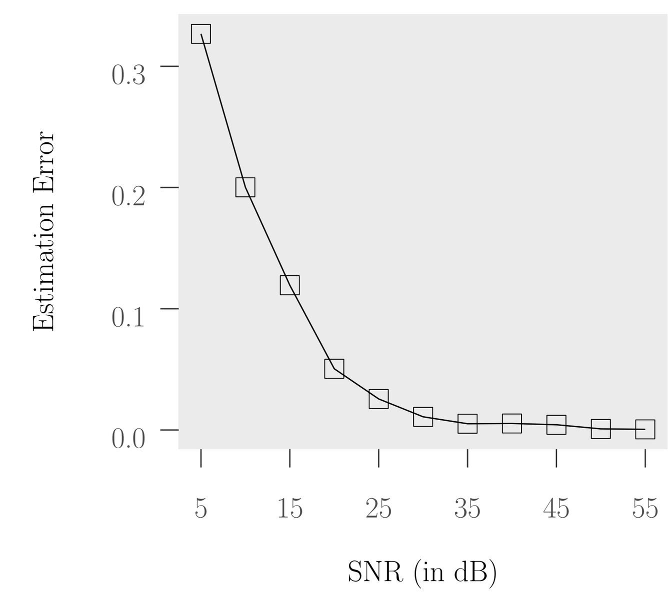
Notation.

- $\operatorname{subg}(x^{(t)})$: subgradient of $\|y - Ax\|_1$ at $x^{(t)}$

Experiments







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