From Unlabeled Compressed Sensing to Sparse Homomorphic Sensing

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Unlabeled Sensing

$$y = \Pi A x \qquad \Pi \in \mathcal{S}_m \qquad x \in \mathbb{R}^n \tag{1}$$

unknown

the set of $m \times m$ permutations

Assume the existence of a solution (Π^*, x^*) to (1), i.e.,

$$y = \Pi^* A x^*$$

Theorem 1. *For $A \in \mathbb{R}^{m \times n}$ generic, x^* is the unique solution to (1) as long as $m \geq 2n$.*[1]

1. J. Unnikrishnan, S. Haghighatshoar and M. Vetterli, "Unlabeled sensing: Solving a linear system with unordered measurements," Allerton 2015.

Theory

$$\min_{x \in \mathbb{R}^n, \Pi \in \mathcal{S}_m} \quad \|x\|_0 \quad \text{s.t.} \quad y = \Pi A x \qquad \Pi \in \mathcal{S}_m \tag{2}$$

Assume the existence of a solution (Π^*, x^*) to (2)

with x^* being k-*sparse*

Theorem 2. *For $A \in \mathbb{R}^{m \times n}$ generic, x^* is the unique solution to (2) as long as $m \geq 2k$.*

Optimization

$$\min_{x\in\mathbb{R}^n,\Pi\in\mathcal{S}_m} \quad \|x\|_0 \quad \text{s.t.} \quad y=\Pi A x \qquad \Pi\in\mathcal{S}_m$$

Assume Π^* is p-sparse in the sense that

$$\|y-Ax^*\|_0\le p$$

$\Rightarrow$

$$\min_{x\in\mathbb{R}^n} \quad \|y-Ax\|_0 \quad \text{s.t.} \quad \|x\|_0\le k$$

$\Rightarrow$

$$\min_{x\in\mathbb{R}^n} \quad \|y-Ax\|_1 \quad \text{s.t.} \quad \|x\|_0\le k$$

Optimization

$$\min_{x \in \mathbb{R}^n} \quad \|y - Ax\|_1 \quad \text{s.t.} \quad \|x\|_0 \leq k$$

$x^{(0)} := 0$

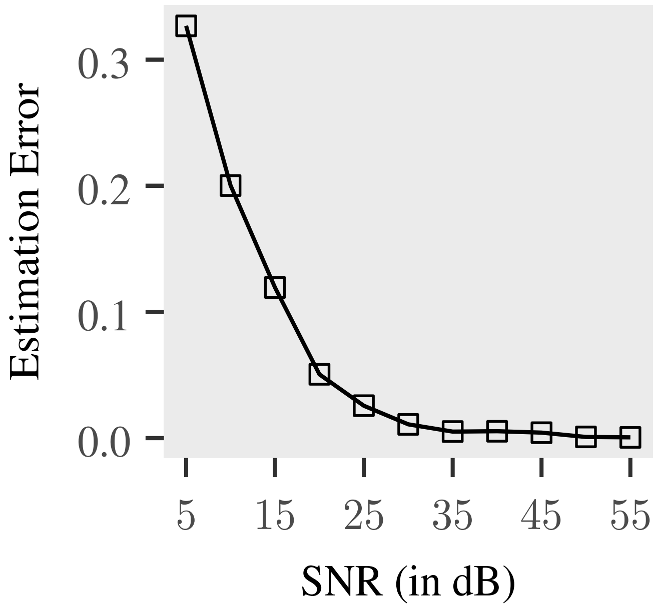
for $t = 0, 1, \ldots, T-1$:

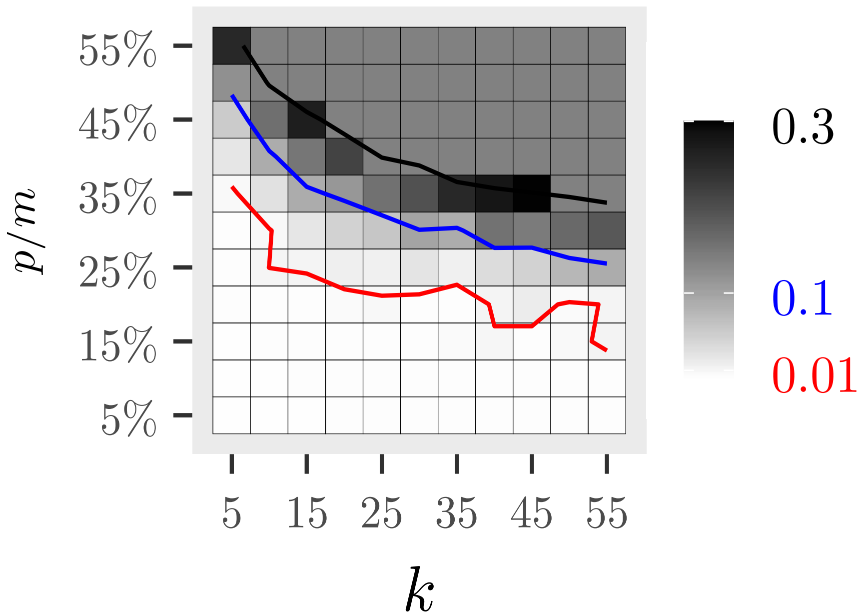
take a projected subgrad step at $x^{(t)}$ to produce a new $x^{(t+1)}$

minimize the objective over the support J of $x^{(t+1)}$, i.e.,

$$x_J^{(t+1)} \in \operatorname{argmin}_{x \in \mathbb{R}^n} \|y - A_J x\|_1$$

Synthetic Experiments





Theory

$$\min_{x \in \mathbb{R}^n, \Pi \in \mathcal{S}_m} \quad \|x\|_0 \quad \text{s.t.} \quad y = TAx \qquad T \in \mathcal{T} \tag{3}$$

a finite set of arbitrary $r \times m$ matrices

Assume the existence of a solution (T^*, x^*) to (3)

with x^* being k-*sparse*

Theorem 3. *For $A \in \mathbb{R}^{m \times n}$ generic, x^* is the unique solution to (3) as long as $\mathcal{T}$ satisfies the rank constraint and quasi-variety constraint.*

Sparse Homomorphic Sensing

Examples

In practice, $\mathcal{T}$ arises as

- the set of $m \times m$ permutation matrices $\mathcal{S}_m$
- the set of $r \times m$ selection matrices $\mathcal{S}_{r,m}$
- the set of $m \times m$ sign matrices $\mathcal{B}_m$
- their combinations $\mathcal{S}_{r,m}\mathcal{B}_m := \{SB : S \in \mathcal{S}_{r,m},\, B \in \mathcal{B}_m\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{S}_{2,3} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{B}_2 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \mathcal{S}_{2,3}\mathcal{B}_3$$

Corollaries

$$\min_{x\in\mathbb{R}^n,\Pi\in\mathcal{S}_m} \quad \|x\|_0 \quad \text{s.t.} \quad y = TAx \qquad T\in\mathcal{T} \tag{4}$$

The following holds true for $A\in\mathbb{R}^{m\times n}$ generic.

Corollary 4. x^* *is the unique solution to (4) if* $\mathcal{T} = \mathcal{S}_{r,m}$ *and* $r \geq 2k$.

Corollary 5. x^* *is the unique (up to sign) solution to (4) as long as* $\mathcal{T} = \mathcal{B}_m$ *with* $m \geq 2k$.

Corollary 6. x^* *is the unique (up to sign) solution to (4) as long as* $\mathcal{T} = \mathcal{S}_{r,m}\mathcal{B}_m$ *and* $r \geq 2k$.

Thank You!

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