SI112: Advanced Geometry Spring 2018

Lecture Note 2 — Feb. 27th, Tuesday

Prof.: Manolis *Scribe*: Liangzu

1 Lecture 1

"*The materials invented 15 years ago are becoming important today, and will be more important after 15 years.*:)"

1.1 Overview of this lecture

1.1.1 What the course is about

This class is a path to *Algebraic Geometry*, where we have to learn *Topology* and *Ring Theory* as a prerequisite. If time permits, we will also introduce Convex *Geometry*, which is a foundation for convex optimization.

The goal of this class is to provide you a formal mathematical training, comprising mathematical intuition, principled thinking, and mathematical tools.

1.1.2 Where this class is applied.

(there may be typos since I do not understand the terminology.)

Topology is used in Data Science (e.g., Pattern Analysis via "Persistent Homology"), Electron Decives ("topological insulator"), Network/Graph Topologies, Molecular Biology (e.g., DNA and protein folding, Krot Theory).

Algebraic Geometry is used in Machine Learning (e.g., Data Clustering, Matrix Completion), Computer Vision (e.g., Structure from Motion, Multi-view Geometry), Robotics (e.g., Control and Planing, the motion space is algebraic), Biology (e.g. Phylogenetics)

1.1.3 Evaluation for the course

There will be no exams for the course, and the homework is occasional. As an alternative, we will have weekly tests, including 2 qustions which you need to solve/prove in 30 minutes.

The course proceeds as follows. 1) You take the class in the week i, 2) there will be a TA session in week $i+1$, where or when we will do again what we did in the week i, 3) you got a new lecture and the quiz in the week $i+2$.

1.1.4 A starting point for Mathematics

To begin mathematics, we have to use some langueges (e.g., we use Chinese to talk). We introduce *set* as a language, or as a primitive notion, to describe mathematics. You know what I mean by *set*, hopefully.

1.2 Math

We are ready to define *function*, as you might already know, it is merely a mapping from one set to another. Formally,

Definition 1.2.1 (function)**.** A function $f : X \to Y$ is a subset $\mathcal{F}$ of $X \times Y$, such that, for each $x \in X$, there is only one element $y \in Y$ satisfying $(x, y) \in \mathcal{F}$.

Usually we say that X is the *domain* of the function f, Y the *target domain* of the function f.

Definition 1.2.2 (image of a function)**.** The image of a function $f : X \to Y$ is defined as follows:

$$im(f) = \{y \in Y | \text{ there is } x \in X : y = f(x)\}. \tag{1}$$

Definition 1.2.3 (inverse image)**.** Let $f : X \to Y$ be a function and T a subset of Y, then

$$f^{-1}(T) = \{x \in X | f(x) \in T\} \tag{2}$$

is called inverse image of T. If T is a singleton set, i.e., $T = \{y\}$ where $y \in Y$. we call $f^{-1}(T) = f^{-1}(\{y\})$ the *fiber* over y.

Definition 1.2.4 (left-invertible and right-invertible)**.** The professor draws pictures to illustrate these two concepts. review the pictures or read the textbook for a reference.

Definition 1.2.5 (invertible function)**.** A function f is invertible if f is both left- and right-invertible.

Question 1.2.6. How to show that a function f is left (right) invertible?

Definition 1.2.7 (injectivity and surjectivity)**.** A function $f : X \to Y$ is called *injective* if whenever $f(x) = f(x')$ for $x, x' \in X$, then $x = x'$. That is, for each $y \in f(X)$ there is only one $x \in X$ such that $f(x) = y$.

A function $f : X \to Y$ is called *surjective* if $Y = f(X)$.

Proposition 1.2.8. *A function $f : X \to Y$ is injective if and only if it is left-invertible.*

proof skeleton. Just follow the definitions of injectivity and left-invertibility.

To show that $f : X \to Y$ is left-invertible, you have to find a function $g : Y \to X$ such that $g(f(x)) = x$ (the definition of left-invertibility).

To show that $f : X \to Y$ is injective, you have to prove that given $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, it must be that $x_1 = x_2$ (the definition of injectivity). $\square$

Exercise 1.2.9. Prove that a function $f : X \to Y$ is surjective if and only if it is right-invertible.

Definition 1.2.10 (Equivalence Relations)**.** A relation R of X is a subset of $X \times X$.

$$(x, y) \in R \iff xRy. \tag{3}$$

An equivalence relation should be reflexive (xRx), symmetric ($xRy \Rightarrow yRx$) and transitive ($xRy, yRz \Rightarrow xRz$).

Question 1.2.11. Can an equivalence relation even be an empty set?

Definition 1.2.12 (equivalence class)**.** Let R be equivalence relation on X, and $x \in X$, then we call $[x] = \{x' \in X | xRx'\}$ is the equivalence class of x.

Proposition 1.2.13. *Let X be a set and $x, y \in X$, then*

$$[x] \cap [y] \neq \emptyset \Rightarrow [x] = [y]. \tag{4}$$

Corollary 1.2.14. *Let X be a set and R a equivalence relation on X, then X is the disjoint union of all equivalence classes.*

proof skeleton. use the proposition above.

$\square$

Example 1.2.15 (examples for understanding equivalence relations)**.** Let $\mathbb{R}$ be the set and $=$ the equivalence relation between real numbers. Then $[x] = \{x\}$.

The connected components in a graph can be viewed as a equivalence class.

Zorn's lemma is important but difficult to understand. Let's do it.

Let X be a set, and R be a partial order relation, in the sense that 1) xRx, 2) $xRy, yRx \Rightarrow x = y$, and 3) $xRy, yRz \Rightarrow xRz$.

Example 1.2.16 (examples for understanding partial order relation)**.** $\leq$ is a partial order relation on $\mathbb{R}$.

Axiom 1.2.17 (Axiom of Choice. v.1)**.** *Let* $(X_i)_{i\in I}$ *be a collection of non-empty sets. Then we can always choose one element from each set.*

Axiom 1.2.18 (Axiom of Choice. v.2)**.** *Let* $(X_i)_{i\in I}$ *be a collection of non-empty sets. Then there exists a choice function* $f : I \to \cup_{i\in I} X_i$*.*

Theorem 1.2.19 (Zorn's Lemma)**.** *Let* $(X, \leq)$ *be a partially ordered set. Suppose that every totally ordered subset* Y *of* X *has an upper bound (i.e.,* $\exists u \in X(u \geq y, \forall y)$*). Then* X *has a maximal element (i.e.* $\exists m \in X(x \geq m \Rightarrow x = m)$*).*

Zorn's Lemma and Axiom of Choice is equivalent, the lemma itself is difficult to prove. We will not prove it here. Refer to Paul Halmos's *Naive Set Theory* if you want to understand the whole story.

Zorn's Lemma can be used to show that

- Every vector space has a basis (in Matrix Analysis course, next semester).
- The product of compact spaces is compact (in this class).
- Every ideal of a ring is contained in a maximal ideal (in this class).

1.3 Further Reading

Mendelson, chapter 1.