Research Statement

Many tasks in machine learning, signal processing, and computer vision can be formulated as inverse problems. The first question to these formulations is the well-posedness, which includes three parts: 1) the existence of the solution, 2) the uniqueness of the solution, and 3) the stability of the solution. The question of existence is usually justified by our belief on the generative procedure of the data and the choice of the model; studying the stability of the solution under noise is of practical concern but it might be investigated only when the uniqueness is guaranteed. Motivated by the above, one main thread of my research is to consider the uniqueness of the solution for a given inverse problem.

If the well-posedness of an inverse problem is settled, I also consider algorithms, as the second thread of my research, to actually solve it. Next I will describe the two threads in detail.

Uniqueness characterization for inverse problems.

Let me further illustrate the role of the uniqueness by an "hello-world" example in machine learning, *linear regression*, where we have a system of linear equations $y = Ax$. Let it be recalled that x can be uniquely recovered from A and y whenever A is of full column rank. With this basic fact comes the following *mental decision boundary*. On the one side of the boundary, where A is of full column rank, we may choose to solve the equations for x via Gaussian elimination or else. On the opposite, in face of infinitely many solutions, a certain regularization is needed. Of the infinitely many, practitioners have decided to take the solution of minimum norm, in the Euclidean or ℓ_0 sense.[1] In this decision, interestingly, the question recurs: is the sparsest solution unique? If so under what conditions?

While the uniqueness for linear regression and compressed sensing has been well-studied in the past decades, there has been an increasing interest in several generalizations of linear regression, for which the uniqueness issue was considered only recently. Such interest was found in modern inverse problems like *linear regression without correspondences*, *unlabeled sensing*, *real phase retrieval*, and *mixed linear regression*, to name a few. Although those problems were motivated by different applications, in different contexts, with the associated uniqueness investigated via different approaches, we (my collaborators + me) found that, from an abstract point of view, the heart of those problems is to solve the linear equations $y = Ax$ for x, where y is still given but A is only known up to a given finite set of matrices. In this general case, then, what would guarantee a unique solution? The answer was given by our recent work, *homomorphic sensing (of subspace arrangements)*, where we provided algebraic-geometric conditions that guarantee the desired uniqueness. Moreover, our conditions can be specialized so that the uniqueness is guaranteed not only for all the problems mentioned above, but also for their sparse variants (where x is sparse). Those results obtained in a unified way are either discovered via different techniques in prior work, or novel themselves to the best of our knowledge.

Our work on homomorphic sensing also opens several possibilities in future research. For example:

- *pure math*) A certain determinantal variety appeared in homomorphic sensing, and it has a tight relation with the Kronecker canonical form (i.e., a generalization of the Jordan canonical form). What is their relation, exactly? The answer would simplify the homomorphic sensing theory.

- *statistics*) We estabilished a theory of local stability for homomorphic sensing under deterministic noise. What if the noise is random? To answer this, we need a probabilistic bound on a specific quadratic form, while the standard bound (e.g., via the Hanson-Wright inequality) is not tight.

[1] It is perhaps the decision of considering the sparsest solution that has cultivated the research field, *compressed sensing*.

- *applied algebraic geometry*) We believe that the algebraic-geometric tools that we used are also essential to other inverse problems where the uniqueness has (not) come into play. We also believe that there could be more instances of homomorphic sensing than discovered.
- *algorithms*) No algorithm exists for sparse variants of linear regression without correspondences and unlabeled sensing, even though ideas from compressed sensing might be borrowed.

Algorithms.

The ultimate goal should be actually obtaining a solution to the inverse problem of interest — correctly and economically. However, a lot of modern inverse problems are found to be *NP-hard*, a property which hardly allows for any algorithm to be effiicent and accurate at the same time, especially when the (intrinsic) data dimension[2] is large. Something must be compromised upon.

Compromising on efficiency: exact algorithms.

Several techniques such as solving polynomial equations or branch-and-bound might be employed to solve computational problems global optimally. Even though they typically have exponential time complexity in terms of data dimension in the worst case, these methods can be quite efficient if the data dimension is small. For example, In our work (Trans. Inf. Theory '20) we used the *action matrix* approach for solving the polynomial system that models the problem of linear regression without correspondences. If the data dimension is ≤ 4, our method is able to handle thousands of samples in milliseconds. Later improvement was by our work (Signal Process. Lett. '20), where we proposed a concave minimization algorithm based on branch-and-bound which can deal with the case where the data dimension is ≤ 8. These two algorithms are state-of-the-art, to our knowledge. Finally, in (ICML '19), we provided two algorithms for the problem of unlabeled sensing. One of them is based on branch-and-bound + dynamic programming and the other algorithm is of RANSAC type. By now, we have not known other working solutions to the unlabeled sensing problem.

Compromising on accuracy: efficient algorithms.

It is also common to trade accuracy for efficiency. The first is to consider the semi-definite relaxation of a given NP-hard problem. While semi-definitie programs (SDPs) are solvable via convex optimization in polynomial time in general, they are still expensive in both time and space. In (Scalable Semidefinite Programming, arXiv '19) Yurtsever et. al. showed that SDPs can be solved at large scale with moderate accuracy. Based on it, I would like to design algorithms for linear programs or the linear assignment problem, or apply it to the instances of homomorphic sensing as mentioned above.

Alternatively, many computational problems might also be tackled via non-convex optimization. Common methods include (sub)-gradient descent, alternating minimization, RANSAC, or reweighted least-squares. While these methodologies are widely applicable, the challange is to provide correctness proofs for the algorithms, in order to show that they can actually solve the problem at hand. The approaches to the proofs are usually statistical, making compromises on data.

Finally, I also consider *point set registration*, an important problem in computer vision.

[2] A popular assumption on data is that they come from a low-dimensional space (e.g., low-rank or sparse assumption). In other words, the intrinsic data dimension is assumed to be low.