SI112: Advanced Geometry		Spring 2018
	Lecture Note 9 — Mar. 27th, Tuesday	
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1 Lecture 9

1.1 Overview of This Lecture

In this lecture we introduce an important concept: *compactness*. After introducing its definition (1.2.4), we devolops theorems, as usual, that relate compactness and other important topological concepts, e.g., compactness in relative topology (1.2.8), neighborhood (1.2.10), continuity (1.2.12) and closeness (1.2.15,1.2.17).

We will spend 2 lectures on compactness (this and the next lecture).

1.2 Proof of Things

Definition 1.2.1 (covering, definition 5.2.1)**.** Let X be a set, B a subset of X, and $\{A_i\}_{i\in I}$ is called a *covering* of B or is said to *cover* B if $B \subset \cup_{i\in I} A_i$. If, in addition, the indexing set I is finite, $\{A_i\}_{i\in I}$ a *finite covering* of B.

Definition 1.2.2 (subcovering, definition 5.2.2)**.** Let X be a set and let $\{A_i\}_{i\in I}, \{B_k\}_{k\in J}$ be two coverings of a subset C of X. If for each $i \in I, A_i = B_k$ for some $k \in J$, then the covering $\{A_i\}_{i\in I}$ is called a *subcovering* if the covering $\{B_k\}_{k\in J}$. Note that this definition is not introduced in the lecture.

Exercise 1.2.3 (open covering, definition 5.2.3)**.** An *open covering* of a set B is a union of open set which covers B. Try to give it a rigorous definition. Or have a look at definition 5.2.3 in Mendelson.

Definition 1.2.4 (definition 5.2.4)**.** A topological space X is said to be *compact* if for each open covering $\{U_i\}_{i\in I}$ of X there is a finite subcovering $U_{i_1}, \ldots, U_{i_n}$.

Remark 1.2.5 (remark for definition 1.2.4)**.** compactness allows to study global properties by looking at a finite number of neighborhood.

Remark 1.2.6 (remark for definition 1.2.4)**.** Given the definition of compactness, how to prove a given set, say X, is compact or not? To prove X is compact, you need to show that **for each** *open* covering of X, there is a *finite* subcovering. To prove that X is not compact, in contrast, you need to give an counterexample, i.e., there exists a open covering of X such that there are no subcovering. Try to prove that the open interval $(0, 1)$ is not compact.

Definition 1.2.7 (definition 5.2.5)**.** A subset C of a topological space X is said to be *compact*, if C is a compact topological space in the relative topology.

Exercise 1.2.8 (theorem 5.2.6)**.** Prove it: A subset C a topological space X is compact if and only if for each open covering $\{U_i\}_{i \in I}$, U_i open in X, there is a finite subcovering $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ of C.

Remark 1.2.9 (remark for exercise 1.2.8)**.** This exercise is theorem 5.2.6 in Mendelson. We skipped it in this lecture. You can prove it by yourself. Use the definition of relative topology and compactness.

Theorem 1.2.10 (theorem 5.2.7)**.** *A topological space X is compact if and only if, whenever for each $x \in X$ a neighborhood N_x of x is given, there is a finite number of points $x_1, x_2, \ldots, x_n$ of X such that $X = \cup_{i=1}^n N_{x_i}$.*

Proof. On the one hand, suppose X is compact. For each $x \in X$ there is a neighborhood N_x of x (why?). Hence for each x, there is an open set U_x such that $x \in U_x \subset N_x$ and $\{U_x\}_{x \in X}$ is an open covering of X. Since X is compact there is a finte subcovering $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$, i.e., $X = \cup_{i=1}^n U_{x_i}$. But $U_{x_i} \subset N_{x_i}$ for each i, hence $X = \cup_{i=1}^n N_{x_i}$.

On the other hand, suppose whenever for each $x \in X$ a neighborhood N_x of x is given, there is a finite number of points $x_1, x_2, \ldots, x_n$ of X such that $X = \cup_{i=1}^n N_{x_i}$. We want to show that X is compact. The below is a **wrong** proof.

For each $x \in X$ there is an open set O_x in X containing x (why?), which is a neighborhood N_x of x, then we have $X = \cup_{x \in X} O_x = \cup_{x \in X} N_x$. By our hypothesis, there are points $x_1, x_2, \ldots, x_n$ of X such that $X = \cup_{i=1}^n N_{x_i} = \cup_{i=1}^n O_{x_i}$. Hence X is compact.

Why is this proof wrong? The problem here is that we have to start with an **arbitrary** open covering $\{U_i\}_{i \in I}$ of X, then we need to show that there is a finite subcovering. Since $\{U_i\}_{i \in I}$ covers X, for each $x \in X$ we have $x \in U_i$ for some $i \in I$. Notice here that different x can be in the same U_i, i.e., it is possible that $x_1, x_2 \in X$ and $x_1 \in U_i, x_2 \in U_i$ for some $i \in I$. To rephrase, for each $x \in X$, there is an $i = i(x)$ such that $x \in U_i$, which is a neighborhood of x. Let $N_x = U_i$, then by our hypothesis, there are points $x_1, x_2, \ldots, x_n$ of X such that $N_{x_i} = U_{i(x_i)}, i = 1, 2, \ldots, n$ covers X, and hence X is compact. □

Theorem 1.2.11 (theorem 5.2.8)**.** *A topological space is compact if and only if whenever a family $\cap_{i \in I} A_i = \emptyset$ of closed sets is such that $\{A_i\}_{i \in I}$ then there is a finite subset of indices $\{i_1, i_2, \ldots, i_n\}$ such that $\cap_{k=1}^n A_{i_k} = \emptyset$.*

proof skeleton. Use the definition of compactness and “the complement of a closed set is open”. □

Theorem 1.2.12 (theorem 5.2.9)**.** *Let $f : X \to Y$ be continuous and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof. This theorem shows that continuous functions perserve compactness.

To show that $F(A)$ is a compact subset of Y, let's start with an arbitrary open covering $\{V_i\}_{i\in I}$ of $f(A)$, i.e., $f(A) \subset \cup_{i\in I} V_i$. Then we have $A \subset f^{-1}(f(A)) \subset \cup_{i\in I} f^{-1}(V)$, which means that $\{f^{-1}(V_i)\}_{i\in I}$ is a covering of A. In addition, since f is continuous and V_i is open for each $i \in I$, $\{f^{-1}(V_i)\}_{i\in I}$ is an open covering of A. Since A is compact, there is a finite subcovering $f^{-1}(V_{i_1}), f^{-1}(V_{i_n}), \ldots, f^{-1}(V_{i_n})$ of A, i.e., $A \subset \cup_{k=1}^{n} f^{-1}(V_{i_k})$.

Remember that we want to show that there is a finite covering of $f(A)$. By theorem 1.2.8, it is enough to show that $f(A) \subset \cup_{k=1}^{n} V_{i_k}$. Does $A \subset \cup_{k=1}^{n} f^{-1}(V_{i_k})$ imply $f(A) \subset \cup_{k=1}^{n} V_{i_k}$? prove it!

□

Corollary 1.2.13 (corollary 5.2.10)**.** *Let the topological spaces X and Y be homeomorphic, then X is compact if and only if Y is compact.*

Example 1.2.14. The open interval $(0, 1)$ is not compact. To show this, we need to construct a covering of $(0, 1)$ that does not have a finite subcovering. (*hint:* construct something like $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$.)

Theorem 1.2.15 (theorem 5.2.11)**.** *Let X be compact and A closed in X. Then A is compact.*

Proof. Let $\{V_i\}_{i\in I}$ be an open covering of the closed set A, i.e., $A \subset \cup_{i\in I} V_i$. Then

$$X = A \cup A^C = \cup_{i\in I} V_i \cup A^C .$$

Since X is compact, there is a subcovering $U_{i_1}, \ldots, U_{i_n}$, i.e., $X = \cup_{k=1}^{n} U_{i_k}$, where for each k, $U_{i_k} = V_i$ for some $i \in I$, or $U_{i_k} = A^C$. Is $U_{i_1}, \ldots, U_{i_k}$ a finite subcovering of A? Why? How can we finish the proof? □

Lemma 1.2.16 (lemma for theorem 1.2.17)**.** *In a topological space X, the intersection of a finite set of neighborhoods of a point x is a neighborhood of x.*

proof skeleton. Immediate. Apply the definition of neighborhood. □

Theorem 1.2.17 (theorem 5.2.12)**.** *Let X be a Hausdorff Space. If a subset F of X is compact, then F is closed.*

Proof. This is a theorem that requires us to prove again that some set F is closed. Review how we prove a set is closed in lecture 4.

So, to prove F is closed, it is enough to show that there are no limit points of F in F^C (why?). Hence we need to prove the following:

$$\forall z \in F^C, \text{ there is a neighborhood } N \text{ of } z \text{ such that } N \cap F = \emptyset. \tag{1}$$

Now let's consider the compact set F. For each $x \in F$, let V_x be an open set containing x, then we have $F = \cup_{x\in F} V_x$, then there is a subcovering $V_{x_1}, \ldots, V_{x_n}$, i.e., $F = \cup_{i=1}^n V_{x_i}$. Hence we need to prove

$$\begin{aligned} & \forall z \in F^C, \text{ there is a neighborhood } N \text{ of } z \text{ such that } N \cap (\cup_{i=1}^n V_{x_i}) = \emptyset \\ \iff & \forall z \in F^C, \text{ there is a neighborhood } N \text{ of } z \text{ such that } \cup_{i=1}^n (N \cap V_{x_i}) = \emptyset \\ \iff & \forall z \in F^C, \text{ there is a neighborhood } N \text{ of } z \text{ such that } N \cap V_{x_i} = \emptyset, \forall i = 1, \ldots, n. \end{aligned} \tag{2}$$

Let $z \in F^C$ be given. For each $i = 1, \ldots, n$, there exists a neighborhood N_{x_i} of z such that $V_{x_i} \cap N_{x_i} = \emptyset$ (why?). Let $N = \cap_{i=1}^n N_{x_i}$, then N is a neighborhood of z (by lemma 1.2.16), and $N \cap V_{x_i} = \emptyset, \forall i = 1, \ldots, n$. This is what equation 2 desires.

□

Definition 1.2.18 (bounded, real line version, definition 5.3.1)**.** A subset A of $\mathbb{R}$ is said to be *bounded* if there is a real number K such that for each $x \in A, |x| \leq K$.

Lemma 1.2.19 (lemma 5.3.2)**.** *If A is a compact subset of $\mathbb{R}$ then A is closed and bounded.*

proof skeleton. It is easy to prove by theorem 1.2.17 that A is closed. To prove A is bounded, you need to construct an open covering of A, which will be reduced to a finite subcovering. Notice that the finite subcovering is basically a collection of open intervals. Now show that A is bounded. □

Lemma 1.2.20 (lemma 5.3.3)**.** *The closed interval $[0, 1]$ is compact.*

Proof. (will be proved on Lecture 10) □

Corollary 1.2.21 (corollary 5.3.4)**.** *Each closed interval $[a, b]$ is compact.*

Theorem 1.2.22 (theorem 3.5)**.** *A subset A of the real line is compact if and only if A is closed and bounded.*

Theorem 1.2.23. *The product of compact spaces is compact.*

Corollary 1.2.24. *$[0, 1]^n$ is compact.*

1.3 Further Reading

5.1-5.4 in Mendelson.