Semidefinite Relaxations of Truncated Least-Squares in Robust Rotation Search: Tight or Not

JOHNS HOPKINS MATHEMATICAL INSTITUTE for DATA SCIENCE

Liangzu Peng, Mahyar Fazlyab, René Vidal

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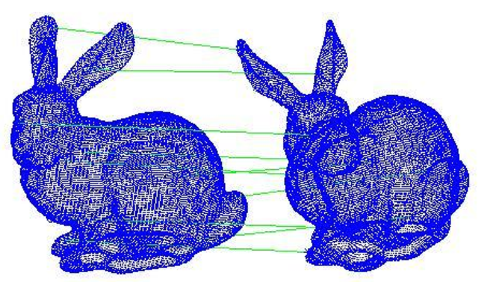
Rotation Search: Problem Setup

Goal:

- Find 3D rotation R_0^* that best aligns 2 point sets

Case without Outliers (Wahba, 1965 [1]):

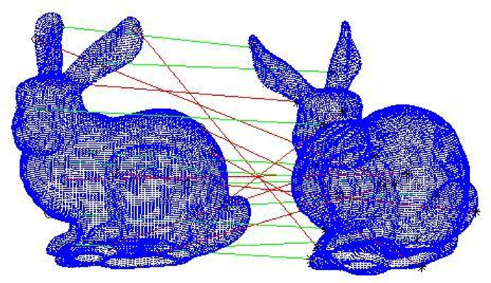
- $y_i \approx R_0^* x_i, \quad i = 1, \dots, l$
- $R_0^* \in SO(3)$
- Optimization problem:



$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \|y_i - R_0 x_i\|_2^2$$

Case with Outliers:

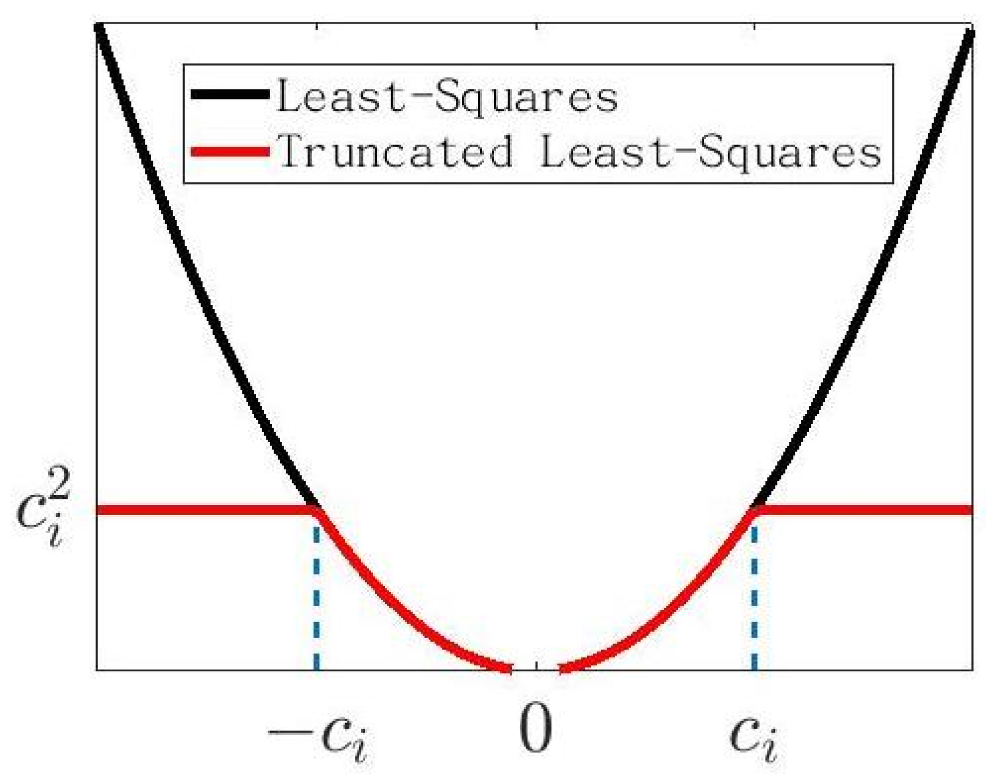
- Inliers: $y_i \approx R_0^* x_i$
- Outliers: (x_i, y_i) arbitrary
- Optimization problem:



Truncated Least-Squares (TLS-R) [1]:

$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \min\left\{ \|y_i - R_0 x_i\|_2^2,\ c_i^2 \right\}$$

(Huber, 1964 [2]) Truncation Parameter



Key challenge:

- Under what conditions can we solve (TLS-R) to global optimality in polynomial time?

From (TLS-R) to Semidefinite Relaxations (SDR)

$$\text{(TLS-R)} \overset{(1)}{\Leftrightarrow} \text{(TLS-Q)} \overset{(2),\ (3)}{\Longleftrightarrow} \text{(QCQP)} \overset{(4)\ \text{lifting}}{\Longrightarrow} \text{(SDR)}$$

Step (1): 3D rotations $R_0 \Leftrightarrow$ unit quaternions w_0

$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \min\left\{ \|y_i - R_0 x_i\|_2^2,\ c_i^2 \right\}$$

$$\min_{w_0 \in \mathbb{S}^3} \sum_{i=1}^{\ell} \min\left\{ w_0^\top Q_i w_0,\ c_i^2 \right\}$$

Step (2): $\min\{a,b\} = \min_{\theta_i \in \{0,1\}} \theta_i a + (1-\theta_i) b$

$$\min_{\substack{w_0 \in \mathbb{S}^3 \\ \theta_i \in \{0,1\}}} \sum_{i=1}^{\ell} \left(\theta_i w_0^\top Q_i w_0 + (1-\theta_i)\ c_i^2 \right)$$

Step (3): $w_i := \theta_i w_0$, so $\theta_i = w_i^\top w_0$ [3]

$$\min_{\substack{w_0 \in \mathbb{S}^3 \\ w_i \in \{0, w_0\}}} \sum_{i=1}^{\ell} \left(w_i^\top Q_i w_0 + (1 - w_i^\top w_0)\ c_i^2 \right)$$

$$\min_{\boldsymbol{w} \in \mathbb{R}^{4(\ell+1)}} \ \operatorname{trace}\left(\boldsymbol{Q}\boldsymbol{w}\boldsymbol{w}^\top\right)$$
$$\text{s.t.}\ [\boldsymbol{w}\boldsymbol{w}^\top]_{0i} = [\boldsymbol{w}\boldsymbol{w}^\top]_{ii}, \quad \forall\, i \in \{1, \dots, \ell\}$$
$$\operatorname{trace}([\boldsymbol{w}\boldsymbol{w}^\top]_{00}) = 1$$

Step (4): From (QCQP) to (SDR)

$$\min_{\boldsymbol{W} \succeq 0} \ \operatorname{trace}\left(\boldsymbol{Q}\boldsymbol{W}\right)$$
$$\text{s.t.}\ [\boldsymbol{W}]_{0i} = [\boldsymbol{W}]_{ii}, \quad \forall\, i \in \{1, \dots, \ell\}$$
$$\operatorname{trace}([\boldsymbol{W}]_{00}) = 1$$

(TLS-R) $\overset{(1)}{\Updownarrow}$ (TLS-Q) $\overset{(2)}{\Updownarrow}$ (TLS-θ_i) $\overset{(3)}{\Updownarrow}$ (TLS-w_i) $\Updownarrow$ (QCQP) $\overset{(4)}{\Downarrow}$ (SDR)

$$\boldsymbol{w} := \begin{bmatrix} w_0 \\ \vdots \\ w_\ell \end{bmatrix}$$

Tightness of (SDR)

Definition (Tightness):
Let $w \in \mathbb{R}^{4(l+1)}$ be a global minimizer of (QCQP).
We say (SDR) is tight if its solution is $W^* = ww^T$.

Main Results:

Positive Result

- *Inlier Assumption:* We have $c_i^2 > f(noise)$ for every inlier point pair (y_i, x_i). Here f is a complicated function of noise with $f(0) = 0$.
- *Outlier Assumption:* Each outlier pair (y_j, x_j) is drawn independently and randomly from $\mathbb{R}^3 \times \mathbb{R}^3$ according to some continuous probability distribution, and c_j^2 is set small enough.
- Under the *Inlier* and *Outlier Assumption*, (SDR) is tight.

Negative Result

(SDR) is not tight for "adversarial" outliers.

Remarks:

- For the exact definitions of noise function f and outlier truncation parameter c_j^2, see our paper.
- "Adversarial" outliers can be point pairs that are defined by a rotation far from the ground-truth.

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References

[1] G. Wahba. A least squares estimate of satellite attitude. SIAM Review, 1965.

[2] P. J. Huber. Robust estimation of a location parameter. Ann. Math. Stat., 1964.

[3] H. Yang & L. Carlone. A quaternion-based certifiably optimal solution to the Wahba problem with outliers. ICCV 2019.