Semidefinite Relaxations of Truncated Least-Squares in Robust Rotation Search: Tight or Not

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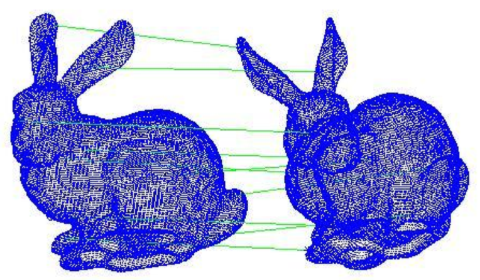
Rotation Search: Problem Setup

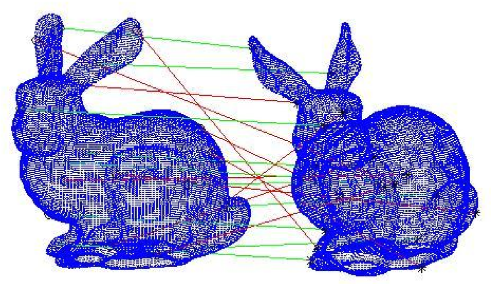
Goal:
- Find 3D rotation R_0^* that best aligns 2 point sets

Outlier-Free Case (Wahba, 1965 [1]):
- $y_i \approx R_0^* x_i, \quad i = 1, \dots, l$
- $R_0^* \in SO(3)$
- Optimization:



$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \|y_i - R_0 x_i\|_2^2$$

Case with Outliers:
- Inliers: $y_i \approx R_0^* x_i$
- Outliers: (x_i, y_i) arbitrary

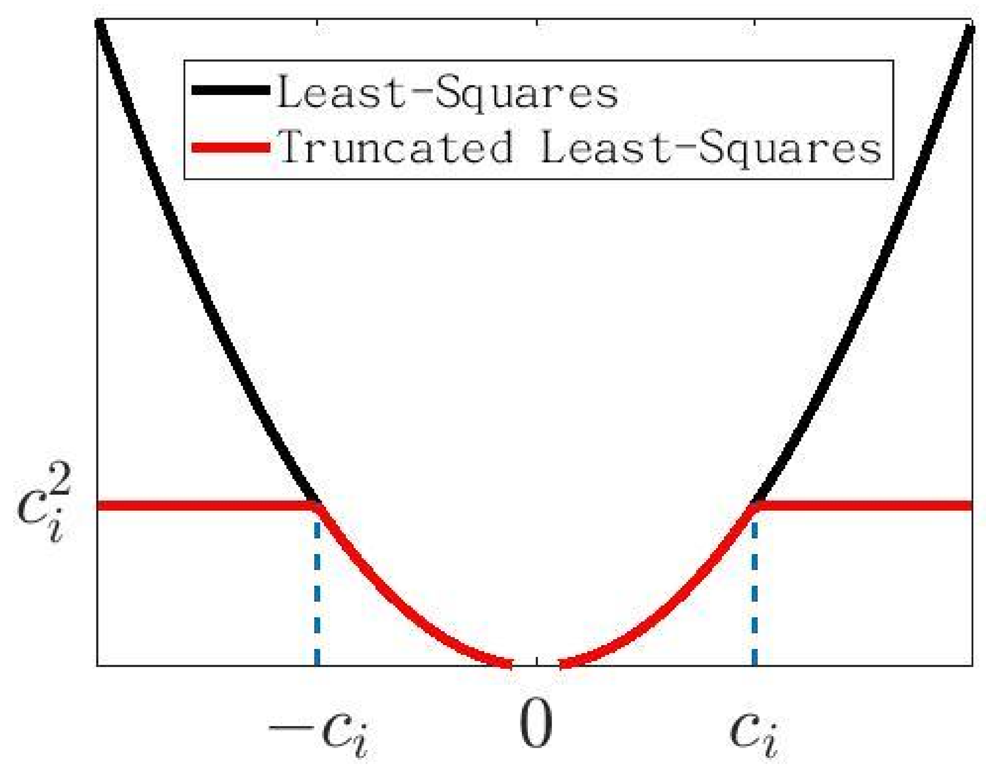


Optimization:

Truncated Least-Squares (TLS-R) [1]:

$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \min\left\{\|y_i - R_0 x_i\|_2^2,\ c_i^2\right\}$$

(Huber, 1964 [2]) Truncation Parameter



From (TLS-R) to Semidefinite Relaxations (SDR)

$$\text{(TLS-R)} \overset{(1)}{\Leftrightarrow} \text{(TLS-Q)} \overset{(2),\ (3)}{\Longleftrightarrow} \text{(QCQP)} \xRightarrow{(4)\text{ lifting}} \text{(SDR)}$$

Step (1): 3D rotations $R_0 \Leftrightarrow$ unit quaternions w_0

Step (2): $\min\{a, b\} = \min_{\theta \in \{0,1\}} \theta a + (1-\theta) b$

Step (3): $w_i := \theta_i w_0$, so $\theta_i = w_i^\top w_0$ [3]

Step (1): From (TLS-R) to (TLS-Q)

$$\min_{R_0 \in \mathrm{SO}(3)} \sum_{i=1}^{\ell} \min\left\{\|y_i - R_0 x_i\|_2^2,\ c_i^2\right\}$$

$$\min_{w_0 \in \mathbb{S}^3} \sum_{i=1}^{\ell} \min\left\{w_0^\top Q_i w_0,\ c_i^2\right\}$$

Step (2), (3): From (TLS-Q) to (QCQP)

$$\min_{\substack{w_0 \in \mathbb{S}^3 \\ \theta_i \in \{0,1\}}} \sum_{i=1}^{\ell} \left(\theta_i w_0^\top Q_i w_0 + (1-\theta_i)\ c_i^2\right)$$

$$\min_{\substack{w_0 \in \mathbb{S}^3 \\ w_i \in \{0, w_0\}}} \sum_{i=1}^{\ell} \left(w_i^\top Q_i w_0 + (1 - w_i^\top w_0)\ c_i^2\right)$$

$$\begin{aligned} \min_{\boldsymbol{w} \in \mathbb{R}^{4(\ell+1)}} \quad & \operatorname{trace}\left(\boldsymbol{Q}\boldsymbol{w}\boldsymbol{w}^\top\right) \\ \text{s.t.} \quad & [\boldsymbol{w}\boldsymbol{w}^\top]_{0i} = [\boldsymbol{w}\boldsymbol{w}^\top]_{ii}, \quad \forall\, i \in \{1, \dots, \ell\} \\ & \operatorname{trace}\left([\boldsymbol{w}\boldsymbol{w}^\top]_{00}\right) = 1 \end{aligned}$$

Step (4): From (QCQP) to (SDR)

$$\begin{aligned} \min_{\boldsymbol{W} \succeq 0} \quad & \operatorname{trace}\left(\boldsymbol{Q}\boldsymbol{W}\right) \\ \text{s.t.} \quad & [\boldsymbol{W}]_{0i} = [\boldsymbol{W}]_{ii}, \quad \forall\, i \in \{1, \dots, \ell\} \\ & \operatorname{trace}\left([\boldsymbol{W}]_{00}\right) = 1 \end{aligned}$$

$$\text{(TLS-R)} \overset{(1)}{\Updownarrow} \text{(TLS-Q)} \overset{(2)}{\Updownarrow} (\text{TLS-}\boldsymbol{\theta_i}) \overset{(3)}{\Updownarrow} (\text{TLS-}\boldsymbol{w_i}) \Updownarrow \text{(QCQP)} \overset{(4)}{\Downarrow} \text{(SDR)}$$

$$\boldsymbol{w} := \begin{bmatrix} w_0 \\ \vdots \\ w_\ell \end{bmatrix}$$

Tightness of (SDR)

Definition (Tightness):
Let $w \in \mathbb{R}^{4(l+1)}$ be a global minimizer of (QCQP).
We say (SDR) is tight if ww^T globally minimizes it.

Main Results:

Theorem 1 (without noise, without outliers): (SDR) is tight.

Theorem 2 (without noise, with outliers): (SDR) is tight for random, but not for adversarial, outliers.

Theorem 3 (with noise, without outliers): (SDR) is tight for small noise.

Theorem 4 (with noise, with outliers): (SDR) is tight for small noise and random outliers.

Remarks:
- Our proof of Theorem 1 is simpler than [3].
- Theorems 2-4 assume truncation parameters c_i^2 are chosen properly; see paper for details.

Acknowledgement: work supported by grants NSF 1704458, NSF 1934979, ONR MURI 503405-78051.

References

[1] G. Wahba. A least squares estimate of satellite attitude. SIAM Review, 1965.

[2] P. J. Huber. Robust estimation of a location parameter. Ann. Math. Stat., 1964.

[3] H. Yang & L. Carlone. A quaternion-based certifiably optimal solution to the Wahba problem with outliers. ICCV 2019.

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