# Homomorphic Sensing of Subspace Arrangements

Liangzu Peng[a], Manolis C. Tsakiris[a,*]

[a] *SIST, ShanghaiTech University, No. 393 Huaxia Middle Road, Pudong Area, Shanghai, China*

**Abstract**

Homomorphic sensing is a recent algebraic-geometric framework that studies the unique recovery of points in a linear subspace from their images under a given collection of linear maps. It has been successful in interpreting such a recovery in the case of permutations composed by coordinate projections, an important instance in applications known as unlabeled sensing, which models data that are out of order and have missing values. In this paper, we provide tighter and simpler conditions that guarantee the unique recovery for the single-subspace case, extend the result to the case of a subspace arrangement, and show that the unique recovery in a single subspace is locally stable under noise. We specialize our results to several examples of homomorphic sensing such as real phase retrieval and unlabeled sensing. In so doing, in a unified way, we obtain conditions that guarantee the unique recovery for those examples, typically known via diverse techniques in the literature, as well as novel conditions for sparse and unsigned versions of unlabeled sensing. Similarly, our noise result also implies that the unique recovery in unlabeled sensing is locally stable.

*Keywords:* homomorphic sensing, unlabeled sensing, linear regression without correspondences, real phase retrieval, mixed linear regression, algebraic geometry.

## 1. Introduction

### 1.1. Homomorphic sensing

The homomorphic sensing problem was posed by [1, 2] as follows. With $\mathbb{H}$ being $\mathbb{R}$ or $\mathbb{C}$ let $\mathcal{V} \subset \mathbb{H}^m$ be a linear subspace and $\mathcal{T}$ a finite set of linear maps $\mathbb{H}^m \to \mathbb{H}^m$. With $v^* \in \mathcal{V}$ and $\tau^* \in \mathcal{T}$ we observe $y := \tau^*(v^*)$. Given $\mathcal{V}, \mathcal{T}$, and $y$, then, can we *uniquely* determine $v^*$ without knowing $\tau^*$? In other words, with $y$ fixed we want to know when the relations

$$y = \tau(v), \qquad \tau \in \mathcal{T}, \qquad v \in \mathcal{V} \tag{1}$$

necessarily imply that $v = v^*$. This motivates the following definition.

**Definition 1** (hsp)**.** *Let $\mathscr{V}$ be a set of vectors and $\mathscr{T}$ a finite set of linear maps. We will say that $\mathscr{V}$ and $\mathscr{T}$ satisfy the "homomorphic sensing property", denoted by* $\operatorname{hsp}(\mathscr{V}, \mathscr{T})$*, whenever the following holds:*

$$\operatorname{hsp}(\mathscr{V}, \mathscr{T}): \quad \forall v_1, v_2 \in \mathscr{V}, \forall \tau_1, \tau_2 \in \mathscr{T}, \quad \tau_1(v_1) = \tau_2(v_2) \Rightarrow v_1 = v_2. \tag{2}$$

*If $\tau_1(v_1) = \tau_2(v_2)$ only implies $v_1 = \pm v_2$, then we will use the notation* $\operatorname{hsp}_{\pm}(\mathscr{V}, \mathscr{T})$*.*

The problem of interest to us is as follows.

**Problem 1 (Homomorphic sensing [1, 2]).** *Find conditions on a finite set $\mathcal{T}$ of linear maps $\mathbb{H}^m \to \mathbb{H}^m$ and an $n$-dimensional linear subspace $\mathcal{V} \subset \mathbb{H}^m$ that imply* $\operatorname{hsp}(\mathcal{V}, \mathcal{T})$*.*

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*Corresponding author

*Email addresses:* `penglz@shanghaitech.edu.cn` (Liangzu Peng), `mtsakiris@shanghaitech.edu.cn` (Manolis C. Tsakiris)

To appreciate Problem 1 we start by looking at several special cases which have been explored via different approaches in the last two decades, e.g., see [3, 4, 5, 6, 7, 8, 9]. The first example is *real phase retrieval* [9], a problem which dates back to the 1910s when the research on *X-ray crystallography* was launched; see [10] for a vivid account. In a mathematical formulation of this problem we let $\mathbb{H} = \mathbb{R}$ and consider the relation $y = B^* A x^*$, where $y$ is an $m$-dimensional vector, $A \in \mathbb{R}^{m\times n}$ is a given matrix, and $B^*$ is known only up to the set $\mathcal{B}_m$ of $m \times m$ sign matrices, i.e., diagonal matrices with $\pm 1$ on the diagonal. Since uniquely recovering a nonzero $x^*$ is impossible[1], we consider unique recovery of $x^*$ up to sign. In other words, with $\mathcal{B}_m A := \{BA : B \in \mathcal{B}_m\}$ we consider the property $\operatorname{hsp}_{\pm}(\mathbb{R}^n, \mathcal{B}_m A)$. In 2006, it was proved by [3] in a frame-theoretical language that $m \geq 2n - 1$ is sufficient for a generic[2] $A \in \mathbb{R}^{m\times n}$ to enjoy $\operatorname{hsp}_{\pm}(\mathbb{R}^n, \mathcal{B}_m A)$, and this is necessary for any $A \in \mathbb{R}^{m\times n}$. If $x^*$ is known to come from the set $\overline{\mathcal{K}_{\mathscr{I}}}$ of all $k$-sparse vectors of $\mathbb{R}^n$, a situation considered in *sparse real phase retrieval* [5], then [4] and [5] have independently showed that a sufficient and necessary condition for $\operatorname{hsp}_{\pm}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{B}_m A)$ is $m \geq \min\{2n-1, 2k\}$ providing $A \in \mathbb{R}^{m\times n}$ is generic. Finally, those results are also true for the problem of *symmetric mixture of two linear regressions* [11], since it bears the same formulation as real phase retrieval in the noiseless case; see, e.g., [12] and [13] for discussions that connect the two problems.

The next example involves the set $\mathcal{S}_{r,m}$ of all rank-$r$ selection matrices, i.e., matrices whose rows are formed by $r$ distinct standard basis vectors of $\mathbb{R}^m$. Motivated by signal processing applications, the property $\operatorname{hsp}(\mathbb{R}^n, \mathcal{S}_{r,m} A)$ was considered in [6, 7] under the name *unlabeled sensing*, and also independently in [8]. Specifically, they proved via different algebraic-combinatorial techniques that $r \geq 2n$ suffices to guarantee $\operatorname{hsp}(\mathbb{R}^n, \mathcal{S}_{r,m} A)$ for $A \in \mathbb{R}^{m\times n}$ generic. For the converse, [8] proved that $r \geq 2n - 1$ is necessary for $\operatorname{hsp}(\mathbb{R}^n, \mathcal{S}_{r,m} A)$ and [6, 7] proved that, if $m$ is odd with $m = r$ and $n \geq 2$, then $r \geq 2n$ is necessary. Finally, if $m = r$, then $\mathcal{S}_m := \mathcal{S}_{m,m}$ is the group of $m \times m$ permutation matrices and the unlabeled sensing problem becomes that of *linear regression without correspondences*. This special case has its origin in applications in statistics such as *record linkage* [14] and the *broken sample problem* [15] (see [16] for detailed discussions); recent development on this topic can be found in, e.g., [17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

An interesting generalization which we call *unsigned unlabeled sensing* was explored in [9] and is a combination of the above two, where we let $\mathcal{S}_{r,m}\mathcal{B}_m := \{SB : S \in \mathcal{S}_{r,m}, B \in \mathcal{B}_m\}$ and the interest is in $\operatorname{hsp}_{\pm}(\mathbb{R}^n, \mathcal{S}_{r,m}\mathcal{B}_m A)$. By extending the approach of [8], it was established in [9] that $r \geq 2n$ is sufficient for $\operatorname{hsp}_{\pm}(\mathbb{R}^n, \mathcal{S}_{r,m}\mathcal{B}_m A)$ for $A \in \mathbb{R}^{m\times n}$ generic and this is necessary if $n \geq 2$.

The homomorphic sensing Problem 1 is an abstraction of the above examples and [1, 2] studied it using algebraic geometry. With linear maps $\tau_1, \tau_2 : \mathbb{H}^m \to \mathbb{H}^m$ let $\overline{\tau}_1, \overline{\tau}_2 : \mathbb{C}^m \to \mathbb{C}^m$ be their complexifications and let $T_1, T_2$ be their matrix representations[3]. Let $\rho$ be a linear projection onto the image $\operatorname{im}(\tau_2)$ of $\tau_2$ with matrix representation $P$ and complexification $\overline{\rho}$. With $w$ a vector of variables, the $2 \times 2$ minors of the $r \times 2$ matrix $[PT_1 w \ T_2 w]$ are polynomials in entries of $w$, so their vanishing locus in $\mathbb{C}^m$ is a complex algebraic variety[4], say $\mathcal{Y}_{\rho\tau_1,\tau_2}$. Removing from $\mathcal{Y}_{\rho\tau_1,\tau_2}$ the union of linear subspaces $\mathcal{Z}_{\rho\tau_1,\tau_2} := \ker(\overline{\rho\tau}_1 - \overline{\tau}_2) \cup \ker(\overline{\rho\tau}_1) \cup \ker(\overline{\tau}_2)$ gives an open set in $\mathcal{Y}_{\rho\tau_1,\tau_2}$ (also called a quasi-variety)

$$\mathcal{U}_{\rho\tau_1,\tau_2} = \mathcal{Y}_{\rho\tau_1,\tau_2} \backslash \mathcal{Z}_{\rho\tau_1,\tau_2}. \tag{3}$$

In [1, 2] it was proved that if for any $\tau_1, \tau_2 \in \mathcal{T}$ it holds that $\operatorname{rank}(\tau_1) := \operatorname{rank}(T_1) \geq 2n$ and $\operatorname{rank}(\tau_2) \geq 2n$, and if there exists a linear projection $\rho$ onto $\operatorname{im}(\tau_2)$ satisfying $\dim(\mathcal{U}_{\rho\tau_1,\tau_2}) \leq m - n$,[5] then a generic subspace $\mathcal{V}$ of dimension $n$ satisfies $\operatorname{hsp}(\mathcal{V}, \mathcal{T})$. They further specialized this result to unlabeled sensing, yielding the same sufficient conditions as in [7] and [8] mentioned above.

A limitation of this result is the presence of the projection $\rho$. It remains unknown whether a $\rho$ that satisfies $\dim(\mathcal{U}_{\rho\tau_1,\tau_2}) \leq m - n$ exists, or even if so how to search for it. One of our main contributions here is

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[1] If $x^*$ is nonzero then we have $B^* A x^* = -B^* A(-x^*)$ but $x^* \neq -x^*$.

[2] By a generic matrix in $\mathbb{R}^{m\times n}$ we mean a non-empty Zariski open subset of $\mathbb{R}^{m\times n}$; see also §2.

[3] We always consider the matrix representation with respect to the standard basis. A linear map $\mathbb{R}^m \to \mathbb{R}^m$ and its complexification have the same matrix representation. See §2.

[4] An algebraic variety is the common root locus (or vanishing locus) of a set of polynomials, see §2.

[5] If $\mathcal{T} = \{\tau_1, \tau_2\}$ then the condition $\operatorname{rank}(\tau_1) \geq 2n$ can be relaxed to $\operatorname{rank}(\tau_1) \geq n$ [2]. See §2 for the definition of dimension.

to dispense with $\rho$. To do so, we consider the complex algebraic variety $\mathcal{Y}_{\tau_1,\tau_2}$ defined by the vanishing of the $2\times 2$ minors of $[T_1 w\ T_2 w]$, the union $\mathscr{Z}_{\tau_1,\tau_2} := \ker(\overline{\tau}_1 - \overline{\tau}_2) \cup \ker(\overline{\tau}_1) \cup \ker(\overline{\tau}_2)$, and the quasi-variety

$$\mathcal{U}_{\tau_1,\tau_2} = \mathcal{Y}_{\tau_1,\tau_2} \backslash \mathscr{Z}_{\tau_1,\tau_2}. \tag{4}$$

We have the following description of the homomorphic sensing phenomenon.

**Theorem 1.** *Suppose* $\operatorname{rank}(\tau) \geq 2n$ *for every* $\tau \in \mathcal{T}$. *Then* $\operatorname{hsp}(\mathcal{V}, \mathcal{T})$ *holds true for a generic subspace* $\mathcal{V}$ *of* $\mathbb{H}^m$ *of dimension* $n$ *whenever*

$$\dim(\mathcal{U}_{\tau_1,\tau_2}) \leq m - n, \quad \forall \tau_1, \tau_2 \in \mathcal{T}. \tag{5}$$

Note that, by definition, $\mathcal{U}_{\tau_1,\tau_2}$ is a subset of $\mathcal{U}_{\rho\tau_1,\tau_2}$, so condition (5) is tighter than that of [1, 2]. Indeed, condition (5) is the tightest possible in the following sense.

**Proposition 1.** *Suppose* $\mathbb{H} = \mathbb{C}$ *and that* (5) *is not true. Then* $\operatorname{hsp}(\mathcal{V}, \mathcal{T})$ *is violated for a generic subspace* $\mathcal{V} \subset \mathbb{H}^m$ *of dimension* $n$.

Using the proof technique of Theorem 1, we get the following extension for $\operatorname{hsp}_{\pm}(\mathcal{V}, \mathcal{T})$.

**Proposition 2.** *Suppose that for every* $\tau \in \mathcal{T}$ *we have* $\operatorname{rank}(\tau) \geq 2n$. *Let* $\mathcal{U}^{\pm}_{\tau_1,\tau_2} := \mathcal{U}_{\tau_1,\tau_2} \backslash \ker(\overline{\tau}_1 + \overline{\tau}_2)$. *Then* $\operatorname{hsp}_{\pm}(\mathcal{V}, \mathcal{T})$ *holds true for a generic subspace* $\mathcal{V}$ *of* $\mathbb{H}^m$ *of dimension* $n$ *whenever*

$$\dim(\mathcal{U}^{\pm}_{\tau_1,\tau_2}) \leq m - n, \quad \forall \tau_1, \tau_2 \in \mathcal{T}. \tag{6}$$

In §1.2, we extend Theorem 1 from a single subspace $\mathcal{V}$ to a *subspace arrangement* (Theorem 2). In §1.3 we consider the local stability of the homomorphic sensing property under noise (Theorem 3). In §1.4, we specialize Theorems 1-3 to the aforementioned applications, e.g., real phase retrieval, unlabeled sensing variants, and their sparse versions. In §2 we give preliminaries. Proofs of all the statements are in §3.

*1.2. Homomorphic sensing of subspace arrangements*

We extend Theorem 1 from a single subspace $\mathcal{V}$ to a subspace arrangement $\mathcal{A} = (\mathcal{V}_1, \dots, \mathcal{V}_\ell)$, the latter being an ordered set of subspaces $\mathcal{V}_i$, $i \in [\ell] := \{1, \dots, \ell\}$ of $\mathbb{H}^m$. With $n_i = \dim(\mathcal{V}_i)$, we refer to $(n_1, \dots, n_\ell)$ as the dimension configuration of $\mathcal{A}$. Thus, by a *generic* subspace arrangement $\mathcal{A}$ with dimension configuration $(n_1, \dots, n_\ell)$ we mean a non-empty Zariski open subset of the product $\operatorname{Gr}_{\mathbb{H}}(n_1, m) \times \cdots \times \operatorname{Gr}_{\mathbb{H}}(n_\ell, m)$ of Grassmannians (see also §2). Consider an ordered set $\mathscr{I} = (\mathcal{I}_1, \dots, \mathcal{I}_s)$ of subsets of $[\ell]$. Each $\mathcal{I}_j$ gives rise to a subspace $\mathcal{V}_{\mathcal{I}_j} := \sum_{i \in \mathcal{I}_j} \mathcal{V}_i$ with dimension upper bounded by $n_{\mathcal{I}_j} := \sum_{i \in \mathcal{I}_j} n_i$, where $\mathcal{V}_\varnothing := 0$. Thus the ordered set $\mathscr{I}$, together with $\mathcal{A}$, induces the *structured* subspace arrangement $\mathcal{A}_{\mathscr{I}} = (\mathcal{V}_{\mathcal{I}_1}, \dots, \mathcal{V}_{\mathcal{I}_s})$. This construction allows us various levels of flexibility that will be exploited later in the paper. For example, $\mathcal{A}_{\mathscr{I}}$ becomes the original $\mathcal{A}$ when $\mathcal{I}_j = \{j\}$ and $s = \ell$, and if in addition $s = 1$, then $\mathcal{A}_{\mathscr{I}}$ becomes a single subspace. We write $\overline{\mathcal{A}_{\mathscr{I}}} := \bigcup_{j \in [s]} \mathcal{V}_{\mathcal{I}_j}$ and consider the property $\operatorname{hsp}(\overline{\mathcal{A}_{\mathscr{I}}}, \mathcal{T})$. We have:

**Theorem 2.** *Suppose* $\operatorname{rank}(\tau) \geq 2n$ *for any* $\tau \in \mathcal{T}$. *Let* $(n_1, \dots, n_\ell)$ *be a dimension configuration and* $\mathscr{I} = (\mathcal{I}_1, \dots, \mathcal{I}_s)$ *an ordered set of subsets of* $[\ell]$ *satisfying* $n_{\mathcal{I}_j} \leq n$ *for any* $j \in [s]$. *Then* $\operatorname{hsp}(\overline{\mathcal{A}_{\mathscr{I}}}, \mathcal{T})$ *holds for a generic subspace arrangement* $\mathcal{A} = (\mathcal{V}_1, \dots, \mathcal{V}_\ell)$ *with* $\dim(\mathcal{V}_i) = n_i$ *whenever* (5) *holds. Similarly,* $\operatorname{hsp}_{\pm}(\overline{\mathcal{A}_{\mathscr{I}}}, \mathcal{T})$ *holds for a generic subspace arrangement* $(\mathcal{V}_1, \dots, \mathcal{V}_\ell)$ *with* $\dim(\mathcal{V}_i) = n_i$ *whenever* (6) *holds.*

*1.3. Noisy homomorphic sensing*

We consider the homomorphic sensing problem with $\mathbb{H} = \mathbb{R}$ in the presence of additive noise $\epsilon \in \mathbb{R}^m$. For $v^* \in \mathcal{V}$ and $\tau^* \in \mathcal{T}$ set $y = \tau^*(v^*)$ and $\overline{y} = y + \epsilon$. Consider the optimization problem

$$(\hat{\tau}, \hat{v}) \in \underset{v \in \mathcal{V}, \tau \in \mathcal{T}}{\operatorname{argmin}} \|\overline{y} - \tau(v)\|_2. \tag{7}$$

What can we say about the optimal solution $\hat{v}$? Under what conditions is $\hat{v}$ close to $v^*$?

For a nonzero vector $u \in \mathbb{R}^m$ and a subspace $\mathcal{W} \subset \mathbb{R}^m$ we define

$$\cos(u, \mathcal{W}) := \max\big\{\langle u, w\rangle / \|u\|_2 : w \in \mathcal{W} \text{ and } \|w\|_2 = 1\big\}. \tag{8}$$

Denote by $\sigma(X)$ the largest singular value of a real matrix $X$. Then we have the following stability result.

**Theorem 3.** *Suppose* $\operatorname{rank}(\tau) \geq 2n$ *for every* $\tau \in \mathcal{T}$ *and that* (5) *holds. Let* $\mathcal{V} \subset \mathbb{R}^m$ *be a subspace of dimension* $n$ *that satisfies* $\operatorname{hsp}(\mathcal{V}, \mathcal{T})$ *and let* $A \in \mathbb{R}^{m\times n}$ *be a matrix that has* $\mathcal{V}$ *as its column space. Let* $(\hat{\tau}, \hat{v})$ *be a solution to* (7) *with* $\hat{T}$ *the matrix representation of* $\hat{\tau}$. *Set* $\mathcal{T}_1 := \{\tau \in \mathcal{T} : y \in \tau(\mathcal{V})\}$. *If* $\mathcal{T} = \mathcal{T}_1$ *or*

$$2\|\epsilon\|_2 < \|y\|_2\Big(1 - \max_{\tau \in \mathcal{T}\setminus\mathcal{T}_1} \cos\big(y, \tau(\mathcal{V})\big)\Big), \tag{9}$$

*then* $\hat{v} - v^* = A(\hat{T}A)^\dagger \epsilon$, *where* $(\hat{T}A)^\dagger$ *is the pseudoinverse of* $\hat{T}A$. *In particular* $\|\hat{v} - v^*\|_2 \leq \sigma(A(\hat{T}A)^\dagger)\|\epsilon\|_2$.

*1.4. Applications of homomorphic sensing theory*

We now consider the applications of Theorems 1-3 to problems mentioned in §1.1, namely linear regression without correspondences ($\mathcal{S}_m$), unlabeled sensing ($\mathcal{S}_{r,m}$), real phase retrieval ($\mathcal{B}_m$), and unsigned unlabeled sensing ($\mathcal{S}_{r,m}\mathcal{B}_m$). Taking $\mathcal{S}_{r,m}$ for example we see that, if $A$ is of full column rank, then $\operatorname{hsp}(\mathbb{R}^n, \mathcal{S}_{r,m}A)$ is equivalent to $\operatorname{hsp}(R(A), \mathcal{S}_{r,m})$, where $R(A)$ is the column space of $A$. Also $\mathcal{S}_{r,m} \subset \mathbb{R}^{r\times m}$ can be viewed as a finite set of linear maps $\mathbb{H}^m \to \mathbb{H}^m$ (via an obvious injection). We then check whether $\mathcal{S}_{r,m}$ satisfies condition (5). Interestingly, whenever the rank constraint $r \geq 2n$ of Theorem 1 on $\mathcal{S}_{r,m}$ is fulfilled, condition (5) is automatically satisfied by $\mathcal{S}_{r,m}$, and similarly for the other types of transformations discussed:

**Proposition 3.** *Let* $\Pi_1, \Pi_2 \in \mathcal{S}_m$, $S_1, S_2 \in \mathcal{S}_{r,m}$, *and* $B_1, B_2 \in \mathcal{B}_m$ *be permutations, rank-r selections, and sign matrices, respectively.*

- $m \geq 2n \Rightarrow \dim(\mathcal{U}_{\Pi_1,\Pi_2}) \leq m - n$.
- $r \geq 2n \Rightarrow \dim(\mathcal{U}_{S_1,S_2}) \leq m - n$.
- $m \geq 2n \Rightarrow \dim(\mathcal{U}^{\pm}_{B_1,B_2}) \leq m - n$.
- $r \geq 2n \Rightarrow \dim(\mathcal{U}^{\pm}_{S_1B_1,S_2B_2}) \leq m - n$.

Combining Proposition 3 with Theorem 1 we get the following results, which have already been obtained in a diverse literature via diverse methods:

**Corollary 1.** *For a generic matrix* $A$ *of* $\mathbb{R}^{m\times n}$, *it holds that*

- $m \geq 2n \Rightarrow \operatorname{hsp}(\mathbb{R}^n, \mathcal{S}_mA)$ *[1, 7, 8, 22].*
- $r \geq 2n \Rightarrow \operatorname{hsp}(\mathbb{R}^n, \mathcal{S}_{r,m}A)$ *[1, 7, 8].*
- $m \geq 2n \Rightarrow \operatorname{hsp}_{\pm}(\mathbb{R}^n, \mathcal{B}_mA)$ *[3, 22].*
- $r \geq 2n \Rightarrow \operatorname{hsp}_{\pm}(\mathbb{R}^n, \mathcal{S}_{r,m}\mathcal{B}_mA)$ *[1, 9].*

Next we consider the sparse counterpart of Corollary 1. This is mostly unexplored territory in prior work and the main player here will be Theorem 2. Consider the standard basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ and the subspace arrangement $\mathcal{K} = (\mathcal{V}_1, \dots, \mathcal{V}_n)$ of $\mathbb{R}^n$ with $\mathcal{V}_i = \operatorname{Span}(e_i)$. Let $s = \binom{n}{k}$ and consider the ordered set $\mathscr{I} = (\mathcal{I}_1, \dots, \mathcal{I}_s)$ of all subsets of $[n]$ of cardinality $k$. It gives the structured subspace arrangement $\mathcal{K}_{\mathscr{I}} = (\mathcal{V}_{\mathcal{I}_1}, \dots, \mathcal{V}_{\mathcal{I}_s})$ where $\mathcal{V}_{\mathcal{I}_j} = \sum_{i\in\mathcal{I}_j} \mathcal{V}_i$. By construction, the union of subspaces $\overline{\mathcal{K}_{\mathscr{I}}} := \cup_{j\in[s]}\mathcal{V}_{\mathcal{I}_j}$ is the set of all $k$-sparse vectors of $\mathbb{R}^n$. With this notation sparse real phase retrieval is equivalent to $\operatorname{hsp}_{\pm}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{B}_mA)$ and this has been studied by [4] and [5], while sparse unlabeled sensing ($\operatorname{hsp}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{S}_{r,m}A)$) and sparse unsigned unlabeled sensing ($\operatorname{hsp}_{\pm}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{S}_{r,m}\mathcal{B}_mA)$) have not been considered yet, to the best of our knowledge. If $A \in \mathbb{R}^{m\times n}$ is a generic measurement matrix and $\tau_A : \mathbb{R}^n \to \mathbb{R}^m$ the corresponding linear transformation, the image $\tau_A(\overline{\mathcal{K}_{\mathscr{I}}})$ of $\overline{\mathcal{K}_{\mathscr{I}}}$ under $\tau_A$ is a generic subspace arrangement. After relating properties of say $\operatorname{hsp}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{S}_{r,m}A)$ to $\operatorname{hsp}(\tau_A(\overline{\mathcal{K}_{\mathscr{I}}}), \mathcal{S}_{r,m})$, Theorem 2 and Proposition 3 give:

**Corollary 2.** *For a generic matrix* $A$ *of* $\mathbb{R}^{m\times n}$ *and* $k \leq n$, *it holds that*

- $m \geq 2k \Rightarrow \operatorname{hsp}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{S}_mA)$.

- $r \geq 2k \Rightarrow \operatorname{hsp}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{S}_{r,m}A)$.
- $m \geq 2k \Rightarrow \operatorname{hsp}_{\pm}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{B}_m A)$ *[4, 5]*.
- $r \geq 2k \Rightarrow \operatorname{hsp}_{\pm}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{S}_{r,m}\mathcal{B}_m A)$.

Our final result is a corollary of Theorem 3. For brevity we only state the result for unlabeled sensing, where $y = S^* A x^*$ for some $S^* \in \mathcal{S}_{r,m}$, $\overline{y} = y + \epsilon$, and the objective function of interest as a special case of (7) is

$$(\hat{S}, \hat{x}) \in \operatorname*{argmin}_{x \in \mathbb{R}^n, S \in \mathcal{S}_{r,m}} \|\overline{y} - SAx\|_2. \tag{10}$$

**Corollary 3.** *Suppose that i) $r \geq 2n$, ii)* (9) *holds for $\mathcal{T} = \mathcal{S}_{r,m}$ and iii) $A \in \mathbb{R}^{m \times n}$ is generic in the sense that* $\operatorname{hsp}(R(A), \mathcal{S}_{r,m})$ *is true. Then $\hat{x} - x^* = (\hat{T}A)^\dagger \epsilon$.*

We note that condition (9) of Corollary 3 defines a non-asymptotic regime where the local stablity of estimating $x^*$ is guaranteed, and this implies the asymptotic result of [7].

## 2. Notations and basic facts

Let $\mathbb{H}$ be equal to $\mathbb{R}$ or $\mathbb{C}$. For $k = 1, 2$ let $\tau_k$ be an $\mathbb{H}$-linear map $\mathbb{H}^m \to \mathbb{H}^m$ and write $T_k \in \mathbb{H}^{m \times m}$ for its matrix representation on the canonical basis of $\mathbb{H}^m$. Denote by $\overline{\tau}_k : \mathbb{C}^m \to \mathbb{C}^m$ the complexification of $\tau_k$. That is $\overline{\tau}_k := \tau_k$ if $\mathbb{H} = \mathbb{C}$, and $\overline{\tau}_k(u + iv) := \tau_k(u) + i\tau_k(v)$ for every $u, v \in \mathbb{R}^m$ if $\mathbb{H} = \mathbb{R}$; here $i = \sqrt{-1}$. Note that if $\mathbb{H} = \mathbb{R}$, then $T_k$ is also a matrix representation for $\overline{\tau}_k$. With $\lambda \in \mathbb{C}$, denote by $\mathscr{E}_{(\tau_1,\tau_2),\lambda}$ the set of all $w \in \mathbb{C}^m$ satisfying $\overline{\tau}_1(w) = \lambda \overline{\tau}_2(w)$. This is a $\mathbb{C}$-subspace of $\mathbb{C}^m$. If $\lambda \in \mathbb{R}$, then $\mathscr{E}_{(\tau_1,\tau_2),\lambda} \cap \mathbb{R}^m$ is an $\mathbb{R}$-subspace of $\mathbb{R}^m$ and we have $\dim_{\mathbb{R}}\left(\mathscr{E}_{(\tau_1,\tau_2),\lambda} \cap \mathbb{R}^m\right) = \dim_{\mathbb{C}}\left(\mathscr{E}_{(\tau_1,\tau_2),\lambda}\right)$, where $\dim_{\mathbb{R}}, \dim_{\mathbb{C}}$ denote real and complex vector space dimension respectively. In the sequel, we will drop the subscript indicating the field, with the convention that by $\dim(\mathcal{W})$ we mean $\dim_{\mathbb{H}}(\mathcal{W})$ whenever $\mathcal{W}$ is a $\mathbb{H}$-subspace of $\mathbb{H}^m$, while $\mathcal{V} \cap \mathscr{E}_{(\tau_1,\tau_2),\lambda}$ will always be treated as an $\mathbb{R}$-subspace whenever $\mathcal{V}$ is such. For simplicity, we write $\mathscr{E}_{\tau_k,\lambda} := \mathscr{E}_{(\tau_k,\mathrm{id}),\lambda}$ for the eigenspace of $\tau_k$ corresponding to eigenvalue $\lambda$, where id is the identity map. For a map of sets $\tau : \mathcal{X} \to \mathcal{Y}$ we let $\tau^{-1}(Q)$ be the inverse image of $\mathcal{Q} \subset \mathcal{Y}$ under $\tau$. Denote by 0 the trivial subspace, the zero vector, and the number zero, to be made clear by the context. The following lemma is elementary but will be called upon frequently and thus is noted for convenience.

**Lemma 1.** *For an $\mathbb{H}$-linear map $\tau : \mathbb{H}^m \to \mathbb{H}^m$ and $\mathcal{B}, \mathcal{C}$ $\mathbb{H}$-subspaces of $\mathbb{H}^m$, we have*

$$\mathcal{B} \cap \ker(\tau) = 0,\ \tau(\mathcal{B}) \cap \mathcal{C} = 0 \Leftrightarrow \mathcal{B} \cap \tau^{-1}(\mathcal{C}) = 0. \tag{11}$$

An algebraic variety is a subset of $\mathbb{H}^m$ defined as the common zero locus of a set of polynomials in $m$ variables with coefficients in $\mathbb{H}$. The *Zariski topology* on $\mathbb{H}^m$ is defined by identifying closed sets with algebraic varieties of $\mathbb{H}^m$. Hence Zariski open sets arise as loci in $\mathbb{H}^m$ of non-simultaneous vanishing of sets of polynomials. An irreducible algebraic variety is one which can not be written as the union of two proper subvarieties of it. Here by subvariety we mean a closed set in the subspace topology. By a *generic* point of an irreducible algebraic variety having some property of interest, we mean that there is a non-empty Zariski open (and thus necessarily dense) subset in the variety, each element of which satisfies the property. We denote by $\mathrm{Gr}_{\mathbb{H}}(n, m)$ the Grassmannian of $n$-dimensional $\mathbb{H}$-subspaces of $\mathbb{H}^m$. One defines a Zariski topology in projective space in a similar fashion as above and under the Plücker embedding [27] $\mathrm{Gr}_{\mathbb{H}}(n, m)$ becomes an irreducible projective variety of dimension $n(m - n)$. Since $\mathbb{H}^{m \times n}$ is also irreducible, we have justified what we mean by a generic matrix of $\mathbb{H}^{m \times n}$ or a generic $n$-dimensional $\mathbb{H}$-subspace of $\mathbb{H}^m$. Another classical irreducible variety that will play a role is the *flag variety* $\mathrm{F}_{\mathbb{H}}(n_0, n, m)$. This lives in the product $\mathrm{Gr}_{\mathbb{H}}(n_0, m) \times \mathrm{Gr}_{\mathbb{H}}(n, m)$ and consists of those pairs $(\mathcal{V}_0, \mathcal{V})$ that satisfy $\mathcal{V}_0 \subset \mathcal{V}$. We will need the following:

**Lemma 2.** *Let $\phi : \mathrm{F}_{\mathbb{H}}(n_0, n, m) \to \mathrm{Gr}_{\mathbb{H}}(n, m)$ be the canonical projection that sends $(\mathcal{V}_0, \mathcal{V})$ to $\mathcal{V}$. If $\mathcal{U}$ is a non-empty Zariski open subset of $\mathrm{F}_{\mathbb{H}}(n_0, n, m)$, then the image $\phi(\mathcal{U})$ contains a non-empty Zariski open subset of $\mathrm{Gr}_{\mathbb{H}}(n, m)$.*

PROOF. We first treat the case $\mathbb{H} = \mathbb{C}$. By Chevalley's theorem [27] $\phi(\mathcal{U})$ is constructible, that is $\phi(\mathcal{U}) = \cup_\nu \mathcal{Y}_\nu \cap \mathcal{U}_\nu$ where the $\mathcal{Y}_\nu$'s are closed in $\mathrm{Gr}_{\mathbb{C}}(n,m)$, the $\mathcal{U}_\nu$'s are open in $\mathrm{Gr}_{\mathbb{C}}(n,m)$, and $\nu$ takes finitely many values. If $\phi(\mathcal{U})$ does not contain any non-empty open set, then necessarily it is contained in the proper closed subset $\mathcal{Y} = \cup_\nu \mathcal{Y}_\nu$. The complement of $\mathcal{Y}$ is a non-empty open subset of $\mathrm{Gr}_{\mathbb{C}}(n,m)$ which does not intersect $\phi(\mathcal{U})$, and thus its inverse image under $\phi$ is also a non-empty open subset of $\mathrm{F}_{\mathbb{C}}(n_0,n,m)$ not intersecting $\mathcal{U}$. This implies that $\mathrm{F}_{\mathbb{C}}(n_0,n,m)$ can be written as a union of two proper closed sets. This is a contradiction because $\mathrm{F}_{\mathbb{C}}(n_0,n,m)$ is irreducible.

Next, we treat the case $\mathbb{H} = \mathbb{R}$. Then the arguments in the previous paragraph apply without change providing we treat $\phi$ as a morphism of finite type of Noetherian schemes over $\mathbb{R}$; see [28, 29, 30]. Thus we write $\overline{\phi} : \overline{\mathrm{F}}_{\mathbb{R}}(n_0,n,m) \to \overline{\mathrm{Gr}}_{\mathbb{R}}(n,m)$, where the overline indicates the scheme structure. By the Jacobson property, the restriction of $\overline{\phi}$ on the $k$-valued points is just $\phi$. The polynomials that define $\mathcal{U}$ also define a corresponding scheme $\overline{\mathcal{U}} \subset \overline{\mathrm{F}}_{\mathbb{R}}(n_0,n,m)$, and the above arguments applied to $\overline{\phi}$ show that $\overline{\phi}(\overline{\mathcal{U}})$ contains a non-empty open subscheme $\overline{\mathcal{V}}$ of $\overline{\mathrm{Gr}}_{\mathbb{R}}(n,m)$. Now $\overline{\mathrm{Gr}}_{\mathbb{R}}(n,m)$ is locally isomorphic to the affine space $\mathbb{A}^{n(m-n)} = \mathrm{Spec}(\mathbb{R}[Z])$, where $Z$ is an $n \times (m-n)$ matrix of variables $z_{ij}$ and $\mathbb{R}[Z]$ is the polynomial ring in the $z_{ij}$'s with coefficients over $\mathbb{R}$. So let $\overline{\mathcal{V}'}$ be an open subscheme of $\overline{\mathrm{Gr}}_{\mathbb{R}}(n,m)$ isomorphic to $\mathbb{A}^{n(m-n)}$. Then $\overline{\mathcal{V}''} = \overline{\mathcal{V}'} \cap \overline{\mathcal{V}}$ is also open in $\overline{\mathrm{Gr}}_{\mathbb{R}}(n,m)$ and in fact non-empty because $\overline{\mathrm{Gr}}_{\mathbb{R}}(n,m)$ is irreducible. Under the isomorphism $\overline{\mathcal{V}'} \cong \mathbb{A}^{n(m-n)}$ we view $\overline{\mathcal{V}''}$ as a non-empty open subscheme of $\mathbb{A}^{n(m-n)}$. Now $\overline{\mathcal{V}''}$ can be written as $\bigcup_p \mathrm{Spec}(k[Z])_p$, with $p \in k[Z]$ and $(k[Z])_p$ the localization of $k[Z]$ at the multiplicatively closed set $\{1, p, p^2, \dots\}$. Since $\overline{\mathcal{V}''}$ is non-empty, not all $p$'s are zero. Hence there is some non-zero $p$ for which $\overline{\mathcal{V}'''} = \mathrm{Spec}(k[Z])_p$ is a non-empty open subscheme of $\mathbb{A}^{n(m-n)}$. Let $\mathcal{U}'$ be the open set of points in $\mathbb{R}^{n(m-n)}$ which are not roots of $p$. Since $\mathbb{R}$ is infinite, $\mathcal{U}'$ is non-empty. Finally, $\mathcal{U}'$ lies in the image of $\phi$. □

The dimension $\dim(\mathcal{Q})$ of an algebraic variety $\mathcal{Q}$ is the maximal length $t$ of the chains $\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_t$ of distinct irreducible algebraic varieties contained in $\mathcal{Q}$. The dimension of any set is the dimension of its *closure*, i.e., the smallest algebraic variety which contains it. By convention $\dim \mathcal{Q} = -1$ if and only if $\mathcal{Q}$ is empty. The following will be frequently used:

**Lemma 3.** *Given $t$ algebraic varieties $\mathcal{Q}_1, \dots, \mathcal{Q}_t$ in $\mathbb{C}^m$ of dimensions $r_1, \dots, r_t$ all passing through the origin, there exists a non-empty Zariski open subset of $\mathrm{Gr}_{\mathbb{C}}(d,m)$ on which every subspace $\mathcal{V}$ satisfies $\dim(\mathcal{Q}_j \cap \mathcal{V}) = \max\{r_j + d - m, 0\}$ for any $j \in [t]$.*

PROOF. Let $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_m]$ be the polynomial ring associated to $\mathbb{C}^m$ and let $I_j$ be an ideal of $\mathbb{C}[x]$ that defines $\mathcal{Q}_j$ as a set, that is $\mathcal{Q}_j$ is the common root locus in $\mathbb{C}^m$ of all the polynomials in $I_j$. With $c = m - d$, a $d$-dimensional subspace $\mathcal{V}$ of $\mathbb{C}^m$ is an algebraic variety defined as the common root locus of linear forms $b_1^\top x, \dots, b_c^\top x$, where $b_i = [b_{i1}, \dots, b_{im}] \in \mathbb{C}^m$ and $b_i^\top x := b_{i1}x_1 + \cdots + b_{im}x_m \in \mathbb{C}[x]$. Here the $b_i$'s can be taken to be a basis for the orthogonal complement of $\mathcal{V}$. Let us also denote by $I_{\mathcal{V}}$ the ideal generated by the $b_i^\top x$'s. Then the dimension of $\mathcal{Q}_j \cap \mathcal{V}$ coincides with the Krull dimension of the ring $\mathbb{C}[x]/(I_j + I_{\mathcal{V}})$. Moreover $I_j + I_{\mathcal{V}}$ is always a proper ideal, since $0 \in \mathcal{Q}_j \cap \mathcal{V}$ and thus $I_j + I_{\mathcal{V}}$ is contained in the ideal generated by the $x_i$'s. Hence $\dim\big(\mathbb{C}[x]/(I_j + I_{\mathcal{V}})\big) \geq 0$. Now let $\mathbb{C}[x,y] := \mathbb{C}[x_1, \dots, x_m, y]$ be another polynomial ring where the new variable $y$ is going to serve as a homogenization variable in the usual way; see [31] for details regarding homogenization. For an ideal $J$ of $\mathbb{C}[x]$ we denote by $J^{\mathrm{hom}}$ its homogenization, which is a homogeneous ideal of $\mathbb{C}[x,y]$. Since $I_{\mathcal{V}}$ is already homogeneous we have that $(I_j + I_{\mathcal{V}})^{\mathrm{hom}} = I_j^{\mathrm{hom}} + I_{\mathcal{V}}$. Moreover, since no polynomial in $I_j$ has non-zero constant term, we always have $\dim\big(\mathbb{C}[x,y]/(I_j^{\mathrm{hom}} + I_{\mathcal{V}})\big) \geq 1$. In fact, by properties of homogenization we have that $\dim\big(\mathbb{C}[x]/(I_j + I_{\mathcal{V}})\big) + 1 = \dim\big(\mathbb{C}[x,y]/(I_j^{\mathrm{hom}} + I_{\mathcal{V}})\big)$. Now, it is a standard fact in commutative algebra that the set of $\mathcal{V} \in \mathrm{Gr}_{\mathbb{C}}(d,m)$ for which $\dim\big(\mathbb{C}[x,y]/(I_j^{\mathrm{hom}} + I_{\mathcal{V}})\big) = \max\{r_j - c + 1, 1\}$ is open. This is the same open set of $\mathcal{V} \in \mathrm{Gr}_{\mathbb{C}}(d,m)$ for which $\dim\big(\mathbb{C}[x]/(I_j + I_{\mathcal{V}})\big) = \max\{r_j - c, 0\}$. It remains to show the non-emptyness of that open set. Let $\mathscr{C}$ be the set of all irreducible components of all $\mathcal{Q}_j$'s. We recall that the dimension of each $\mathcal{Q}_j$ is equal to the maximal dimension of its irreducible components. We also recall that if $\mathcal{Q}$ is an irreducible algebraic variety of dimension $r$ in $\mathbb{C}^m$ and $\mathcal{F}$ a hypersurface not containing $\mathcal{Q}$, then every irreducible component of $\mathcal{Q} \cap \mathcal{F}$ has dimension $r - 1$; see Exercise 1.8 in [32]. Take any $\mathcal{Q} \in \mathscr{C}$. If $\mathcal{Q}$ is different from $\{0\}$, we fix a non-zero point $\xi_{\mathcal{Q}} \in \mathcal{Q}$. If $\mathcal{H}$ is a hyperplane that does not contain $\xi_{\mathcal{Q}}$ then $\dim \mathcal{Q} \cap \mathcal{H} = \dim \mathcal{Q} - 1$. Now let $\mathcal{H}_1$ be a hyperplane through the origin that does not contain $\xi_{\mathcal{Q}}$ for every $\mathcal{Q} \in \mathscr{C}$. Such a hyperplane certainly exists because $\mathscr{C}$ has finite cardinality. Then $\dim \mathcal{Q}_j \cap \mathcal{H}_1 = \max\{\dim \mathcal{Q}_j - 1, 0\}$ for every $j \in [t]$. Next, let $\mathscr{C}$ be the set of the irreducible components of all $\mathcal{Q}_j \cap \mathcal{H}_1$'s and define $\mathcal{H}_2$ in a similar fashion, by avoiding a non-zero point for every $\{0\} \neq \mathcal{Q} \in \mathscr{C}$ and making sure that $\mathcal{H}_1 \cap \mathcal{H}_2$ has codimension 2. Then $\dim \mathcal{Q}_j \cap \mathcal{H}_1 \cap \mathcal{H}_2 = \max\{\dim \mathcal{Q}_j - 2, 0\}$ and repeating this step $c$ times in total gives a $\mathcal{V} = \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_c$ in the aforementioned open set.

## 3. Proofs

### *3.1. Proof of Theorem 1*

For every $\tau_1, \tau_2 \in \mathcal{T}$ it suffices to exhibit a non-empty Zariski open subset of $\mathrm{Gr}_{\mathbb{H}}(n, m)$ on which every subspace $\mathcal{V}$ satisfies $\mathrm{hsp}(\mathcal{V}, \{\tau_1, \tau_2\})$, which will imply $\mathrm{hsp}(\mathcal{V}, \mathcal{T})$ since $\mathcal{T}$ is a finite set and the intersection of finitely many non-empty Zariski open subsets of $\mathrm{Gr}_{\mathbb{H}}(n, m)$ is also non-empty and open.

We will divide the proof of Theorem 1 into two cases $\dim(\mathscr{E}_{(\tau_1,\tau_2),1}) \leq m - n$ and $\dim(\mathscr{E}_{(\tau_1,\tau_2),1}) > m - n$. Assume that we are in the first case. Then we have the following proposition whose proof is placed at §3.1.1.

**Proposition 4.** *In addition to the hypotheses of Theorem 1, further assume* $\dim(\mathscr{E}_{(\tau_1,\tau_2),1}) \leq m - n$. *Then there is an $n$-dimensional subspace $\mathcal{V}'$ of $\mathbb{H}^m$ which satisfies* $\dim(\tau_1(\mathcal{V}') + \tau_2(\mathcal{V}')) = 2n$.

With the subspace $\mathcal{V}'$ of Proposition 4 we get that the set $\mathbb{U}_1$ of subspaces $\mathcal{V} \in \mathrm{Gr}_{\mathbb{H}}(n, m)$ for which $\dim(\tau_1(\mathcal{V}) + \tau_2(\mathcal{V})) = 2n$ for every $\mathcal{V} \in \mathbb{U}_1$ is non-empty. Let $A \in \mathbb{H}^{m \times n}$ have $\mathcal{V} \in \mathbb{U}_1$ as its column space. Then we see that $\dim(\tau_1(\mathcal{V}) + \tau_2(\mathcal{V})) = 2n$ is equivalent to $\mathrm{rank}[T_1 A \ T_2 A] = 2n$. It follows that $\mathbb{U}_1$ is a Zariski open subset of $\mathrm{Gr}_{\mathbb{H}}(n, m)$ defined by the non-vanishing of some $2n \times 2n$ minor of $[T_1 A \ T_2 A]$. We next show that $\mathrm{hsp}(\mathcal{V}, \{\tau_1, \tau_2\})$ holds for every $\mathcal{V} \in \mathbb{U}_1$. Indeed, let $v_1, v_2 \in \mathcal{V}$ be such that $\tau_1(v_1) = \tau_2(v_2)$. But $\dim(\tau_1(\mathcal{V}) + \tau_2(\mathcal{V})) = 2n$ implies that $\tau_1(\mathcal{V}) \cap \tau_2(\mathcal{V}) = 0$ and that $\dim(\tau_1(\mathcal{V})) = \dim(\tau_2(\mathcal{V})) = \dim(\mathcal{V}) = n$. So $\ker(\tau_1) \cap \mathcal{V} = 0$ and $\ker(\tau_2) \cap \mathcal{V} = 0$. We conclude that $\tau_1(v_1) = \tau_2(v_2) = 0$ and moreover $v_1 = v_2 = 0$.

We tackle the second case $\dim(\mathscr{E}_{(\tau_1,\tau_2),1}) > m - n$ by the following proposition (proved in §3.1.2).

**Proposition 5.** *In addition to the hypotheses of Theorem 1, further assume* $\dim(\mathscr{E}_{(\tau_1,\tau_2),1}) = m - n_0 > m - n$. *There are two subspaces $\mathcal{V}_0 \subset \mathcal{V}$ of $\mathbb{H}^m$ of dimension $n_0$ and $n$ respectively so that* $\dim(\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{V})) = n_0 + n$.

With $\mathcal{V}_0$ and $\mathcal{V}$ of Proposition 5 we know that

$$\mathbb{U}_2 := \{(\mathcal{V}_0, \mathcal{V}) \in \mathrm{F}_{\mathbb{H}}(n_0, n, m) : \dim_{\mathbb{H}}(\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{V})) = n_0 + n\} \tag{12}$$

is not empty. Similar to the argument that $\mathbb{U}_1$ is Zariski open, $\mathbb{U}_2$ is also Zariski open in $\mathrm{F}_{\mathbb{H}}(n_0, n, m)$.

The final step is to show that we have $\mathrm{hsp}(\mathcal{V}, \{\tau_1, \tau_2\})$ for any $\mathcal{V} \in \phi(\mathbb{U}_2)$. Let $\tau_1(v_1) = \tau_2(v_2)$ with $v_1, v_2 \in \mathcal{V}$. There is some $\mathcal{V}_0$ such that $(\mathcal{V}_0, \mathcal{V}) \in \mathbb{U}_2$. So $\dim(\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{V})) = n_0 + n$, and in particular $\dim(\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{V}_0)) = 2n_0$. Then we have $\mathcal{V}_0 \cap \mathscr{E}_{(\tau_1,\tau_2),1} = 0$. Since $\dim_{\mathbb{R}}(\mathscr{E}_{(\tau_1,\tau_2),1} \cap \mathbb{R}^m) = \dim_{\mathbb{C}}(\mathscr{E}_{(\tau_1,\tau_2),1})$, the intersection of $\mathcal{V}$ and $\mathscr{E}_{(\tau_1,\tau_2),1}$ has dimension at least $n + m - n_0 - m = n - n_0$ (regardless of $\mathbb{H}$), and we get $\dim(\mathcal{V} \cap \mathscr{E}_{(\tau_1,\tau_2),1}) = n - n_0$. With $\mathcal{V}_0 \cap \mathscr{E}_{(\tau_1,\tau_2),1} \cap \mathcal{V} = 0$ we now have that $\mathcal{V}$ is a direct sum of $\mathcal{V}_0$ and $\mathscr{E}_{(\tau_1,\tau_2),1} \cap \mathcal{V}$. Write $v_1$ as a sum of two vectors $v_0$ and $w$ in $\mathcal{V}_0$ and $\mathcal{V} \cap \mathscr{E}_{(\tau_1,\tau_2),1}$ respectively. Then $\tau_1(v_1) = \tau_2(v_2)$ implies $\tau_1(v_0) = \tau_2(v_2) - \tau_1(w) = \tau_2(v_2 - w)$. Since $v_0 \in \mathcal{V}_0$ and $(v_2 - w) \in \mathcal{V}$ and $(\mathcal{V}_0, \mathcal{V}) \in \mathbb{U}_2$, the definition of $\mathbb{U}_2$ implies that $v_0 = v_2 - w = 0$. That is, $v_1 = w + v_0 = w = v_2$. $\square$

#### *3.1.1. Proof of Proposition 4*

For $\mathbb{H} = \mathbb{R}$ it suffices to show that $\mathrm{rank}[T_1 A \ T_2 A] = 2n$ with $T_1, T_2 \in \mathbb{R}^{m \times m}$ for some $A \in \mathbb{R}^{m \times n}$, that is, some $2n \times 2n$ minor of $[T_1 A \ T_2 A]$ is a nonzero polynomial with real coefficients in entries of $A$. This holds true whenever there is some $A^* \in \mathbb{C}^{m \times n}$ at which the evaluation of some $2n \times 2n$ minor of $[T_1 A \ T_2 A]$ is non-zero. Hence it suffices to prove Proposition 4 for $\mathbb{H} = \mathbb{C}$.

Let us first define[6] a series of subspaces and discover their properties. Write $\mathcal{R}_0 := \mathbb{C}^m$, $\mathcal{F}_0 := \mathbb{C}^m$. For any non-negative integer $j$ define

$$\begin{aligned}
\mathcal{G}_{j+1} &= \tau_1(\mathcal{R}_j \cap \mathcal{F}_j) \cap \tau_2(\mathcal{R}_j \cap \mathcal{F}_j),\\
\mathcal{R}_{j+1} &= \tau_1^{-1}(\mathcal{G}_{j+1}) \cap \mathcal{R}_j \cap \mathcal{F}_j,\\
\mathcal{F}_{j+1} &= \tau_2^{-1}(\mathcal{G}_{j+1}) \cap \mathcal{R}_j \cap \mathcal{F}_j.
\end{aligned} \tag{13}$$

The next lemma shows that what were defined in (13) are three chains of subspaces.

**Lemma 4.** *We have $\mathcal{R}_{j+1} \subset \mathcal{R}_j \cap \mathcal{F}_j \subset \mathcal{R}_j$ and $\mathcal{F}_{j+1} \subset \mathcal{R}_j \cap \mathcal{F}_j \subset \mathcal{F}_j$ and $\mathcal{G}_{j+2} \subset \mathcal{G}_{j+1}$ and $\tau_1(\mathcal{R}_{j+1}) = \tau_2(\mathcal{F}_{j+1}) = \mathcal{G}_{j+1}$ for any non-negative integer $j$.*

Proof. By definition (13) we have $\mathcal{R}_{j+1} \subset \mathcal{R}_j \cap \mathcal{F}_j \subset \mathcal{R}_j$ and $\mathcal{F}_{j+1} \subset \mathcal{R}_j \cap \mathcal{F}_j \subset \mathcal{F}_j$. Note also $\mathcal{R}_{j+1} \cap \mathcal{F}_{j+1} \subset \mathcal{R}_j \cap \mathcal{F}_j$. This further implies that

$$\mathcal{G}_{j+2} = \tau_1(\mathcal{R}_{j+1} \cap \mathcal{F}_{j+1}) \cap \tau_2(\mathcal{R}_{j+1} \cap \mathcal{F}_{j+1}) \subset \tau_1(\mathcal{R}_j \cap \mathcal{F}_j) \cap \tau_2(\mathcal{R}_j \cap \mathcal{F}_j) = \mathcal{G}_{j+1}. \tag{14}$$

Also noting that $\mathcal{G}_{j+1} \subset \tau_1(\mathcal{R}_j \cap \mathcal{F}_j) \subset \mathrm{im}(\tau_1)$ and $\mathcal{R}_{j+1} \subset \tau_1^{-1}(\mathcal{G}_{j+1})$ for any non-negative integer $j$, we have $\tau_1(\mathcal{R}_{j+1}) \subset \tau_1(\tau_1^{-1}(\mathcal{G}_{j+1})) = \mathcal{G}_{j+1}$. To show $\mathcal{G}_{j+1} \subset \tau_1(\mathcal{R}_{j+1})$ we let $z \in \mathcal{G}_{j+1} = \tau_1(\mathcal{R}_j \cap \mathcal{F}_j) \cap \tau_2(\mathcal{R}_j \cap \mathcal{F}_j)$. In particular $z \in \tau_1(\mathcal{R}_j \cap \mathcal{F}_j)$. So there is some $w \in \mathcal{R}_j \cap \mathcal{F}_j$ such that $\tau_1(w) = z$. Then $w \in \tau_1^{-1}(z) \cap \mathcal{R}_j \cap \mathcal{F}_j$. But $\tau_1^{-1}(z) \subset \tau_1^{-1}(\mathcal{G}_{j+1})$, and this suggests $w \in \tau_1^{-1}(\mathcal{G}_{j+1}) \cap \mathcal{R}_j \cap \mathcal{F}_j = \mathcal{R}_{j+1}$. With $\tau_1(w) = z$ we see that $z \in \tau_1(\mathcal{R}_{j+1})$. To conclude, we get $\tau_1(\mathcal{R}_{j+1}) = \mathcal{G}_{j+1}$. A similar derivation gives $\tau_2(\mathcal{F}_{j+1}) = \mathcal{G}_{j+1}$. $\square$

It is expected that those chains will be stable eventually at some point and yield some special property.

**Lemma 5.** *There is a non-negative integer $\alpha$ for which $\mathcal{R}_\alpha = \mathcal{F}_\alpha$ and $\tau_1(\mathcal{R}_\alpha) = \tau_2(\mathcal{R}_\alpha)$.*

Proof (Lemma 5). From Lemma 4 we see two chains $\cdots \subset \mathcal{R}_{j+1} \subset \mathcal{R}_j \subset \cdots \subset \mathcal{R}_0$ and $\cdots \subset \mathcal{F}_{j+1} \subset \mathcal{F}_j \subset \cdots \subset \mathcal{F}_0$. Since the subspaces $\mathcal{R}_0$ and $\mathcal{F}_0$ are of finite dimension $m$, these two chains stabilize respectively, that is, there exist some non-negative integers $\alpha_1$ and $\alpha_2$ such that for any integers $j_1 \geq \alpha_1$ and $j_2 \geq \alpha_2$ it holds that $\mathcal{R}_{j_1} = \mathcal{R}_{j_1+1}$ and $\mathcal{F}_{j_2} = \mathcal{F}_{j_2+1}$. Let $\alpha := \max\{\alpha_1, \alpha_2\}$. We then have $\mathcal{R}_\alpha = \mathcal{R}_{\alpha+1}$ and $\mathcal{F}_\alpha = \mathcal{F}_{\alpha+1}$. With Lemma 4 we obtain $\mathcal{R}_{\alpha+1} \subset \mathcal{R}_\alpha \cap \mathcal{F}_\alpha \subset \mathcal{R}_\alpha = \mathcal{R}_{\alpha+1}$. This implies $\mathcal{R}_\alpha = \mathcal{R}_\alpha \cap \mathcal{F}_\alpha$. Similarly we get $\mathcal{F}_\alpha = \mathcal{R}_\alpha \cap \mathcal{F}_\alpha$. It follows that $\mathcal{R}_\alpha = \mathcal{F}_\alpha$. Lemma 4 then gives $\tau_1(\mathcal{R}_\alpha) = \tau_2(\mathcal{R}_\alpha)$. $\square$

Lemma 5 suggests to focus on the chain acsending from $\mathcal{R}_\alpha$ say

$$\mathcal{R}_\alpha \subset \mathcal{R}_{\alpha-1} \cap \mathcal{F}_{\alpha-1} \subset \mathcal{R}_{\alpha-1} \subset \cdots \subset \mathcal{R}_1 \subset \mathcal{R}_0 \cap \mathcal{F}_0 \subset \mathcal{R}_0 = \mathbb{C}^m. \tag{15}$$

Our proof rests on chain (15) and the strategy is as follows. We start by exhibiting a subspace say $\mathcal{W}_j$ inside some subspace $\mathcal{R}_j$ on chain (15) such that $\dim(\tau_1(\mathcal{W}_j) + \tau_2(\mathcal{W}_j)) = 2\dim(\mathcal{W}_j)$. While $\mathcal{W}_j$ may have smaller dimension than the required $n$, we provide devices with which we are capable of ascending the chain and recursively extending $\mathcal{W}_j$ to a larger subspace say $\mathcal{W}_{j-1}$ contained in $\mathcal{R}_{j-1}$, and at the same time keeping the property $\dim(\tau_1(\mathcal{W}_{j-1}) + \tau_2(\mathcal{W}_{j-1})) = 2\dim(\mathcal{W}_{j-1})$, eventually obtaining the desired subspace $\mathcal{V} = \mathcal{W}_0$.

The first step is to find some subspace $\mathcal{W}_j$ to start with. The next lemma provides such one.

**Lemma 6.** *In addition to the hypotheses of Proposition 4, suppose $\dim(\mathcal{R}_\alpha) > m - n$ , then there is a subspace $\mathcal{W}_\alpha$ of $\mathcal{R}_\alpha$ of dimension $[\dim(\mathcal{R}_\alpha) - (m-n)]$ such that $\dim(\tau_1(\mathcal{W}_\alpha) + \tau_2(\mathcal{W}_\alpha)) = 2\dim(\mathcal{W}_\alpha)$.*

[6] Defintion (13) is based on our observation on the problem, with the main motivation of finding two subspaces whose images are the same under $\tau_1, \tau_2$ (Lemma 5). We found that this definition is similar to, yet more complicated than, the Wong sequence, invented in [33] and later used in e.g., [34, 35]. We name a detailed comparison of the two constructions as future work.

Proof (Lemma 6). The first implication of $\mathcal{R}_\alpha = \mathcal{F}_\alpha$ and $\tau_1(\mathcal{R}_\alpha) = \tau_2(\mathcal{R}_\alpha) = \mathcal{G}_\alpha$ is that

$$\dim(\ker(\tau_1) \cap \mathcal{R}_\alpha) = \dim(\mathcal{R}_\alpha) - \dim(\mathcal{G}_\alpha) = \dim(\ker(\tau_2) \cap \mathcal{R}_\alpha). \tag{16}$$

This implies that $(m-n) + \dim(\mathcal{G}_\alpha) - \dim(\mathcal{R}_\alpha) = \mathrm{rank}(\tau_1) - n > 0$. Note that $\mathcal{U}_{\tau_1,\tau_2}$ has the same dimension as its closure[7] $\overline{\mathcal{U}}_{\tau_1,\tau_2}$ and $\mathscr{E}_{(\tau_1,\tau_2),1}$ is of dimension at most $(m-n)$. By Lemma 3 there is a subspace $\mathcal{H}$ of $\mathcal{R}_\alpha$ of dimension $\dim(\mathcal{G}_\alpha)$ which intersects both $\ker(\tau_1)$ and $\ker(\tau_2)$ only at zero, and such that

$$\dim(\overline{\mathcal{U}}_{\tau_1,\tau_2} \cap \mathcal{H}) \leq (m-n) + \dim(\mathcal{G}_\alpha) - \dim(\mathcal{R}_\alpha), \tag{17}$$
$$\dim(\mathscr{E}_{(\tau_1,\tau_2),1} \cap \mathcal{H}) \leq (m-n) + \dim(\mathcal{G}_\alpha) - \dim(\mathcal{R}_\alpha). \tag{18}$$

Let $\tau_1|_{\mathcal{H}}$ and $\tau_2|_{\mathcal{H}}$ be restrictions of $\tau_1$ and $\tau_2$ on $\mathcal{H}$ respectively. Since $\ker(\tau_1) \cap \mathcal{H} = \ker(\tau_2) \cap \mathcal{H} = 0$, we have that $\tau_1|_{\mathcal{H}}$ and $\tau_2|_{\mathcal{H}}$ are isomorphisms from $\mathcal{H}$ to $\mathcal{G}_\alpha$. Recalling that $\mathcal{H}$ is a subspace of $\mathcal{R}_\alpha$ and $\tau_1(\mathcal{R}_\alpha) = \tau_2(\mathcal{R}_\alpha)$, we have $\tau_1(\mathcal{H}) = \mathcal{G}_\alpha = \tau_2(\mathcal{H})$. So $\tau_{\mathcal{H}} := (\tau_1|_{\mathcal{H}})^{-1}\tau_2|_{\mathcal{H}}$ is an isomorphism of $\mathcal{H}$. Since $\dim(\mathcal{H}) = \dim(\mathcal{G}_\alpha) = \dim(\mathcal{R}_\alpha) - \dim(\ker(\tau_1) \cap \mathcal{R}_\alpha) \geq \dim(\mathcal{R}_\alpha) - (m - \mathrm{rank}(\tau_1)) > \dim(\mathcal{R}_\alpha) - (m-n)$, $\mathcal{H}$ can contain a subspace of dimension $[\dim(\mathcal{R}_\alpha) - (m-n)]$. Recalling that $\mathscr{E}_{\tau,\lambda}$ denotes the eigenspace of a linear map $\tau$ of $\mathbb{C}^m$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$, we will need Lemma 3 of [2].

**Lemma 7 (Lemma 5 of [2], restated).** *Suppose for any $\lambda \in \mathbb{C}$ and some linear map $\tau$ of $\mathcal{H}$ we have $\dim(\mathscr{E}_{\tau,\lambda}) \leq \dim(\mathcal{H}) - [\dim(\mathcal{R}_\alpha) - (m-n)]$ and $\dim(\mathcal{H}) \geq 2[\dim(\mathcal{R}_\alpha) - (m-n)]$. There is a subspace $\mathcal{W}_\alpha$ of $\mathcal{H}$ of dimension $[\dim(\mathcal{R}_\alpha) - (m-n)]$ such that $\dim(\mathcal{W}_\alpha + \tau(\mathcal{W}_\alpha)) = 2\dim(\mathcal{W}_\alpha)$.*

Next we will show that $\tau_{\mathcal{H}}$ satisfies the conditions of Lemma 7. First note that

$$\begin{aligned}
\dim(\mathcal{H}) \geq 2[\dim(\mathcal{R}_\alpha) - (m-n)] &\Leftrightarrow 2m - 2n \geq \dim(\mathcal{R}_\alpha) + \dim(\mathcal{R}_\alpha) - \dim(\mathcal{G}_\alpha) && (19)\\
&\Leftrightarrow 2m - 2n \geq \dim(\mathcal{R}_\alpha) + \dim(\ker(\tau_1) \cap \mathcal{R}_\alpha) && (20)\\
&\Leftarrow 2m - 2n \geq \dim(\mathcal{R}_\alpha) + \dim(\ker(\tau_1)) && (21)\\
&\Leftrightarrow (\mathrm{rank}(\tau_1) - 2n) + (m - \dim(\mathcal{R}_\alpha)) \geq 0. && (22)
\end{aligned}$$

Another condition of Lemma 7 is that for any $\lambda \in \mathbb{C}$ it holds that

$$\dim(\mathscr{E}_{\tau_{\mathcal{H}},\lambda}) \leq \dim(\mathcal{H}) - [\dim(\mathcal{R}_\alpha) - (m-n)], \tag{23}$$

which is true for the following reason. When $\lambda = 0$, the eigenspace $\mathscr{E}_{\tau_{\mathcal{H}},0}$ is exactly $\ker(\tau_{\mathcal{H}})$, which is zero since $\tau_{\mathcal{H}}$ is an isomorphism. When $\lambda \neq 0$, note that $\mathscr{E}_{\tau_{\mathcal{H}},\lambda}$ is exactly the set of all points $v$'s of $\mathbb{C}^m$ satisfying $\tau_1|_{\mathcal{H}}(v) = \lambda\tau_2|_{\mathcal{H}}(v)$. That is, $\tau_1(v) = \lambda\tau_2(v)$ and $v \in \mathcal{H}$. Recalling the definition of $\mathcal{Y}_{\tau_1,\tau_2}$ we get $v \in \mathcal{Y}_{\tau_1,\tau_2} \cap \mathcal{H}$. But the definitions of $\mathcal{H}$ and $\mathcal{U}_{\tau_1,\tau_2}$ imply $v \in \overline{\mathcal{U}}_{\tau_1,\tau_2} \cap \mathcal{H}$ or $v \in \mathscr{E}_{(\tau_1,\tau_2),1} \cap \mathcal{H}$. It follows that $\mathscr{E}_{\tau_{\mathcal{H}},\lambda}$ is a subset of $\overline{\mathcal{U}}_{\tau_1,\tau_2} \cap \mathcal{H}$ or $\mathscr{E}_{(\tau_1,\tau_2),1} \cap \mathcal{H}$, both of which have dimension at most $(m-n) + \dim(\mathcal{G}_\alpha) - \dim(\mathcal{R}_\alpha)$ as per (17) and (18). Together, we have proved (23) for any $\lambda \in \mathbb{C}$. Then Lemma 7 is applicable and we get the subspace $\mathcal{W}_\alpha \subset \mathcal{H}$ of dimension $[\dim(\mathcal{R}_\alpha) - (m-n)]$ such that $\mathcal{W}_\alpha + \tau_{\mathcal{H}}(\mathcal{W}_\alpha)$ has dimension $2[\dim(\mathcal{R}_\alpha) - (m-n)]$.

Since $\tau_1|_{\mathcal{H}}$ is an isomorphism from $\mathcal{H}$ to $\mathcal{G}_\alpha$ and $\mathcal{W}_\alpha + \tau_{\mathcal{H}}(\mathcal{W}_\alpha)$ is a subspace of $\mathcal{H}$, we see that $\tau_1|_{\mathcal{H}}(\mathcal{W}_\alpha + \tau_{\mathcal{H}}(\mathcal{W}_\alpha)) = \tau_1|_{\mathcal{H}}(\mathcal{W}_\alpha) + \tau_2|_{\mathcal{H}}(\mathcal{W}_\alpha)$ also has dimension $2[\dim(\mathcal{R}_\alpha) - (m-n)]$. But note that $\tau_1|_{\mathcal{H}}(\mathcal{W}_\alpha) + \tau_2|_{\mathcal{H}}(\mathcal{W}_\alpha) = \tau_1(\mathcal{W}_\alpha) + \tau_2(\mathcal{W}_\alpha)$. We finished the proof. □

Note that Lemma 6 places in addition the dimension constraint $\dim(\mathcal{R}_\alpha) > m - n$ on $\mathcal{R}_\alpha$. When $\alpha = 0$ we get $\dim(\mathcal{R}_\alpha) = m$, and we finished the proof of Proposition 4 by Lemma 6. Assume $\alpha > 0$ in what follows.

Then, this dimension constraint on $\mathcal{R}_\alpha$ might be voilated because by construction (Lemma 5) it is quite possible for $\mathcal{R}_\alpha$ to be the trivial subspace 0. On the other hand, the converse $\dim(\mathcal{R}_\alpha) \leq m - n$ implies a dimension transition on chain (15) in the sense that there exist some non-negative integers $\beta$ or $\gamma$ satisfying $\dim(\mathcal{R}_{\beta+1}) \leq m - n < \dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta)$ or $\dim(\mathcal{R}_\gamma \cap \mathcal{F}_\gamma) \leq m - n < \dim(\mathcal{R}_\gamma)$. It is at this transition that we can obtain the subspace of interest as an alternative starting point, via the next two lemmas.

[7] The vanishing ideal of $\overline{\mathcal{U}}_{\tau_1,\tau_2}$ is homogeneous, so $\overline{\mathcal{U}}_{\tau_1,\tau_2}$ contains 0; See Lemma 13. Hence Lemma 3 can be applied.

**Lemma 8 (Dimension Transition-1).** *In addition to the hypotheses of Proposition 4, suppose for some non-negative integer* $\beta$ *that* $\dim(\mathcal{R}_{\beta+1}) \leq m-n < \dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta)$*, then there exists a subspace* $\mathcal{Z}_\beta$ *of* $\mathcal{R}_\beta \cap \mathcal{F}_\beta$ *of dimension* $[\dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta) - (m-n)]$ *such that* $\dim(\tau_1(\mathcal{Z}_\beta) + \tau_2(\mathcal{Z}_\beta)) = 2\dim(\mathcal{Z}_\beta)$*.*

**Lemma 9 (Dimension Transition-2).** *In addition to the hypotheses of Proposition 4, suppose for some non-negative integer* $\gamma$ *that* $\dim(\mathcal{R}_\gamma \cap \mathcal{F}_\gamma) \leq m-n < \dim(\mathcal{R}_\gamma)$*. Then there exists a subspace* $\mathcal{W}_\gamma$ *of* $\mathcal{R}_\gamma$ *of dimension* $[\dim(\mathcal{R}_\gamma) - (m-n)]$ *such that* $\dim(\tau_1(\mathcal{W}_\gamma) + \tau_2(\mathcal{W}_\gamma)) = 2\dim(\mathcal{W}_\gamma)$*.*

Proof (Lemma 8). Note that we have

$$[\dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta) - (m-n)] + \dim(\ker(\tau_1)) = \dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta) + (n - \operatorname{rank}(\tau_2)) < \dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta) \tag{24}$$

and similarly

$$[\dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta) - (m-n)] + \dim(\ker(\tau_2)) \leq \dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta). \tag{25}$$

Also note that

$$[\dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta) - (m-n)] + \dim(\mathcal{R}_{\beta+1}) = \dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta) + [\dim(\mathcal{R}_{\beta+1}) - (m-n)] \leq \dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta). \tag{26}$$

Lemma 3 implies that there is a subspace $\mathcal{Z}_\beta$ of $\mathcal{R}_\beta \cap \mathcal{F}_\beta$ of dimension $[\dim(\mathcal{R}_\beta \cap \mathcal{F}_\beta) - (m-n)]$ which intersects $\ker(\tau_1)$ and $\ker(\tau_2)$ and $\mathcal{R}_{\beta+1}$ only at zero, respectively. This gives $\dim(\tau_1(\mathcal{Z}_\beta)) = \dim(\tau_2(\mathcal{Z}_\beta)) = \dim(\mathcal{Z}_\beta)$.

It now suffices to prove $\tau_1(\mathcal{Z}_\beta) \cap \tau_2(\mathcal{Z}_\beta) = 0$. Let $\tau_1(v_1) = \tau_2(v_2)$ for some $v_1, v_2 \in \mathcal{Z}_\beta$. Hence $\tau_1(v_1)$ is contained in $\tau_1(\mathcal{R}_\beta \cap \mathcal{F}_\beta)$ and $\tau_1(v_1) = \tau_2(v_2)$ is contained in $\tau_2(\mathcal{R}_\beta \cap \mathcal{F}_\beta)$. In sum we have

$$\tau_1(v_1) \in \tau_1(\mathcal{R}_\beta \cap \mathcal{F}_\beta) \cap \tau_2(\mathcal{R}_\beta \cap \mathcal{F}_\beta) = \mathcal{G}_{\beta+1}. \tag{27}$$

This implies $v_1 \in \tau_1^{-1}(\mathcal{G}_{\beta+1})$ and so

$$v_1 \in \mathcal{Z}_\beta \cap \tau_1^{-1}(\mathcal{G}_{\beta+1}) = \mathcal{Z}_\beta \cap \tau_1^{-1}(\mathcal{G}_{\beta+1}) \cap \mathcal{R}_\beta \cap \mathcal{F}_\beta = \mathcal{R}_{\beta+1} \cap \mathcal{Z}_\beta = 0. \tag{28}$$

That is, $v_1 = 0$. Then $0 = \tau_2(v_2)$, which implies $v_2 \in \ker(\tau_2) \cap \mathcal{Z}_\beta = 0$. We proved $\tau_1(\mathcal{Z}_\beta) \cap \tau_2(\mathcal{Z}_\beta) = 0$. ☐

Proof (Lemma 9). Clearly $\gamma \neq 0$. Note that we have

$$[\dim(\mathcal{R}_\gamma) - (m-n)] + \dim(\ker(\tau_1)) = \dim(\mathcal{R}_\gamma) + (m - \operatorname{rank}(\tau_1)) - (m-n) < \dim(\mathcal{R}_\gamma) \tag{29}$$

and similarly

$$[\dim(\mathcal{R}_\gamma) - (m-n)] + \dim(\ker(\tau_2)) \leq \dim(\mathcal{R}_\gamma). \tag{30}$$

Also note that

$$[\dim(\mathcal{R}_\gamma) - (m-n)] + \dim(\mathcal{R}_\gamma \cap \mathcal{F}_\gamma) \leq \dim(\mathcal{R}_\gamma) + [\dim(\mathcal{R}_\gamma \cap \mathcal{F}_\gamma) - (m-n)] \leq \dim(\mathcal{R}_\gamma). \tag{31}$$

Consequently, by Lemma 3, there exists a subspace $\mathcal{W}_\gamma$ of $\mathcal{R}_\gamma$ of dimension $[\dim(\mathcal{R}_\gamma) - (m-n)]$ which intersects $\ker(\tau_1)$ and $\ker(\tau_2)$ and $\mathcal{R}_\gamma \cap \mathcal{F}_\gamma$ only at zero, respectively. By Lemma 4 we get $\tau_1(\mathcal{W}_\gamma) \subset \tau_1(\mathcal{R}_\gamma) = \mathcal{G}_\gamma$. Recalling definition (13) and $\mathcal{W}_\gamma \subset \mathcal{R}_\gamma$ we obtain

$$\begin{aligned}
\mathcal{W}_\gamma \cap \tau_2^{-1}(\tau_1(\mathcal{W}_\gamma)) &\subset \mathcal{W}_\gamma \cap \tau_2^{-1}(\mathcal{G}_\gamma) && (32)\\
&= \mathcal{W}_\gamma \cap \tau_2^{-1}(\mathcal{G}_\gamma) \cap \mathcal{R}_\gamma && (33)\\
&= \mathcal{W}_\gamma \cap \tau_2^{-1}(\mathcal{G}_\gamma) \cap \tau_1^{-1}(\mathcal{G}_\gamma) \cap \mathcal{R}_{\gamma-1} \cap \mathcal{F}_{\gamma-1} && (34)\\
&= \mathcal{W}_\gamma \cap \tau_1^{-1}(\mathcal{G}_\gamma) \cap \mathcal{F}_\gamma && (35)\\
&\subset \mathcal{W}_\gamma \cap \mathcal{F}_\gamma = \mathcal{W}_\gamma \cap \mathcal{R}_\gamma \cap \mathcal{F}_\gamma = 0. && (36)
\end{aligned}$$

In short $\mathcal{W}_\gamma \cap \tau_2^{-1}(\tau_1(\mathcal{W}_\gamma)) = 0$, and it follows from Lemma 1 that $\tau_2(\mathcal{W}_\gamma) \cap \tau_1(\mathcal{W}_\gamma) = 0$. Recalling that $\mathcal{W}_\gamma \cap \ker(\tau_1) = 0$ and $\mathcal{W}_\gamma \cap \ker(\tau_2) = 0$, we conclude with $\dim(\tau_1(\mathcal{W}_\gamma) + \tau_2(\mathcal{W}_\gamma)) = 2\dim(\mathcal{W}_\gamma)$. ☐

Table 1: Three different cases and Lemmas that address them.

| Cases | Lemmas |
|---|---|
| $m-n<\dim(\mathcal{R}_\alpha)$ | Lemma 6 |
| $\dim(\mathcal{R}_{\beta+1})\le m-n<\dim(\mathcal{R}_\beta\cap\mathcal{F}_\beta)$ | Lemma 8 |
| $\dim(\mathcal{R}_\gamma\cap\mathcal{F}_\gamma)\le m-n<\dim(\mathcal{R}_\gamma)$ | Lemma 9 |

As summarized in Table 1, we have obtained three subspaces $\mathcal{W}_\alpha$, $\mathcal{Z}_\beta$, and $\mathcal{W}_\gamma$ contained in $\mathcal{R}_\alpha$, $\mathcal{R}_\beta\cap\mathcal{F}_\beta$ , and $\mathcal{R}_\gamma$, respectively, depending on whether the aforementioned dimension transition exists (Lemma 6) or if so where it happens (Lemmas 8 and 9). Note that $\mathcal{Z}_\beta$ is a subspace of $\mathcal{Z}_\beta\subset\mathcal{R}_\beta\cap\mathcal{F}_\beta$ satisfying

$$\mathscr{P}(\mathcal{Z}_\beta):\ \dim(\mathcal{Z}_\beta)=[\dim(\mathcal{R}_\beta\cap\mathcal{F}_\beta)-(m-n)]\ \text{ and }\ \dim(\tau_1(\mathcal{Z}_\beta)+\tau_2(\mathcal{Z}_\beta))=2\dim(\mathcal{Z}_\beta). \tag{37}$$

Let $\mu$ be either $\alpha$ or $\gamma$ then we see that the subspace $\mathcal{W}_\mu$ of $\mathcal{R}_\mu$ satisfies

$$\mathscr{P}(\mathcal{W}_\mu):\ \dim(\mathcal{W}_\mu)=[\dim(\mathcal{R}_\mu)-(m-n)]\ \text{ and }\ \dim(\tau_1(\mathcal{W}_\mu)+\tau_2(\mathcal{W}_\mu))=2\dim(\mathcal{W}_\mu). \tag{38}$$

Thus the three cases in Table 1 give rise to two possible chains say

$$\begin{array}{l}\mathcal{R}_\beta\cap\mathcal{F}_\beta\subset\mathcal{R}_\beta\subset\cdots\subset\mathcal{R}_0=\mathbb{C}^m\\ \quad\cup\\ \quad\mathcal{Z}_\beta\end{array} \tag{39}$$

and

$$\begin{array}{l}\mathcal{R}_\mu\subset\mathcal{R}_{\mu-1}\cap\mathcal{F}_{\mu-1}\subset\cdots\subset\mathcal{R}_0=\mathbb{C}^m\\ \cup\\ \mathcal{W}_\mu\end{array} \tag{40}$$

where we added $\mathcal{Z}_\beta$ or $\mathcal{W}_\mu$ into chain (15).

The next step is to extend $\mathcal{Z}_\beta$ or $\mathcal{W}_\gamma$ in chain (39) or (40), recursively if necessary.

**Lemma 10 (Extension-1).** *In addition to the hypotheses of Proposition 4, suppose for some non-negative integer $j$ that $\dim(\mathcal{R}_j\cap\mathcal{F}_j)>m-n$ and that there exists a subspace $\mathcal{Z}_j$ of $\mathcal{R}_j\cap\mathcal{F}_j$ satisfying*

$$\mathscr{P}(\mathcal{Z}_j):\ \dim(\mathcal{Z}_j)=[\dim(\mathcal{R}_j\cap\mathcal{F}_j)-(m-n)]\ \textit{ and }\ \dim(\tau_1(\mathcal{Z}_j)+\tau_2(\mathcal{Z}_j))=2\dim(\mathcal{Z}_j). \tag{41}$$

*Then there exists a subspace $\mathcal{W}_j$ of $\mathcal{R}_j$ satisfying $\mathcal{Z}_j\subset\mathcal{W}_j$ and*

$$\mathscr{P}(\mathcal{W}_j):\ \dim(\mathcal{W}_j)=[\dim(\mathcal{R}_j)-(m-n)]\ \textit{ and }\ \dim(\tau_1(\mathcal{W}_j)+\tau_2(\mathcal{W}_j))=2\dim(\mathcal{W}_j). \tag{42}$$

**Lemma 11 (Extension-2).** *In addition to the hypotheses of Proposition 4, suppose for some non-negative integer $j$ that $\dim(\mathcal{R}_{j+1})>m-n$, and that there exists a subspace $\mathcal{W}_{j+1}$ of $\mathcal{R}_{j+1}$ satisfying*

$$\mathscr{P}(\mathcal{W}_{j+1}):\ \dim(\mathcal{W}_{j+1})=[\dim(\mathcal{R}_{j+1})-(m-n)]\ \textit{ and }\ \dim(\tau_1(\mathcal{W}_{j+1})+\tau_2(\mathcal{W}_{j+1}))=2\dim(\mathcal{W}_{j+1}). \tag{43}$$

*Then there exists a subspace $\mathcal{Z}_j$ of $\mathcal{R}_j\cap\mathcal{F}_j$ satisfying $\mathcal{W}_{j+1}\subset\mathcal{Z}_j$ and*

$$\mathscr{P}(\mathcal{Z}_j):\ \dim(\mathcal{Z}_j)=[\dim(\mathcal{R}_j\cap\mathcal{F}_j)-(m-n)]\ \textit{ and }\ \dim(\tau_1(\mathcal{Z}_j)+\tau_2(\mathcal{Z}_j))=2\dim(\mathcal{Z}_j). \tag{44}$$

Proof (Lemma 10). Clearly $\mathcal{R}_j\cap\mathcal{F}_j\subset\mathcal{R}_j$. If $\mathcal{R}_j\cap\mathcal{F}_j=\mathcal{R}_j$ we are done by letting $\mathcal{W}_j=\mathcal{Z}_j$. In what follows we assume $\dim(\mathcal{R}_j)>\dim(\mathcal{R}_j\cap\mathcal{F}_j)$. This implies $j\neq 0$.

Note that $[\dim(\mathcal{R}_j) - \dim(\mathcal{R}_j \cap \mathcal{F}_j)] + \dim(\mathcal{R}_j \cap \mathcal{F}_j) = \dim(\mathcal{R}_j)$. The inverse image $\tau_1^{-1}(\tau_1(\mathcal{Z}_j) + \tau_2(\mathcal{Z}_j))$ has dimension at most $(m - \operatorname{rank}(\tau_1)) + 2[\dim(\mathcal{R}_j \cap \mathcal{F}_j) - (m-n)]$. So the summation $[\dim(\mathcal{R}_j) - \dim(\mathcal{R}_j \cap \mathcal{F}_j)] + \dim(\tau_1^{-1}(\tau_1(\mathcal{Z}_j) + \tau_2(\mathcal{Z}_j)))$ is at most

$$\dim(\mathcal{R}_j) + [2n - \operatorname{rank}(\tau_1)] + [\dim(\mathcal{R}_j \cap \mathcal{F}_j) - m] \leq \dim(\mathcal{R}_j). \tag{45}$$

Hence, by Lemma 3, there is a subspace $\mathcal{W}'_j$ of $\mathcal{R}_j$ of dimension $[\dim(\mathcal{R}_j) - \dim(\mathcal{R}_j \cap \mathcal{F}_j)]$ which intersects both $\mathcal{R}_j \cap \mathcal{F}_j$ and $\tau_1^{-1}(\tau_1(\mathcal{Z}_j) + \tau_2(\mathcal{Z}_j))$ only at zero, respectively. The latter with Lemma 1 implies that $\tau_1(\mathcal{W}'_j) \cap [\tau_1(\mathcal{Z}_j) + \tau_2(\mathcal{Z}_j)] = 0$ and $\mathcal{W}'_j \cap \ker(\tau_1) = 0$. So $[\tau_1(\mathcal{Z}_j) + \tau_2(\mathcal{Z}_j)] + \tau_1(\mathcal{W}'_j)$ is of dimension $2[\dim(\mathcal{R}_j \cap \mathcal{F}_j) - (m-n)] + [\dim(\mathcal{R}_j) - \dim(\mathcal{R}_j \cap \mathcal{F}_j)]$. In other words,

$$\dim(\tau_1(\mathcal{Z}_j + \mathcal{W}'_j) + \tau_2(\mathcal{Z}_j)) = \dim(\mathcal{R}_j) + \dim(\mathcal{R}_j \cap \mathcal{F}_j) - 2(m-n). \tag{46}$$

Since $\mathcal{Z}_j \subset \mathcal{R}_j \cap \mathcal{F}_j$ we see that $\tau_2(\mathcal{Z}_j) \subset \tau_2(\mathcal{F}_j) = \mathcal{G}_j$. With $\tau_1(\mathcal{Z}_j + \mathcal{W}'_j) \subset \tau_1(\mathcal{R}_j) = \mathcal{G}_j$, we obtain that $\tau_1(\mathcal{Z}_j + \mathcal{W}'_j) + \tau_2(\mathcal{Z}_j)$ is a subset of $\mathcal{G}_j$, and consequently

$$\begin{aligned}
\mathcal{W}'_j \cap \tau_2^{-1}(\tau_1(\mathcal{Z}_j + \mathcal{W}'_j) + \tau_2(\mathcal{Z}_j)) &\subset \mathcal{W}'_j \cap \tau_2^{-1}(\mathcal{G}_j) && (47)\\
&= \mathcal{W}'_j \cap \tau_2^{-1}(\mathcal{G}_j) \cap \mathcal{R}_j && (48)\\
&= \mathcal{W}'_j \cap \tau_2^{-1}(\mathcal{G}_j) \cap \tau_1^{-1}(\mathcal{G}_j) \cap \mathcal{R}_{j-1} \cap \mathcal{F}_{j-1} && (49)\\
&= \mathcal{W}'_j \cap \tau_1^{-1}(\mathcal{G}_j) \cap \mathcal{F}_j && (50)\\
&\subset \mathcal{W}'_j \cap \mathcal{F}_j && (51)\\
&= \mathcal{W}'_j \cap \mathcal{F}_j \cap \mathcal{R}_j = 0. && (52)
\end{aligned}$$

From (47) to (48) we used that $\mathcal{W}'_j$ is a subset of $\mathcal{R}_j$, and from (48) to (50) we used the definitions of $\mathcal{R}_j$ and $\mathcal{F}_j$. In short we have $\mathcal{W}'_j \cap \tau_2^{-1}(\tau_1(\mathcal{Z}_j + \mathcal{W}'_j) + \tau_2(\mathcal{Z}_j)) = 0$, which with Lemma 1 yields $\tau_2(\mathcal{W}'_j) \cap [\tau_1(\mathcal{Z}_j + \mathcal{W}'_j) + \tau_2(\mathcal{Z}_j)] = 0$ and $\mathcal{W}'_j \cap \ker(\tau_2) = 0$. Recalling (46) it follows that $[\tau_1(\mathcal{Z}_j + \mathcal{W}'_j) + \tau_2(\mathcal{Z}_j)] + \tau_2(\mathcal{W}'_j)$ is of dimension $[\dim(\mathcal{R}_j) + \dim(\mathcal{R}_j \cap \mathcal{F}_j) - 2(m-n)] + [\dim(\mathcal{R}_j) - \dim(\mathcal{R}_j \cap \mathcal{F}_j)]$, that is,

$$\dim(\tau_1(\mathcal{Z}_j + \mathcal{W}'_j) + \tau_2(\mathcal{Z}_j + \mathcal{W}'_j)) = 2\dim(\mathcal{R}_j) - 2(m-n). \tag{53}$$

By letting $\mathcal{W}_j = \mathcal{Z}_j + \mathcal{W}'_j$ we finished the proof. □

Proof (Lemma 11). Recalling $\mathcal{R}_{j+1} = \tau_1^{-1}(\mathcal{G}_{j+1}) \cap \mathcal{R}_j \cap \mathcal{F}_j$, we see that $\mathcal{R}_{j+1} \subset \mathcal{R}_j \cap \mathcal{F}_j$. If $\mathcal{R}_{j+1} = \mathcal{R}_j \cap \mathcal{F}_j$ we are done by letting $\mathcal{Z}_j = \mathcal{W}_{j+1}$. Hence we assume $\dim(\mathcal{R}_j \cap \mathcal{F}_j) > \dim(\mathcal{R}_{j+1})$ in what follows.

Note that $[\dim(\mathcal{R}_j \cap \mathcal{F}_j) - \dim(\mathcal{R}_{j+1})] + \dim(\mathcal{R}_{j+1}) = \dim(\mathcal{R}_j \cap \mathcal{F}_j)$. Since the inverse image $\tau_2^{-1}(\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1}))$ has dimension at most $(m - \operatorname{rank}(\tau_2)) + 2[\dim(\mathcal{R}_{j+1}) - (m-n)]$, we can see that $[\dim(\mathcal{R}_j \cap \mathcal{F}_j) - \dim(\mathcal{R}_{j+1})] + \dim(\tau_2^{-1}(\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1})))$ is no more than

$$\dim(\mathcal{R}_j \cap \mathcal{F}_j) + [\dim(\mathcal{R}_{j+1}) - m] + [2n - \operatorname{rank}(\tau_2)] \leq \dim(\mathcal{R}_j \cap \mathcal{F}_j). \tag{54}$$

Consequently, by Lemma 3, there exists a subspace $\mathcal{Z}'_j$ of $\mathcal{R}_j \cap \mathcal{F}_j$ of dimension $[\dim(\mathcal{R}_j \cap \mathcal{F}_j) - \dim(\mathcal{R}_{j+1})]$ which intersects both $\mathcal{R}_{j+1}$ and $\tau_2^{-1}(\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1}))$ only at zero, respectively. The later with Lemma 1 implies that $\mathcal{Z}'_j \cap \ker(\tau_2) = 0$ and $\tau_2(\mathcal{Z}'_j) \cap [\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1})] = 0$. Hence $\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j) = [\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1})] + \tau_2(\mathcal{Z}'_j)$ is of dimension $2[\dim(\mathcal{R}_{j+1}) - (m-n)] + [\dim(\mathcal{R}_j \cap \mathcal{F}_j) - \dim(\mathcal{R}_{j+1})]$. Simplifying it, we get

$$\dim(\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j)) = \dim(\mathcal{R}_{j+1}) + \dim(\mathcal{R}_j \cap \mathcal{F}_j) - 2(m-n). \tag{55}$$

Since $\mathcal{W}_{j+1}$ is a subspace of $\mathcal{R}_{j+1}$, we have $\tau_1(\mathcal{W}_{j+1}) \subset \tau_1(\mathcal{R}_{j+1}) = \mathcal{G}_{j+1} \subset \tau_2(\mathcal{R}_j \cap \mathcal{F}_j)$. Since $\mathcal{W}_{j+1} + \mathcal{Z}'_j$ is a subspace of $\mathcal{R}_j \cap \mathcal{F}_j$, we get $\tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j) \subset \tau_2(\mathcal{R}_j \cap \mathcal{F}_j)$. Together we obtain that $\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j) \subset$

$\tau_2(\mathcal{R}_j \cap \mathcal{F}_j)$ and thus $\tau_1^{-1}(\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j))$ is a subspace of $\tau_1^{-1}(\tau_2(\mathcal{R}_j \cap \mathcal{F}_j))$. Since $\mathcal{Z}'_j$ is a subspace of $\mathcal{R}_j \cap \mathcal{F}_j$, we see that

$$\begin{aligned}
\mathcal{Z}'_j \cap \tau_1^{-1}(\tau_2(\mathcal{R}_j \cap \mathcal{F}_j)) &= \mathcal{Z}'_j \cap \tau_1^{-1}(\tau_2(\mathcal{R}_j \cap \mathcal{F}_j)) \cap \mathcal{R}_j \cap \mathcal{F}_j && (56)\\
&= \mathcal{Z}'_j \cap \tau_1^{-1}(\tau_1(\mathcal{R}_j \cap \mathcal{F}_j) \cap \tau_2(\mathcal{R}_j \cap \mathcal{F}_j)) \cap \mathcal{R}_j \cap \mathcal{F}_j && (57)\\
&= \mathcal{Z}'_j \cap \tau_1^{-1}(\mathcal{G}_{j+1}) \cap \mathcal{R}_j \cap \mathcal{F}_j = \mathcal{Z}'_j \cap \mathcal{R}_{j+1} = 0. && (58)
\end{aligned}$$

This in particular implies $\mathcal{Z}'_j \cap \tau_1^{-1}(\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j)) = 0$, which with Lemma 1 we know that $\tau_1(\mathcal{Z}'_j) \cap [\tau_1(\mathcal{W}_{j+1}) + \tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j)] = 0$ and $\ker(\tau_1) \cap \mathcal{Z}'_j = 0$. With (55) it follows that $\tau_1(\mathcal{W}_{j+1} + \mathcal{Z}'_j) + \tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j)$ has dimension $[\dim(\mathcal{R}_{j+1}) + \dim(\mathcal{R}_j \cap \mathcal{F}_j) - 2(m-n)] + [\dim(\mathcal{R}_j \cap \mathcal{F}_j) - \dim(\mathcal{R}_{j+1})]$. After simplification we get

$$\dim(\tau_1(\mathcal{W}_{j+1} + \mathcal{Z}'_j) + \tau_2(\mathcal{W}_{j+1} + \mathcal{Z}'_j)) = 2[\dim(\mathcal{R}_j \cap \mathcal{F}_j) - (m-n)]. \tag{59}$$

Letting $\mathcal{Z}_j = \mathcal{W}_{j+1} + \mathcal{Z}'_j$ we finished the proof. ☐

We are ready to summarize the proof of Proposition 4. To repeat we can construct a chain of subspaces say (39) or (40), where $\mathcal{Z}_\beta$ or $\mathcal{W}_\mu$ satisfies $\mathscr{P}(\mathcal{Z}_\beta)$ or $\mathscr{P}(\mathcal{W}_\mu)$, respectively. Lemmas 10 and 11 can then be used iteratively to extend chain (39) or (40), with (39) yielding the chain

$$\begin{array}{ccccccccc}
\mathcal{R}_\beta \cap \mathcal{F}_\beta & \subset & \mathcal{R}_\beta & \subset & \mathcal{R}_{\beta-1} \cap \mathcal{F}_{\beta-1} & \subset & \cdots & \subset & \mathcal{R}_0 = \mathbb{C}^m \\
\cup & & \cup & & \cup & & & & \cup \\
\mathcal{Z}_\beta & \subset & \mathcal{W}_\beta & \subset & \mathcal{Z}_{\beta-1} & \subset & \cdots & \subset & \mathcal{W}_0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathscr{P}(\mathcal{Z}_\beta) & & \mathscr{P}(\mathcal{W}_\beta) & & \mathcal{P}(\mathcal{Z}_{\beta-1}) & & \cdots & & \mathscr{P}(\mathcal{W}_0)
\end{array} \tag{60}$$

or (40) giving rise to

$$\begin{array}{ccccccccc}
\mathcal{R}_\gamma & \subset & \mathcal{R}_{\gamma-1} \cap \mathcal{F}_{\gamma-1} & \subset & \mathcal{R}_{\gamma-1} & \subset & \cdots & \subset & \mathcal{R}_0 = \mathbb{C}^m \\
\cup & & \cup & & \cup & & & & \cup \\
\mathcal{W}_\gamma & \subset & \mathcal{Z}_{\gamma-1} & \subset & \mathcal{W}_{\gamma-1} & \subset & \cdots & \subset & \mathcal{W}_0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathscr{P}(\mathcal{W}_\gamma) & & \mathscr{P}(\mathcal{Z}_{\gamma-1}) & & \mathcal{P}(\mathcal{W}_{\gamma-1}) & & \cdots & & \mathscr{P}(\mathcal{W}_0)
\end{array} \tag{61}$$

where both in (60) and (61) each $\mathcal{W}_j$ satisfies $\mathscr{P}(\mathcal{W}_j)$ defined in (38). In both cases $\mathcal{W}_0$ satisfies $\dim(\mathcal{W}_0) = [\dim(\mathcal{R}_0) - (m-n)] = n$ and $\dim(\tau_1(\mathcal{W}_0) + \tau_2(\mathcal{W}_0)) = 2\dim(\mathcal{W}_0) = 2n$. The proof is complete. ☐

*3.1.2. Proof of Proposition 5*

Note that $\operatorname{rank}(\tau_1) \geq 2n > 2n_0$ and $\operatorname{rank}(\tau_2) \geq 2n > 2n_0$ and $\dim(\mathcal{U}_{\tau_1,\tau_2}) \leq m-n < m-n_0$. Invoking Proposition 4, we get a subspace $\mathcal{V}_0$ of $\mathrm{Gr}_{\mathbb{H}}(n_0, m)$ which satisfies $\dim(\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{V}_0)) = 2n_0$. The dimension of the subspace $\tau_2^{-1}(\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{V}_0))$ is at most $(m - \operatorname{rank}(\tau_2)) + 2n_0$, and

$$(n-n_0) + [(m - \operatorname{rank}(\tau_2)) + 2n_0] = m + (n + n_0 - \operatorname{rank}(\tau_2)) < m + 2n - \operatorname{rank}(\tau_2) \leq m. \tag{62}$$

Thus, there is a subspace $\mathcal{W}$ of $\mathbb{H}^m$ of dimension $n-n_0$ such that $\mathcal{W}$ intersects the subspace $\tau_2^{-1}(\tau_1(\mathcal{V}_0)+\tau_2(\mathcal{V}_0))$ only at zero. With Lemma 1 we get $\mathcal{W} \cap \ker(\tau_2) = 0$ and $\tau_2(\mathcal{W}) \cap [\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{V}_0)] = 0$. Hence $\mathcal{W}$ intersects $\mathcal{V}_0$ only at zero, $\dim(\mathcal{W} + \mathcal{V}_0) = n$, and

$$\dim(\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{W} + \mathcal{V}_0)) = \dim(\tau_2(\mathcal{W})) + \dim(\tau_1(\mathcal{V}_0) + \tau_2(\mathcal{V}_0)) = n - n_0 + 2n_0 = n + n_0. \tag{63}$$

Letting $\mathcal{V} = \mathcal{W} + \mathcal{V}_0$ we are done. ☐

*3.2. Proof of Theorem 2*

The following lemma is elementary.

**Lemma 12.** *For $\mathcal{I} \subset [\ell]$ with $n_{\mathcal{I}} \leq m$, there is a non-empty Zariski open subset of $\prod_{t\in\mathcal{I}} \mathrm{Gr}_{\mathbb{H}}(n_t, m)$ whose element $(\mathcal{V}_1, \ldots, \mathcal{V}_{|\mathcal{I}|})$ consists of independent subspaces, i.e., $\mathcal{V}_j \cap \mathcal{V}_{\mathcal{I}\setminus\{j\}} = 0$, $\forall j \in \mathcal{I}$.*

Similar to the proof of Theorem 1 it suffices to consider two linear maps $\tau_1$ and $\tau_2$ of $\mathcal{T}$ and prove $\mathrm{hsp}(\mathcal{A}_{\mathscr{I}}, \{\tau_1, \tau_2\})$. And for the same reason we need only to prove $\mathrm{hsp}(\mathcal{V}_{\mathcal{I}_1} \cup \mathcal{V}_{\mathcal{I}_2}, \{\tau_1, \tau_2\})$.

Since $n_{\mathcal{I}_1} + n_{\mathcal{I}_2} \leq 2n \leq m$, by Lemma 12 there is a non-empty Zariski open subset $\mathbb{U}_0$ of $\prod_{t\in\mathcal{I}_1\cup\mathcal{I}_2} \mathrm{Gr}_{\mathbb{H}}(n_t, m)$ whose element consists of independent subspaces. Let $\mathbb{U}_1$ contain all elements $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2}$ of $\mathbb{U}_0$ satisfying

$$\dim(\tau_2^{-1}(\tau_1(\mathcal{V}_{\mathcal{I}_1})) + \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2} + \mathcal{V}_{\mathcal{I}_2\setminus\mathcal{I}_1}) = \dim(\tau_2^{-1}(\tau_1(\mathcal{V}_{\mathcal{I}_1})) + \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2}) + n_{\mathcal{I}_2\setminus\mathcal{I}_1}. \tag{64}$$

Note that $\mathbb{U}_1$ is Zariski open since it is defined by the non-vanishing of the maximal minors of $[A_1, A_2]$, where $A_1$ and $A_2$ are basis matrices of $\tau_2^{-1}(\tau_1(\mathcal{V}_{\mathcal{I}_1})) + \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2}$ and $\mathcal{V}_{\mathcal{I}_2\setminus\mathcal{I}_1}$ respectively. Moreover, This Zariski open subset $\mathbb{U}_1$ of $\mathbb{U}_0$ is not-empty since the right-hand side of (64) is at most

$$m - r_2 + n_{\mathcal{I}_1} + n_{\mathcal{I}_1\cap\mathcal{I}_2} + n_{\mathcal{I}_2\setminus\mathcal{I}_1} \leq m - r_2 + n_{\mathcal{I}_1} + n_{\mathcal{I}_2} \leq m - r_2 + 2n \leq m, \tag{65}$$

and thus we can always choose a $\mathcal{V}_{\mathcal{I}_2\setminus\mathcal{I}_1}$ such that it intersects $\tau_2^{-1}(\tau_1(\mathcal{V}_{\mathcal{I}_1})) + \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2}$ only at zero. Consequently $\mathbb{U}_1$ is a non-empty Zariski open subset of $\prod_{t\in\mathcal{I}_1\cup\mathcal{I}_2} \mathrm{Gr}_{\mathbb{H}}(n_t, m)$, on which every element $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2}$ satisfies (64), which implies $\mathcal{V}_{\mathcal{I}_2\setminus\mathcal{I}_1} \cap [\tau_2^{-1}(\tau_1(\mathcal{V}_{\mathcal{I}_1})) + \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2}] = 0$. Changing the role of $\mathcal{I}_1$ and $\mathcal{I}_2$ we obtain another non-empty Zariski open subset $\mathbb{U}_2 \in \prod_{t\in\mathcal{I}_1\cup\mathcal{I}_2} \mathrm{Gr}_{\mathbb{H}}(n_t, m)$, on which every element $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2}$ satisfies $\mathcal{V}_{\mathcal{I}_1\setminus\mathcal{I}_2} \cap [\tau_2^{-1}(\tau_1(\mathcal{V}_{\mathcal{I}_2})) + \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2}] = 0$. Denote by $\mathbb{U}_3$ the intersection of $\mathbb{U}_1$ and $\mathbb{U}_2$, which is non-empty open.

Then we show that, for any $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2} \in \mathbb{U}_3$, the relation $\tau_1(v_1) = \tau_2(v_2)$ where $v_1 \in \mathcal{V}_{\mathcal{I}_1}$ and $v_2 \in \mathcal{V}_{\mathcal{I}_1} \cup \mathcal{V}_{\mathcal{I}_2}$ implies $v_2 \in \mathcal{V}_{\mathcal{I}_1}$. Suppose for the sake of contradiction that $v_2 \in \mathcal{V}_{\mathcal{I}_2}\setminus\mathcal{V}_{\mathcal{I}_1}$. Since $\mathcal{V}_{\mathcal{I}_2} = \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2} + \mathcal{V}_{\mathcal{I}_2\setminus\mathcal{I}_1}$, we get $v_2 = w_0 + w_2$ for some $w_0 \in \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2}$ and $w_2 \in \mathcal{V}_{\mathcal{I}_2\setminus\mathcal{I}_1}$. This implies $w_0 + w_2 \in \tau_2^{-1}(\tau_1(\mathcal{V}_{\mathcal{I}_1}))$ and $w_2 \in \tau_2^{-1}(\tau_1(\mathcal{V}_{\mathcal{I}_1})) + \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2}$. By the definition of $\mathbb{U}_3$ we have $w_2 = 0$. That is, $v_2 = w_0 \in \mathcal{V}_{\mathcal{I}_1\cap\mathcal{I}_2} \subset \mathcal{V}_{\mathcal{I}_1}$, a contradiction. Similarly we can prove for any $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2} \in \mathbb{U}_3$ that the relation $\tau_1(v_1) = \tau_2(v_2)$ where $v_1 \in \mathcal{V}_{\mathcal{I}_2}$ and $v_2 \in \mathcal{V}_{\mathcal{I}_1} \cup \mathcal{V}_{\mathcal{I}_2}$ implies $v_2 \in \mathcal{V}_{\mathcal{I}_2}$. To conclude, for any $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2} \in \mathbb{U}_3$, the property $\mathrm{hsp}(\mathcal{V}_{\mathcal{I}_1} \cup \mathcal{V}_{\mathcal{I}_2}, \{\tau_1, \tau_2\})$ reduces to $\mathrm{hsp}(\mathcal{V}_{\mathcal{I}_1}, \{\tau_1, \tau_2\})$ and $\mathrm{hsp}(\mathcal{V}_{\mathcal{I}_2}, \{\tau_1, \tau_2\})$, we next show the former for $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2}$ in some non-empty Zariski open subset of $\mathbb{U}_3$, from which the latter will follow by symmetry.

Since $n_{\mathcal{I}_1} \leq n \leq m/2$, Theorem 1 implies that there is a non-empty Zariski open set $\mathbb{O}_1$ of $\mathrm{Gr}_{\mathbb{H}}(n_{\mathcal{I}_1}, m)$ on which every $\mathcal{V} \in \mathbb{O}_1$ satisfies $\mathrm{hsp}(\mathcal{V}, \{\tau_1, \tau_2\})$. Consider the surjective polynomial map from $\mathbb{U}_0$ to $\mathrm{Gr}_{\mathbb{H}}(n_{\mathcal{I}_1}, m)$ which sends the independent subspaces $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2}$ to the sum $\mathcal{V}_{\mathcal{I}_1}$. The inverse image of $\mathbb{O}_1$ under this surjective polynomial map is also Zariski open and non-empty. The intersection $\mathbb{U}$ of this inverse image with $\mathbb{U}_3$ is non-empty Zariski open in $\prod_{t\in\mathcal{I}_1} \mathrm{Gr}_{\mathbb{H}}(n_t, m)$, and $\mathrm{hsp}(\mathcal{V}_{\mathcal{I}_1}, \{\tau_1, \tau_2\})$ holds for $(\mathcal{V}_t)_{t\in\mathcal{I}_1\cup\mathcal{I}_2} \in \mathbb{U}$.

*3.3. Proof of Proposition 3*

It has been shown in [1, 2] that for every $S_1, S_2 \in \mathcal{S}_{r,m}$ there is some projection $P$ onto the column space of $S_2$ satisfying $\dim(\mathcal{U}_{PT_1,T_2}) \leq m - \lfloor r/2 \rfloor$. With $r \geq 2n$ we have $\dim(\mathcal{U}_{S_1,S_2}) \leq \dim(\mathcal{U}_{PS_1,S_2}) \leq m - r$, and this proves the first two statements of Proposition 3. The last statment follows similarly from [2]. For the third statement, let $B_1, B_2 \in \mathcal{B}_m$. Then for every $w \in \mathcal{U}^{\pm}_{B_1,B_2}$ we have $B_1 w = \lambda B_2 w$ for some $\lambda$, that is, $Bw = \lambda w$ where $B = (B_2)^{-1} B_1$ is a sign matrix. Hence $w$ is an eigenvector of $B$ corresponding to the eigenvalue $\lambda$. The only possibility is that $\lambda = \pm 1$. That is, $w$ is in the null space of $B_1 \pm B_2$, which does not intersect $\mathcal{U}_{B_1,B_2}$. It is only possible that $\mathcal{U}_{B_1,B_2} = \varnothing$.

*3.4. Proof of Corollary 1*

The first statement is a special case of the second, and here we prove the second. Theorem 1 and Proposition 3 imply that when $r \geq 2n$ there is a non-empty Zariski open subset $\mathbb{O}$ of $\mathrm{Gr}_{\mathbb{R}}(n, m)$ on which every subspace $\mathcal{V}$ satisfies $\mathrm{hsp}(\mathcal{V}, \mathcal{S}_{r,m})$. The inverse image $\mathbb{U}$ of $\mathbb{O}$ under the polynomial map from the non-empty set of all full column rank matrices of $\mathbb{R}^{m\times n}$ to $\mathrm{Gr}_{\mathbb{R}}(n, m)$ that sends a matrix $A$ to its column space $R(A)$ is also non-empty and Zariski open. Then for every $m \times n$ matrix $A \in \mathbb{U}$ we have $\mathrm{hsp}(R(A), \mathcal{S}_{r,m})$. Since every $A \in \mathbb{U}$ is of full column rank, we also have $\mathrm{hsp}(\mathbb{R}^n, A)$. We proved the first two statements. Using Propositions 2 and 3 the last two statements can be proved similarly.

*3.5. Proof of Corollary 2*

We present the proof for the second statement, which implies the first, and from which the last two statements follow similarly. Let $r \geq 2k$. Then Propositon 3 implies that $\dim(\mathcal{U}_{S_1,S_2}) \leq m-k$ for any rank-$r$ selections $S_1, S_2 \in \mathcal{S}_{r,m}$. Let $s = \binom{n}{k}$ and let $\mathscr{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_s)$ be an ordered set of all subsets of $[n]$ of cardinality $k$. Applying Theorem 2 we get a non-empty Zariski open subset $\mathbb{O}$ of $\prod_{j\in[n]} \mathrm{Gr}_{\mathbb{R}}(1,m)$ such that for any $\mathcal{A} = (\mathcal{V}_1, \ldots, \mathcal{V}_n) \in \mathbb{O}$, the property $\mathrm{hsp}(\overline{\mathcal{A}_{\mathscr{I}}}, \mathcal{S}_{r,m})$ holds.

Consider the map from $\mathbb{R}^{m\times n}$ to $\prod_{j\in[n]} \mathrm{Gr}_{\mathbb{R}}(1,m)$ which sends the $j$-th column of a matrix to its column space. This map is surjective, and so the inverse image $\mathbb{U}_1$ of $\mathbb{O}$ under this map is also non-empty Zariski open. Let $\mathbb{U}_2$ be the set of matrices $A$'s of $\mathbb{R}^{m\times n}$ such that any $2k$ different columns of $A$ are linearly independent (if $n < 2k$ then let $\mathbb{U}_2$ be the set of all full column rank matrices of $\mathbb{R}^{m\times n}$). Let $\mathbb{U} = \mathbb{U}_1 \cap \mathbb{U}_2$. Then $\mathbb{U}$ is a non-empty Zariski open subset of $\mathbb{R}^{m\times n}$.

Let $A \in \mathbb{U}$. We will show $\mathrm{hsp}(\overline{\mathcal{K}_{\mathscr{I}}}, \mathcal{S}_{r,m}A)$. Let us view $A$ as a linear map $\tau_A$ such that $\tau_A(x) = Ax$. The choice of $\mathbb{U}_1$ (and so of $\mathbb{U}$) implies that $\mathrm{hsp}(\tau_A(\overline{\mathcal{K}_{\mathscr{I}}}), \mathcal{S}_{r,m})$ holds true. That is, for any $k$-sparse vectors $x, x' \in \overline{\mathcal{K}_{\mathscr{I}}}$ and $S, S' \in \mathcal{S}_{r,m}$ satisfying $SAx = S'Ax'$, we have $Ax = Ax'$. For some $J \subset [n]$ use $A_J \in \mathbb{R}^{m\times|J|}$ to denote the sub-matrix of $A$ formed by $|J|$ columns of $A$ indexed by $J$, and also write $x_J$ and $x'_J$ for the sub-vectors of $x$ and $x'$ indexed by $J$, respectively. Then, for $J_1$ and $J_2$ two index sets corresponding to the locations of nonzero entries of $x$ and $x'$ respectively, we have that $A_{J_1}x_{J_1} = A_{J_2}x'_{J_2}$. That is, $A_{J_1\cap J_2}(x_{J_1\cap J_2} - x'_{J_1\cap J_2}) + A_{J_1-J_2}x_{J_1-J_2} - A_{J_2-J_1}x'_{J_2-J_1} = 0$. But $|J_1\cap J_2| + |J_1 - J_2| + |J_2 - J_1| \leq 2k$ and any $2k$ distinct columns of $A$ are linearly independent, so we must have $x_{J_1\cap J_2} = x'_{J_1\cap J_2}$ and $x_{J_1-J_2} = 0$ and $x'_{J_2-J_1} = 0$. That is, $x = x'$. We finished the proof. □

*3.6. Proof of Theorem 3*

Since the conditions of Theorem 1 are fulfilled, there is a non-empty Zariski open subset of $\mathrm{Gr}_{\mathbb{R}}(n,m)$ on which every subspace $\mathcal{V}'$ satisfies $\mathrm{hsp}(\mathcal{V}', \mathcal{T})$. Let $\mathcal{V}$ be a subspace in that Zariski open set. We first rewrite (7) into a convenient form. Note that

$$\hat{\tau} = \operatorname*{argmin}_{\tau\in\mathcal{T}} \min_{v\in\mathcal{V}} \|\overline{y} - \tau(v)\|_2 \tag{66}$$

$$= \operatorname*{argmin}_{\tau\in\mathcal{T}} \min_{w\in\tau(\mathcal{V})} \|\overline{y} - w\|_2^2 \tag{67}$$

$$= \operatorname*{argmin}_{\tau\in\mathcal{T}} \min_{w\in\tau(\mathcal{V})} \{\|w\|_2^2 - 2\langle\overline{y}, w\rangle\} \tag{68}$$

$$= \operatorname*{argmin}_{\tau\in\mathcal{T}} \min_{\lambda>0} \min_{w\in\tau(\mathcal{V}):\|w\|_2=\lambda} \{\lambda^2 - 2\langle\overline{y}, w\rangle\} \tag{69}$$

$$= \operatorname*{argmin}_{\tau\in\mathcal{T}} \min_{\lambda>0} \{\lambda^2 - 2\lambda\|\overline{y}\|_2 \max_{w\in\tau(\mathcal{V}):\|w\|_2=\lambda} \frac{\langle\overline{y}, w\rangle}{\|\overline{y}\|_2\|w\|_2}\} \tag{70}$$

$$= \operatorname*{argmin}_{\tau\in\mathcal{T}} \min_{\lambda>0} \{\lambda^2 - 2\lambda\|\overline{y}\|_2 \cos(\overline{y}, \tau(\mathcal{V}))\} \tag{71}$$

$$= \operatorname*{argmax}_{\tau\in\mathcal{T}} \cos(\overline{y}, \tau(\mathcal{V})). \tag{72}$$

We then prove $\hat{\tau} \in \mathcal{T}_1$. It suffices to show for any $\tau_2 \in \mathcal{T} \setminus \mathcal{T}_1$ that there is some $\tau_1 \in \mathcal{T}_1$ so that

$$\cos(\overline{y}, \tau_1(\mathcal{V})) > \cos(\overline{y}, \tau_2(\mathcal{V})), \tag{73}$$

which surely holds, if the following stronger condition

$$\frac{\langle\overline{y}, y\rangle}{\|\overline{y}\|_2\|y\|_2} > \cos(\overline{y}, \tau_2(\mathcal{V})) \tag{74}$$

is satisfied. Letting $w_2 \in \tau_2(\mathcal{V})$ with $\|w_2\|_2 = 1$ be such that $\langle\overline{y}, w_2\rangle/\|\overline{y}\|_2 = \cos(\overline{y}, \tau_2(\mathcal{V}))$ and noticing that $\overline{y} = y + \epsilon$, condition (74) is equivalent to

$$\frac{\langle\overline{y}, y\rangle}{\|\overline{y}\|_2\|y\|_2} > \frac{\langle\overline{y}, w_2\rangle}{\|\overline{y}\|_2} \Leftrightarrow \frac{\langle\overline{y}, y\rangle}{\|y\|_2^2} > \frac{\langle\overline{y}, w_2\rangle}{\|y\|_2} \Leftrightarrow 1 > \frac{\langle y, w_2\rangle}{\|y\|_2} + \frac{\langle\epsilon, w_2\rangle}{\|y\|_2} - \frac{\langle\epsilon, y\rangle}{\|y\|_2^2}. \tag{75}$$

Note that $\frac{\langle y, w_2\rangle}{\|y\|_2} \leq \cos(y, \tau_2(\mathcal{V}))$, $\langle \epsilon, w_2\rangle \leq \|\epsilon\|_2$, and $-\langle \epsilon, y\rangle \leq \|\epsilon\|_2\|y\|_2$. For (75) to be true, it is enough to satisfy the following condition

$$1 > \cos(y, \tau_2(\mathcal{V})) + 2\frac{\|\epsilon\|_2}{\|y\|_2} \Leftrightarrow \|y\|_2(1 - \cos(y, \tau_2(\mathcal{V}))) > 2\|\epsilon\|_2, \tag{76}$$

which is already fulfilled by (9). Hence $\hat{\tau} \in \mathcal{T}_1$. So we have $y = \tau^*(v^*) = \hat{\tau}(v)$ for some $v \in \mathcal{V}$. This implies $v = v^*$, and thus $y = \hat{\tau}(v^*)$. On the other hand, according to (7), we have

$$\hat{v} = \operatorname*{argmin}_{v\in\mathcal{V}} \|y + \epsilon - \hat{\tau}(v)\|_2. \tag{77}$$

Thus, for $\hat{x} \in \mathbb{R}^n$ and $x^* \in \mathbb{R}^n$ satisfying $\hat{v} = A\hat{x}$ and $v^* = Ax^*$, we get that

$$\hat{x} = \operatorname*{argmin}_{x\in\mathbb{R}^n} \|y + \epsilon - \hat{T}Ax\|_2 = (\hat{T}A)^{\dagger}(y + \epsilon), \tag{78}$$

where we used the fact that $\hat{T}A$ is of full column rank. Recalling $y = \hat{\tau}(v^*) = \hat{T}Ax^*$, we obtain

$$\hat{x} = (\hat{T}A)^{\dagger}(\hat{T}Ax^* + \epsilon) = x^* + (\hat{T}A)^{\dagger}\epsilon, \tag{79}$$

and consequently $\hat{v} = v^* + A(\hat{T}A)^{\dagger}\epsilon$. □

## 4. Appendix

### *4.1. Proof of Proposition 1*

Any $\mathcal{V}' \in \mathrm{Gr}_{\mathbb{C}}(n, m)$ that intersects $\mathcal{U}_{\tau_1,\tau_2}$ violates $\mathrm{hsp}(\mathcal{V}', \mathcal{T})$. So it suffices to show $\mathcal{V} \cap \mathcal{U}_{\tau_1,\tau_2}$ is not empty for a generic $\mathcal{V} \in \mathrm{Gr}_{\mathbb{C}}(n, m)$. This can be seen from Lemma 13 by substituting $\mathcal{U}_{\tau_1,\tau_2}$ into $Y\backslash Z$.

**Lemma 13.** *Let $Z \subset Y$ be two algebraic varieties of $\mathbb{C}^m$ defined by homogeneous ideals $J$ and $I$ of $\mathfrak{R} := \mathbb{C}[w_1, \ldots, w_m]$, respectively, where $J \not\subset (w_1, \ldots, w_m)$. If $\dim(Y\backslash Z) > m - n$, then a generic subspace $\mathcal{V} \subset \mathbb{C}^m$ of dimension $n$ intersects $Y\backslash Z$.*

Proof (Lemma 13). Let $U = Y\backslash Z$. Then the vanishing ideal of the closure $\overline{U}$ of $U$ is $\mathfrak{a} := I : J^{\infty}$, where $\mathfrak{a}$ is the saturation of $I$ with respect to $J$. Hence we have $\dim(\mathfrak{R}/\mathfrak{a}) = \dim(\overline{U}) = \dim(U) > m - n$. Since $\mathfrak{a}$ is homogeneous, we have for $m - n$ general linear forms $\ell_1, \ldots, \ell_{m-n}$ of $\mathfrak{R}$ that (see, e.g., Lemma 1 of [1])

$$\dim(\mathfrak{R}/\mathfrak{a} + (\ell_1, \ldots, \ell_{m-n})) = \max\{\dim(\mathfrak{R}/\mathfrak{a}) - (m - n), 0\} = \dim(\mathfrak{R}/\mathfrak{a}) - (m - n). \tag{80}$$

If $U = \overline{U}$ we are done. Assume that $X := \overline{U}\backslash U$ is not empty. So $X$ is closed in $\overline{U}$. Suppose for the sake of contradiction that $\dim(X) = \dim(\overline{\mathcal{U}})$. Then the maximal variety in the maximal-length chain of irreducible varieties of $X$ is also maximal in $\overline{U}$. But this maximal variety does not intersect $U$, thus violating the definition of $\overline{U}$ as the closure of $U$. We must have $\dim(X) < \dim(\overline{U})$. Let $\mathfrak{b}$ be the vanishing ideal of $X$ then we have $\dim(\mathfrak{R}/\mathfrak{a}) > \dim(\mathfrak{R}/\mathfrak{b})$. For any $z \in X \subset \overline{U} \subset Y$, we have $\lambda' z \in \overline{U}$ for any $\lambda' \in \mathbb{C}$. Assume for the sake of contradiction that $\lambda z \in U$ for some $\lambda \in \mathbb{C}$. Hence $\lambda \neq 0$. But $z \notin U$ implies $z \in Z$ and so $\lambda z \in Z$, a contradiction. This proves that $\mathfrak{b}$ is homogeneous. As a result, we have

$$\dim(\mathfrak{R}/\mathfrak{b} + (\ell_1, \ldots, \ell_{m-n})) = \max\{\dim(\mathfrak{R}/\mathfrak{b}) - (m - n), 0\}. \tag{81}$$

Combining (80), (81) with $\dim(\mathfrak{R}/\mathfrak{a}) > \dim(\mathfrak{R}/\mathfrak{b})$ we get

$$\dim(\mathfrak{R}/\mathfrak{a} + (\ell_1, \ldots, \ell_{m-n})) > \dim(\mathfrak{R}/\mathfrak{b} + (\ell_1, \ldots, \ell_{m-n})). \tag{82}$$

Geometrically, this implies $\dim(\overline{U} \cap \mathcal{V}) > \dim(X \cap \mathcal{V})$ for a generic $\mathcal{V} \in \mathrm{Gr}_{\mathbb{C}}(n, m)$. This $\mathcal{V}$ intersects $U$. □

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