Dealership or Marketplace: A Dynamic Comparison

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Dealership or Marketplace: A Dynamic Comparison

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Abstract

We consider two business models for a two-sided economy under uncertainty: dealership and marketplace. Although both business models can bridge the gap between demand and supply, it is not clear which model is better for the firm or for the consumers. We show that while the two models differ substantially in pricing power, inventory risk, fee structure, and fulfillment time, both models share several important features, with the revenues to the firm from the two models converging when the markets are thick. We also show that for thick markets there is a one-to-one mapping between their corresponding optimal policies. We provide guidelines and insights as to which business model is preferable under different conditions when the markets are not thick.

1. Introduction

Intermediaries are often needed to facilitate transactions in two-sided markets, which consist of the supply side (sellers) and the demand side (buyers). There are two common business models for the intermediary in two-sided markets: dealership and marketplace. A dealer sets bid (ask) prices to buy from (sell to) the supply (demand) side of the market. The two sides of the market interact only indirectly through the dealer who runs an inventory. Dealerships have existed for a long time, especially in the automobile industry. For example, CarMax, a dealer company that

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buys and resells used cars, bridges consumers who want to buy used cars and car owners who want to sell used cars but do not want to deal with the hassle of direct sales. In a marketplace (e.g. retail platforms eBay, Amazon and Taobao, or ride-sharing platforms such as Uber and Didi) the firm provides a platform that connects the two sides directly. In return for providing the infrastructure, the platform often charges a fee per transaction. See, e.g., Parker et al. (2016) and Cusumano et al. (2019).

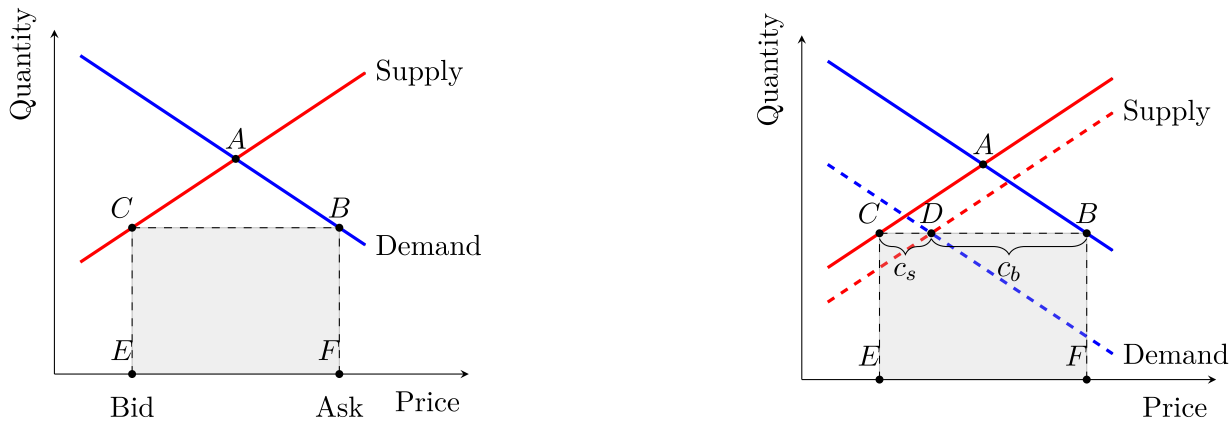
1.1. Background

The two business models seem to be drastically different. For consumers, a dealer offers immediate fulfillment eliminating waiting costs, but the transaction might cost more relative to a marketplace; choosing the marketplace might be associated with a tiring process of looking for counterparties and negotiating prices. From a firm's perspective, becoming a dealer means exercising the full pricing power, managing its own inventory and bearing the inventory risk. The differences between the two models are summarized in Table 1.

	Dealership	Marketplace
Pricing power	The dealer	Consumers
Holding inventory	Yes	No
Revenue source	Bid-ask spread	Transaction fees
Benefit for consumers	Immediate transaction	No middle man

Table 1: The differences between dealership and marketplaces.

We will next argue that the two models are essentially identical under a static, deterministic demand and supply model as illustrated in Figure 1. Point A, in figures 1a and 1b, is the equilibrium at which demand and supply intersect in a frictionless market without an intermediary. For a dealer, with the bid and ask prices represented by E and F in Figure 1a, the resulting supply and demand are, respectively, B and C. To make a sustainable business, the prices should be set so that BC forms a horizontal line with balanced demand and supply. Thus the earned bid-ask spread is represented by $|BC|$ per unit of good. In a marketplace (Figure 1b), demand and supply curves are shifted horizontally by the transaction fees charged from both sides, c_s for the sellers and c_b for the buyers. Thus point D is the equilibrium in the marketplace. The firm's earned fee is $c_s + c_b = |BC|$ per transaction. The associated quantity is the y value of D, or equivalently that of B or C. This



(a) Dealership

(b) Marketplace

Figure 1: The fundamental connection between the dealership and the marketplace.

demonstrates a one-to-one correspondence between the policies of the two models. Moreover, since the matched demand and supply in both business models are the y value of B or C, the total profit/revenue is represented by the the area of the rectangle formed by $BCEF$. As a result, both models aim to maximize the same target in the static versions of the two models.

A natural question arises: if the revenues of the two business models are comparable under a static model, why is one particular business model preferred under certain market conditions? We provide several cases. First, according to a market survey of the luxry market[1], the market share of second-hand luxury goods is well above 20% in developed countries such as Japan, U.S., and France. In those markets, consumers have seen the coexistence of dealership and marketplace in the secondary market. In developing countries like China, the number is only 2% and the dealership model is dominant. The thickness of the market seems to play a role. Second, Gautier et al. (2016) empirically find that Amazon is more likely to sell products as a middleman (dealer) when the market size of the demand side is relatively large. The market composition of buyers and sellers seems to affect the adoption of business models. Third, haoche51, a Chinese online platform for used cars, started in 2014 as a marketplace to match sellers and buyers directly, and later transformed to a model that is more similar to dealership in 2016. The CEO explained why the first business model failed in an interview[2] stating that many buyers and sellers are unwilling to wait, and the new C2B2C (dealership) model better accommodates their needs. The impatience

[1] `http://baogao.chinabaogao.com/baihuo/392626392626.html`
[2] `https://m.qctt.cn/news/120581`

of consumers waiting from a matching also factors in. The static model represented by Figure 1 is unable to capture the factors that make one business model better than the other.

For this reason, it is important to extend the analysis to the realm of stochastic dynamic models. It is much closer to the market environment faced by the intermediaries: the supply and demand are random and fluctuate over time due to intertemporal effects such as inventory costs in the dealership model and sellers waiting for buyers in the marketplace. Our goal in this paper is to study stochastic dynamic models that incorporate inventory costs and waiting times to shed light into the differences between the two business models. Following the large literature[3] after Rochet (2003); Rochet and Tirole (2006), we focus on the two-sided non-financial markets[4]. In this paper, we want to address the following issues:

1. Given the business model, how do the market participants (the intermediary and consumers) react to the transient supply/demand imbalances and changes in other market parameters such as the market sizes? This is answered by Proposition 2.1 and Proposition 3.1.

2. Are there simple heuristics that perform well for the firm? This is answered by Proposition 2.2 and discussions after Proposition 3.2.

3. To what extend does the static connection between the two business models of Figure 1 holds in practice in the presence of randomness? This is answered by Theorem 4.2.

1.2. Our Contribution

In this paper, we build dynamic models for the dealership and the marketplace as monopolies. We first study a dealer that sets dynamic ask/bid prices based on the on-hand inventory to maximize the long-run discounted profit, when the supply and demand arrive as Poisson processes. Managing the inventory level through pricing is crucial. Having too little inventory, the firm may miss potential future demand and revenues if consecutive buyers arrive; having too much inventory affects the cash flow of the business through the discount factor. We then study a marketplace platform that determines optimal transaction fees. Sellers set prices dynamically based on the

[3]The two papers have Google citation counts about 4200 and 2600, respectively, as of 09/2019.

[4]There are at least two main differences between platforms for non-financial markets and for financial markets. First, non-financial markets tend to have less liquidity; thus, game theoretical models are perhaps needed for non-financial markets rather than asset pricing models. Secondly, in non-financial platforms typically only sellers put their orders on order books; in financial platforms both buyers and sellers put orders on order books.

transient supply/demand imbalance and their waiting costs in a dynamic game among themselves; we characterize the Markov perfect equilibrium of the dynamic game.

The contribution of this paper is threefold:

(1) We build the first *dynamic* models for two-sided markets to capture the demand and supply fluctuations and the intertemporal effects. The two models are based on the same market setups and subject to "apple-to-apple" comparison. In the marketplace model, we provide a framework to capture the trade-off between monetary and time values. Unlike many queueing models, we leverage the competitive nature of heterogeneous sellers, which allows us to analyse the *transient* strategic behavior in a dynamic game. We establish the existence and uniqueness of the Markov perfect equilibrium in the dynamic game, and use the framework to explain various market phenomena.

(2) We derive the optimal policies of the two models and explore their structural properties. Despite the differences summarized in Table 1, the optimal policies share several salient features: (i) The firms always benefit from higher inventory levels or more sellers but the marginal value is diminishing. (ii) More inventory or sellers both lead to a lower transaction price, though in the dealership market the transaction price is determined by the monopolistic dealer and in the marketplace it is an equilibrium outcome of the price war among sellers. (iii) The quantities measuring *transient supply relative to demand* exhibit "mean-reversion", demonstrating the self-regulatory nature of the two models. For the dealer, it is the inventory level; in the marketplace, it is measured by the number of sellers waiting to be matched.

(3) We show that either model can be more profitable depending on the market conditions. In particular, if the sellers are impatient, or the market size of buyers is relatively small, then the dealership model is more profitable, due to the high waiting cost of the sellers in the marketplace. It may explain why many online retailers like Amazon started with a dealership model at their early stage when the customer base has not been fully built. When both sides of the market become thicker, a fundamental connection of the two models emerges. The optimal bid/ask prices of the dealer and the optimal commissions charged by the platform converge to the quantities illustrated in Figure 1, confirming the intuition (the rigorous notion of convergence will be specified later). As a result, the optimal revenues of the two business models converge to the same value, when both sides of the market become thicker.

1.3. Necessity of Building Dynamic Models

There are several reasons that motivate us to build dynamic models. The first reason is that static models are not easy to incorporate the intertemporal effects that are important to analyze strategic behavior of the intermediary as well as consumers to transient market conditions. Through dynamic models, we discover new operational implications, some of which are perhaps surprising. For example, a dealer will reduce both the bid and ask prices when its current inventory becomes higher, via which its inventory mean-reverts to a preferred position; other parameters being fixed, the sellers existing in the marketplace may not benefit from a larger size of the demand side. One reason for the counter-intuitive phenomenon in the marketplace is that a surge in the demand side may cause the next arriving seller to be more likely to join the platform, thereby intensifying competition among sellers in the future, which shall be explained in a dynamic context.

Second, static models usually assume that each individual seller or buyer is non-atomic (meaning in the language of probability theory that they have measure 0); consequently, static models can only study the impact of market composition (i.e. the ratio between the market sizes of buyers and sellers) but not the number of sellers and buyers. However, this assumption does not hold for a thin market. In our model, it is shown that the market thickness also plays an important role. For example, both the dealer's (bid and ask) prices and preferred inventory level increase when the market sizes of demand and supply grow; in the marketplace, the market thickness affects the endogenous capacity of sellers that a platform can accommodate. We have also demonstrated that the more profitable business model depends both on the market composition and thickness.

Third, static models only focus on the equilibrium and its conditions, but without showing how the equilibrium is achieved. For example, from Figure 1, we only know that the equilibrium condition is the balance between demand and supply, while through dynamic models, we show that the balance is due to the self-regulatory nature of the two models. Furthermore, dynamic models help verify the robustness of the results derived from static models. For example, by the thick market analyses, we show in which pattern (in distribution or almost surely, etc) the equilibrium of dynamic models converges to a static one.

Finally, and perhaps more importantly is the question of how much of the static profits are eaten by the dynamic effect of randomness for each of these business models, which can only be addressed in a dynamic model.

1.4. Literature Review

The study of two-sided markets has generated literature from multiple disciplines. First, in the literature of game theoretical models there are papers studying two-sided markets by building static models to investigate governance structures, platform competitions (Rochet, 2003), indirect network effect (Caillaud and Jullien, 2003), single-or-multi homing (Armstrong, 2006), usage and membership externalities (Rochet and Tirole, 2006), platform investment (Njoroge et al., 2014), MFN clauses (Johnson, 2017), quality selection (Johari et al., 2019), etc. Gautier et al. (2016) construct search models to compare two business modes, focusing on the impact of market composition. Both Rust and Hall (2003) and Li et al. (2019) consider the competition among dealers, while assuming that each dealer is of measure 0. Li et al. (2019) show how the dealer's pricing and order decisions are affected by its inventory. With similar results, we further illustrate how its decisions are affected by market thickness and provide an asymptotic optimal pricing policy. There are also dynamic models studying the matching and bargaining of two-sided markets, such as Rubinstein and Wolinsky (1985); Gale (1987); Mortensen and Wright (2002); Duffie et al. (2005); Satterthwaite and Shneyerov (2007). See the references in Manea (2016) for works on trading in networked markets. Compared to this line of literature, our dynamic model focuses on the decisions of a dealer and a platform to maximize revenues, and we consider buyers and sellers of heterogeneous valuations; indeed, our model converges to a special equilibrium associated with an optimization problem.

Second, our dealership model is related to the literature of dynamic pricing. In particular, our model has a similar setting to Gallego and van Ryzin (1994), but the main differences are our model has two-sided markets (for buyers and sellers), the inventory can be replenished by attracting sellers, the horizon is infinite, and the objective is to maximize the present value of future transactions. Our model is also related to an inventory system with joint inventory and pricing decisions under stochastic demand (Chen and Simchi-Levi, 2004a,b; Balseiro et al., 2014; Chen and Gallego, 2019). The main difference in our model is that the inventory is not replenished by ordering a certain quantity from a manufacturer; rather, it is replenished by setting a dynamic price to buy from individual sellers, who arrive in a stochastic fashion. Such symmetry between the demand and supply sides of the inventory gives rise to the bid-ask spread, and other unique structural properties in our model.

Third, there are recent papers studying platforms in the sharing economy. Cohen and Zhang (2017) use a discrete choice model to study two-sided platforms' competition and coordination. Birge et al. (2019) introduce a model to study a platform's optimal commission and subscription policy in a networked marketplace, where not all types of buyers and sellers are compatible. Taylor (2018); Bai et al. (2019) use queueing models to examine the operations of on-demand platforms. Besbes et al. (2018); Bimpikis et al. (2019) introduce the spatial or geographical elements into the pricing and matching of the platforms. The optimization problem related to Figure 1 is studied in Hu and Zhou (2017), which study a platform's optimal policy to set price, wage and commission; their focus is on the equilibrium state (without intertemporal fluctuations in demand and supply), which, interestingly, can be viewed as special cases of the limit of our models in a thick market. Benjaafar et al. (2018) consider an on-demand service platform with riders sensitive to the waiting time, strategic drivers and a profit-seeking intermediary. Although with a different model, their observation that a larger size of the opposite side does not necessarily increase the welfare coincides with ours.

Compared to this line of literature, our work differs in the following two regards. First, we focus on the connection and comparison of the two business models. To the best of our knowledge, no previous studies have analysed the pricing/commission policies of the two business models in a *dynamic* setting and discovered their fundamental connection. Second, from the modeling perspective, our framework provides a novel approach to capture the trade-off between monetary and time values of sellers on a platform. Most previous papers resort to queueing models, which usually only allows for the steady-state analysis and cannot handle the transient strategic behaviors. Our framework leverages the competitive nature of the agents and links the waiting cost to the state of the dynamic game. It allows for tractable analysis of the transient dynamics of the game.

Fourth, in the boarder management community, including information system and business strategy, a growing number of papers are focusing on the analysis of business models in two-sided markets. They consider the impacts such as the complementarity and information asymmetry (Hagiu, 2007), the capability of marketing activities (Hagiu and Wright, 2015), channel structures (Abhishek et al., 2016), and third-party information (Kwark et al., 2017). However, the seller side is usually an upstream manufacturer which responds to the order request of the intermediary passively or a retailer who can sell directly. Instead, we focus on the exchange of goods or services between individuals in a C2C market, so the two sides are more or less symmetric.

This paper is organized as follows, Section 2 introduces the dynamic formulation for the dealer's optimal policy. In Section 3, we define and investigate the dynamic game in the marketplace. We show in Section 4 that despite the functional distinctions the two models have a high-level connection, precisely related to Figure 1 above. In Section 5, we give numerical examples. Economic insights are discussed in Section 6.

2. The Dealership Market

Consider a dealership for a single product with two crowds of consumers, buyers and sellers. We assume that consumers arrive according to two independent Poisson processes with rates λ_b and λ_s, respectively, for buyers and sellers. The random private valuations of the buyers (sellers) associated with one unit of the product are denoted by Ω_b (Ω_s). The expect demands and supply rates at price p are given, respectively, by $d_b(p) \triangleq \mathsf{P}(\Omega_b \geq p)$ and $d_s(p) \triangleq \mathsf{P}(\Omega_s \leq p)$. We assume that buyers and sellers cannot trade directly, but do it indirectly through the dealer. In particular, the dealer determines the ask and bid prices dynamically. Buyers with valuations above the ask price (p_b) would buy from the dealer, whereas sellers with valuations below the bid price (p_s) would sell to the dealer. We assume that consumers make decisions upon arrival, and do not strategically wait for a different price. Thus the arrivals of buyers and sellers who transact with the dealer follow two time varying Poisson processes, $N_b(t)$ and $N_s(t)$, with rate $\lambda_b d_b(p_b(t))$ and $\lambda_s d_s(p_s(t))$, respectively, where $p_b(t)$ and $p_s(t)$ are the ask and the bid prices at time $t \geq 0$.

To simplify notations, we respectively define $\mathcal{R}_b(p, z) \triangleq \lambda_b d_b(p)(p - z)$ as the expected profit for the seller at ask price p, when his unit cost is z, and $\mathcal{R}_b(z) \triangleq \sup_{p \geq z} \mathcal{R}_b(p, z)$ as the optimal expected profit at unit cost z. Similarly, we respectively define $\mathcal{R}_s(p, z) \triangleq \lambda_s d_s(p)(z - p)$ and $\mathcal{R}_s(z) \triangleq \sup_{0 \leq p \leq z} \mathcal{R}_s(p, z)$ as the expected surplus for a buyer at bid price p, when his valuation is z, and the optimal expected surplus at valuation z.

Assumption 2.1. *(1) The market environment is stationary, i.e., $d_b(\cdot)$ and $d_s(\cdot)$ do not change over time. (2) $d_b(p)$ ($d_s(p)$) is continuous and strictly decreasing (increasing) for $p \in [0, \infty)$, with*

$d_b(\infty) = d_s(0) = 0$. *(3) For any z, both $\mathcal{R}_b(p,z)$ and $\mathcal{R}_s(p,z)$ are unimodal and have a finite maximizer in $p \geq 0$.*

The first assumption is common in the literature of dynamic pricing, e.g., Gallego and van Ryzin (1994). The second assumption guarantees the existence of a unique equilibrium price $p_e > 0$ such that $\lambda_b d_b(p_e) = \lambda_s d_s(p_e)$; in a competitive equilibrium without dealers, it is given by the intersection of demand and supply curves. The last technical assumption is that the supremums are attained in the definitions of $\mathcal{R}_b$ and $\mathcal{R}_s$. A sufficient condition is that $d_b(p)$ and $d_s(p)$ are upper semi-continuous in $p \geq 0$ and $d_b(p) = o(1/p)$. Most classic demand (supply) functions satisfy this condition, e.g., linear demand and exponential demand. For any $z > 0$, define $P_b(z) \triangleq \operatorname{argmax}_{p\geq 0} \mathcal{R}_b(p,z)$ and $P_s(z) \triangleq \operatorname{argmax}_{p\geq 0} \mathcal{R}_s(p,z)$. Elementary comparative statics reveal that $P_b(z)$ and $P_s(z)$ are nonnegative and increasing in $z \geq 0$.

2.1. General Properties

When the current inventory is $x \in \mathbb{N}$, the dealer uses bid and ask prices $p_b(t)$ and $p_s(t)$ to maximize the expected revenue over an infinite horizon discounted at rate r, which can be expressed as:

$$V_{\mathcal{D}}(x) \triangleq \max_{p_b(t),p_s(t)} \mathsf{E}\left[\int_0^\infty e^{-rt}(p_b(t)dN_b(t) - p_s(t)dN_s(t))\right]$$
$$\text{subject to} \quad x - N_b(t) + N_s(t) \geq 0, \quad \forall t \geq 0,$$

where the constraint means that the dealer can only sell if its inventory is positive.

Let $\Delta V_{\mathcal{D}}(x) \triangleq V_{\mathcal{D}}(x) - V_{\mathcal{D}}(x-1)$ represent the marginal value of inventory for integer $x \geq 1$, then $V_{\mathcal{D}}(x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation[5] :

$$\begin{aligned} rV_{\mathcal{D}}(x) &= \lambda_b \mathcal{R}_b(\Delta V_{\mathcal{D}}(x)) + \lambda_s \mathcal{R}_s(\Delta V_{\mathcal{D}}(x+1)) \quad x \geq 1, \\ rV_{\mathcal{D}}(0) &= \lambda_b \mathcal{R}_b(\Delta V_{\mathcal{D}}(1)) \end{aligned} \tag{1}$$

[5]Consider the next infinitesimal period dt. The probability of an arrival of a buyer (seller) during $[t, t+dt)$ is $\lambda_b d_b(p_b)dt$ ($\lambda_s d_s(p_s)dt$). Hence we have

$$V_{\mathcal{D}}(x) = (1 - \lambda_b d_b(p_b)dt - \lambda_s d_s(p_s)dt)(1 - rdt)V_{\mathcal{D}}(x) + \lambda_b d_b(p_b)dt(V_{\mathcal{D}}(x-1) + p_b) + \lambda_s d_s(p_s)dt(V_{\mathcal{D}}(x+1) - p_s).$$

Note that when $x = 0$, the dealer sets p_b to a choke price ($d(p_b) = 0$). Simplifying the above equation and letting $dt \to 0$ yields the HJB equation. For the uniqueness of a bounded solution to the HJB equation, see the online supplement or Section 1 in Bertsekas (1995).

The optimal bid and ask prices are functions of x, denoted as $p_s^*(x)$ and $p_b^*(x)$:

$$p_s^*(x) = P_s(\Delta V_{\mathcal{D}}(x+1)),\ x \geq 0; \quad p_b^*(x) = \begin{cases} P_b(\Delta V_{\mathcal{D}}(x)), & \text{if } x \geq 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

The *reflective* boundary condition in (1) at $x = 0$ highlights the inventory risk faced by the dealer. The dealer tries to avoid having zero inventory, as potential arrivals of demand (and thus revenue) would be missed if $x = 0$. This may nevertheless happen due to randomness; in this case, the inventory is "reflected" to $x = 1$ at the next transaction due to a purchase by the dealer.

Proposition 2.1. *[Figure 3 and Figure 4] Under the optimal pricing policy, we have the following results.*

(1) The value function $V_{\mathcal{D}}(x)$ is increasing concave in x.

(2) The existence of bid-ask spread: $p_b^*(x) \geq \Delta V_{\mathcal{D}}(x) \geq \Delta V_{\mathcal{D}}(x+1) \geq p_s^*(x)$ for all $x \geq 0$.

(3) Monotonicity of bid and ask prices: (a) $p_s^*(x)$ and $p_b^*(x)$ are decreasing in x. Besides, $\lim_{x\to\infty} p_s^*(x) = 0$ and $\lim_{x\to\infty} p_b^*(x) = \operatorname{argmax}_p\{d_b(p)p\}$. (b) $p_s^*(x)$ and $p_b^*(x)$ decrease if r increases or λ_b and λ_s decrease in proportion.

(4) Mean-reversion of inventory: There exists $0 \leq x_p < +\infty$, such that when $x > x_p$, $\lambda_b d_b(p_b^*(x)) > \lambda_s d_s(p_s^*(x))$; when $x < x_p$, $\lambda_b d_b(p_b^*(x)) < \lambda_s d_s(p_s^*(x))$. Moreover, x_p increases if r decreases or λ_b and λ_s increase in proportion.

The existence of bid-ask spread has been observed in centralized markets and analysed in the literature (see e.g. Amihud and Mendelson (1980), Miao (2006)). In our model, the bid-ask spread arises from the concavity of $V_{\mathcal{D}}(x)$ as it implies that $p_b^*(x) \geq \Delta V_{\mathcal{D}}(x) \geq \Delta V_{\mathcal{D}}(x+1) \geq p_s^*(x)$, i.e., the marginal value of inventory must be straddled by the spread. Note that the bid-ask spread does not necessarily straddle the equilibrium price p_e, at which demand and supply cross, i.e., $\lambda_b d_b(p_e) = \lambda_s d_s(p_e)$. The monotonicity of $p_s^*(x)$ in the dealer's inventory level is intuitive: As x increases, the marginal value of inventory decreases and the dealer is unwilling to buy more, thus the bid price decreases. Combining with $\lim_{x\to\infty} p_s^*(x) = 0$, it implies that if the lowest valuation of sellers is above 0, there are no sellers willing to sell to the dealer when the dealer's inventory is higher than a threshold. It resembles the classic base stock policy in the inventory management literature (Porteus, 2002). The monotonicity of $p_b^*(x)$ in the dealer's inventory level follows from

Gallego and van Ryzin (1994): The ask price increases in the "scarcity" of inventory, as $\Delta V_{\mathcal{D}}(x)$ decreases in x. However, the bid-ask spread is *not* monotone in x in general. With unlimited inventory, the dealer does not need to replenish the inventory from the sellers and simply optimizes the revenue rate from the buyers. When the interest rate increases (or the market sizes decrease), the dealer lowers the bid and ask prices to avoid the high opportunity cost, or maintain the turnover rate. The mean reversion of the inventory is consistent with the findings in Amihud and Mendelson (1980); Li et al. (2019). This phenomenon occurs due to the inventory risk: The dealer is unwilling to keep a very low level of inventory, as potential buyers might be turned away if $x = 0$ due to stochastic fluctuation; on the other hand, he is unwilling to stock too much, as the opportunity cost is high because of the discounting. As the market size increases (λ_b and λ_s increase), the dealer needs more inventory as a "buffer" to counter the increasing stochastic fluctuation. This explains the monotonicity of x_p. It implies that for a thicker market, the dealer is willing to keep more inventory to hedge the inventory risk.

2.2. Thick Markets and Fixed-Price Policies

Frequently the market sizes are large in the sense that the arrival rates of both buyers and sellers result in low coefficients of variation for both buyers and sellers (which are of order of $1/\sqrt{\lambda_b}$ and $1/\sqrt{\lambda_s}$). This is the case for popular product categories (e.g. popular used books on Amazon). In this section, we study this "thick-market" regime and show that the fixed-pricing rule corresponding to the fluid approximation where demands are replaced by their expectations has desirable properties.

We consider the fluid approximation where the Poisson processes, $N_b(t)$ and $N_s(t)$, are replaced by their means. More precisely, with ask price p_b and bid price p_s, in the next time interval of dt, the dealer sells $\lambda_b d_b(p_b)dt$ units and buys $\lambda_s d_s(p_s)dt$ units exactly. (Hence the product is infinitely divisible and the inventory is a continuous variable.) Such fluid approximation is used widely in dynamic pricing as it sheds light on the first-order effects. Denote $\bar{V}_{\mathcal{D}}(x)$ as the optimal discounted revenue for the fluid approximation. The dealer attempts to maximize the discounted cash flow

$$\bar{V}_{\mathcal{D}}(x) = \max_{p_b(t), p_s(t)} \int_0^\infty e^{-rt}(\lambda_b p_b(t) d_b(p_b(t)) - \lambda_s p_s d_s(p_s(t)))dt$$
$$\text{subject to } \int_0^t (\lambda_b d_b(p_b(s)) - \lambda_s d_s(p_s(s)))ds \le x, \quad \forall t \ge 0. \tag{3}$$

Replacing $\Delta V_{\mathcal{D}}$ by the derivative $\bar{V}'_{\mathcal{D}}(x)$ in (1), we obtain the following HJB equation for $\bar{V}_{\mathcal{D}}(x)$:

$$r\bar{V}_{\mathcal{D}}(x) = \mathcal{R}_b(\bar{V}'_{\mathcal{D}}(x)) + \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(x)), \quad x > 0. \tag{4}$$

Next, we need to find the related boundary condition which exhibits an entirely different pattern from that of $V_{\mathcal{D}}(0)$. In the fluid approximation, by the continuity of $d_b(\cdot)$ and $d_s(\cdot)$ and the infinitesimal deterministic demand/supply flow, the inventory will always reach an equilibrium level $\bar{x}_p$ under the optimal pricing policy, such that $\lambda_b d_b(\bar{p}^*_b(\bar{x}_p)) = \lambda_s d_s(\bar{p}^*_s(\bar{x}_p))$, and not deviate from the equilibrium afterwards. Without uncertainty, the equilibrium $\bar{x}_p$ must equal to zero, since stocking a positive inventory is meaningless after reaching the equilibrium, as it could have been turned into profit. Therefore, $\bar{V}_{\mathcal{D}}(0)$ should satisfy the following *absorbing* (instead of *reflective*) boundary condition:

$$r\bar{V}_{\mathcal{D}}(0) = \max_{\lambda_b d_b(p_b) = \lambda_s d_s(p_s)} \{\lambda_b p_b d_b(p_b) - \lambda_s p_s d_s(p_s)\}. \tag{5}$$

After the inventory reaches zero, it stays in the absorbing state $x = 0$. This is a significant difference from the original problem, which has a positive preferred inventory x_p, to which the inventory process reverts. Note that the source of the inventory risk, uncertain demand, is entirely eliminated in the fluid approximation. It can be shown that $\bar{V}_{\mathcal{D}}(x)$ is increasing and concave, and serves as an upper bound of $V_{\mathcal{D}}(x)$ (see Proposition B.2 in the appendix). In other words, eliminating uncertainty and the inventory risk increases the dealer's profit. Moreover, the ratio $\bar{V}_{\mathcal{D}}(x)/V_{\mathcal{D}}(x)$ tends to one as $x \to \infty$.

A Fixed-Pricing Policy for the Fluid Model

In the revenue management literature (see, e.g., Gallego and van Ryzin (1994)), it is found that the fluid approximation and the associated fixed pricing heuristic works well in a "scaled" system. Here we consider a similar setting of the thick-market regime, in which we scale demand and supply simultaneously by n, i.e., $(\lambda_b^{(n)}, \lambda_s^{(n)}) = n(\lambda_b, \lambda_s)$.

Proposition 2.2. *[Figure 4] Consider a fixed-pricing policy* $\{\hat{p}_b, \hat{p}_s\}$ *that solves the absorbing boundary condition in* (5)*, namely*

$$\max_{p_s, p_b} \quad \frac{1}{r}\lambda_s d_s(p_s)(p_b - p_s), \ \textit{subject to } \lambda_s d_s(p_s) = \lambda_b d_b(p_b), \tag{6}$$

and let $\tilde{V}_{\mathcal{D}}^{(n)}(x)$ *is the expected discounted revenue of the dealer using the fixed price policy* $\{\hat{p}_b, \hat{p}_s\}$*. Then for any* $x \in \mathbb{N}$*, we have*

$$\frac{\tilde{V}_{\mathcal{D}}^{(n)}(x)}{V_{\mathcal{D}}^{(n)}(x)} \geq \frac{\tilde{V}_{\mathcal{D}}^{(n)}(x)}{\bar{V}_{\mathcal{D}}^{(n)}(x)} = 1 - O(\frac{1}{\sqrt{n}}) \textit{ and } \lim_{n\to\infty} \frac{\tilde{V}_{\mathcal{D}}^{(n)}(x)}{n} = \frac{1}{r}\lambda_s d_s(\hat{p}_s)(\hat{p}_b - \hat{p}_s).$$

Due to the dynamic effect of randomness, the relative loss of profit under the heuristic will be $O(1/\sqrt{n})$, yielding that the fixed-price heuristic is asymptotically optimal. Intuitively, this holds because, when the market becomes thick, the initial inventory is negligible compared to the mass stream of supply and demand from consumers. We will discuss a fundamental connection of the fixed-pricing policy to the marketplace in Section 4.

3. The Marketplace

In a marketplace, the intermediate firm provides a platform that facilitates the *direct* transaction between buyers and sellers, instead of serving as a middle man, and usually charges fees per transaction from both sides. While sellers can potentially make more profits from transactions, they have to wait for counterparties, instead of immediately transacting with a dealer. The buyers may also get better prices because of competition among sellers. In this section, we introduce a dynamic continuous-time framework to capture those trade-offs and analyse the optimal fees charged by the platform (firm).

3.1. Basic Settings

Suppose the platform charges sellers (buyers) c_s (c_b) per transaction[6]. We assume that c_s and c_b are set at time zero and fixed over time, because in practice transaction fees are usually stated upfront before consumers come to the platform and kept unchanged for a long time. For example, most platforms such as eBay inform the clients about the fee policy when they register. In the marketplace, there are sellers posting prices for their homogeneous goods. Both sides are assumed to have one unit of product to sell or buy. We use "he" to refer to the platform and "she" to refer to a seller or a buyer.

The buyers. Buyers arrive at the marketplace at a Poisson process with rate λ_b. Their valuations are drawn from Ω_b independently whose tail distribution is $d_b(\cdot)$, as explained in Section 2. When a buyer arrives, she observes the sellers' quoted prices and purchases from the seller who offers the lowest price, provided that this price plus the transaction fee c_b is lower than her valuation of the product. If two or more sellers quote the same lowest price, then an arbitrary tie-breaking rule is introduced (e.g. random selection). Finally, a buyer who finds the lowest price plus c_b higher than her valuation leaves the marketplace permanently without purchasing — that is, she does not wait, which is consistent with the behavior in the dealership model. Consequently, when the lowest price is p, the random time elapsed before a buyer arrives and purchases a product follows an exponential distribution with rate $\lambda_b d_b(p + c_b)$, in which case a transaction occurs and the number of sellers in the marketplace decreases by one.

The sellers. Sellers arrive in a Poisson process with rate λ_s, whose valuations have i.i.d. distributions with CDF $d_s(\cdot)$. An arriving seller observes m, the number of sellers, and their posted prices. Similar to the dealership model, a seller makes a take-it-or-leave-it decision upon her arrival at the platform: joining the platform and posting a price, or leaving the market permanently if the expected utility of joining the platform is less than her valuation. For the sellers, the expected utility of joining the marketplace is

$$f_m = \mathbb{E}[\tilde{p} - c_s - \bar{w}\tau],$$

[6]In Section C.4 of the online supplement, we also analyse the case that the platform charges a fee proportional to the transaction price. All the results in this section hold except that the platform's expected discounted revenue may decrease, when there are more sellers on the platform. Intuitively, this is because a lower transaction fee caused by more sellers on the platform may outweigh the benefit from more transactions.

where $\tilde{p}$ is the (random) future transaction price, τ is the (random) time elapsed between joining the platform and the time her product is sold; both $\tilde{p}$ and τ will be determined endogenously later. Since comparing to the platform, sellers care more about the short-term return, we use linear waiting costs to characterize their time disutility instead of discounting, with unit cost $\bar{w}$. If the seller decides to join, then the number of sellers in the marketplace is increased by one. We assume that once joining, the seller does not withdraw and leaves the marketplace only after her product is sold.[7] The sellers in the marketplace can adjust their posted prices at any time without cost, a practice adopted by many platforms, e.g. Craigslist.

The Dynamics. Since an arriving buyer only transacts with the seller offering the lowest price, the sellers continuously undercut the current lowest price, leading to a price war. The price war ends immediately.[8] As a result all m sellers are indifferent between missing the next transaction and selling the product at the lowest price. At the moment of the next transaction, the seller offering the lowest price, denoted as p_m, receives $p_m - c_s$ and leaves the market, while the future expected utility of the remaining $m-1$ sellers becomes f_{m-1}. By the discussion above, we must have $p_m - c_s = f_{m-1}$ for $m \geq 2$. For $m = 1$, the single seller does not face any competition and maximizes her utility by setting the optimal price p_1.

To understand the dynamics of the marketplace, consider the events that could happen in the next infinitesimal time period of length δt. For $m > 1$, the transition to $m-1$ is caused by a buyer arriving at the marketplace whose valuation is higher than the lowest price p_m plus c_b. In the following, we will refer to $p_m + c_b$ as the *payment* made by buyers when there are m sellers in the marketplace. Therefore, the transition probability is approximately $\lambda_b d_b(p_m + c_b)\delta t$. After the transition, the seller who has sold her item or not receives payoff $p_m - c_s$ or f_{m-1}, which are equal. On the other hand, the transition to $m+1$ is caused by an arriving seller, provided that her expected utility of joining the marketplace is higher than her valuation. Note that her expected utility is equal to the expected future utility of the $m+1$ sellers on the platform after she joins, which is f_{m+1}. Therefore, the transition probability is approximately $\lambda_s d_s(f_{m+1})\delta t$. For convenience, let

[7] A seller's valuation incorporates the hassle cost before her item can be sold on the market. Some online second-hand luxury marketplaces (such as Realreal) require sellers to ship their items to stores first. Used car sellers need to perform repairs and safety inspections before reselling on the platform. These hassle costs are sunk after joining the marketplace, so sellers are reluctant to give up.

[8] In some platforms, e.g. eBay, the buyers also engage in an auction for a single product, similar to the price war of sellers. We do not consider this case in the model.

$f_0 = p_1 - c_s$ and $w = c_s + c_b$.

Thus, given the number of sellers m at any time, the dynamics of the marketplace can be modeled as a continuous-time Markov chain with state m, i.e., a state-dependent birth-death process. More precisely, consider a time point when there are m sellers on the platform. They engage in an immediate price war, leading to a competitive price $p_m = c_s + f_{m-1}$. If a buyer arrives and her valuation is above $p_m + c_b$, which occurs with rate $\lambda_b d_b(p_m + c_b) = \lambda_b d_b(f_{m-1} + w)$, then a transaction happens and m becomes $m - 1$. If a seller arrives and her valuation is below f_{m+1}, which occurs with rate $\lambda_s d_s(f_{m+1})$, then m becomes $m+1$. Thus the utility function f_m will satisfy the following recursive equations[9]:

$$
\begin{aligned}
&f_1 = \max_{f_0}\{\frac{\lambda_b d_b(f_0 + w)f_0 + \lambda_s d_s(f_2)f_2}{\lambda_b d_b(f_0 + w) + \lambda_s d_s(f_2)} - \frac{\bar{w}}{\lambda_b d_b(f_0 + w) + \lambda_s d_s(f_2)}\},\\
(7)\qquad &f_m = \frac{\lambda_b d_b(f_{m-1} + w)f_{m-1} + \lambda_s d_s(f_{m+1})f_{m+1}}{\lambda_b d_b(f_{m-1} + w) + \lambda_s d_s(f_{m+1})} - \frac{\bar{w}}{\lambda_b d_b(f_{m-1} + w) + \lambda_s d_s(f_{m+1})}, \quad m \geq 2.
\end{aligned}
$$

where the first term of the right hand is the expected payoff at the next transition, and the second term is the expected waiting cost before the transition.

Although the recursive system for f_m is derived for all $m > 0$, it does not necessarily mean that there will be infinite number of sellers in the marketplace (the Markov chain has infinite number of states). Indeed, according to Assumption 2.1, the lower bound of the support of Ω_s is zero. Define $M = \min\{m : f_{m+1} \leq 0\}$. When there are M sellers in the marketplace, the expected utility of joining for an arriving seller is $f_{M+1} \leq 0$, which is always less than her valuation of the product. Therefore, no more sellers are willing to join the marketplace and the number of sellers in the marketplace never exceeds M. Because M is endogenously determined by solving the recursive system in (7), it is referred to as the *endogenous capacity* of the platform. We impose the following assumption on M.

Assumption 3.1. *The endogenous capacity M is finite.*

In practice, the marketplace cannot accommodate an infinite number of sellers. Assumption 3.1 is imposed merely for technical purposes to rule out unrealistic solutions to (7).

Platform's Objective. Let $V_{\mathcal{M}}(m)$ be the platform's expected discounted revenue when there

[9] Observe that only the sum $w = c_s + c_b$ matters here. Intuitively this is because, as long as w is fixed, changing c_s and c_b only shifts the demand and supply curves horizontally without changing other quantities.

are m sellers in the marketplace, at discount rate r. That is,

$$V_{\mathcal{M}}(m) = \int_{t=0}^{\infty} e^{-rt} w dN_t, \quad \text{given } m \text{ sellers in the marketplace at } t = 0,$$

where N_t is the counting process of the number of transactions up to time t. Similar to the derivation of f_m, we have

$$\begin{aligned}
rV_{\mathcal{M}}(0) &= \lambda_s d_s(f_1) \Delta V_{\mathcal{M}}(1), \\
rV_{\mathcal{M}}(m) &= \lambda_s d_s(f_{m+1}) \Delta V_{\mathcal{M}}(m+1) + \lambda_b d_b(f_{m-1} + w)(w - \Delta V_{\mathcal{M}}(m)), \quad 1 \leq m \leq M-1, \\
rV_{\mathcal{M}}(M) &= \lambda_b d_b(f_{M-1} + w)(w - \Delta V_{\mathcal{M}}(M)),
\end{aligned} \tag{8}$$

where $\Delta V_{\mathcal{M}}(m) = V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ is the marginal value of an additional seller. The platform's objective is to choose c_s and c_b to maximize $V^*_{\mathcal{M}}(m) \triangleq \max_{c_s, c_b} V_{\mathcal{M}}(m)$.

3.2. Equilibrium Analysis

The type of equilibrium in the marketplace is a Markov perfect equilibrium. The dynamic game is stationary and modulated by the state of the Markov chain m, representing the number of sellers in the marketplace. Given m, the value of f_m that solves (7), the associated Markov process is a state-dependent birth-death process. Due to Assumption 3.1, the birth-death process has a finite number of states and thus a steady-state distribution. Let π_m denote the steady-state probability of state m. The equilibrium is characterized below.

Definition 3.1. *As all sellers maximzie their expected utilities $\mathbb{E}[\tilde{p} - c_s - \bar{w}\tau]$ and compete in the price war, the expected utility f_m only depends on the number of sellers in the marketplace and solves* (7) *. The equilibrium of the dynamic game in the marketplace can thus be characterized by a state-dependent birth-death process, (1) The birth rate from state m to $m+1$ is $\lambda_s d_s(f_{m+1})$, $m = 0, \ldots, M-1$. (2) The death rate from state m to $m-1$ is $\lambda_b d_b(p_m + c_b) = \lambda_b d_b(f_{m-1} + w)$, $m = 1, \ldots, M$. (3) The steady-state distribution satisfies*

$$\frac{\pi_m}{\pi_{m-1}} = \frac{\lambda_s d_s(f_m)}{\lambda_b d_b(f_{m-1} + w)}. \tag{9}$$

Despite the difficulty of analysing dynamic games, we are able to establish the existence and uniqueness of the equilibrium, as a main result of the paper.

Theorem 3.1. *Suppose Assumption 2.1 and 3.1 hold. There exists a unique solution to* (7) *and thus a unique Markov perfect equilibrium of the dynamic game.*

The proof designs a special algorithm that always outputs the exact solution f_m to (7). Finding the solution to nonlinear equations and proving the uniqueness is highly non-trivial, especially when the number of equations M in (7) is endogenous and part of the solution as well.

Next we provide some properties of the equilibrium.

Proposition 3.1. *[Figure 5 and Figure 7]In the equilibrium:*

(1) The expected utilities of sellers f_m as well as the transaction prices $p_m = f_{m-1} + c_s$, are decreasing and convex in m.

(2) There exists $0 \leq M^* < \infty$ such that when $m > M^*$, $\lambda_b d_b(p_m + c_b) \geq \lambda_s d_s(f_m)$; when $m \leq M^*$, $\lambda_b d_b(p_m + c_b) < \lambda_s d_s(f_m)$.

(3) The endogenous capacity M is decreasing in $\bar{w}$, λ_s, w, and increasing in λ_b. Moreover, there are sellers joining the marketplace if and only if $\max_p\{d_b(p+w)p\} \geq \bar{w}/\lambda_b$.

(4) Given $w = c_s + c_b$, $V_{\mathcal{M}}(m)$ is increasing and concave in m and $\Delta V_{\mathcal{M}}(m) \leq w$.

The Sellers' Utility and Transaction Prices

The state m reflects the transient supply/demand imbalance. A larger m implies an excessive level of supply and each seller in the marketplace faces more fierce competition, which reduces the expected utility. The convexity of f_m implies that the adverse effect of competition is diminishing as m increases. This is because the sellers are selling homogeneous products. Intuitively, one can think of a single unit of demand being shared evenly among the sellers and the share $1/m$ is decreasing and convex in m.

The Mode of the Steady-State Distribution

Because f_m is decreasing, the effective arrival rate of a new seller, $\lambda_s d_s(f_{m+1})$, is decreasing in m; on the other hand, the arrival rate of a new buyer, $\lambda_b d_b(f_{m-1}+w)$, is increasing. It implies that the equilibrium dynamics exhibit "mean-reversion". That is, there is a state $M^* \triangleq \max\{m : \lambda_s d_s(f_m) \geq$

$\lambda_b d_b(f_{m-1}+w)\}$ so that for $m > M^*$, the number of sellers m tends to decrease on average and for $m < M^*$ the number of sellers tends to increase. In other words, the platform is self-regulating the amount of supply because of the economic decisions made by the buyers and sellers. Combined with the property of π_m in (9), M^* is also the "mode" of the steady-state distribution.

The Endogenous Capacity

The endogenous capacity M is the maximal number of sellers the platform can potentially accommodate. By Proposition 3.1-(3), the endogenous capacity M is decreasing in the waiting cost $\bar{w}$, the market size of sellers λ_s, the total transaction fees w, and increasing in the market size of buyers λ_b. Consistent with the economic intuition, when the cost of selling the product decreases (low waiting cost or transaction fees) or the demand side becomes relatively large, the platform can accommodate more sellers. In extreme cases, for example, when the market size of buyers is considerably small, the endogenous capacity $M = 0$ and no sellers are willing to join the marketplace. The platform cannot make any revenue. Note that this is not the case in the dealership market: the dealer can always make profits no matter how small λ_b is.

A Surprising Effect of Externality

For the comparative statics of other market parameters, we find some unexpected results: Given the platform's commission fees, when the demand side grows relative to the supply side (λ_s decreases or λ_b increases), the transaction price p_m may end up being lower for a fixed m, or equivalently, the existing sellers in the marketplace may be worse off (f_m decreases). This is inconsistent with the economic intuition that stronger demand is usually tied to higher prices.

There are two reasons behind the counter-intuitive phenomena related to the dynamic nature of the model: (1) The endogenous capacity may change as the parameters change. The comparative statics for a fixed m is not the correct way to understand the interactions. (2) The changing parameter may alter the complex dynamics of the system, causing the seller of the next arrival more likely to join the platform and intensifying the competition for existing sellers in the short term. That is, the increase in f_{m+1} causes the surge in the arrival of potential sellers, leads to more competition and the decrease in f_m. The first reason can be explained by a traditional free-entry model, but the second reason needs to be analysed in a dynamic context.

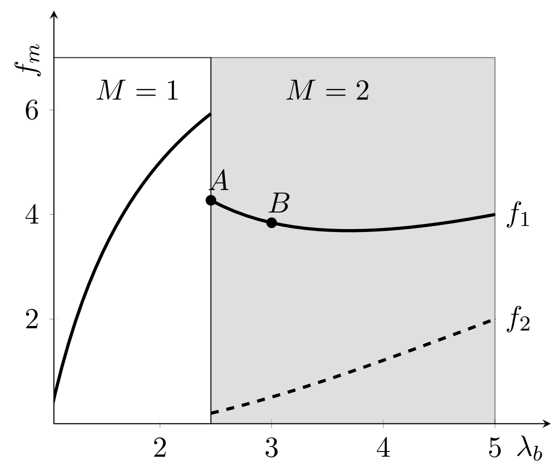


Figure 2: The changes of f_1 (solid line) and f_2 (dashed line) in λ_b. We let $\lambda_b d_b(p) = \lambda_b \mathbb{I}_{\{p \leq 10\}}$, $\lambda_s d_s(p) = 5p\mathbb{I}_{\{p \geq 0.2\}}$, $w = c_s + c_b = 0$, $\bar{w} = 10$. The endogenous capacity in the white and shaded regions are one and two, respectively.

To illustrate, consider the example in Figure 2. At point A, the endogenous capacity changes from $M = 1$ to $M = 2$ as the market size of buyers increases. A second seller is now willing to join the marketplace, compared to the left region where the platform only admits a monopolistic seller. Thus the existing seller in the marketplace faces a potential competitor, and hence her utility, f_1, drops. At point B, the endogenous capacity is unchanged and the utility of the existing seller f_1 may still decrease. This is because as f_2 increases, joining the marketplace becomes more attractive to future arriving sellers. For the existing seller, the competition is intensified as potential sellers join the marketplace at a higher rate. It outweighs the benefits from the increased demand, and hence drives down f_1. These counter-intuitive phenomena demonstrate the complexity of the problem.

The impact on consumers' utilities by the market size is often referred to as the externality. As we have seen, the externality can be either positive or negative for a platform. The example implies platforms should reconsider when cultivating the market size of a single side. In particular, the growth of buyers may cause the proliferation of seller entry, intensify the competition and thus hurt existing sellers on the platform, which may turn away future participants and cause long-term detrimental damage to its business. To avoid this dilemma, the unilateral strategy needs to be replaced by careful balancing of both sides, through filtering, controlling, and limiting the access of users to the platform.

The Platform's Revenue

This optimization problem is intractable, as $V_{\mathcal{M}}(m)$ depends on f_m, which cannot be solved explicitly and the dependence on the transaction fees is highly nonlinear (recall the counter-intuitive exterbal effect results). However, we can provide qualitative structural results about the platform's expected revenue, which are summarized in Proposition 3.1-(4).

Although the platform does not keep its own inventory like the dealer does, the number of sellers m in the marketplace is connected to the inventory in the dealership model at a high level. In particular, the magnitudes of both quantities signal the excessive level of supply relative to demand and lead to lower transaction prices (see Proposition 2.1-(3) and Proposition 3.1-(1)). Interestingly, the value functions demonstrate similar structural properties (Proposition 2.1-(1) and Proposition 3.1-(4)). The platform makes more revenue as the number of sellers increases in the marketplace. This is because the firm's revenue only depends on the number of transactions regardless of the transaction price; more sellers endogenously lead to more transactions. Moreover, the marginal value is decreasing and bounded by w: an additional seller leads to at most one more transaction, and one transaction generates revenue w for the firm.

3.3. The Thick Market

Consider a sequence of systems indexed by n. In the nth system, $(\lambda_b^{(n)}, \lambda_s^{(n)}) = (n\lambda_b, n\lambda_s)$[10]. We characterize the limit of the dynamic game as $n \to \infty$, and derive the optimal strategy of the platform in the asymptotic regime.

As the market becomes thicker, transactions occur more frequently and the waiting time between the time points a seller joins the marketplace and her product gets sold becomes negligible. Therefore, sellers are more willing to join the platform and the endogenous capacity tends to infinity. Another important implication of a thick market is that the state transition rates of the dynamic game (birth-death process) becomes larger. As a result, the state of the game, m, resembles the state of a fluid system: the normalized inflow to the system is at the rate that a seller joins the marketplace $\lambda_s d_s(f_{m+1})$ and the outflow is at the rate that a buyer makes a purchase $\lambda_b d_b(f_{m-1} + w)$. Because the waiting cost diminishes, one has $f_m \approx f_{m+1} \approx f_{m-1}$. Therefore,

[10] Most quantities have the superscript (n) in this section. We omit it if there is no confusion.

define $\hat{f} \triangleq \left\{f : \lambda_s d_s(\hat{f}) = \lambda_b d_b(\hat{f}+w)\right\}$ to be the expected utility of a seller when the marketplace is in such a state that inflow equals the outflow. Intuitively, $\hat{f}$ is closely tied to the mode, M^*, of the distribution. Indeed, when there are M^* sellers in the marketplace, the inflow roughly matches the outflow and the expected utility f_{M^*} is very close to $\hat{f}$.

Proposition 3.2. *[Figure 6] In the thick-market regime: (1) For any finite $k \in \mathbb{Z}$, $\lim_{n\to\infty} f^{(n)}_{M^{*(n)}+k} = \hat{f}$. (2) For any $\epsilon > 0$, we have $\lim_{n\to\infty} \sum_{m:|f^{(n)}_m - \hat{f}|>\epsilon} \pi_m = 0$.*

Note that the standard techniques (*i.e.* fluid approximation) in revenue management cannot be used to prove the proposition, due to the price war among sellers. The first part shows that for a wide range of m, the expected utility of a seller, f_m, does not deviate much from $\hat{f}$. It is the key to characterizing the asymptotic steady-state distribution of the dynamic game. Because M^* is the mode of the steady state distribution, the above result states that the expected utility of sellers is "flat" when the number of sellers is around the mode. Furthermore, the second part shows that the expected utility of a seller to join the marketplace is concentrated around $\hat{f}$ with high probability.[11] By Proposition 3.2, the expected utility of the sellers in the thick market can be approximated by $\hat{f}$ in the steady state. Hence the transaction prices are almost equal to $\hat{p} \triangleq \hat{f} + c_s$. The rate at which a transaction occurs is approximately $\lambda_b^{(n)} d_b(\hat{f} + c_s + c_b) = \lambda_b^{(n)} d_b(\hat{f}+w) = \lambda_s^{(n)} d_s(\hat{f})$, and thus the revenue rate is approximately $\lambda_s^{(n)} d_s(\hat{f}) w$.

When the market becomes thick, it is reasonable to focus on the long-run discounted revenue in the steady state, which is defined as $V_{\mathcal{M}} \triangleq \sum_{m=0}^{M} \pi_m V_{\mathcal{M}}(m)$. By (8), we have

$$
\begin{aligned}
(10)\qquad rV_{\mathcal{M}} &= \sum_{m=0}^{M-1} \pi_m \lambda_s d_s(f_{m+1}) \Delta V_{\mathcal{M}}(m+1) + \sum_{m=1}^{M} \pi_m \lambda_b d_b(f_{m-1}+w)(w - \Delta V_{\mathcal{M}}(m)) \\
&= w \sum_{m=1}^{M} \pi_{m-1} \lambda_s d_s(f_m) = w \sum_{m=1}^{M} \pi_m \lambda_b d_b(f_{m-1}+w),
\end{aligned}
$$

where we have used $\pi_{m-1}\lambda_s d_s(f_m) = \pi_m \lambda_b d_b(f_{m-1}+w)$ due to (9). Therefore, the optimal transaction fees of the firm are $\mathrm{argmax}_{c_s,c_b} \frac{V^{(n)}_{\mathcal{M}}}{n} = \mathrm{argmax}_{c_s,c_b}\{(c_s+c_b)\sum_{m=1}^{M^{(n)}} \pi_{m-1}\lambda_s d_s(f^{(n)}_m)\}$. This optimization is intractable for a finite n because $f^{(n)}_m$ and π_m do not have explicit forms and their dependence on c_s and c_b is highly nonlinear. However, given $w = c_s + c_b$, in the thick market

[11] Note that Proposition 3.2 does not imply that $f_m \approx \hat{f}$ for all the states $m < M$. There are still infinitely many states m such that f_m is very different from $\hat{f}$ when $n \to \infty$.

the firm's long-run average revenue converges, $\lim_{n\to\infty} \frac{V_{\mathcal{M}}^{(n)}}{n} = \frac{\lambda_s d_s(\hat{f})w}{r}$ (see Proposition C.1 in the online supplement). To maximize the revenue in the thick market, the firm sets c_s and c_b to maximize

$$\max_{c_s, c_b} \quad \frac{1}{r}\lambda_s d_s(\hat{f})w, \text{ subject to } \lambda_s d_s(\hat{f}) = \lambda_b d_b(\hat{f}+w). \tag{11}$$

The value $\hat{f}$ satisfies the "flow-balance" constraint. Surprisingly, the formulation has a fundamental connection to the optimization problem of the dealer in the thick market, which we discuss in the next section.

4. Business Model Comparison

The dynamic models capture the differences in the two business models highlighted in Table 1. In the dealership market, the dealer has a monopolistic pricing power by setting bid and ask prices dynamically. In the marketplace, the firm only charges transaction fees that are set up upfront, and the transaction price is completely determined by the price war among sellers in the marketplace. The dealer manages his own inventory, and is thus exposed to the inventory risk. This is one of the reasons why the prices are set dynamically. In comparison, the platform does not keep its own inventory. The dealer makes a profit by earning the bid-ask spread, which straddles the marginal value of inventory. The platform earns revenues as long as buyers and sellers engage in transactions. For sellers, the benefit of selling to the dealer is the guaranteed immediate transactions, although it usually means that they are charged a premium by the dealer. Conversely, they are directly matched to the counterparty in the marketplace, after a (random) period of waiting. For buyers, they face a single dealer reselling the product in the dealership market, whereas the prices of the product is competitively set by the sellers in the marketplace.

4.1. A General Comparison between the two Business Models

The profitability of the two business models can be quite different depending on the market environments. In particular, we can show that

Theorem 4.1. *[Figure 8 and Table 2] Other parameters being fixed, when the waiting cost of sellers*

$\bar{w}$ is sufficiently large (above a certain threshold), or the market size of buyers λ_b is sufficiently small (below a certain threshold), the dealer earns more profits than the platform; when $\bar{w}$ is sufficiently small (below a certain threshold), the platform earns more profits than the dealer.

When sellers are impatient, they are unwilling to wait for long on the platform. As a result, not many sellers join and they are unable to offer a competitive price, which limits the transaction volume and thus the profitability of the marketplace relative to the dealership. When the sellers are patient enough, a firm will transfer the inventory risk to the sellers and then earn more profits as a platform. Similarly, when not many buyers are populating the other side of the market, the marketplace looks less attractive to potential sellers, while the dealer can still wield its pricing power to partially balance the demand and supply. For other market conditions, we have found scenarios that either business model can dominate the other in terms of revenues.

Despite the distinctions, the two models share several important features. We may regard the inventory held by the dealer and the number of sellers in the marketplace as similar indicators for transient excess supply in the two models. More inventory/sellers imply that the firm can meet short-term demand surge without worrying about the supply side. We have shown that the firm always benefits from higher inventory or more sellers but the marginal payoff is diminishing (see Proposition 2.1-(1) and Proposition 3.1-(4)). More inventory or sellers both lead to a lower transaction price (see Proposition 2.1-(3) and Proposition 3.1-(1)), though in the dealership market the transaction price is determined by the monopolistic dealer and in the marketplace it is an equilibrium outcome. Both quantities display mean reversion (see Proposition 2.1-(4) and Proposition 3.1-(2)), demonstrating the self-regulatory nature of the two models.

4.2. A Fundamental Connection of the two Business Models in the Thick Market

For the dealer, (6) demonstrates the asymptotic optimality of constant bid and ask prices when the market is thick. For the platform, in the thick market the transaction fees are designed to maximize (11). The optimization problems exactly reflect the intuitions illustrated in Figure 1. In the thick market regime, both the inventory risk of the dealer and the waiting cost of the sellers in the marketplace are negligible. As a result, the demand and supply fluctuations and the intertemporal effects become second-order and the systems can be described by static optimization problems.

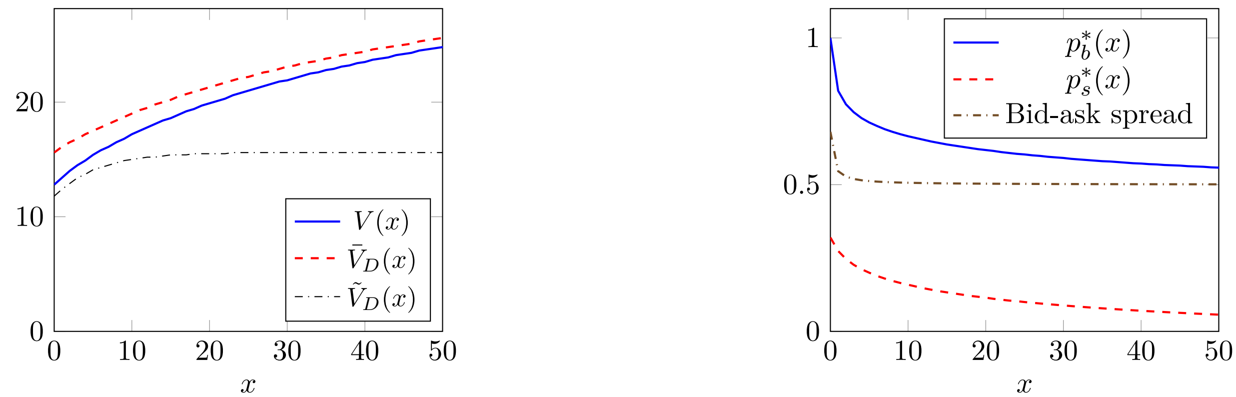
In Figure 1a, the dealer decides the bid and ask prices, i.e., p_s and p_b represented by E and F, respectively. The constraint $\lambda_s d_s(p_s) = \lambda_b d_b(p_b)$ implies that BC must be horizontal, i.e., the demand equals supply under the bid and ask prices. The objective $\lambda_s d_s(p_s)(p_b - p_s)/r$ is the area of the rectangle $BCEF$ divided by r. In Figure 1b, the platform charges the transaction fees c_s and c_b, causing the product to be cheaper and more expensive for sellers and buyers. It shifts the supply and demand curves to the new positions, represented by the dashed lines. The new equilibrium, in which the shifted demand equals the shifted supply, is represented by D. Interpret $\hat{f}$ as the x value of E or C. The constraint $\lambda_s d_s(\hat{f}) = \lambda_b d_b(\hat{f} + c_s + c_b)$ states that D is the new equilibrium after shift and BC is horizontal. The objective $\lambda_s d_s(\hat{f})(c_s + c_b)/r$ is the area of the rectangle $BCEF$ divided by r. In essence, in the thick market, the dealer and the platform are maximizing the same objective; their decisions, $\{p_s, p_b\}$ and $\{c_s, c_b\}$, also coincide $c_s + c_b = p_b - p_s$ at optimality and their revenues are equivalent after normalization. This observation leads to the next theorem.

Theorem 4.2. *[Figure 8] In the limits of thick markets, two business models (*(6) *and* (11)*) share the following properties: (1) The revenues (normalized by the market size) are equivalent. (2) The optimal bid-ask spread of the dealer $p_b - p_s$ coincides with the commission fee of the platform $c_b + c_s$. (3) The optimal bid price set by the dealer coincides with the expected utility of sellers in the marketplace. (4) The optimal ask price set by the dealer coincides with the average payment made by buyers in the marketplace.*

Our model studies a homogeneous product where both sides consist of individual consumers. Therefore, in a thick market, the impact of each individual's decision on the entire market is negligible. This is one of the key reasons behind the equivalence. For example, the equivalence fails to hold in Abhishek et al. (2016), in which the supply side has only two manufactures, and the dealer can soften the competition of the two manufactures, compared to a marketplace. However, it is interesting to find conditions under which the equivalence may still hold in a thick market with multiple types of products; we leave this for a future study.

5. Numerical Examples

In this section, we conduct a comprehensive numerical study based on linear demand and supply functions: $\lambda_b d_b(p) = \lambda_b(1 - \alpha p)^+$ and $\lambda_s d_s(p) = \lambda_s \beta p$. Without loss of generality, we always



(a) The value functions for the optimal policy, the fluid approximation, and the fixed-pricing policy

(b) The optimal pricing policy

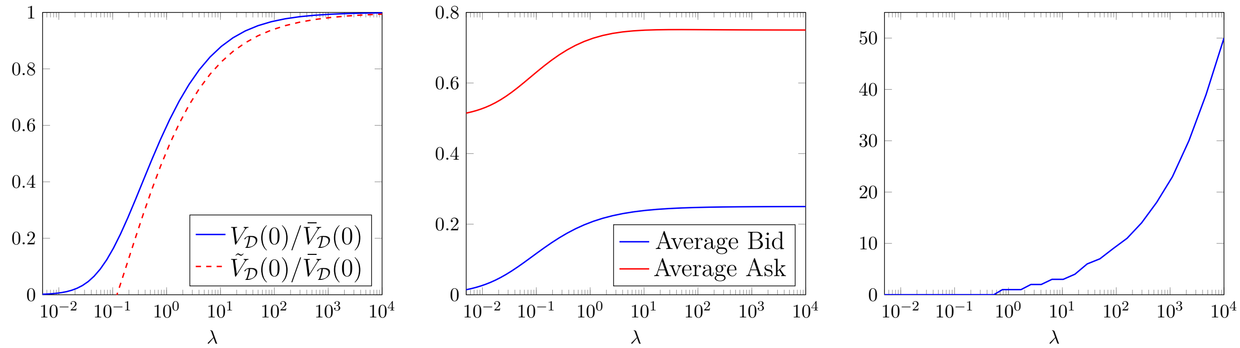
Figure 3: The optimal solution for the dealership model with linear demand and supply $\lambda_b d_b(p) = 5(1-p)^+$, $\lambda_s d_s(p) = 5p$ and $r = 0.04$.

set $\lambda_b = \lambda_s = \lambda$ which represents the market size. The main goal of this section is to explore some structural properties that cannot be investigated in a general model. Moreover, we test the robustness of our major insights for medium market sizes.

The Dealership Model

Figure 3 illustrates the structures of the optimal pricing policy. The underlying parameters are shown below. The buyers and sellers have the same price sensitivity, $\alpha = \beta = 1$. The discount factor per time unit for the dealer is $e^{-r} = 96.1\%$. On average, there are $\lambda = 5$ buyers and sellers per time unit arriving at the market. In a frictionless market, the equilibrium price is given by the intersection of the demand and supply curves $p_e = 0.5$. From panel (a), the value function of the fluid approximation is accurate for large x, while the fixed pricing policy performs well for small x. The bid-ask spread is decreasing and converging to 0.5 rapidly, although $p_b^*(x)$ and $p_s^*(x)$ are converging to their limits relatively slowly. The preferred inventory position (defined in Proposition 2.1) is given by the intersection of the arrival rates of buyers and sellers, $x_p = 3$.

Next we investigate the performance of the fixed-pricing policy and other quantities of the optimal pricing policy for different market sizes λ. The parameters in Figure 4 are the same as those in Figure 3. Figure 4a demonstrates $V(0)/\bar{V}_D(0)$ and $\tilde{V}_D(0)/\bar{V}_D(0)$. When the market is sufficiently thick ($\lambda > 1000$), both the optimal pricing policy and fixed-pricing policy lead to a



(a) The value functions of the optimal and fixed-pricing policies relative to the fluid approximation

(b) The average bid and ask price in the steady state

(c) The preferred inventory level: x_p

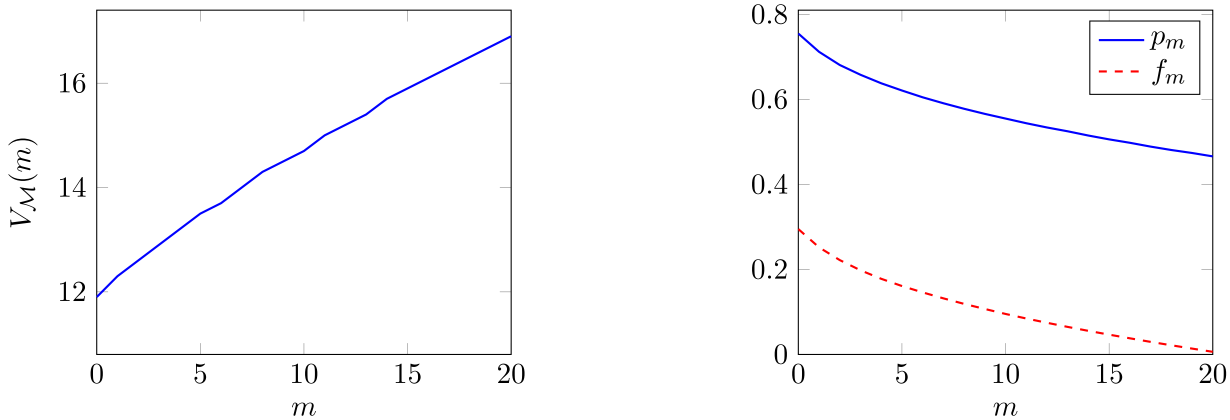
Figure 4: Some performance measures for the dealership model with $\lambda_b d_b(p) = \lambda(1-p)^+$, $\lambda_s d_s(p) = \lambda p$ and $r = 0.04$.

similar revenue relative to that of the fluid approximation. In Figure 4b,we plot the average bid and ask prices in the steady state[12]. Both prices increase in the market size and finally converge. Figure 4c shows that the preferred inventory position of the dealer increases as the market becomes thicker. A thick market leads to stronger stochastic fluctuations, and requires the dealer to have a larger buffer.

The Marketplace

We use the same set of parameters r, λ, α and β as in Figure 3. The sellers have waiting cost $\bar{w} = 0.02$ per time unit before their items are sold. It implies that relative to the equilibrium price in a frictionless market ($p_e = 0.5$), the discount factor per time unit of the sellers is $1 - \bar{w}/p_e = 96\%$, which is comparable to the choice of r. Figure 5 illustrates the structure of the optimal pricing policy. The optimal fee is approximately 0.46. When the charged fee is low, the platform makes little profit per transaction; when charged fee is high, the expected number of transactions decreases. The optimal transaction fees charged by the platform is close to the thick-market policy 0.5, suggested by (11). From Figure 5a, we can see under the optimal policy, the platform's revenue $V_{\mathcal{M}}(m)$ is increasing and concave in the number of sellers in the marketplace m. The monotonicity and convexity of the seller's expected utility f_m and the payment made by the buyers in m can

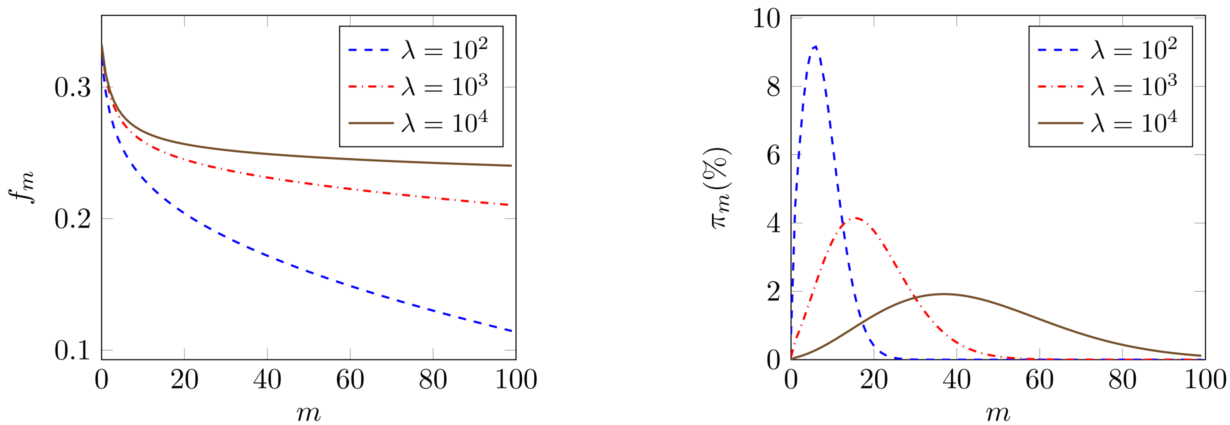
[12] Since the bid and ask prices are set based on the current inventory, we take average when the inventory reaches the steady state under the optimal policy.



(a) The platform's revenue under the optimal policy

(b) The seller's expected utility and the payment by buyers under the optimal policy

Figure 5: The optimal solution for the marketplace with linear demand $\lambda_b d_b(p) = 5(1-p)^+$, $\lambda_s d_s(p) = 5p$, $r = 0.04$ and $\bar{w} = 0.02$.



(a) The seller's expected utility f_m

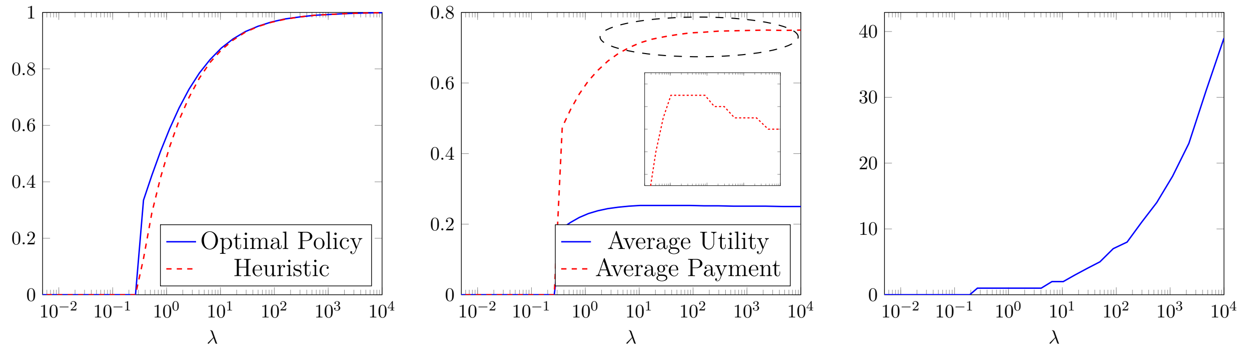
(b) The steady-state distribution $\pi_m(\%)$

Figure 6: Illustration of the convergence in the marketplace in the thick market with $\lambda_b d_b(p) = \lambda(1-p)^+$, $\lambda_s d_s(p) = \lambda p$, $r = 0.04$ and $\bar{w} = 0.02$. Note for $\lambda = 10^4$ the mode $M^* = 39$ and $\hat{f} \approx 0.25$.

be observed in Figure 5b. The mode of the steady-state distribution of sellers is given by the intersection of the arrival rates of buyers and sellers, i.e., $M^* = 1$.

Figure 6 illustrates the convergence (see Proposition 3.2) when the market sizes grow. From the figures, it is clear that the steady-state distribution does not become more concentrated. It is the combination of "flatter" f_m and π_m that leads to our technical characterization of the convergence in Proposition 3.2.

We proceed to investigate the performance of the thick-market heuristic and other quantities



(a) The long-run average revenue of the optimal policy and the thick-market heuristic relative to the thick market limit

(b) The average utility of a seller $\sum_m \pi_m f_m$ and the average payment made by buyers $\sum_m \pi_m (p_m + c_b)$.

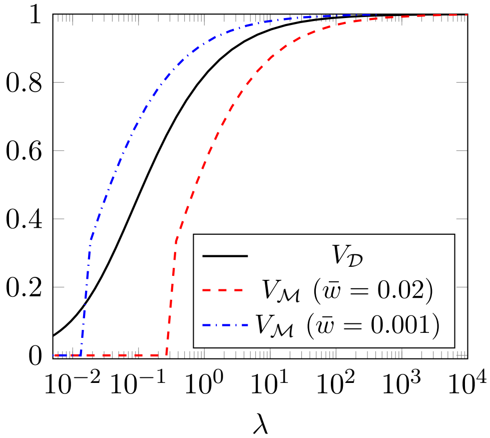
(c) The long-run mean of the number of sellers

Figure 7: Various quantities in the marketplace for different market sizes. $\lambda_b d_b(p) = \lambda(1-p)^+$, $\lambda_s d_s(p) = \lambda p$, $r = 0.04$ and $\bar{w} = 0.02$.

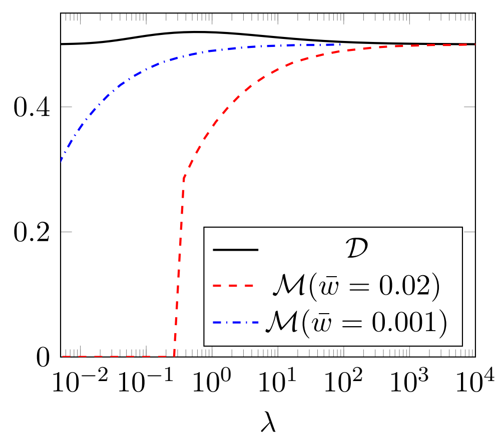
for different market sizes λ. Figure 7a demonstrates the platform's long-run average revenue of the optimal policy and the thick-market policy, which performs well even for medium market sizes. The pattern is similar to Figure 4a and validates Theorem 4.2 that both business models are equally profitable in a thick market. As shown in Figure 7b, both the seller's average utility and the average payment of a buyer converge as the market size grows. However, the seller is not necessarily better off when λ increases. Figure 7c shows that the average number of sellers also increases, as they see more potential profits from the increasing number of buyers. This is consistent with Figure 4c where the dealer also stocks more inventory as λ increases.

Finally, we compare the two business models under the optimal policies. The underlying parameters of Figure 8 and Table 2 are the same as those in Figure 7. As the theory predicts, when the market is thick, the firm's revenues in both models converge to the same limit. For the set of parameters ($\bar{w} = 0.02$), the firm is always better off acting as a dealer due to the waiting cost $\bar{w}$, which is a free parameter the dealership model does not have. If we set $\bar{w} = 0.001$ instead, then the revenue of the marketplace dominates the dealer's profit except for a very small λ.

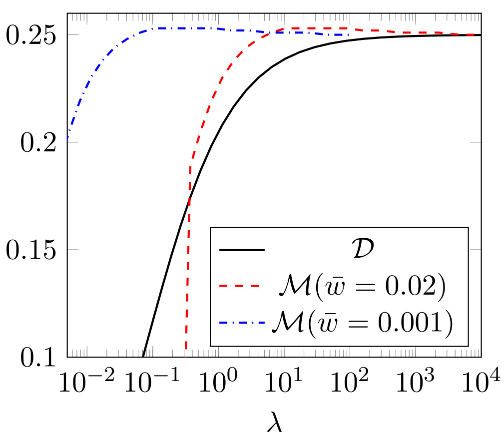
In Figure 8b, we compare the bid-ask spread of the dealer and the optimal fee of a platform. Both quantities converge to the same value when the market size grows. The optimal transaction fees are always increasing in the market size, however, the average bid-ask spread is not necessarily monotone. In Figure 8c and Figure 8d, we compare a few measures related to the surplus of



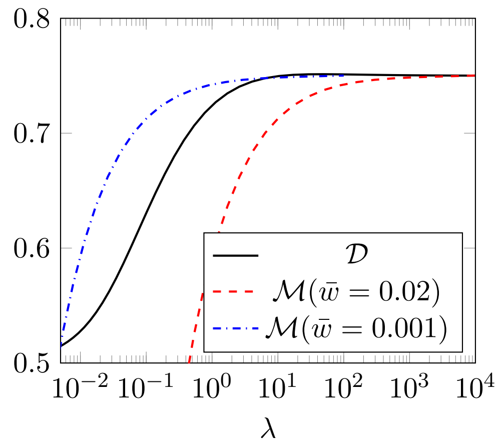
(a) The average revenues for the firm in the steady state in both models relative to the thick-market limit



(b) The average optimal bid-ask spread (dealership; solid line) and the optimal transaction fees (marketplace; dashed line and dash dotted line)



(c) The average bid price (dealership) and utility of a seller (marketplace)



(d) The average ask price (dealership) and payment made by buyers (marketplace)

Figure 8: Comparison of two models with $\lambda_b d_b(p) = \lambda(1-p)^+$, $\lambda_s d_s(p) = \lambda p$ and $r = 0.04$.

consumers: In the dealership model we plot the average bid and ask prices experienced by buyers and sellers; in the marketplace, we plot the average utility of a seller (f_m averaged over the steady state) and the average payment made by buyers (the cost of the product plus the transaction fee paid to the platform). We can see that in general, the prices are rising when the market size grows or the sellers become more patient.

In Table 2, we show which business model is more profitable with $\bar{w} = 0.001$, for different market sizes and compositions of buyers and sellers. Consistent with Theorem 4.1 and Theorem 4.2, when the market size increases, both models are equally profitable. When the market size of buyers is

$\frac{\lambda_b+\lambda_s}{2}$ \ λ_b/λ_s	0.001	0.01	0.1	1	10	100	1000
0.001	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
0.003	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.01	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.03	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.1	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.3	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
1	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
3	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
10	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
30	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
100	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$
300	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$
1000	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$

Table 2: The more profitable business model, depending on the market size $\lambda_b+\lambda_s$ and the composition of buyers and sellers λ_b/λ_s. The notation $\mathcal{D}/\mathcal{M}$ indicates that both models are equally profitable, with revenue difference less than 2%.

relatively small, the dealership model is more profitable. When the market size of sellers is relatively small, the marketplace is more profitable.

6. Case Studies and Economic Insights

We shall present some case studies to illustrate economic insights.[13]

Second-hand Luxury Goods Market. According to a market survey[14], the market share of second-hand luxury goods is well above 20% in developed countries such as Japan, U.S., and France. In China, the number is only 2% but growing rapidly in the last few years. The different stages of the market also lead to different business models. In developed countries, consumers have seen the coexistence of dealership and marketplace. BrandOff and Komehyo operate as dealers: They buy high-end luxury goods from consumers and resell it, controlling the price and inventory on their own. HEWI London, Videdressing, VerstiaireCollective, and Poshmark operate as marketplaces:

[13] Our theoretical results are also consistent with empirical finding in the finance literature, although the boundary between dealership and marketplace in financial markets is blurred and the limit order books in financial markets are in general two-sided. Mayhew (2002) analyses the data of equity option prices from 1986 to 1997, and find that dealers such as designated primary marketmaker works in thin markets, while traditional open outcry platform works well for liquid markets Mayhew (2002) show that for high-volume options the difference between the two market structures becomes slighter. This is consistent with our finding (Theorem 4.2), as for high-volume options, the market is thick and the two models are fundamentally the same.

[14] `http://baogao.chinabaogao.com/baihuo/392626392626.html`

They allow customers to list their used luxury products on the websites and to be directly matched with buyers. There are companies operating in a hybrid mode: RealReal and Thredup ask sellers to ship goods to the company, take full charge of the selling process, and share the revenues when the product gets sold. In contrast, in the Chinese market, the dealership model is dominant, including MilanStation and PonhuLuxury.

This is consistent with our theory: Theorem 4.1 predicts that when the market size is small, especially that of buyers, then the dealership model is more profitable, as is the case in China. Theorem 4.2 predicts that when the market is thick, then both models are equivalent and may coexist, as is the case in developed countries. In fact, the CEO of PonhuLuxury pointed out explicitly in an interview[15] that in the current Chinese market, a marketplace is hard to survive because of the difficulty to attract customers and make the market thick.

Used-Car Market. The used-car market is a major component of the automobile industry. It is estimated that more than 42 million used cars were sold in 2014 (Manheim, 2015), and 2/3 of them were through dealers. The marketplace of used cars is growing, but eclipsed by the market share of dealership. Besides the concern for quality and information asymmetry, which is not captured by our model, Theorem 4.1 explains the dominance of dealership in the used-car market from the angle of waiting cost. The unit prices of cars are usually substantially higher than other second-hand products. This can be translated to higher waiting cost for sellers. From Theorem 4.1, the dealership model is more profitable when consumers are impatient. For instance, haoche51, a Chinese online platform for used cars, started in 2014 as a marketplace to match sellers and buyers directly, and later transformed to a model that is more similar to dealership in 2016. The CEO explained why the first business model failed in an interview[16], stating that many buyers and sellers are unwilling to wait, and the new C2B2C (dealership) model better accommodates their needs.

P2P Lending Markets. In money markets, we can view lenders as sellers and borrowers as buyers. Traditional banking attracts deposits and makes loans to individuals or small and medium enterprises (SMEs). This can be viewed as a dealership model, as a bank manages its own "inventory" and makes profits from the spread. Over the past decade, technology has revolutionized the financial-service sectors. Thanks to the emergence of efficient online communication technology and

[15] `https://36kr.com/p/5112915`
[16] `https://m.qctt.cn/news/120581`

data and tools available for credit modeling , P2P lending platforms, in which individual borrowers (or owners of SMEs) are matched with individual investors, has become popular since 2005, when the UK-based Zopa opened its doors. The industry has grown to \$3.5bn, and continues to gain attraction[17]. These platforms are designed as a marketplace model. For example, LendingClub, a US-based P2P lending company, allows borrowers to post their loan needs on the website with a target rate they would pay, and then individual lenders can browse the loan listings and select one to invest; thus, the roles of buyers/sellers are flipped, as the buyers/borrowers are placing orders, as opposed to sellers on traditional platforms. The recent emergence of this business model is consistent with the numerical study in Table 2, in which we show that the marketplace model becomes relatively profitable for larger market size. Indeed, the rapid increase of the consumer base is crucial for P2P lending platforms, which is only made possible by recent technology. Interestingly, when the consumer base becomes saturated, we find some companies turning to a hybrid model, consistent with our thick market analysis. For example, after eight years from its foundation, LendingClub began partnering with banks to enable them to purchase loans directly through the LendingClub platform or offer LendingClub products to their customers[18].

7. Conclusion

We build dynamic models for the two most common business models in two-sided markets: dealership and marketplace. For the marketplace, we use Markovian dynamic games to capture the waiting cost of sellers without resorting to queueing models. The competitive nature of the sellers allows the framework to yield tractable transient analysis of the equilibrium. Although having distinct features and facing different trade-offs, the two business models exhibit a fundamental connection, illustrated by Figure 1. In particular, when the two sides of the market get thicker, the optimal revenue earned from the two models converge and the optimal pricing/commission policies can be simply translated to each other. Our study provides new economic insights when comparing the two business models.

There are several future research directions. (1) The arrivals of consumers may form a compound Poisson process, as buyers (sellers) may demand (supply) multiple units. Such extensions to the one-

[17] `https://igniteoutsourcing.com/fintech/peer-to-peer-lending-industry-trends/`
[18] `https://www.lendingclub.com/investing/institutional/banks`

sided market have been studied in Gallego and Topaloglu (2019) while the implication on two-sided markets is unclear. (2) The market may have more than one dealer, more than one marketplace, and may have both dealers and marketplaces. It is unclear if the impact of competition on dynamic two-sided markets would be the same as that on one-sided markets. (3) The dealer or marketplace may deal with multiple products that can be either complements or substitutes. Multiple products add complexity to the system as choice models need to be brought into play with optimal solutions becoming intractable. Near-optimal solutions can be found for single-side markets, and we expect the same to be true for two-sided markets. (4) In practice, consumers and dealers, or buyers and sellers in the marketplace, may engage in bargaining during the transaction. Instead of take-it-or-leave-it bid and ask prices, the Nash bargaining equilibrium naturally arises. The potential research topic is the impact of the negotiation power of different parties on the resulting equilibrium and the allocation of revenues or social welfare.

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Online Supplement: Dealership or Marketplace, A Dynamic Comparison

A. A Summary of Notations

Notation	Descriptions
λ_b, λ_s	Arrival rate of the buyers or sellers
Ω_b, Ω_s	The random valuation of a buyer or seller
d_b, d_s	$d_b(p) = P(\Omega_b \geq p)$ and $d_s(p) = P(\Omega_s \leq p)$
$x^{(n)}$	The quantity x in the n-th system, where $(\lambda_b^{(n)}, \lambda_s^{(n)}) = n(\lambda_b, \lambda_s)$
p_e	The equilibrium price, where $\lambda_b d_b(p_e) = \lambda_s d_s(p_e)$
r	Discount factor
p_b, p_s	The ask and bid prices to the buyers and sellers, respectively
p_b^*, p_s^*	The optimal ask and bid prices to the buyers and sellers, respectively
$N_b(t), N_s(t)$	The number of buyers and sellers who transact with the dealer at (include) time t
P_b, P_s	$P_b(z) \triangleq \text{argmax}_{p\geq 0}\{d_b(p)(p-z)\}$ and $P_s(z) \triangleq \text{argmax}_{p\geq 0}\{d_s(p)(z-p)\}$
$\mathcal{R}_b, \mathcal{R}_s$	$\mathcal{R}_b(p,z) \triangleq \lambda_b d_b(p)(p-z)$, $\mathcal{R}_s(p,z) \triangleq \lambda_s d_s(p)(z-p)$, $\mathcal{R}_b(z) \triangleq \max_{p\geq z} \mathcal{R}_b(p,z)$ and $\mathcal{R}_s(z) \triangleq \max_{0\leq p\leq z} \mathcal{R}_s(p,z)$
x	The inventory level of the dealer
x_p	The mean reverted position of inventory in the dealership model
c_b, c_s	The fixed fees charged by a platform from the buyers and sellers per transaction
w	The sum of transaction fees, i.e. $w = c_s + c_b$
$\bar{w}$	Sellers' waiting cost per unit of time
m	The number of sellers in the marketplace
f_m	The future utility of a seller, given m sellers in the marketplace
p_m	The lowest price offered by the sellers, given m sellers in the marketplace. $p_m = f_{m-1} + c_s$ due to the price war
π_m	The steady-state distribution with m sellers in the marketplace
$\hat{f}$	$\hat{f} \triangleq \left\{ f : \lambda_s d_s(\hat{f}) = \lambda_b d_b(\hat{f} + w) \right\}$
M	The endogenous capacity of the marketplace
M^*	The mean reverted number of sellers in the marketplace
$V_{\mathcal{D}}(x), V_{\mathcal{M}}(m)$	A firm's discount revenue in the dealership or marketplace model
$\bar{V}_{\mathcal{D}}(x), \tilde{V}_{\mathcal{D}}(x)$	The dealer's optimal revenue with a fluid approximation or a fixed price policy
$\bar{V}_{\mathcal{D}}'(x)$	The first order derivative of the dealer's optimal revenue in fluid approximation
$\Delta V_{\mathcal{D}}(x), \Delta V_{\mathcal{M}}(m)$	The marginal value of an additional inventory in the dealership model or an additional seller in the marketplace
$V_{\mathcal{M}}$	The long-run average revenue of the platform in the steady state, $V_{\mathcal{M}} \triangleq \sum_{m=0}^{M} \pi_m V_{\mathcal{M}}(m)$.

Table 3: A summary of notations.

B. The Dealership Market

B.1. Additional Results and Proofs in Section 2.1

We first introduce a lemma that is used for the results in Section 2 about the dealership market.

Lemma B.1. *For any $z > 0$, $\mathcal{R}_b'(z) < 0$, $\mathcal{R}_s'(z) > 0$. The functions $P_b(z)$ and $P_s(z)$ are increasing in z.*

Proof of Lemma B.1: From Assumption 2.1, it is easy to see that $P_b(z) > 0$ and $P_s(z) > 0$ when $z > 0$. To show $\mathcal{R}_b'(z) < 0$, $\mathcal{R}_s'(z) > 0$, first note that from the envelope theorem, $\mathcal{R}_b'(z) = -\lambda_b d_b(P_b(z)) < 0$ and $\mathcal{R}_s'(z) = \lambda_s d_s(P_s(z)) > 0$.

Next we show that $P_b(z)$ and $P_s(z)$ are increasing. Let $z_2 > z_1 > 0$. By Assumption 2.1, $d_b(p)(p - z_1)$ is continuous and unimodal in $p \in [0, \infty)$ and then

$$\frac{d[d_b(p)(p - z_1)]}{dp} \,|_{p=P_b(z_1)} \quad = 0.$$

Therefore,

$$\frac{d[d_b(p)(p - z_2)]}{dp} \,|_{p=P_b(z_1)} \quad = \frac{d[d_b(p)(p - z_1)]}{dp} \,|_{p=P_b(z_1)} + d_b'(P_b(z_1))(z_1 - z_2) > 0.$$

For unimodel functions, if the derivative is less than zero, then it must be larger than the stationary point. As a result, $P_b(z_2) > P_b(z_1)$. Similarly, $P_s(z_2) > P_s(z_1)$. Then we complete the proof. ■

Next we establish the uniqueness of the HJB equation.

Uniqueness of the HJB Equation (1)*:* We prove the uniqueness by contradiction. First, notice that when the dealer has infinite inventory, he does not buy from sellers, and thus only maximizes the revenue from buyers.

$$\lim_{x\to\infty} V_{\mathcal{D}}(x) = \mathbb{E}\left[\int_0^\infty e^{-rt} p_b dN_b(t)\right] = \mathbb{E}\left[\int_0^\infty e^{-rt} p_b \lambda_b d_b(p_b) dt\right] = \max_{p_b} \frac{\lambda_b d_b(p_b) p_b}{r},$$

where the second equality can be found in Brémaud (1981).

Now suppose there are two distinct solutions with $V_{\mathcal{D}}^1(0) > V_{\mathcal{D}}^2(0)$. From $rV_{\mathcal{D}}(0) = \mathcal{R}_s(\Delta V_{\mathcal{D}}(1))$, as $\mathcal{R}_s'(\cdot) > 0$ from Lemma B.1, we have $\Delta V_{\mathcal{D}}^1(1) > \Delta V_{\mathcal{D}}^2(1)$. Combining with $V_{\mathcal{D}}^1(0) > V_{\mathcal{D}}^2(0)$, we have $V_{\mathcal{D}}^1(1) > V_{\mathcal{D}}^2(1)$.

Next we use induction to show that $\lim_{x\to\infty}(V_{\mathcal{D}}^1)(x) - V_{\mathcal{D}}^2(x)) > 0$. Suppose $V_{\mathcal{D}}^1(x) > V_{\mathcal{D}}^2(x)$ and $\Delta V_{\mathcal{D}}^1(x) > \Delta V_{\mathcal{D}}^2(x)$ hold for some x. From the HJB equation, we have

$$V_{\mathcal{D}}(x) - \mathcal{R}_b(\Delta V_{\mathcal{D}}(x)) = \mathcal{R}_s(\Delta V_{\mathcal{D}}(x+1)).$$

As $\mathcal{R}_b'(\cdot) < 0$ from Lemma B.1, we have $\mathcal{R}_s(\Delta V_{\mathcal{D}}^1(x+1)) > \mathcal{R}_s(\Delta V_{\mathcal{D}}^2(x+1))$, which implies $\Delta V_{\mathcal{D}}^1(x+1) > \Delta V_{\mathcal{D}}^2(x+1)$. Combining with the fact that $V_{\mathcal{D}}^1(x) > V_{\mathcal{D}}^2(x)$, we have $V_{\mathcal{D}}^1(x+1) > V_{\mathcal{D}}^2(x+1)$. As a result, if $V_{\mathcal{D}}^1(0) > V_{\mathcal{D}}^2(0)$, then $\Delta V_{\mathcal{D}}^1(x) > \Delta V_{\mathcal{D}}^2(x)$ for all x. However, note that

$$\lim_{x\to\infty}(V_{\mathcal{D}}^1(x) - V_{\mathcal{D}}^1(x)) = V_{\mathcal{D}}^1(0) + \sum_x \Delta V_{\mathcal{D}}^1(x) - V_{\mathcal{D}}^2(0) - \sum_x \Delta V_{\mathcal{D}}^2(x) > V_{\mathcal{D}}^1(0) - V_{\mathcal{D}}^2(0) > 0,$$

leading to a contradiction, as $\lim_{x\to\infty} V_{\mathcal{D}}(x)$ must equal to $\max_{p_b} \frac{\lambda_b d_b(p_b) p_b}{r}$ as derived above. Thus we complete the proof. ■

Proposition B.1. *The value function $V_{\mathcal{D}}(x)$ satisfies the following properties:*

1. *$V_{\mathcal{D}}(x)$ is increasing and concave in x.*

2. *$V_{\mathcal{D}}(x)$ decreases in r and increases in λ_b or λ_s.*

3. *$\Delta V_{\mathcal{D}}(x)$ decreases if r increases or λ_b and λ_s decrease in proportion.*

The first result shows that more inventory leads to higher revenues for the dealer. Furthermore, the concavity of $V_{\mathcal{D}}(x)$ implies that the marginal value of inventory, $\Delta V_{\mathcal{D}}(x)$, is decreasing. Notice that having more inventory only benefits the dealer by lowering the probability of hitting $x = 0$, i.e. the inventory risk. Thus, as x increases and gets further from the boundary, the inventory risk and thus the marginal value of inventory is diminishing. The second result characterizes the potential revenue when the dealer has unlimited inventory. The third result states that a larger market size of either side benefits the dealer, and a higher opportunity cost (a higher interest rate) hurts the dealer. In fact, the effect of r is exactly the opposite to that of λ_b and λ_s. If we scale r, λ_b and λ_s by the same constant, $V_{\mathcal{D}}(x)$ remains unchanged from (1). The implication of the fourth result is not clear for now, but as shown in (2), the marginal value of inventory is the key to decoding the structure of the optimal pricing policy. In (2), the optimal prices depend on $\Delta V_{\mathcal{D}}(x)$ directly, and the dealer is effectively trading the marginal value of inventory by the bid/ask prices. Therefore, the properties of $\Delta V_{\mathcal{D}}(x)$ yield various empirical implications for the optimal pricing policy as well as the inventory control strategy.

Proof of Proposition B.1: For part one, note that any pricing policy for initial inventory x is feasible for $x+1$. Therefore, the latter achieves a larger revenue than the former and $V_{\mathcal{D}}(x+1) > V_{\mathcal{D}}(x)$. The concavity of $V_{\mathcal{D}}(x)$ is equivalent to show

$$2V_{\mathcal{D}}(x) \geq V_{\mathcal{D}}(x+1) + V_{\mathcal{D}}(x-1), \quad x \geq 1. \tag{12}$$

We use a sample path coupling argument to prove the claim. Considering two systems, system one starting with $x+1$ units of inventory, system two with x units. Assume that system one follows its optimal policies. For system two, we use the sub-optimal policy designed for system one, and couple the Poisson processes with those of system one. The coupling guarantees that the difference of inventory between two systems is always one, for all sample paths of the arrival processes before the inventory of system two reaches 0. We break the coupling when the inventories of two systems drop to x and $x-1$, respectively and then let each follows its optimal policy. During coupling, the realized discounted revenue of two systems are equal. Therefore the revenue difference of two systems is $d(V_{\mathcal{D}}(x) - V_{\mathcal{D}}(x-1))$, where $d < 1$ is the expected discounted factor at the moment coupling is broken and $V_{\mathcal{D}}(x) - V_{\mathcal{D}}(x-1)$ is the revenue difference after breaking the coupling. Since system two follows sub-optimal policies for a period, we can conclude that $V_{\mathcal{D}}(x+1) - V_{\mathcal{D}}(x)$ is not larger than this difference, namely $V_{\mathcal{D}}(x+1) - V_{\mathcal{D}}(x) \leq d(V_{\mathcal{D}}(x) - V_{\mathcal{D}}(x-1)) \leq V_{\mathcal{D}}(x) - V_{\mathcal{D}}(x-1)$. Hence (12) is proved.

For part two, we again use the sample path coupling argument. Consider $V_{\mathcal{D}}(x; r_1)$ and $V_{\mathcal{D}}(x; r_2)$ for $r_1 < r_2$. Let system one set the same price as the optimal pricing policy of system two. Since $V_{\mathcal{D}}(x; r_2) > 0$, the revenue earned by system one is discounted less and thus $V_{\mathcal{D}}(x; r_1) > V_{\mathcal{D}}(x; r_2)$. The monotonicity in λ_b and λ_s can be obtained similarly.

For part three, note that if we scale r, λ_b and λ_s by the same constant, then $V_{\mathcal{D}}(x)$ remains unchanged from Equation (1). So we just need to show the monotonicity of $\Delta V_{\mathcal{D}}(x)$ in r by showing the following three claims step by step.

Step (1) For all $x > 0$, $\frac{d[rV_{\mathcal{D}}(x)]}{dr} \geq 0$ and $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} \geq 0$ cannot hold simultaneously.

Suppose for some x, the claim above is not true. Then from Equation (1),

$$\frac{d[rV_{\mathcal{D}}(x)]}{dr} = \mathcal{R}_b'[\Delta V_{\mathcal{D}}(x))]\frac{d\Delta V_{\mathcal{D}}(x)}{dr} + \mathcal{R}_s'[\Delta V_{\mathcal{D}}(x+1)]\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr}.$$

As $\frac{d[rV_{\mathcal{D}}(x)]}{dr} \geq 0$, $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} \geq 0$, $\mathcal{R}_b'[\Delta V_{\mathcal{D}}(x))] < 0$ and $R_s'[\Delta V_{\mathcal{D}}(x+1))] > 0$, we must have

$$\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} \geq 0.$$

Moreover, $\frac{d[r\Delta V_{\mathcal{D}}(x+1)]}{dr} = \Delta V_{\mathcal{D}}(x+1) + r\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr}$, the inequality above implies

$$\frac{d[rV_{\mathcal{D}}(x+1)]}{dr} > \frac{d[rV_{\mathcal{D}}(x)]}{dr} \geq 0.$$

By induction, $\frac{d[rV_{\mathcal{D}}(y)]}{dr} > \frac{d[rV_{\mathcal{D}}(x)]}{dr} \geq 0$ and $\frac{d\Delta V_{\mathcal{D}}(y)}{dr} \geq 0$ for all $y > x$. However, $\frac{d\lim_{y\to\infty}[rV_{\mathcal{D}}(y)]}{dr} = \frac{d\mathcal{R}_b(0)}{dr} = 0$, leading to a contradiction.

Step (2) $\frac{d[rV_{\mathcal{D}}(0)]}{dr} < 0$.

We prove this claim by contradiction. Suppose $\frac{d[rV(0)]}{dr} \geq 0$, then from Equation (1),

$$\frac{d[rV_{\mathcal{D}}(0)]}{dr} = R_s'[\Delta V_{\mathcal{D}}(1)]\frac{d\Delta V_{\mathcal{D}}(1)}{dr} \geq 0,$$

which implies $\frac{d\Delta V_{\mathcal{D}}(1)}{dr} \geq 0$. As a result, $\frac{d[r\Delta V_{\mathcal{D}}(1)]}{dr} = \Delta V_{\mathcal{D}}(1) + r\frac{d\Delta V_{\mathcal{D}}(1)}{dr} > 0$, leading to

$$\frac{d[rV_{\mathcal{D}}(1)]}{dr} > \frac{d[rV_{\mathcal{D}}(0)]}{dr} \geq 0.$$

However, from the first step, we know $\frac{d[rV_{\mathcal{D}}(1)]}{dr} > 0$ and $\frac{d\Delta V_{\mathcal{D}}(1)}{dr} \geq 0$ cannot hold simultaneously. Therefore, we must have $\frac{d[rV_{\mathcal{D}}(0)]}{dr} < 0$.

Step (3) $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$.

Note that by (1), we have $\frac{d[rV_{\mathcal{D}}(0)]}{dr} = \mathcal{R}_s'[\Delta V_{\mathcal{D}}(1)]\frac{d\Delta V_{\mathcal{D}}(1)}{dr}$. Because $\frac{d[rV_{\mathcal{D}}(0)]}{dr} < 0$ shown in the second step, we have $\frac{d\Delta V_{\mathcal{D}}(1)}{dr} < 0$.

Next we prove the claim by induction: suppose that for some $x \geq 1$, $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$, then we must have $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} < 0$. We prove it by contradiction. Suppose $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$ but $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} \geq 0$, then

$$\frac{d[r\Delta V_{\mathcal{D}}(x+1)]}{dr} = \Delta V_{\mathcal{D}}(x+1) + r\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} > 0.$$

which implies that

$$\frac{d[rV_{\mathcal{D}}(x+1)]}{dr} > \frac{d[rV_{\mathcal{D}}(x)]}{dr} = \mathcal{R}_b'[\Delta V_{\mathcal{D}}(x))]\frac{d\Delta V_{\mathcal{D}}(x)}{dr} + \mathcal{R}_s'[\Delta V_{\mathcal{D}}(x+1)]\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr}$$

by (1). As $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$ and $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} \geq 0$, then combined with the facts that $\mathcal{R}_b' < 0$ and $\mathcal{R}_s' > 0$ by Lemma B.1, we have the last expression above being nonnegative and thus $\frac{d[rV_{\mathcal{D}}(x+1)]}{dr} > 0$. However, $\frac{d[rV_{\mathcal{D}}(x+1)]}{dr} \geq 0$ and $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} > 0$ contradict step (1). Therefore $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} < 0$. Therefore, by induction, we have $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$ for any x. ■

Proof of Proposition 2.1: Part one is just same as part one of Proposition B.1.

Recall that $p_b^*(x) = \text{argmax}_{p\geq 0}\{\lambda_b d_b(p)(p - \Delta V_{\mathcal{D}}(x))\}$, and $p_s^*(x) = \text{argmax}_{p\geq 0}\{\lambda_s d_s(p)(\Delta V_{\mathcal{D}}(x+1) - p)\}$. For part two, it is straightforward as $p_s^*(x) \leq \Delta V_{\mathcal{D}}(x+1) \leq \Delta V_{\mathcal{D}}(x) \leq p_b^*(x)$.

For part three, as $\lim_{x\to\infty} \Delta V_{\mathcal{D}}(x) = 0$, $\lim_{x\to\infty} p_s^*(x) = 0$ and $\lim_{x\to\infty} p_b^*(x) = \text{argmax}_p\{d_b(p)p\}$. From Proposition B.1, $\Delta V_{\mathcal{D}}(x)$ is decreasing in x and $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$. Then by Lemma B.1, we have the monotonicity of $p_b^*(x)$ and $p_s^*(x)$. The dependence on the market size when λ_b and λ_s change in proportion, is just the opposite to that of r.

The properties about the preferred inventory level follows directly from the monotonicity of $p_s^*(x)$ and $p_b^*(x)$. ■

B.2. Additional Results and Proofs in Section 2.2:

We next show the properties of the value function of the fluid approximation.

Proposition B.2. *The value function $\bar{V}_{\mathcal{D}}(x)$ satisfies the following properties:*

1. *$\bar{V}_{\mathcal{D}}(x)$ is increasing and concave in x.*

2. *$\bar{V}_{\mathcal{D}}(x)$ is an upper bound of $V_{\mathcal{D}}(x)$ and $\lim_{x\to\infty} \bar{V}_{\mathcal{D}}(x) = \lim_{x\to\infty} V_{\mathcal{D}}(x) = \mathcal{R}_b(0)/r$.*

Proof of Proposition B.2: By the coupling argument, it is easy to prove that $\bar{V}_{\mathcal{D}}(x)$ is strictly increasing in x. For concavity, note that we must have $\lambda_b d_b(\bar{p}_b^*(x)) \geq \lambda_s d_s(\bar{p}_s^*(x))$ for all $x \geq 0$, i.e., the inventory is always being depleted. This is because in the fluid approximation, there is no stochasticness and it is never optimal to stock more inventory. Because $\bar{V}_{\mathcal{D}}(x)$ is strictly increasing in x, we must have that for any $x > 0$,

$$0 < \frac{dr\bar{V}_{\mathcal{D}}(x)}{dx} = \frac{d[\mathcal{R}_b(\bar{V}'_{\mathcal{D}}(x)) + \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(x))]}{dx} = [\lambda_s d_s(\bar{p}_s^*(x)) - \lambda_b d_b(\bar{p}_b^*(x))]V''_{\mathcal{D}}(x),$$

where the second equality follows from the envelop theorem. Therefore, $\bar{V}''_{\mathcal{D}}(x) < 0$ and $\bar{V}_{\mathcal{D}}(x)$ is increasing concave.

For part two, it is easy to show that $\lim_{x\to\infty} \bar{V}_{\mathcal{D}}(x) = \mathcal{R}_b(0)/r$.

Next we show that $\bar{V}(x)$ is an upper bound for $V(x)$. Because $\bar{V}(x)$ is concave, $\Delta\bar{V}_{\mathcal{D}}(x) > \bar{V}'_{\mathcal{D}}(x) > \Delta\bar{V}_{\mathcal{D}}(x+1)$. By the property of $\mathcal{R}_b$ and $\mathcal{R}_s$ in Lemma B.1, we have

$$r\bar{V}_{\mathcal{D}}(x) = \mathcal{R}_b(\bar{V}'_{\mathcal{D}}(x)) + \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(x)) \geq \mathcal{R}_b(\Delta\bar{V}_{\mathcal{D}}(x)) + \mathcal{R}_s(\Delta\bar{V}_{\mathcal{D}}(x+1)),$$

for $x \in \mathbb{Z}^+$. Moreover, by the continuous differentiability of $\bar{V}_{\mathcal{D}}$, we have

$$r\bar{V}_{\mathcal{D}}(0) = \mathcal{R}_b(\bar{V}'_{\mathcal{D}}(0)) + \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(0)) \geq \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(0)) \geq \mathcal{R}_s(\Delta\bar{V}_{\mathcal{D}}(1)).$$

The HJB equation in (1) can be formulated as linear programming,

$$\begin{aligned} V_{\mathcal{D}}(y) = \min_{F(\cdot)} \quad & F(y) \\ \text{subject to} \quad & rF(0) \geq \mathcal{R}_s(\Delta F(1)), \\ & rF(x) \geq \mathcal{R}_b(\Delta F(x)) + \mathcal{R}_s(\Delta F(x+1)), \text{ for any } x \in \mathbb{Z}^+. \end{aligned}$$

Clearly, $\bar{V}_{\mathcal{D}}(x)$ is a feasible solution to the problem above, then $V_{\mathcal{D}}(x) \leq \bar{V}_{\mathcal{D}}(x)$. ■

Proof of Proposition 2.2: When the dealer adopts the fixed pricing policies in (6), the dynamics of the inventory follows that of a state independent M/M/1 queue. Denote $\tilde{V}_{\mathcal{D}}(x)$ as the expected discounted profit for this policy with x units of initial inventory. By the Markovian structure, we have

$$
\begin{aligned}
(r + \lambda_b d_b(\hat{p}_b) + \lambda_s d_s(\hat{p}_s))\tilde{V}_{\mathcal{D}}(x) &= \lambda_b d_b(\hat{p}_b)\tilde{V}_{\mathcal{D}}(x-1) + \lambda_s d_s(\hat{p}_s)\tilde{V}_{\mathcal{D}}(x+1) \\
&\quad + \hat{p}_b \lambda_b d_b(\hat{p}_b) - \hat{p}_s \lambda_s d_s(\hat{p}_s) \quad x > 0, \\
(r + \lambda_s d_s(\hat{p}_s))\tilde{V}_{\mathcal{D}}(0) &= \lambda_s d_s(\hat{p}_s)\tilde{V}_{\mathcal{D}}(1) - \hat{p}_s \lambda_s d_s(\hat{p}_s). \qquad (13)
\end{aligned}
$$

From the first equation above, $\tilde{V}_{\mathcal{D}}(x)$ must be of the form

$$\tilde{V}_{\mathcal{D}}(x) = C_0 s_0^x + C_1 s_1^x + \frac{1}{r}(\hat{p}_b \lambda_b d_b(\hat{p}_b) - \hat{p}_s \lambda_s d_s(\hat{p}_s)),$$

where $0 < s_0 < 1 < s_1$ and s_0 and s_1 are the two solutions to $\lambda_s d_s(\hat{p}_s)x^2 - (r + \lambda_b d_b(\hat{p}_b) + \lambda_s d_s(\hat{p}_s))x + \lambda_b d_b(\hat{p}_b) = 0$, with

$$s_0 = \frac{r + \lambda_b d_b(\hat{p}_b) + \lambda_s d_s(\hat{p}_s) - \sqrt{(r + \lambda_b d_b(\hat{p}_b) + \lambda_s d_s(\hat{p}_s))^2 - 4\lambda_s \lambda_b d_s(\hat{p}_s) d_b(\hat{p}_b)}}{2\lambda_s d_s(\hat{p}_s)} \in (0,1).$$

Since $\tilde{V}_{\mathcal{D}}(x)$ is bounded above by the same argument as in Proposition B.1, we must have $C_1 = 0$. By the boundary condition, the last equation in (13),

$$C_0 = -\frac{\hat{p}_b \lambda_b d_b(\hat{p}_b)}{r + \lambda_s d_s(\hat{p}_s)(1 - s_0)} < 0.$$

Now scale demand and supply simultaneously by n, i.e., $(\lambda_b^n, \lambda_s^n) = n(\lambda_b, \lambda_s)$ and denote $V_D^{(n)}(x)$,$\bar{V}_D^{(n)}(x)$,$\tilde{V}_D^{(n)}(x)$ as the corresponding value function in the original problem, fluid approximation and the fixed pricing policy, respectively. Denote $d_n = n\lambda_b d_b(\hat{p}_b) = n\lambda_s d_s(\hat{p}_s)$. Then

$$|\tilde{V}_{\mathcal{D}}^{(n)}(x) - \bar{V}_{\mathcal{D}}^{(n)}(0)| = |C_0| s_0^x \le |C_0| = \frac{\hat{p}_b d_n}{r + \left(\sqrt{r^2 + 4rd_n} - r\right)/2} = O(\sqrt{n}).$$

As $\bar{V}_{\mathcal{D}}^{(n)}(x) = \Omega(n)$, $\lim_{n\to\infty} \frac{\tilde{V}_{\mathcal{D}}^{(n)}(x)}{\bar{V}_{\mathcal{D}}^{(n)}(0)} = 1$ and $\tilde{V}_{\mathcal{D}}^{(n)}(x) \le V_{\mathcal{D}}^{(n)}(x) \le \bar{V}_{\mathcal{D}}^{(n)}(x)$, we have the result. ■

C. The Marketplace

C.1. Proof of Theorem 3.1

We first introduce some notations, and a few auxiliary results used in the proof. Define f^* as the unique solution to $\mathcal{R}_b(f^*) = \max_{p\geq 0}\{\lambda_b d_b(p+w)(p-f^*)\} = \bar{w}$.[19] Define f^{**} as the smallest positive solution to $\lambda_b d_b(f^{**}+w)f^{**} = \bar{w}$ if there exists such a solution or $+\infty$ otherwise. Note that when $f^* > 0$, $f^{**} < +\infty$ exists and $\frac{d\lambda_b d_b(p+w)p}{dp}\mid_{p=f^{**}} > 0$.[20] We also define the following function

$$R(x; y, z) \triangleq \lambda_b d_b(x+w)(x-y) - \bar{w} - \lambda_s d_s(z)(y-z). \tag{14}$$

Using this function, the recursive system (7) can be rewritten as

$$\begin{aligned} &\sup_x R(x; f_1, f_2) = 0, \\ &R(f_{m-1}; f_m, f_{m+1}) = 0, m \geq 2. \end{aligned} \tag{15}$$

Lemma C.1. *Suppose that $\tilde{x}_1 \geq y_1$ satisfying $R(\tilde{x}_1; y_1, z_1) = 0$ and $\frac{dR(x;y_1,z_1)}{dx}|_{x=\tilde{x}_1} > 0$, and $\tilde{x}_2 \geq y_2$ satisfying $R(\tilde{x}_2; y_2, z_2) = 0$. If $y_2 - z_2 \geq y_1 - z_1 \geq 0$ and $y_2 - y_1 \geq z_2 - z_1 > 0$, then $\tilde{x}_2 - \tilde{x}_1 > y_2 - y_1 > 0$.*

Proof of Lemma C.1. From the conditions, we have

$$\begin{aligned} 0 &= R(\tilde{x}_1; y_1, z_1) - R(\tilde{x}_2; y_2, z_2) \\ &= \lambda_b[d_b(\tilde{x}_1+w)(\tilde{x}_1-y_1) - d_b(\tilde{x}_2+w)(\tilde{x}_2-y_2)] + \lambda_s[d_s(z_2)(y_2-z_2) - d_s(z_1)(y_1-z_1)]. \end{aligned}$$

If $y_2 - z_2 \geq y_1 - z_1 \geq 0$ and $y_2 - y_1 \geq z_2 - z_1 \geq 0$, then the last term in the last expression above is nonnegative. Therefore, the first term is nonpositive

$$d_b(\tilde{x}_1+w)(\tilde{x}_1-y_1) \leq d_b(\tilde{x}_2+w)(\tilde{x}_2-y_2). \tag{16}$$

[19]To show that such f^* always exists, note that $\lim_{z\to+\infty}\mathcal{R}_b(z) = 0$ and $\lim_{z\to-\infty}\mathcal{R}_b(z) = +\infty$, and that $\mathcal{R}_b(z)$ is continuous. To show the uniqueness, note that by the similar proof in Lemma B.1, $\mathcal{R}_b(\cdot)$ is monotone. Moreover, $\mathcal{R}_b'(z)|_{z=f^*} = -\lambda_b d_b(P_b(f^*)+w) < 0$, so $\mathcal{R}_b(z)$ is strictly decreasing at $z = f^*$. This implies that f^* is unique.

[20]If $f^* > 0$, then $\max_p\{\lambda_b d_b(p+w)p\} > \max_p\{\lambda_b d_b(p+w)(p-f^*)\} = \bar{w}$. Therefore, f^{**} exists. Moreover, by Assumption 2.1, $\lambda_b d_b(p+w)p$ is unimodal in $p \in [0,\infty)$. Because f^{**} is the smaller solution, we have $\frac{d\lambda_b d_b(p+w)p}{dp}\mid_{p=f^{**}} > 0$.

Since $y_2 > y_1$, the inequality above implies

$$d_b(\tilde{x}_1 + w)(\tilde{x}_1 - y_1) < d_b(\tilde{x}_2 + w)(\tilde{x}_2 - y_1).$$

By Assumption 2.1, $d_b(x+w)(x-y_1)$ and $R(x; y_1, z_1)$ are unimodal in x. Since $\frac{dR(x;y_1,z_1)}{dx}|_{x=\tilde{x}_1} > 0$, $\tilde{x}_1$ is on the left side of the mode. Therefore, the above inequality implies $\tilde{x}_1 < \tilde{x}_2$. As a result, $d_b(\tilde{x}_1 + w) > d_b(\tilde{x}_2 + w)$ and thus the inequality in (16) will lead to $\tilde{x}_1 - y_1 < \tilde{x}_2 - y_2$. That is $\tilde{x}_2 - \tilde{x}_1 > y_2 - y_1 > 0$. ■

Now we are ready to prove Theorem 3.1. Note that from the definitions of f^* and f^{**}, we only need to establish the proof for three separate cases: $f^* \leq 0$, $0 < f^* \leq f^{**}$ and $f^* > f^{**} > 0$. The proof is organized as follows: In Section C.1.1, we prove the existence of solutions to (7) when $f^* \leq 0$ or $0 < f^* \leq f^{**}$. In Section C.1.2, we prove the existence of solutions to (7) when $f^* > f^{**} > 0$. In Section C.1.3, we demonstrate the properties that any solution to (7) will satisfy. Using these properties, we then prove that the solutions that have been found in Section C.1.1 and Section C.1.2 are the unique solutions to (7). Thus there exists a unique Markov perfect equilibrium of the dynamic game.

C.1.1. Existence: $f^* \leq 0$ or $0 < f^* \leq f^{**}$

When $f^* \leq 0$ or $0 < f^* \leq f^{**}$, then we can show that

$$\begin{cases} f_1 = f^* \\ f_m = f_{m-1} - \frac{\bar{w}}{\lambda_b d_b(f_{m-1}+w)} \quad m \geq 2 \end{cases} \tag{17}$$

is a solution to (7). In fact, if $f^* \leq 0$, then $f_2 \leq f_1 = f^* \leq 0$. Therefore, $M = 0$ and (7) degenerates to a single equation $\bar{w} = \max_{f_0} \{\lambda_b d_b(f_0 + w)(f_0 - f_1)\}$, which holds when $f_1 = f^*$ by the definition of f^*.

If $0 < f^* \leq f^{**}$, then from the definition of f^{**} (by Footnote 20, f^{**} is less than the mode), $f^* \leq f^{**} < p^* \triangleq \text{argmax}_{p \geq 0}\{\lambda_b d_b(p+w)p\}$. By Assumption 2.1, $\lambda_b d_b(p+w)p$ is increasing in $[0, p^*)$ and thus $\lambda_b d_b(f^*+w)f^* \leq \lambda_b d_b(f^{**}+w)f^{**} = \bar{w}$ (by definition of f^{**}). Therefore, $f_2 = f^* - \frac{\bar{w}}{\lambda_b d_b(f^*+w)} \leq 0$, so $M = 1$ and (7) again degenerates to a single equation $\bar{w} = \max_{f_0} \{\lambda_b d_b(f_0 + w)(f_0 - f_1)\}$, INPUT: a

Set $n \leftarrow 1$;

Set $X_1 \leftarrow a$ and $X_0 \leftarrow X_1 - \frac{\bar{w}}{\lambda_b d_b (X_1 + w)}$;

while true:

    Set $R_n \leftarrow \max_x R(x; X_n, X_{n-1})$;

    if $R_n > 0$:

        Solve $X_{n+1} > 0$ such that $R(X_{n+1}; X_n, X_{n-1}) = 0$ and $\frac{dR(x;X_n,X_{n-1})}{dx} |_{x=X_{n+1}} > 0$;

        $n \leftarrow n + 1$;

    else:

        Break;

Set $N \leftarrow n$;

Return: R_N, N, $[X_N, \cdots, X_0]$.

Algorithm 1: Algorithm to find a solution to (7) when $f^* \geq f^{**}$.

which holds for $f_1 = f^*$.

C.1.2. Existence: $f^* > f^{**} > 0$

To show the existence of solutions to (7) when $f^* > f^{**} > 0$, we design Algorithm 1. The algorithm is based on the following intuition: Assign a value to f_M, then recursively compute f_{M-1}, f_{M-2}, ..., f_1 using the Mth, $(M-1)$th, ..., second equation of (7) . Then check whether the first equation of (7) holds. Since M is not known in advance, in the algorithm, we use X_n to denote f_{M+1-n}. When backwards calculating X_{n+1}, we use $R(X_{n+1}; X_n, X_{n-1}) = 0$ due to (15). The additional constraint in the computation $\frac{dR(x;X_n,X_{n-1})}{dx} |_{x=X_{n+1}} > 0$ will be specified in Lemma C.3 as a necessary condition for any solution to (7) .

The algorithm takes an input a, which is a candidate solution for f_M. If the output is $R_N = 0$,

$N < \infty$, $X_0 \leq 0$ and $X_i \geq 0$ for $i > 0$, then $[X_N, \cdots, X_1, X_0]$ must satisfy

$$\sup_x R(x; X_N, X_{N-1}) = 0,$$

$$R(f_{n+1}; f_n, f_{n-1}) = 0, 1 \leq n < N.$$

It solves (15), or equivalently (7) , by substituting $M = N$, $f_1 = X_N$, $\cdots$, $f_M = X_1$, $f_{M+1} = X_0$.

Denote $N(a)$ as the output of N when the input is a. Similarly, we introduce $X_n(a)$ and $R_n(a)$ for fixed n. First we point out some facts for the proposed algorithm, based on Assumption 2.1 and 3.1 and the following conditions: $f^* \geq f^{**} > 0$ and $a \in [0, f^{**}]$. These facts are then used to prove the existence of solutions to (7) when $f^* > f^{**} > 0$ based on Algorithm 1.

Fact 1: $X_0(a) \leq 0$, $X_0(a) < X_1(a)$ and $\frac{d\lambda_b d_b(x+w)(x-X_0(a)}{dx}|_{x=X_1(a)} > 0$.

This fact states that if $f_M = a \in [0, f^{**}]$, then $f_{M+1} = X_0(a) \leq 0$. So the choice of a is proper.

Proof of Fact 1. By their definitions in the algorithm, $X_1(a) = a \geq 0$ and $X_0(a) < X_1(a)$. Let $p^* \triangleq \text{argmax}_{p \geq 0}\{\lambda_b d_b(p+w)p\}$. By the definition (Footnote 20), $f^{**} < p^*$. By Assumption 2.1, $\lambda_b d_b(p+w)p$ is increasing in $[0, p^*)$. Then as $a \leq f^{**} < p^*$, $\lambda_b d_b(a+w)a \leq \lambda_b d_b(f^{**}+w)f^{**} = \bar{w}$ and thus $X_0(a) = a - \frac{\bar{w}}{\lambda_b d_b(a+w)} \leq 0$, where the equality holds only when $a = f^{**}$. When $a = f^{**}$, $\frac{d\lambda_b d_b(x+w)(x-X_0(a)}{dx}|_{x=X_1(a)} = \frac{d\lambda_b d_b(x+w)x}{dx}|_{x=f^{**}} > 0$, where the inequality is due to the definition of f^{**}. When $a < f^{**}$, $X_0(a) < 0$ and thus

$$\lambda_b d_b(X_1(a)+w)(X_1(a)-X_0(a)) = \bar{w} = \lambda_b d_b(f^{**}+w)f^{**} < \lambda_b d_b(f^{**}+w)(f^{**}-X_0(a)),$$

where the first equality is due to the definition of $X_0(a)$ in the algorithm and the second equality is due to the definition of f^{**}. Due to Assumption 2.1, $\lambda_b d_b(x+w)(x-X_0(a))$ is unimodal in x and as $X_1(a) = a < f^{**}$, the inequality above leads to $\frac{d\lambda_b d_b(x+w)(x-X_0(a))}{dx}|_{x=X_1(a)} > 0$. ■

Fact 2: If $R_n(a) > 0$, $X_{n+1}(a)$ exists and is unique. Moreover, $X_{n+1}(a) \geq X_n(a) + \frac{\bar{w}}{\lambda_b d_b(w)} > X_n(a) \geq 0$.

This fact states that for any input $a \in [0, f^{**}]$, there is a unique output by the algorithm.

Proof of Fact 2. As $X_1(a) = a \geq 0 \geq X_0(a)$, $R(0; X_1(a), X_0(a)) < 0$. By Assumption 2.1, $R(x; X_1(a), X_0(a))$ is unimodal in $x \in [0, \infty)$. Then if $R_1(a) = \sup_{x \geq 0} R(x; X_1(a), X_0(a)) > 0$, there exists a unique $X_2(a) > 0$ satisfying $R(X_2(a); X_1(a), X_0(a)) = 0$ and $\frac{dR(x;X_1(a),X_0(a))}{dx} |_{x=X_2(a)} > 0$. As $R(X_2(a); X_1(a), X_0(a)) = \lambda_b d_b(X_2(a)+w)(X_2(a)-X_1(a)) - \bar{w} - \lambda_s d_s(X_0(a))(X_1(a)-X_0(a)) = 0$ and $X_1(a) > X_0(a)$, we have $\lambda_b d_b(X_2(a) + w)(X_2(a) - X_1(a)) - \bar{w} > 0$. That is, $X_2(a) - X_1(a) > \frac{\bar{w}}{\lambda_b d_b(X_2(a)+w)} > \frac{\bar{w}}{\lambda_b d_b(w)} > 0$. By induction, if $R_n(a) > 0$, $X_{n+1}(a)$ exists and is unique. Besides, $X_{n+1}(a) \geq X_n(a) + \frac{\bar{w}}{\lambda_b d_b(w)} > X_n(a) \geq 0$. ■

Fact 3: $N(f^{**}) = N(0) - 1 > 0$, $R_{N(0)}(0) = R_{N(0)-1}(f^{**}) \leq 0$ and $R_{N(0)-1}(0) > 0$.

This fact checks the property of two extremes of the input $a \in [0, f^{**}]$.

Proof of Fact 3. By the step in the algorithm, $R_1(0) = \sup_x R(x; 0, -\frac{\bar{w}}{\lambda_b d_b(w)}) = \max_p\{\lambda_b d_b(p + w)p\} - \bar{w} > 0$, where the inequality is due to $f^* > 0$. From the algorithm, it will lead to $N(0) > 1$. Clearly, $X_0(f^{**}) = 0$, $X_1(f^{**}) = f^{**}$, $X_1(0) = 0$ and $X_2(0) = f^{**}$. Therefore, there is always a lag of one when using the input $a = 0$ and $a = f^{**}$. One can check that $X_n(0) = X_{n-1}(f^{**})$ and thus $N(f^{**}) = N(0) - 1 > 0$. From the condition that the algorithm stops, we have $R_{N(0)}(0) = R_{N(f^{**})}(f^{**}) = R_{N(0)-1}(f^{**}) \leq 0$ and $R_{N(0)-1}(0) > 0$. ■

Fact 4: $N(a) < +\infty$.

Proof of Fact 4. From *Fact 2*, $X_n(a) \geq X_1(a) + \frac{(n-1)\bar{w}}{\lambda_b d_b(w)}$. If $N(a)$ is infinite, then when $n \to \infty$, $X_n(a) \geq X_{n-1}(a) = +\infty$, which will lead to $R_n < 0$, in contradiction with the condition that the algorithm stops. ■

Lemma C.2. *Under Assumption 2.1 and 3.1, when $f^* \geq f^{**} > 0$, there exists a unique $a^* \in (0, f^{**}]$ such that when the input is a^*, Algorithm 1 outputs $R_N = 0$, $N < \infty$, $X_0 \leq 0$ and $X_i \geq 0$ for $i > 0$. That is, one solution to* (7) *can be found by setting $M = N(a^*)$, $f_1 = X_N(a^*)$, $\cdots$, $f_M = X_1(a^*)$, $f_{M+1} = X_0(a^*)$.*

Proof of Lemma C.2. We divide the proof into several steps.

Increase the input $a \to a+\Delta a$ by an infinitesimal positive amount Δa. Denote $\tilde{N} = \min\{N(a), N(a+\Delta a)\}$ and $X_n(a + \Delta a) = X_n(a) + \Delta X_n$ for $n \leq \tilde{N}$. With a slight abuse of notation, we use X_n to refer to $X_n(a)$ below.

Step (1): $\Delta X_{\tilde{N}} > \cdots > \Delta X_n > \cdots > \Delta X_1 > \Delta X_0 > 0$.

Note that $X_0 = X_1 - \frac{\bar{w}}{\lambda_b d_b(X_1+w)}$ and $X_0 + \Delta X_0 = X_1 + \Delta X_1 - \frac{\bar{w}}{\lambda_b d_b(X_1+\Delta X_1+w)}$ from Step 3 in the algorithm when the input is a and $a + \Delta a$ respectively. These equalities can be rewritten as $\lambda_b d_b(X_1 + w)(X_1 - X_0) = \bar{w}$ and $\lambda_b d_b(X_1 + \Delta X_1 + w)((X_1 + \Delta X_1) - (X_0 + \Delta X_0)) = \bar{w}$. Therefore

$$\begin{aligned}0 =& \lambda_b d_b(X_1 + \Delta X_1 + w)((X_1 + \Delta X_1) - (X_0 + \Delta X_0)) - \lambda_b d_b(X_1 + w)(X_1 - X_0) \\ =& \frac{d\lambda_b d_b(x + w)(x - X_0)}{dx} |_{x=X_1} \Delta X_1 - \lambda_b d_b(X_1 + w)\Delta X_0\end{aligned}$$

where the second equality follows from a Taylor expansion. From *Fact 1*, $\frac{d\lambda_b d_b(x+w)(x-X_0)}{dx} |_{x=X_1} > 0$. Then due to $\Delta X_1 = \Delta a > 0$, the equation above will lead to $\Delta X_0 > 0$. Besides, the equation above can be rewritten as

$$\lambda_b d_b'(X_1 + w)(X_1 - X_0)\Delta X_1 + \lambda_b d_b(X_1 + w)(\Delta X_1 - \Delta X_0) = 0.$$

From *Fact 1*, $X_1 > X_0$. As $\lambda_b d_b'(X_1 + w) < 0$ and $\Delta X_1 > 0$, the first term in the left hand above is negative, and thus the second term in the left hand above is positive, which leads to $\Delta X_1 > \Delta X_0 > 0$.

Now from $\Delta X_1 > \Delta X_0$ and $X_1 > X_0$, one can show that $(X_1+\Delta X_1)-(X_0+\Delta X_0) > X_1-X_0 > 0$, $(X_1 + \Delta X_1) - X_1 > (X_0 + \Delta X_0) - X_0 > 0$. Then from Lemma C.1, by setting $y_2 = X_1 + \Delta X_1$, $z_2 = X_0 + \Delta X_0$, $y_1 = X_1$ and $z_1 = X_0$, we have $\Delta X_2 - \Delta X_1 > 0$. By induction, for all $1 \le n \le \tilde{N}$, $\Delta X_n > \Delta X_{n-1} > \cdots > \Delta X_1 > 0$.

Step (2): $N(a + \Delta a) \le N(a)$ and $R_n(a + \Delta a) < R_n(a)$ for all $1 \le n \le N(a + \Delta a)$.

By the definition in the algorithm $R_n(a) = \mathcal{R}_b(X_n(a)) - \bar{w} - \lambda_s d_s(X_{n-1}(a))(X_n(a) - X_{n-1}(a))$. Then for all $1 \le n \le \tilde{N}$, by Taylor's expansion, $R_n(a + \Delta a) = R_n(a) + \mathcal{R}_b'(X_n + w)\Delta X_n - [\lambda_s d_s(X_{n-1})(\Delta X_n - \Delta X_{n-1}) + \lambda_s d_s'(X_{n-1})(X_n - X_{n-1})\Delta X_{n-1}]$. As $X_n \ge X_{n-1}$ by *Fact 2* and $\Delta X_n \ge \Delta X_{n-1} > 0$ by Step (1), we have $R_n(a + \Delta a) < R_n(a)$.

Next we prove $N(a + \Delta a) \le N(a)$ by contradiction. Suppose $N(a + \Delta a) > N(a)$. Then by the condition that the algorithm stops, $R_{N(a)}(a + \Delta a) > 0$ and $R_{N(a)}(a) \le 0$, contradicting $R_{N(a)}(a + \Delta a) < R_{N(a)}(a)$ we have just proved. As a result, $N(a + \Delta a) \le N(a)$.

Step (3): There exists a unique value, $a^* \in (0, f^{**}]$, such that $R_{N(a^*)}(a^*) = 0$, $N(a^*) < \infty$, $X_0(a^*) \le 0$ and $X_i(a^*) \ge 0$ for $i > 0$.

From Step (2), $N(a)$ is non-increasing in a. From *Fact 3*, $N(f^{**}) = N(0) - 1$. As a result, $N(0) \geq N(a) \geq N(f^{**}) = N(0) - 1$ when $a \in (0, f^{**}]$. That is, when $a \in (0, f^{**}]$, the algorithm terminates with $N(a) \in \{N(0) - 1, N(0)\}$. Next we show that $N(a^*) = N(0) - 1$. Suppose that the algorithm terminates with $N = N(0)$. From Step (2), the termination of the algorithm must satisfy $R_N < R_{N(0)}(0) \leq 0$. As a result, if the output of algorithm satisfies $R_{N(a^*)}(a^*) = 0$, we must have $N(a^*) = N(0) - 1$. From *Fact 3*, $R_{N(0)-1}(f^{**}) \leq 0$ and $R_{N(0)-1}(0) > 0$. Then because of Step (2), $R_{N(0)-1}(a)$ is decreasing in a and thus there exists a unique value $a^* \in (0, f^{**}]$ such that $R_{N(a^*)}(a^*) = 0$. By *Fact 1*, *Fact 2* and *Fact 4*, $N(a^*) < \infty$, $X_0(a^*) \leq 0$ and $X_i(a^*) \geq 0$ for $i > 0$.

By the discussion of the intuition to design the algorithm, one solution to (7) can be derived by setting $M = N(a^*)$, $f_1 = X_N(a^*)$, $\cdots$, $f_M = X_1(a^*)$, $f_{M+1} = X_0(a^*)$. ■

C.1.3. Structural Properties

In this subsection, we show the structural properties of any solution to (7) , using the definition of the last section.

Lemma C.3. *Under Assumption 2.1 and 3.1, any solution to* (7) *satisfies the following properties, (1) f_m is decreasing and convex in m. (2) $\frac{d}{dx}[\lambda_b d_b(x+w)(x-f_m)]|_{x=f_{m-1}} > 0$ for $1 < m \leq M+1$. (3) $f_1 \leq f^*$ and $f_M \leq f^{**}$. (4) $M = 0$ if and only if $f^* \leq 0$; $M = 1$ if and only if $0 < f^* \leq f^{**}$; $M > 1$ if and only if $f^* > f^{**}$.*

Proof of Lemma C.3: To show part (1), we first show f_m is decreasing in m. For $m \geq M$, according to the m-th equation in (7) , $f_{m-1} - f_m = \frac{\bar{w}}{\lambda_b d_b(f_{m-1}+w)} > 0$. Suppose there exists $m < M$ such that $f_{m-1} - f_m \leq 0$. Then according to the mth equation in (7) , $\lambda_s d_s(f_{m+1})(f_m - f_{m+1}) + \bar{w} = \lambda_b d_b(f_{m-1} + w)(f_{m-1} - f_m) \leq 0$. Therefore, $f_m \leq f_{m+1}$. By induction, from the mth to the $(M-1)$th equations in (7) , we have $f_{M-1} \leq f_M$. However, from the M-th equation in (7) , $f_{M-1} - f_M = \frac{\bar{w}}{\lambda_b d_b(f_{M-1}+w)} > 0$. This leads to a contradiction and proves that f_m must be decreasing.

Next we show that $f_m - f_{m+1}$ is decreasing in m in part (1). For $m \geq M$, $f_m - f_{m+1} = \frac{\bar{w}}{\lambda_b d_b(f_m+w)}$, which is clearly decreasing in m as f_m is decreasing in m and $d_b(\cdot)$ is a decreasing function on its support. For $m < M$, starting from the $(M-1)$th and Mth equations in (7) ,

$$\lambda_s d_s(f_M)(f_{M-1} - f_M) + \bar{w} = \lambda_b d_b(f_{M-2} + w)(f_{M-2} - f_{M-1}),$$
$$\bar{w} = \lambda_b d_b(f_{M-1} + w)(f_{M-1} - f_M).$$

we have $\lambda_b d_b(f_{M-2}+w)(f_{M-1}-f_M) < \lambda_b d_b(f_{M-1}+w)(f_{M-1}-f_M) = \bar{w} < \lambda_b d_b(f_{M-2}+w)(f_{M-2}-f_{M-1})$, where the first inequality follows from $f_{M-2} > f_{M-1}$, the equality follows from the second equation above, and the second inequality follows from the first equation above and $f_{M-1} \geq f_M$. As a result, $f_{M-1} - f_M < f_{M-2} - f_{M-1}$. Now suppose $f_{m-1} - f_m < f_{m-2} - f_{m-1}$ for some $m < M$. Then

$$\begin{aligned}
&\lambda_b d_b(f_{m-2} + w)(f_{m-2} - f_{m-1}) = \lambda_s d_s(f_m)(f_{m-1} - f_m) + \bar{w} \\
<&\lambda_s d_s(f_m)(f_{m-2} - f_{m-1}) + \bar{w} < \lambda_s d_s(f_{m-1})(f_{m-2} - f_{m-1}) + \bar{w} \\
=&\lambda_b d_b(f_{m-3} + w)(f_{m-3} - f_{m-2}) < \lambda_b d_b(f_{m-2} + w)(f_{m-3} - f_{m-2}),
\end{aligned}$$

where the equalities follow from (7), the first inequality follows from $f_{m-1} - f_m < f_{m-2} - f_{m-1}$, the second inequality follows from $f_{m-1} > f_m$ and the last inequality follows from $f_{m-2} < f_{m-3}$. As a result, $f_{m-2} - f_{m-1} < f_{m-3} - f_{m-2}$. By induction, we have shown the claim.

Next we show part (2): $\frac{d}{dx}[\lambda_b d_b(x + w)(x - f_m)]|_{x=f_{m-1}} > 0$ for $1 < m \leq M$.

Note that

$$\begin{aligned}
\lambda_b d_b(f_{m-1} + w)(f_{m-1} - f_m) =&\bar{w} + \lambda_s d_s(f_{m+1})(f_m - f_{m+1}) < \bar{w} + \lambda_s d_s(f_m)(f_{m-1} - f_m) \\
=&\lambda_b d_b(f_{m-2} + w)(f_{m-2} - f_{m-1}) < \lambda_b d_b(f_{m-2} + w)(f_{m-2} - f_m),
\end{aligned}$$

where the equalities follow from (7), the first inequality follows from $f_{m+1} < f_m$ and $f_m - f_{m+1} < f_{m-1} - f_m$, and the last inequality is due to $f_m < f_{m-1}$. Let $x^* = \text{argmax}_x\, d_b(x + w)(x - f_m)$. By Assumption 2.1, $d_b(x + w)(x - f_m)$ is unimodal and increasing in $x \in [0, x^*)$. Since $f_{m-1} < f_{m-2}$, the inequality above indicates that $f_{m-1} < x^*$. Thus, $\frac{d}{dx}[\lambda_b d_b(x + w)(x - f_m)]|_{x=f_{m-1}} > 0$.

Next we show part (3): $f_1 \leq f^*$ and $f_M \leq f^{**}$. Note that $\max_p\{\lambda_b d_b(p + w)(p - f_1)\} = \lambda_s d_s(f_2)(f_1 - f_2) + \bar{w} \geq \bar{w} = \max_p\{\lambda_b d_b(p + w)(p - f^*)\}$, where the first equality is due to (7) and the second equality is due to the definition of f^*, and the inequality is due to the fact that f_m is decreasing in m. As $\max_p\{\lambda_b d_b(p+w)(p-z)\}$ is nonincreasing in z, $f_1 \le f^*$. To show $f_M \le f^{**}$, we only need to focus on the case where f^{**} is finite. According to the definition of f^{**}, $\frac{d}{dx}[\lambda_b d_b(x+w)x]|_{x=f^{**}} \ge 0$. Besides,

$$\lambda_b d_b(f^{**}+w)f^{**} = \bar{w} = \lambda_b d_b(f_M+w)(f_M - f_{M+1}) \ge \lambda_b d_b(f_M+w)f_M,$$

where the first equality is due to the definition of f^{**}, the second equality is due to the $(M+1)$th equation in (7), and the inequality is due to $f_{M+1} \le 0$. Also, since $\frac{d}{dx}[\lambda_b d_b(x+w)(x-f_{M+1})]|_{x=f_M} \ge 0$ by part (2), we have $\frac{d}{dx}[\lambda_b d_b(x+w)x]|_{x=f_M} \ge 0$. As $\lambda_b d_b(x+w)x$ is unimodal by Assumption 2.1, the inequality above indicates $f_M \le f^{**}$.

Finally we show part (4): $M=0$ if and only if $f^* \le 0$; $M=1$ if and only if $0 < f^* \le f^{**}$; $M>1$ if and only if $f^* > f^{**}$.

Necessary Conditions:

a) If $M=0$, $f_1 = f^* \le 0$.

b) If $M=1$, due to the monotonicity of f_m, it requires that $f_0 > f_1 = f^* > 0 > f_2$. Then

$$\lambda_b d_b(f_1+w)f_1 \le \bar{w} = \lambda_b d_b(f_0+w)(f_0-f_1) < \lambda_b d_b(f_0+w)f_0,$$

where the first inequality is due to $f_2 = f_1 - \frac{\bar{w}}{\lambda_b d_b(f_1+w)} \le 0$, the equality is due to the first equation in (7), and the last inequality is due to $f_1 > 0$. By Assumption 2.1, $\lambda_b d_b(p+w)p$ is unimodal. As $f_1 < f_0$, the inequality above leads to $\frac{d}{dp}[\lambda_b d_b(p+w)p]|_{p=f_1} \ge 0$. Besides, $\lambda_b d_b(f^{**}+w)f^{**} = \bar{w} \ge \lambda_b d_b(f_1+w)f_1$, where the equality is due to the definition of f^{**} and the inequality is due to $f_2 = f_1 - \frac{\bar{w}}{\lambda_b d_b(f_1+w)} \le 0$. Then we have $f^{**} \ge f_1 = f^*$.

c) If $M>1$, we know $\lambda_b d_b(f_1+w)(f_1-f_2) = \bar{w} + \lambda_s d_s(f_3)(f_2-f_3) \ge \bar{w}$, where the equality is due to (7) and the inequality is due to $f_2 > f_3$. As $f_2 > 0$ for $M>1$, it leads to $\lambda_b d_b(f_1+w)f_1 > \bar{w}$. Then by the definition of f^{**}, $\lambda_b d_b(f_1+w)f_1 > \lambda_b d_b(f^{**}+w)f^{**}$. As $\frac{d}{dx}[\lambda_b d_b(x+w)x]|_{x=f^{**}} \ge 0$ and $\lambda_b d_b(x+w)x$ is unimodal in x, we have $f_1 > f^{**}$. Thus by part (3) $f^* > f^{**}$.

Sufficient Conditions:

a) For $f^* < 0$, since $f_1 \le f^* \le 0$ by part (3), we have $M=0$.

b) For $0 < f^* \leq f^{**}$, suppose $M > 1$, we must have $f^* \geq f_1 > f^{**}$ from the necessary condition, leading to contradiction. Suppose $M = 0$, then $f_1 = f^* \leq 0$, leading to contradiction. Therefore, $M = 1$.

c) For $f^* > f^{**}$, suppose $M \leq 1$, we must have $f^* \leq 0$ or $0 < f^* \leq f^{**}$ from the necessary condition, leading to contradiction. Therefore, $M > 1$. ■

Proof of Theorem 3.1. We establish the existence in Section C.1.1 and Section C.1.2. We then need the following steps to show the uniqueness.

Step (1): The solution to (7) is unique when $f^* \leq 0$ or $0 < f^* \leq f^{**}$.

By Lemma C.3, $M \leq 1$. Then the unique solution to (7) can be directly solved, having the form in (17).

Step (2): The solution to (7) is unique when $f^* > f^{**}$.

We will show that any solution can be derived from the proposed algorithm with the output $R_N = 0$. Let $[f_1, \cdots, f_M, \cdots]$ be a solution to (7) . By part (2) in Lemma C.3, for $1 < m \leq M$, $\frac{dR(x; f_m, f_{m+1})}{dx} \,|_{x=f_{m-1}} = \frac{d}{dx}[\lambda_b d_b(x + w)(x - f_m)]|_{x=f_{m-1}} > 0$. Due to (7) , for $1 < m \leq M$, $R(f_{m-1}; f_m, f_{m+1}) = 0$, then $\sup_x R(x; f_m, f_{m+1}) > 0$. Besides, the first equation in (7) can be rewritten as $\sup_x R(x; f_1, f_2) = 0$. Therefore if we set the input a to be f_M in the solution, then the output of the algorithm satisfies $R_N = 0$, $N = M$, $X_0 = f_{M+1}$, $X_1 = f_M$, $\cdots$, $X_N = f_1$. Due to $f_M \leq f^{**}$ by part (3) in Lemma C.3 and Lemma C.2, the solution to (7) is uniquely output by Algorithm 1. This proves Theorem 3.1. ■

C.2. Proof of Proposition 3.1

It is straightforward to have part one and part two by part one in Lemma C.3.

For part three, suppose M is unchanged when we change $\bar{w}$. We will examine how f_m changes with respect to $\bar{w}$. Taking derivative with respect to $\bar{w}$ on both sides of the recursive system (7) yields,

$$
\begin{aligned}
\lambda_s d_s(f_2)(f_1' - f_2') + a_1 f_2' + 1 &= b_1 f_1', \\
\lambda_s d_s(f_{m+1})(f_m' - f_{m+1}') + a_m f_{m+1}' + 1 &= b_m f_{m-1}' + \lambda_b d_b(f_{m-1} + w)(f_{m-1}' - f_m'), \\
& \qquad M - 1 \geq m \geq 2, \\
1 &= b_M f_{M-1}' + \lambda_b d_b(f_{M-1} + w)(f_{M-1}' - f_M'),
\end{aligned} \tag{18}
$$

where $f_i' = \frac{df_i}{d\bar{w}}$, $a_m = \lambda_s d_s'(f_{m+1})(f_m - f_{m+1})$, and

$$
b_m = \begin{cases} \frac{d}{df_1}[\max_{f_0} \{\lambda_b d_b(f_0 + w)(f_0 - f_1)\}], & m = 1, \\ \lambda_b d_b'(f_{m-1} + w)(f_{m-1} - f_m), & m > 1. \end{cases}
$$

From the properties of $d_b(\cdot)$, $d_s(\cdot)$ and the monotonicity of f_m, we have $a_m > 0$, $b_m < 0$ for any m. As $\frac{d}{dx}[\lambda_b d_b(x + w)(x - f_m)]|_{x=f_{m-1}} > 0$ by part (2) in Lemma C.3, rewriting this inequality, we have $b_m + \lambda_b d_b(f_{m-1} + w) > 0$.

Next we prove $f_M' < 0$ by contradiction. Suppose $f_M' \geq 0$. From the last equation in (18), we have

$$
1 = (b_M + \lambda_b d_b(f_{M-1} + w))f_{M-1}' - \lambda_b d_b(f_{M-1} + w)f_M'.
$$

As $b_M + \lambda_b d_b(f_{M-1} + w) > 0$, we must have $f_{M-1}' > 0$. Then as $b_M < 0$ and $f_{M-1}' > 0$, this equation also shows

$$
1 = b_M f_{M-1}' + \lambda_b d_b(f_{M-1} + w)(f_{M-1}' - f_M') \leq \lambda_b d_b(f_{M-1} + w)(f_{M-1}' - f_M').
$$

Therefore $f_{M-1}' > f_M' \geq 0$. By recursively applying the same argument to the other equations in (18) (replacing the left-hand side 1 by $\lambda_s d_s(f_{m+1})(f_m' - f_{m+1}') + a_m f_{m+1}' + 1$, which is still positive), we have $f_1' > f_2' > 0$. This cannot happen, as in the first equation of (18), if $f_1' > f_2' > 0$, as $a_1 > 0$, the left hand is positive; however, as $b_1 < 0$, the right hand is negative, leading to a contradiction. As a result, $f_M' < 0$.

The above proof shows that when $\bar{w}$ increases, f_M decreases. When it becomes zero, then the endogenous capacity M decreases by one. Therefore, M is decreasing in $\bar{w}$. Similarly we can prove how f_M changes in λ_b and λ_s. The endogenous capacity is decreasing (increasing, decreasing, decreasing) in $\bar{w}$ (λ_b, λ_s, w).

Due to part four in Lemma C.3, $M \geq 1$ if and only if $\max_p\{d_b(p+w)p\} \geq \bar{w}/\lambda_b$.

For part four, we divide the proof into several steps.

(1) $V_{\mathcal{M}}(m)$ is increasing in m.

From the first equation in (8), $V_{\mathcal{M}}(0) < V_{\mathcal{M}}(1)$. We prove the claim for $m > 0$ by contradiction. Suppose $V_{\mathcal{M}}(l-1) < V_{\mathcal{M}}(l)$ holds for $l \leq m$, but $V_{\mathcal{M}}(m) \geq V_{\mathcal{M}}(m+1)$ for some $m \geq 1$. Then we have

$$\begin{aligned}
&[r + \lambda_s d_s(f_{m+1}) + \lambda_b d_b(f_{m-1} + w)]V_{\mathcal{M}}(m) \\
=&\lambda_s d_s(f_{m+1})V_{\mathcal{M}}(m+1) + \lambda_b d_b(f_{m-1} + w)[V_{\mathcal{M}}(m-1) + w] \\
<&\lambda_s d_s(f_{m+1})V_{\mathcal{M}}(m) + \lambda_b d_b(f_{m-1} + w)[V_{\mathcal{M}}(m) + w].
\end{aligned}$$

where the equality is due to (8), and the inequality is due to our induction hypothesis that $V_{\mathcal{M}}(m) \geq V_{\mathcal{M}}(m+1)$ and $V_{\mathcal{M}}(m-1) \leq V_{\mathcal{M}}(m)$. After arranging the terms, we have $rV_{\mathcal{M}}(m) < \lambda_b d_b(f_{m-1} + w)w$. Due to the monotonicity of f_m, $rV_{\mathcal{M}}(m) < \lambda_b d_b(f_m + w)w$. Since $V_{\mathcal{M}}(m) \geq V_{\mathcal{M}}(m+1)$, we have $rV_{\mathcal{M}}(m+1) < \lambda_b d_b(f_m + w)w$. As a result,

$$\begin{aligned}
&(\lambda_s d_s(f_{m+2}) + \lambda_b d_b(f_m + w))V_{\mathcal{M}}(m+1) + \lambda_b d_b(f_m + w)w \\
>&(\lambda_s d_s(f_{m+2}) + \lambda_b d_b(f_m + w) + r)V_{\mathcal{M}}(m+1) \\
=&\lambda_s d_s(f_{m+2})V_{\mathcal{M}}(m+2) + \lambda_b d_b(f_m + w)[V_{\mathcal{M}}(m) + w] \\
\geq&\lambda_s d_s(f_{m+2})V_{\mathcal{M}}(m+2) + \lambda_b d_b(f_m + w)(V_{\mathcal{M}}(m+1) + w).
\end{aligned}$$

where the equality is due to (8), the first inequality is due to $rV_{\mathcal{M}}(m+1) < \lambda_b d_b(f_m + w)w$, the second inequality is due to $V_{\mathcal{M}}(m+1) \leq V_{\mathcal{M}}(m)$. Arranging the terms, we have $V_{\mathcal{M}}(m+1) \geq V_{\mathcal{M}}(m+2)$. Moreover, as $rV_{\mathcal{M}}(m+1) < \lambda_b d_b(f_m + w)w$ and $f_m > f_{m+1}$, we have $rV_{\mathcal{M}}(m+2) < \lambda_b d_b(f_m + w)w < \lambda_b d_b(f_{m+1} + w)w$. By induction, we can show

$$V_{\mathcal{M}}(M-1) \geq V_{\mathcal{M}}(M) \quad \text{and} \quad rV_{\mathcal{M}}(M) < \lambda_b d_b(f_{M-1} + w)w.$$

However, this leads to contradiction as the last equation in (8) can not hold. Thus, $V_{\mathcal{M}}(m)$ is increasing in m.

(2) $V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ is decreasing in m.

By the last two equations of 8, we have

$$
\begin{aligned}
&\lambda_b d_b(f_{M-2}+w)[V_{\mathcal{M}}(M-1) - V_{\mathcal{M}}(M-2) - w] \\
=& - rV_{\mathcal{M}}(M-1) + \lambda_s d_s(f_M)[V_{\mathcal{M}}(M) - V_{\mathcal{M}}(M-1)] \\
>& - rV_{\mathcal{M}}(M) = \lambda_b d_b(f_{M-1}+w)[V_{\mathcal{M}}(M) - V_{\mathcal{M}}(M-1) - w]
\end{aligned}
$$

where the inequality is due to $V_{\mathcal{M}}(M) > V_{\mathcal{M}}(M-1)$. Since f_m is decreasing, $\lambda_b d_b(f_{M-1}+w) > \lambda_b d_b(f_{M-2}+w)$, and the inequality above leads to

$$V_{\mathcal{M}}(M-1) - V_{\mathcal{M}}(M-2) > V_{\mathcal{M}}(M) - V_{\mathcal{M}}(M-1).$$

Now we use induction. Suppose $V_{\mathcal{M}}(m-1) - V_{\mathcal{M}}(m-2) > V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ for some $m \le M$. Then

$$
\begin{array}{rl}
& \lambda_b d_b(f_{m-3}+w)[V_{\mathcal{M}}(m-2) - V_{\mathcal{M}}(m-3) - w] \\
= & -rV_{\mathcal{M}}(m-2) + \lambda_s d_s(f_{m-1})[V_{\mathcal{M}}(m-1) - V_{\mathcal{M}}(m-2)] \\
> & -rV_{\mathcal{M}}(m-1) + \lambda_s d_s(f_m)[V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)] \\
= & \lambda_b d_b(f_{m-2}+w)[V_{\mathcal{M}}(m-1) - V_{\mathcal{M}}(m-2) - w]
\end{array},
$$

where we have used $\lambda_s d_s(f_{m-1}) \ge \lambda_s d_s(f_m)$ and $V_{\mathcal{M}}(m-1) - V_{\mathcal{M}}(m-2) > V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ in the inequality and the equalities are due to (8). Similarly, as $d_b(f_{m-2}+w) > d_b(f_{m-3}+w)$, we have

$$V_{\mathcal{M}}(m-2) - V_{\mathcal{M}}(m-3) > V_{\mathcal{M}}(m-1) - V_{\mathcal{M}}(m-2).$$

This completes the proof.

(3) $V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1) \le w$.

As f_m is decreasing in m, the arrival rate of the sellers is bounded by $\lambda_s d_s(f_1)$, then we must have that for any m, $V_{\mathcal{M}}(m) \le \frac{\lambda_s d_s(f_1)w}{r}$. We then prove the claim by contradiction.

Suppose $V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1) > w$ for some m. Then due to the concavity of $V_{\mathcal{M}}(m)$, we have $V_{\mathcal{M}}(1) - V_{\mathcal{M}}(0) > w$. As $(r+\lambda_s d_s(f_1))V_{\mathcal{M}}(0) = \lambda_s d_s(f_1)V_{\mathcal{M}}(1)$ from (8), then $(r+\lambda_s d_s(f_1))V_{\mathcal{M}}(0) >$

$\lambda_s d_s(f_1)[V_{\mathcal{M}}(0)+w]$. That is $V_{\mathcal{M}}(0) > \frac{\lambda_s d_s(f_1)w}{r}$. This leads to contradiction.

C.3. Additional Results and Proofs in Section 3.3

Lemma C.4. $\lim_{n\to\infty} M^{(n)} = +\infty$.

Proof of Lemma C.4. As n increases, from the definition of f^{**} in Section C.1, it satisfies $n\lambda_b d_b(f^{**}+w)f^{**} = \bar{w}$. Clearly, $\lim_{n\to\infty} f^{**} = 0$. By the fact that $f_{M^{(n)}} \leq f^{**}$ derived in part (3) in Lemma C.3, we must have $\lim_{n\to\infty} f_{M^{(n)}} = 0$. From the $M^{(n)}$th equation of (7) , $\lim_{n\to\infty} d_b(f_{M^{(n)}-1}+w)f_{M^{(n)}-1} = 0$. It can only happen when $\lim_{n\to\infty} f_{M^{(n)}-1} = 0$ or $\lim_{n\to\infty} f_{M^{(n)}-1} = +\infty$. Because of part (2) in Lemma C.3 and the unimodality, we have $\lim_{n\to\infty} f_{M^{(n)}-1} = 0$. By induction, for any finite k, $\lim_{n\to\infty} f_{M^{(n)}-k} = 0$. Suppose $M^{(n)}$ is finite, then $\lim_{n\to\infty} f_1 = 0$. That is, there will be no sellers joining the marketplace even there are many buyers, which cannot be true. ∎

Proof of Proposition 3.2. For part (1), from (7) , we have

$$d_s(f^{(n)}_{M^*+2})(f^{(n)}_{M^*+1}-f^{(n)}_{M^*+2})+\frac{\bar{w}}{n\lambda_s} = \frac{\lambda_b}{\lambda_s}d_b(f^{(n)}_{M^*}+w)(f^{(n)}_{M^*}-f^{(n)}_{M^*+1}) > \frac{\lambda_b}{\lambda_s}d_b(f^{(n)}_{M^*}+w)(f^{(n)}_{M^*+1}-f^{(n)}_{M^*+2}).$$

Arranging the terms of the two sides of the inequality, we have

$$\lim_{n\to\infty}(\frac{\lambda_b}{\lambda_s}d_b(f^{(n)}_{M^*}+w)-d_s(f^{(n)}_{M^*+2}))((f^{(n)}_{M^*+1}-f^{(n)}_{M^*+2})) \leq 0.$$

We then prove $\lim_{n\to\infty}(f^{(n)}_{M^*+1}-f^{(n)}_{M^*+2}) = 0$ by contradiction. If for a subsequence $n_k \to \infty$, $\lim_{k\to\infty}(f^{(n_k)}_{M^*+1}-f^{(n_k)}_{M^*+2}) > 0$, to make the equation above hold, we have

$$\lim_{k\to\infty}\left(\frac{\lambda_b}{\lambda_s}d_b(f^{(n_k)}_{M^*}+w)-d_s(f^{(n_k)}_{M^*+2})\right) \leq 0.$$

However, from the definition of M^*, we have $\frac{\lambda_b}{\lambda_s}d_b(f^{(n_k)}_{M^*}+w) \geq d_s(f^{(n_k)}_{M^*+1}) \geq d_s(f^{(n_k)}_{M^*+2})$. Then we must have $\lim_{k\to\infty}(\frac{\lambda_b}{\lambda_s}d_b(f^{(n_k)}_{M^*}+w)-d_s(f^{(n_k)}_{M^*+2})) = 0$. As a result, $\lim_{k\to\infty}(d_s(f^{(n_k)}_{M^*+1})-d_s(f^{(n_k)}_{M^*+2})) = 0$ and thus $\lim_{k\to\infty}(f^{(n_k)}_{M^*+1}-f^{(n_k)}_{M^*+2}) = 0$, leading to contradiction. Therefore, $\lim_{n\to\infty}(f^{(n)}_{M^*+1}-f^{(n)}_{M^*+2}) = 0$. Then using (7) , by taking the limit of $n\to\infty$, the waiting cost is negligible and one can easily prove that for any finite k, $\lim_{n\to\infty}[f^{(n)}_{M^*\pm k}-f^{(n)}_{M^*}] = 0$. As $f^{(n)}_{M^*+1} \leq \hat{f}$ and $f^{(n)}_{M^*-1} \geq \hat{f}$, $\lim_{n\to\infty} f^{(n)}_{M^*\pm k} = \hat{f}$.

For part (2), we need several steps to complete the proof.

1) $\lim_{n\to\infty} M^{*(n)} = +\infty$.

From the first equation in (7) ,

$$\lambda_s d_s(f_2^{(n)})(f_1^{(n)} - f_2^{(n)}) + \frac{\bar{w}}{n} = \max_{f_0}\{\lambda_b d_b(f_0 + w)(f_0 - f_1^{(n)})\}.$$

If the claim does not hold, then from part (1), we have $\lim_{n\to\infty} f_1^{(n)} = \lim_{n\to\infty} f_2^{(n)} = \hat{f}$ (for a subsequence of n with slight abuse of notation), then the left-hand side of the equation above is 0, whereas the right-hand side is positive, which leads to contradiction.

2) For any positive ϵ, $\lim_{n\to\infty} \pi_{n_1} = \lim_{n\to\infty} \pi_{n_2} = 0$, where $n_1 = \max\{m : f_m^{(n)} > \hat{f} + \epsilon\}$ and $n_2 = \min\{m : f_m^{(n)} < \hat{f} - \epsilon\}$.

Recall that π_m is the steady-state distribution of state m, which depends on n as well. Clearly, $n_1 < M^*$. Since $\lim_{n\to\infty} f_{M^*\pm k}^{(n)} = \hat{f}$ for finite k, $\lim_{n\to\infty}(M^* - n_1) = \infty$. Besides, for any $m \in [n_1, M^*]$, we know $\pi_m \geq \pi_{n_1}$, because of the definition of π_m in (9), the monotonicity of f_m and the definition of M^*. Then

$$1 \geq \lim_{n\to\infty} \sum_{m=n_1}^{M^*} \pi_m \geq \lim_{n\to\infty} \sum_{m=n_1}^{M^*} \pi_{n_1} = \lim_{n\to\infty} (M^* - n_1)\pi_{n_1}.$$

As $\lim_{n\to\infty}(M^* - n_1) = \infty$, $\lim_{n\to\infty} \pi_{n_1} = 0$. Similarly, $\lim_{n\to\infty} \pi_{n_2} = 0$.

3) $\lim_{n\to\infty} \sum_{m:|f_m^{(n)} - \hat{f}| > \epsilon} \pi_m = 0$.

For $m \in [0, n_1]$, because of (9) and $f_m \leq f_{m-1}$, we have $\pi_{m-1} < a\pi_m$, where $a_n \leq \frac{\lambda_b d_b(f_{n_1-1}^{(n)}+w)}{\lambda_s d_s(f_{n_1+1}^{(n)})}$. As a result, we have

$$\lim_{n\to\infty} \sum_{m:f_m^{(n)} > \hat{f}+\epsilon} \pi_m \leq \lim_{n\to\infty} \sum_{n=0}^{\infty} a_n^n \pi_{n_1} = \lim_{n\to\infty} \frac{1}{1-a_n} \pi_{n_1}.$$

As $\lim_{n\to\infty} \frac{\lambda_b d_b(f_{n_1-1}^{(n)}+w)}{\lambda_s d_s(f_{n_1}^{(n)})} < \lim_{n\to\infty} \frac{\lambda_b d_b(f_{M^*}^{(n)}+w)}{\lambda_s d_s(\hat{f}+\epsilon)} = \frac{\lambda_b d_b(\hat{f}+w)}{\lambda_s d_s(\hat{f}+\epsilon)} < \frac{\lambda_b d_b(\hat{f}+w)}{\lambda_s d_s(\hat{f})} = 1$, we have $a_n < 1 - \delta$ being bounded away from 1 and thus $\lim_{n\to\infty} \sum_{m:f_m^{(n)} > \hat{f}+\epsilon} \pi_m = 0$ because of step 2) above. Similarly, $\lim_{n\to\infty} \sum_{m:f_m^{(n)} < \hat{f}-\epsilon} \pi_m = 0$. Thus we complete the proof. ■

Proposition C.1. *Given $w = c_s + c_b$, in the thick market, the firm's long-run average revenue converges:* $\lim_{n\to\infty} \frac{V_{\mathcal{M}}^{(n)}}{n} = \frac{\lambda_s d_s(\hat{f})w}{r}$.

Proof of Proposition C.1. For any positive ϵ, define $n_1 = \max\{m : f_m^{(n)} > \hat{f} + \epsilon\}$ and $n_2 = \min\{m : f_m^{(n)} < \hat{f} - \epsilon\}$. Then

$$\begin{aligned}\frac{rV_{\mathcal{M}}^{(n)}}{nw} &= \sum_{m=1}^{M^{(n)}} \pi_m \lambda_b d_b(f_{m-1}^{(n)} + w) \\ &= \sum_{m=1}^{n_1} \pi_m \lambda_b d_b(f_{m-1}^{(n)} + w) + \sum_{m=n_1+1}^{n_2-1} \pi_m \lambda_b d_b(f_{m-1}^{(n)} + w) + \sum_{m=n_2}^{M^{(n)}} \pi_m \lambda_b d_b(f_{m-1}^{(n)} + w).\end{aligned}$$

For the first term,

$$\lim_{n\to\infty} \sum_{m=1}^{n_1} \pi_m \lambda_b d_b(f_{m-1}^{(n)} + w) \le \lim_{n\to\infty} \sum_{m=1}^{n_1} \pi_m \lambda_b d_b(f_{n_1-1}^{(n)} + w) \le \lambda_b d_b(\hat{f} + \epsilon + w) \lim_{n\to\infty} \sum_{m=1}^{n_1} \pi_m = 0.$$

where we have used $\lim_{n\to\infty} \sum_{m=1}^{n_1} \pi_m = 0$ from Proposition 3.2. Similarly, the last term is also zero when $n \to \infty$. For the second term, as for any $m \in [n_1 + 1, n_2 - 1]$, $d_b(\hat{f} + \epsilon + w) \le d_b(f_{m-1}^{(n)} + w) \le d_b(\hat{f} - \epsilon + w)$, we have

$$\lambda_b d_b(\hat{f}+\epsilon+w) \lim_{n\to\infty} \sum_{m=n_1+1}^{n_2-1} \pi_m \le \lim_{n\to\infty} \sum_{m=n_1+1}^{n_2-1} \pi_m \lambda_b d_b(f_{m-1}^{(n)}+w) \le \lambda_b d_b(\hat{f}-\epsilon+w) \lim_{n\to\infty} \sum_{m=n_1+1}^{n_2-1} \pi_m.$$

From Proposition 3.2, $\lim_{n\to\infty} \sum_{m=n_1+1}^{n_2-1} \pi_m = 1$. Therefore,

$$\lambda_b d_b(\hat{f} + \epsilon + w) \le \lim_{n\to\infty} \frac{rV_{\mathcal{M}}^{(n)}}{nw} \le \lambda_b d_b(\hat{f} - \epsilon + w).$$

Since it holds for an arbitrary $\epsilon > 0$, we have completed the proof. ■

C.4. Proportional Fee Structure

Other things being unchanged, suppose the platform charges a proportion $(1 - \beta)$ per transaction. At the moment of the next transaction, the seller offering the lowest price, denoted as p_m, receives βp_m and leaves the market, while the future expected utility of the remaining $m-1$ sellers becomes f_{m-1}. Due to the price war among sellers, $\beta p_m = f_{m-1}$ for $m \ge 2$. For convenience, let $f_0 = \beta p_1$. Similar to the dynamics when the platform charges fixed fees, the utility function f_m will satisfy the following recursive equations:

$$
\begin{aligned}
f_1 &= \max_{f_0}\{\frac{\lambda_b d_b(\frac{f_0}{\beta})f_0 + \lambda_s d_s(f_2)f_2}{\lambda_b d_b(\frac{f_0}{\beta}) + \lambda_s d_s(f_2)} - \frac{\bar{w}}{\lambda_b d_b(\frac{f_0}{\beta}) + \lambda_s d_s(f_2)}\}, \\
f_m &= \frac{\lambda_b d_b(\frac{f_{m-1}}{\beta})f_{m-1} + \lambda_s d_s(f_{m+1})f_{m+1}}{\lambda_b d_b(\frac{f_{m-1}}{\beta}) + \lambda_s d_s(f_{m+1})} - \frac{\bar{w}}{\lambda_b d_b(\frac{f_{m-1}}{\beta}) + \lambda_s d_s(f_{m+1})}, \quad m \geq 2.
\end{aligned} \tag{19}
$$

Similarly, the value function of the platform will satisfy:

$$
\begin{aligned}
rV_{\mathcal{M}}(0) &= \lambda_s d_s(f_1)\Delta V_{\mathcal{M}}(1), \\
rV_{\mathcal{M}}(m) &= \lambda_s d_s(f_{m+1})\Delta V_{\mathcal{M}}(m+1) + \lambda_b d_b(\frac{f_{m-1}}{\beta})(\frac{(1-\beta)f_{m-1}}{\beta} - \Delta V_{\mathcal{M}}(m)), \quad 1 \leq m \leq M-1, \\
rV_{\mathcal{M}}(M) &= \lambda_b d_b(\frac{f_{M-1}}{\beta})(\frac{(1-\beta)f_{M-1}}{\beta} - \Delta V_{\mathcal{M}}(M)),
\end{aligned}
$$

where $\Delta V_{\mathcal{M}}(m) = V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ is the marginal value of an additional seller.

Equilibrium Analysis. Comparing (19) with (7), $d_b(\cdot) \to d_b(\cdot/\beta)$ and $w \to 0$. Thus the dynamic game still admits a unique equilibrium. Similarly, we have the convexity of f_m, the mode M^*, the monotonicity of M over various parameters and the convergence of f_m and π_m. However, $V_{\mathcal{M}}(m)$ may not be increasing in m any more. Consider that $\lambda_b d_b(p) = 5I_{p<10}$, $\lambda_s d_s(p) = 5p$, $r = 1$, $\bar{w} = 10$ and $\beta = 1/2$, then $M = 2$, $f_1 = \frac{7}{3}$ and $f_2 = \frac{1}{3}$. Moreover, $V_{\mathcal{M}}(0) = 15.5522 < V_{\mathcal{M}}(1) = 16.8852 > V_{\mathcal{M}}(2) = 16.0154$.

The Thick Market. By the HJB equation of $V_{\mathcal{M}}(m)$, we have

$$
\begin{aligned}
rV_{\mathcal{M}} &= \sum_{m=0}^{M-1} \pi_m \lambda_s d_s(f_{m+1})\Delta V_{\mathcal{M}}(m+1) + \sum_{m=1}^{M} \pi_m \lambda_b d_b(\frac{f_{m-1}}{\beta})(\frac{(1-\beta)f_{m-1}}{\beta} - \Delta V_{\mathcal{M}}(m)) \\
&= (1-\beta)\sum_{m=1}^{M} \pi_m \lambda_b d_b(\frac{f_{m-1}}{\beta})\frac{f_{m-1}}{\beta},
\end{aligned}
$$

where we have used $\pi_{m-1}\lambda_s d_s(f_m) = \pi_m \lambda_b d_b(\frac{f_{m-1}}{\beta})$. Then as $d_b(x)x$ is bounded up, similar to the proof of Proposition C.1, for arbitrary $\epsilon > 0$,

$$
\lambda_b d_b(\frac{\hat{f}+\epsilon}{\beta})\frac{\hat{f}-\epsilon}{\beta} \leq \lim_{n\to\infty} \frac{rV_{\mathcal{M}}^{(n)}}{n(1-\beta)} \leq \lambda_b d_b(\frac{\hat{f}-\epsilon}{\beta})\frac{\hat{f}+\epsilon}{\beta},
$$

where $\lambda_s d_s(\hat{f}) = \lambda_b d_b(\frac{\hat{f}}{\beta})$. Therefore given β, in the thick market the firm's long-run average revenue converges, $\lim_{n\to\infty}\frac{V_{\mathcal{M}}^{(n)}}{n}=\frac{1-\beta}{r}\lambda_b d_b(\frac{\hat{f}}{\beta})\frac{\hat{f}}{\beta}$. To maximize the revenue in the thick market, the firm sets β to maximize

$$\max_{\beta}\quad \frac{1-\beta}{r}\lambda_b d_b(\frac{\hat{f}}{\beta})\frac{\hat{f}}{\beta},\quad \text{subject to}\quad \lambda_s d_s(\hat{f})=\lambda_b d_b(\frac{\hat{f}}{\beta}). \tag{20}$$

Elementary analysis shows that Problem (20) is equivalent to Problem (11).

D. Proofs for Results in Section 4

Proof of Theorem 4.1. By Proposition 3.1-(3), when the market size of buyers is relatively small or the cost of selling a product is too large, the marketplace is infeasible, i.e., $M=0$. Therefore the dealer earns more profits than the platform.

When $\bar{w}\to 0$, by a similar proof to Proposition 3.2 and Proposition C.1, we have $\lim_{\bar{w}\to 0}V_{\mathcal{M}}=\frac{\lambda_s d_s(\hat{f})w}{r}$, which is the optimal value of problem (11) after optimizing w. From Proposition B.2, $V_{\mathcal{D}}(0)$ is upper bounded by $\bar{V}_{\mathcal{D}}(0)$, the optimal value of problem (6). In Theorem 4.2, we will show that the optimal values in problems (6) and (11) are equivalent. Therefore, $V_{\mathcal{D}}(0)<\lim_{\bar{w}\to 0}V_{\mathcal{M}}^{*}$. That is, the platform earns more profits than the dealer. ■

Proof of Theorem 4.2. The convergences to the static optimization problems (6) and (11) are established in Section 2 and 3. Therefore, we only need to compare 6 and 11 in the theorem. It is easy to see that the decision variables of (6), p_b and p_s, can be transformed to those of (11), c_s+c_b by the following equations:

$$\hat{f}=p_s\quad \hat{f}+c_s+c_b=p_b.$$

There is a one-to-one correspondence between (6) and (11). Part (1) and part (2) immediately follow from this observation. Part (3) and part (4) follow from the additional observation that the average utility of sellers is $\hat{f}$ and the average utility ov buyers is $\hat{f}+w$, from Proposition 3.2. ■