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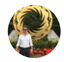
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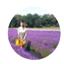
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A Sequential Recommendation and Selection Model

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Abstract

We propose a sequential recommendation-selection model where a seller recommends sets of products to consumers over multiple stages. consumers are heterogeneous in the patience levels, characterized by a certain number of stages that a consumer is willing to go through when making a purchase. Consumers view the products stage by stage. If a consumer can find a satisfactory product before exhausting her patience, she will purchase the product and leave the system immediately. Otherwise, the consumer stays till the last stage within her patience level but ends up without purchasing. The seller's objective is to maximize his expected overall revenue by optimizing the recommendation sequence or the products' prices. We note that the seller can learn the consumers' patience levels as well as their utilities through the recommendation process, and thus can adjust his future recommendations accordingly. However, a static sequential recommendation strategy would suffice. Therefore, we derive a set of results: 1) For the pure recommendation order problem, the optimal solution possesses a sequential revenue-ordered property, which can be efficiently discovered by dynamic programming. We also find that a crude heuristic – only offering one set of products at a single stage – will earn a tight 50% of the optimal revenue. 2) In the single-leg dynamic capacity control problem, the optimal recommendations admit an inclusion property. 3) The optimal pricing policy under a fixed recommendation order is unique, which can be efficiently found by a binary search. 4) However, the joint recommendation and pricing problem is NP-hard, while recommending all products only at a single stage and optimizing their prices accordingly will earn a tight 88% of the optimal revenue.

Our results also characterize the reason that the assortment in stores is always same on different date[1].

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[1]A store provides one assortment on each date. Consumers make sequential decisions on consecutive dates, but consumers who first visit the store on different date may have different market sizes and different distributions of patience levels. The results are robust even when the market sizes and distributions of patience levels are unknown.

1. Introduction

Products recommendation strategies are the keys to the success of platforms such as Amazon, Taobao and Expedia. Nowadays, with the explosion of product variety, consumers' cost of searching "the perfect" product dramatically increases. Thus, the conventional assumption that an arrived consumer will patiently analyze all products' utilities and choose the best one becomes less practical. More specifically, it is inappropriate in the following situations: i) There is a menu in front of a consumer, but either the menu is too long or she has a limited memory capacity. The consumer cannot easily view all the alternatives. A typical example is in online retail, after a consumer types in a search word, there are hundreds of pages displayed. consumers evaluate the products page by page without anticipation of what products are recommended in later pages. ii) A seller recommends his products over multiple stages but the consumers are impatiently waiting for all recommendations. For example, when scheduling a healthcare appointment over the phone, to avoid overwhelming the patient, the service provider offers appointment slots sequentially. The patient may make a final decision before examining all the slots. iii) A firm distributes its new products on different dates and consumers may be unlikely to wait to the last date. In these situations, that the display order in the menu or the recommendation order by the seller will cause the consumers gradually aware of product availability serves as a key factor affecting the seller's decisions.

Motivated by the real behaviors of consumers, we propose a sequential recommendation-selection choice model aiming to capture two key ingredients underlying consumers' choices: limited patience levels and the satisficing choice rule. Limited patience levels refer to that consumers will abandon without purchasing if there are no satisfactory products yet after a certain number of stages of recommendations. This phenomenon has been widely observed in daily life. For example, Kim et al. (2010) demonstrate that consumers typically only view up to 10 to 15 items in some product categories in online retailing. While the notation of "satisficing" was first introduced by Simon (1955), a term combining "satisfactory" and "sufficing" and it refers to that before the consumer's patience gets exhausted, namely before a certain stage, a consumer will immediately purchase the current best if there are products that are satisfactory (according to a criterion to be defined) at the current stage. Satisficing choice rule has been supported by several experimental studies, including

Caplin et al. (2011), Reutskaja et al. (2011), and Stüttgen et al. (2012).

Building upon the cascade model, we incorporate the limited patience and the satisficing choice behaviors of consumers. Particularly, in our model, the seller makes recommendations over multiple stages. In each stage, a new assortment is provided. A consumer makes choices based on a "satisficing" rule (Simon, 1955): the consumers evaluates assortments stage by stage and makes a purchase decision once the best at the current stage has utility above some threshold, i.e., the no-purchase utility, while the threshold can be heterogeneous across consumers. Besides, we assume that the consumers have heterogenous patience levels as well. That is, each consumer will at most view up to a certain stage once none of products recommended are attractive for her so far. And the patience level of a consumer is randomly drawn from a distribution. As a result, our model accounts for the abandonment behaviors of consumers. Our goal is to find the optimal recommendation strategy such that the seller's expected revenue is maximized.

Our results and contributions are as follows:

i) The proposed model relates to and serves as an extension of the classic consumer choice models and considers a sequence of choices, referred to as sequential selection model. Building upon the Gumbel distribution assumption, our model can be considered as a generalization of Multinomial Logit model (MNL) while alleviates the independence of irrelevant alternatives (IIA) property of MNL model and allows the sequential selection behaviors of consumers. To the best of our knowledge, our work is the first one that takes the no-purchase utility as the consumer's endogenous value that remains fixed over the multiple stages. Thus, we help the seller dynamically learn the consumer's no-purchase utility through the recommendation process, and adjust his future recommendations accordingly. Interestingly, we show that a static recommendation strategy would suffice.

ii) For the pure recommendation problem where the seller cannot adjust the products' prices but can decide which set of products to offer at each stage, we derive some insightful results. Under the sequential selection model, the recommendation order can be efficiently optimized, as the optimal solution admits a nested structures: the revenues of products recommended at earlier stages will be higher than those recommended later or not recommended. Besides, we show that if the seller only optimally recommend products at the first stage but not at latter stages, it will achieve half of the optimal revenue. For the dynamic single-leg capacity control problem, we find that the optimal recommendation strategy admits an inclusion property. More precisely, if the marginal value of the capacity reduces, the union of products recommended up to any stage will be larger. We also provide some extensions to the sequential selection model.

iii) In the pure pricing problem, the recommendation of products is fixed and the seller can only adjust the prices. We show that the optimal pricing policy of the sequential selection model will be unique and can be efficiently found by a a binary search. Surprisingly, unlike the common belief that a product with a higher quality or recommended at an earlier stage should be priced higher at optimality, under the sequential selection model, this intuition may no longer be true. Though, the optimal pricing policy still have certain properties: all the products recommended at the same stage will be priced at the same markup and the optimal prices across different stages have a certain relationship with the exogenous abandonment probability at each stage.

iv) For the joint recommendation order and pricing problem, the seller can adjust the products's prices and decide which set of products to offer at each stage. Even though, the individual problem, i.e. the pure recommendation or the pure pricing problem is solvable, the joint problem is shown to be NP-hard. We use the continuous relaxation to derive a tight upper bound of the seller's expected revenue. Afterwards, a simple heuristic is proposed achieving a constant performance guarantee, even relative to the upper bound under the continuous relaxation.

1.1. Related Literature

Our work falls into the literature on consumer choice models and assortment optimization. The most famous one, MNL, was first developed by McFadden (1978), prescribing a utility maximizing consumer who chooses a best product after realizing all available products' utilities. Although embedded with its elegance in presentation and efficiency in operation management decisions (Talluri and van Ryzin, 2004; Gallego et al., 2004), MNL is frequently criticized for its restrictive substitution pattern, known as the independence of irrelevant alternatives (IIA) property: the ratio of the choice probabilities between any two alternatives is independent of a third one (see, e.g., Debreu (1952)). Therefore, many extensions on MNL are then proposed to relax the IIA property, such as the mixed MNL models (McFadden and Train, 2000), nested logit models (Train, 2009), and extensions with perception priority levels (Flores et al., 2019). We refer the interested readers to Strauss et al. (2018) for a comprehensive review on the various extensions on MNL.

All of the above mentioned choice models fundamentally assume that each consumer views all products and maximizes her utilities by choosing the best one, but ignore the bounded rationality, for example, the limited attention or bounded patience of consumers. Therefore, consideration set based choice models (e.g., Aouad et al. (2016); Jagabathula and Rusmevichientong (2017); Wang and Sahin (2018)) are proposed and have been widely accepted as consideration set prescribes a set of products that consumers considers before making a purchase decision and the existence of consideration set perfectly explains the limited attention/ patience of consumers. However, existing literatures on consideration set based models focus less on the impact of framing/ display/ recommendation decisions on the formation of consideration sets, but more on the impact of products' intrinsic values, for example, products' prices or qualities (see Weitzman (1979)).

In reality, consumers may not be aware of the availability of all products. So their consideration sets of products are more subject to the seller's framing decisions, for example, the products' display order from an online web page. Having noticed the framing effect, many recent works have turned their attentions to the seller's recommendation decision. Not only the offer set matters, but the recommendation order as well. Among the recent works, the click models have gained its popularity in operation management community. For example, Gao et al. (2018) study the pricing problem in a general cascade click model. Compared to this work, our model is derived fundamentally based on consumers' rational behaviors and we provide comprehensive analyses of the recommendation order, pure pricing and joint recommendation problems. Based on the Weitzman model, (Chu et al., 2019) propose a sequential choice model where a platform uses position ranking to manipulate the sequential availability of products to consumers. The consumer's choice rule in their work is close to ours. However, without consumers recalls, we allow consumer abandonments and instead of a single product at each stage, in our model, a consumer is faced with an assortment. Gallego et al. (2016) propose a model where products are recommended over multiple stages, and each consumer picks a number of the first few stages to view and then simultaneously evaluates all the products considered. Distinctively, in our model, each consumer picks a certain stage that she will at most view up to, but then evaluates the products and make a purchase or not decision stage by stage. In the work of Liu et al. (2019), the consumers also evaluate the products stage by stage. But different from ours, the utility of the outside option in their model can be independent across stages. As a result, in our model, the seller is learning the consumer's information, such as the non-purchase utility, over stages, in order to adjust his future recommendations. Xu and Wang (2018) consider an assortment optimization where consumers are present and make a purchase decision at each stage, while we consider the situation that each consumer only demands at most one unit, but not purchases repeatedly. Liu et al. (2019) consider a service provider who offers assortments of appointment slots to a consumer in multiple stages. Instead of a binary value in their setting, the product utility can take any value drawn from a distribution in our model. To summarize, the key differences of our model compared with existing literatures are two-fold: 1. consumers have limited and heterogeneous patient levels; 2. consumers follow a satisficing choice rule, so they will leave as soon as they find a product exceeds a threshold within their patience level; and the thresholds can be heterogeneous among consumers but remain the same for a single consumer across different stages. Recently, Liu and Cooper (2015) and Lobel (2017) study a multi-stage single-product pricing problem, which essentially combine these two features. In their models, a consumer makes a purchase decision as long as the price drops below a target threshold, while a consumer may also leave the system without purchasing if she has waited for a certain number of stages but none of the prices provided are satisfactory. In comparison, in our model, each consumer faces with an assortment at each stage.

1.2. Organization

The remaining of this paper is organized as follows. In Section 2, we describe the sequential recommendation-selection model and the seller's problem. In Sections 3, 4, and 5, we analyse the seller's pure recommendation order problem, pure pricing problem, and the joint recommendation order and pricing problem, respectively. In Section 6, we provide some extensions which allow multiple consumer types. We conclude the paper in Section 8.

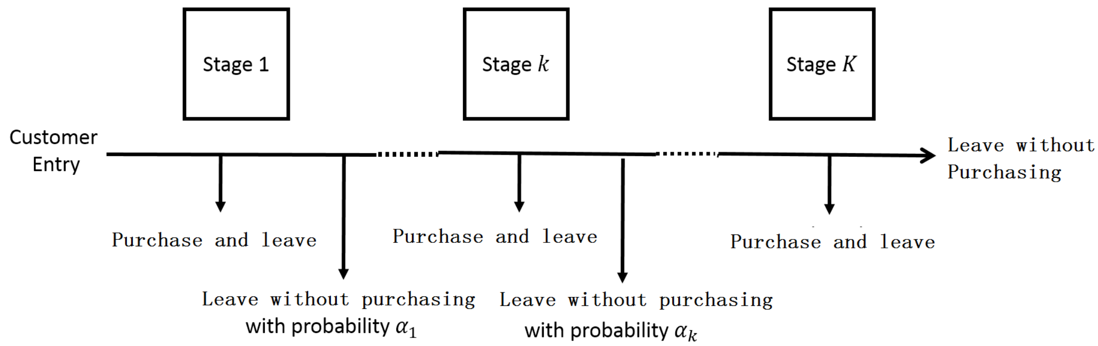


Figure 1: An illustration of the sequential choice process.

2. The Model

Consider a seller (referred to as "he") recommending products to a consumer (referred to as "she") sequentially over at most K stages. If $K \to \infty$, there is no constraint on the number of recommendatiat stages. At each stage, the seller offers an assortment selected from the universe of products $N \triangleq \{1, 2, ..., n\}$, which is denoted by $S_k \subseteq N$, $1 \leq k \leq K$. The marginal cost and price associated with product i are c_i and p_i, respectively. Therefore the revenue of product i is $r_i \triangleq p_i - c_i$. To study the firm's recommendation strategy, we first describe the choice process of a random consumer given the order of recommendation sets of products as $S_1, \dots, S_K$. For notational convenience, let $S_{K+1} \triangleq N \setminus \cup_{m=1}^{K} S_m$ be the set of products not recommended.

2.1. The consumer's Choice Process

An arriving consumer demands at most one unit of the products. Her net utility associated with product i is given by $U_i \triangleq \mu_i + \epsilon_i$, where μ_i is the expected utility of the product and ϵ_i stands for the random utility realized for the consumer. This is similar to a standard random utility model. The consumer also garners a utility $U_0 \triangleq \epsilon_0$ if she determines to leave without purchasing. In other words, different consumers have different realizations of the random utilities $\{\epsilon_i\}_{i=0}^{n}$, which have an i.i.d. probability density function[2].

Initially, the consumer realizes U_0 but does not anticipate what products will be offered at each stage. At the moment stage k is revealed, she immediately observes all products $i \in S_k$ and their realized utilities $\{U_i : i \in S_k\}$. She makes one of the following decisions:

[2] It implies that the utilities do not tie with probability one.

- If the best product at the current stage is more attractive than the no-purchase option $\max_{i \in S_k} U_i > U_0$, then she will buy the product and leave the system.

- If no product at the current stage is more attractive than the no-purchase option (or no products are displayed), then she leaves the system without purchasing with probability α_k, independent of everything else. Otherwise, with probability $1 - \alpha_k$, she moves on to stage $k + 1$ and repeats the process. By convention, $\alpha_K = 1$.

As a result, the consumers do not recall products at previous stages: once she has reached stage k, it must imply that all products displayed before stage k are less attractive than the outside option U_0.

We provide an interpretation to rationalize the behavior of consumers in our model. Suppose each arriving consumer is equipped with a threshold U_0 and a maximal number of stages to browse $\hat{k}$, where U_0 and $\hat{k}$ are private information. The threshold U_0 characterizes how "picky" a consumer is. A picky consumer is harder to satisfy, and is likely to browse more stages to find a product that meets her high threshold. The value $\hat{k}$ characterizes how "patient" a consumer is. A patient consumer is willing to wait for more stages of recommendations if no products are satisfactory. In particular, at stage $k \leq \hat{k}$, the consumer would immediately purchase product i as long as product i is the best choice at the current stage, namely $U_i = \max_{j \in S_k} U_j$, and its utility is above the target threshold, namely $U_i > U_0$. If none of the products at the current stage meet the threshold, then the consumer moves on to the next stage, given that she has gone through less than $\hat{k}$ stages. If no product is more attractive than U_0 up to stage $\hat{k}$, then the consumer leaves without purchasing anything. In this setup, we let $\mathsf{P}(\hat{k} = k) = \alpha_k \prod_{m=1}^{k-1}(1 - \alpha_m)$ and α_k can capture an arbitrary distribution of patience levels $\hat{k}$ in the population. For example, if most consumers are willing to browse more than one stages, then we can simply set $\alpha_1 \approx 0$.

This threshold choice behavior is similar to the concept of "satisficing" introduced in Simon (1955), a term combining "satisfactory" and "sufficing". consumers evaluate products sequentially. If the current best is satisfactory (in our model the criterion is whether its utility is above the threshold utility U_0), the consumer chooses it and leaves the system. If not, the consumer keeps evaluating. In the following context, we will refer to a satisficing product as one with utility above the threshold utility. It is worth mentioning that a "satisficing" choice model implicitly captures the search cost and limited attention of consumers at a high level. A similar framework is adopted in Chu et al. (2019).

In practice, the leaving probability α_k can be efficiently estimated from the data in online retailing. For example, if an online seller monitors the traffic and clicks of his webpage, then he may estimate α_k by the fraction of traffic that is lost after page k, i.e., the number of consumers who visit stage k but are then lost and do not visit stage $k+1$ divided by the number of consumers who visit but do not purchase anything up to stage k.

2.2. The Seller's Expected Profit

Next we study the expected profit of the seller from a random consumer given the order of recommendation sets as $S_1, \ldots, S_K$, where $S_k \subseteq N$. The seller does not observe an arriving consumer's realized random utilities but knows their distributions as well as the distribution of the patience level $\hat{k}$, equivalently, the values of α_k's.

If a consumer moves on to stage k but does not buy anything at this stage, then the seller knows:

- The utility of the no-purchase option of the consumer is more likely to be high;
- The utilities of the products recommended at stage k must be lower than U_0.

The information is then used to update the conditional purchase probability of the consumer at stage $k' > k$. In a realistic setting, each product is only priced once for a certain consumer. As a result, the second piece of information is not useful, since the seller would never recommend the same product again: If product i recommended at stage k is not purchased by the consumer, then it implies $U_i < U_0$ as well at latter stages and thus there is no need to recommend the same product at stage k'. Therefore, we can simply set $S_k \cap S_{k'} = \emptyset$ for $k \neq k'$.

Let $H_k \triangleq \bigcup_{m=1}^{k} S_m$ be the union of all products recommended over the first k stages and A_k be the event that a consumer moves on to stage $k+1$ after finding none of the products at stage k are satisficing, which happens with probability $\mathsf{P}(A_k) = 1 - \alpha_k$. The event that a consumer browses the products at stage k is equivalent to

$$B_k \triangleq \left\{U_0 > \max_{i \in H_{k-1}} U_i\right\} \bigcap A_1 \bigcap A_2 \bigcap \cdots \bigcap A_{k-1}.$$

Therefore, the expected profit of stage k is

$$\sum_{i \in S_k} r_i \mathsf{P}(U_i > U_0, U_i \geq \max_{j \in S_k} U_j, B_k) = \pi_k \sum_{i \in S_k} r_i \mathsf{P}(U_i > U_0, U_i \geq \max_{j \in H_k} \mu_j, U_0 > \max_{j \in H_{k-1}} U_j),$$

where we introduce the notation $\pi_k \triangleq \prod_{m=1}^{k-1}(1-\alpha_m)$ and by convention $\pi_1 \triangleq 1$. Notice that the purchase probability depends on the joint distribution of the random utilities $\{\epsilon_i\}_{i=0}^n$, but does not depend on the recommendation order of products displayed up to stage k. This is because the consumer moves on to stage k, only if none of products are satisficing at earlier stages, i.e., none of them surpass the threshold utility. Now the expected profit of the seller can be written as follows,

$$\sum_{k=1}^{K} \sum_{i \in S_k} r_i \pi_k \mathsf{P}(U_i > U_0, U_i \geq \max_{j \in H_k} U_j, U_0 > \max_{j \in H_{k-1}} U_j). \tag{1}$$

There are two levers the seller may control to increase the profit: the set of products recommended at each stage and their prices. There are two possible recommendation strategies:

- Static recommendation: the seller designs a fixed selling strategy in advance before the arrival of the consumer and does not update the policy over time.

- Dynamic recommendation: the seller designs what to display at stage k and/or their prices adaptively, if the consumer does not make a purchase yet.

In general, dynamic policies perform better than static policies in sequential decision-making, because they can utilize the information generated in the process. However, in this problem we can show that:

Lemma 1. *The optimal static recommendation generates the same profit as the optimal dynamic recommendation.*

We provide some intuition behind the lemma. The seller can anticipate three potential outcomes after the consumer moving on to stage $k-1$: the consumer purchases a product at stage $k-1$; the consumer leaves without purchasing; the consumer moves on to stage k. Among the three outcomes, the first two cannot be used to refine the recommendation at stage k because the consumer leaves the system forever. As a result, what to recommend at stage k only matters if the third outcome occurs. However, this event can be anticipated at the beginning and therefore the purchase probability will not be updated. In other words, the dynamic optimization only needs to take into account the branch of outcomes that the consumer keeps browsing and reaches the same expected profit as the static recommendation. The lemma implies that given the consumer's choice behavior as stated in our model, each consumer will face with the same assortment at each stage if they have not purchased anything before. From now on, we only need to focus on the static formulation.

3. Recommendation Problem

In this section, we study the seller's recommendation problem, where the prices of the products that the seller can recommend are exogenously fixed. Moreover, we assume that $\{\epsilon_i\}_{i=0}^n$ follow i.i.d. Gumbel distribution. It leads to the well-known multinomial logit model with choice probability $\mathsf{P}(U_i \geq \max_{j\in\{0\}\cup S} U_j) = \frac{\exp(\mu_i)}{1+\sum_{j\in S}\exp(\mu_j)}$. In Section 6, we relax this assumption and study more general distributions. For notational convenience, we let $v_i \triangleq \exp(\mu_i)$ refer to the attractiveness of product i. The key quantity in (1), the purchase probability of product i at the k-th stage can be expressed as

$$\pi_k \mathsf{P}(U_i > U_0, U_i \geq \max_{j\in H_k} U_j, U_0 > \max_{j\in H_{k-1}} U_j) = \frac{\pi_k}{1+\sum_{j\in H_{k-1}} v_j} \frac{v_i}{1+\sum_{j\in H_k} v_j}. \tag{2}$$

The derivation is available in the appendix. This probability is a product of two terms: The first term is the probability that a consumer does not purchase over the first $k-1$ stages and moves on to the k-th stage. The second term is the probability of the event that product i is satisficing and has the largest utility at stage k conditional on the event that there are no satisficing products over the first $k-1$ stages.

The choice probability of (2) has a connection to the well-known MNL model. If $K = 1$, then the choice process boils down to the standard MNL model. In our setting, the consumer is making a sequential choice, referred to as the sequential selection model. This sequential selection model possesses some salient properties compared to the MNL model. As widely studied, the MNL model exhibits a restrictive substitution pattern, known as the independence of IIA property: the ratio of the choice probabilities between any two alternatives is independent of a third one (see, e.g., Debreu (1952)). The sequential selection model can alleviate the IIA property and allow more flexible substitution patterns, as the choice probability of a product will also depend on which stage it lies in. Fixing the recommendation orders of other products, if a product is recommended at an earlier stage, its choice probability increases. It is obvious that the IIA property only holds withat the same stage.

Now we can express the seller's recommendation order problem as follows:

$$
\begin{aligned}
\max_{S_1,\cdots,S_K} \quad & \sum_{k=1}^{K} \frac{\pi_k}{1+\sum_{j\in\cup_{m=1}^{k-1}S_m} v_j} \frac{\sum_{i\in S_k} v_i r_i}{1+\sum_{j\in\cup_{m=1}^{k}S_m} v_j}, \\
\text{subject to} \quad & S_k \subseteq N, 1 \le k \le K, \\
& S_m \cap S_n = \varnothing, \quad m \neq n.
\end{aligned} \tag{3}
$$

Because the optimization is discrete, one can always enumerate all possible sequential recommendations and evaluate their expected revenues. Then the optimal recommendation policy can be found. However, note that the combinatorial nature makes enumeration almost infeasible: there are $(K+1)^n$ possible ways to arrange n products into K stages, including the products not displayed. Next, we study the structure of the optimization problem to find efficient solutions to (3).

3.1. Structures of the Optimal Recommendation Policy

To analyze the structure, we first focus on the continuous relaxation of the optimization problem (Rusmevichientong et al., 2010). The continuous relaxation is based on a reformulation of (3) into an integer programming. If we use $x_{ij} \in \{0,1\}$ to represent whether product i is recommended at stage j, then (3) can be equivalently cast as

$$\begin{aligned} \max_{x_{ij}\in\{0,1\}} \quad & \sum_{k=1}^{K} \frac{\pi_k}{1+\sum_{i=1}^{n}\sum_{j=1}^{k-1} x_{ij}v_j} \frac{\sum_{i=1}^{n} v_i r_i x_{ik}}{1+\sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}v_j}, \\ \text{subject to} \quad & \sum_{j=1}^{K} x_{ij} \le 1, \quad i \in N. \end{aligned}$$

The continuous relaxation replaces the integer constraint $x_{ij} \in \{0,1\}$ by $x_{ij} \in [0,1]$. In other words, each product now becomes infinitely divisible and thus can be placed at multiple stages, as long as the total fraction does not exceed one, i.e., $\sum_{j=1}^{K} x_{ij} \le 1$. Denote $q_k \triangleq \sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}v_j$ as the total attractiveness over the first k stages. The continuous relaxation can be formulated in two stages. We first fix $q_1, \cdots, q_K$, and maximize over x_{ij}:

$$\begin{aligned} R(q_1,\cdots,q_K) \triangleq \max_{x_{ij}\in[0,1]} \quad & \sum_{k=1}^{K} \frac{\pi_k}{1+q_{k-1}} \frac{\sum_{i=1}^{n} v_i r_i x_{ik}}{1+q_k}, \\ \text{subject to} \quad & \sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}v_i = q_k, \quad 1 \le k \le K, \\ & \sum_{j=1}^{K} x_{ij} \le 1, i \in N. \end{aligned} \tag{4}$$

The first-stage problem may be of independent interest: if the firm has a total "attraction budget" at each stage and all products are divisible, then it provides the optimal solution to how to allocate the products to meet the budget and maximize the total revenue. In the second stage, we maximize over the attraction budget q_k:

$$R^* = \max_{(q_1,\cdots,q_K)\in Q} R(q_1,\cdots,q_K), \tag{5}$$

where $Q = \{(q_1,\cdots,q_K) : 0 \le q_1 \le q_2 \cdots \le q_K \le \sum_{i\in N} v_i\}$.

The optimal value of the continuous relaxation provides an upper bound for problem (3). Moreover, if an optimal solution to the continuous relaxation happens to be integers, then it is also the optimal solution to problem (3). Because the two problems are closely related, next we study the structure of the continuous relaxation, which may generate insights for the original problem. We first introduce the following definition.

Definition 1. *A recommendation policy x_{ij}, $i = 1, \cdots, n$ and $j = 1, \cdots, K$, for the continuous relaxation is (strongly) sequentially revenue-ordered, if it satisfies* $\min\{r_i : x_{ik} > 0\} \geq (>) \max\{r_j : x_{j,k+1} > 0\}$, $\forall k \in \{1, \cdots, K\}$.

Note that here we have adopted the convention that $x_{i,K+1} = 1 - \sum_{j=1}^{K} x_{ij}$ represents the fraction of the product not being displayed at any stage. The definition also applies to integer x_{ij} of the original problem.

Lemma 2. *1. Given $(q_1, \cdots, q_K)$, the optimal solution to problem (4) is sequentially revenue-ordered.*

2. The set Q is a lattice and $R(q_1, \cdots, q_K)$ is supermodular for $(q_1, \cdots, q_K) \in Q$.

In the second part of Lemma 2, the supermodularity of $R(\cdot)$ implies that for any two vectors $\mathbf{q}_1, \mathbf{q}_2 \in Q$, we have $R(\mathbf{q}_1) + R(\mathbf{q}_2) \leq R(\mathbf{q}_1 \vee \mathbf{q}_2) + R(\mathbf{q}_1 \wedge \mathbf{q}_2)$, where $\vee$ and $\wedge$ are element-wise maximum and minimum. This property is useful in comparing the optimal recommendations under different parameters, such as different marginal value of additional capacity in a dynamic single-leg capacity control problem analysed in Section 3.3. The revenue-ordered structure is shown by contradiction. If a product of lower revenue is displayed before a product of higher revenue, then by switching the positions of the two products, the firm can garner higher revenue. This structure of (4) is inherited by the second-stage problem (5). An implication of this structure is that, for the optimal solution x_{ij} to the continuous relaxation, if $x_{ij} > 0$ and $x_{ij'} > 0$ for $j' > j$, then for all products displayed between stage j and j', their revenues must equal to r_i.

However, the revenue-ordered structure cannot be directly translated to apply to the original problem (3). The divisible products in the optimal solution to the continuous relaxation may span multiple stages as long as the revenue-ordered property is satisfied. The next lemma bridges the gap:

Lemma 3. *For any optimal solution to problem (5), if consecutive stages contain fractions of products of the same revenue, then moving all of them to one of the stages generates the same (optimal) revenue.*

By Lemma 3, one can transform an optimal solution to the continuous relaxation and obtain a feasible solution to the original problem (3). The transformation does not lower the revenue, and

Algorithm 1 Recommendation Order Optimization Algorithm.

Input: $(v_1, r_1), \cdots, (v_n, r_n)$.
Initialize the indexes such that $r_i \geq r_j$ if $i > j$.
Set $A_0 = \underbrace{\{\{\infty\}, \cdots, \{\infty\}\}}_{K+1 \text{ elements}}$.
for $m = 1, \cdots, K$ **do**
  for $i = 1, \cdots, n$ **do**
    $l = \mathrm{argmax}_{1 \leq j \leq i} \mathcal{R}(A^j_{m-1} \cup \underbrace{\{r_i, \cdots, r_i\}}_{K-m+1 \text{ elements}})$.
    Set $A^i_m = A^l_{m-1} \cup \{r_i\}$.
  end for
end for
$j^* = \mathrm{argmax}_j \mathcal{R}(A^j_K)$.
Output: $A^{j^*}_K$.

thus the transformed solution is optimal for (3). More importantly, we can always guarantee that one transformed solution is strongly sequentially revenue-ordered. The following example illustrates Lemma 3.

Example 1. *Suppose $K = 2$ and there are five products with $r_1 > r_2 > r_3 > r_4 = r_5$. Let $x_{11} = 1$, $x_{21} = 0.5$, $x_{22} = 0.5$, $x_{32} = 1$, $x_{42} = 0.2$, $x_{43} = 0.8$, $x_{53} = 1$ be optimal for problem (5). Lemma 3 implies that the following recommendations are also optimal:*

1. $x_{11} = 1$, $x_{21} = 1$, $x_{32} = 1$, $x_{42} = 1$, $x_{52} = 1$,

2. $x_{11} = 1$, $x_{22} = 1$, $x_{32} = 1$, $x_{43} = 1$, $x_{53} = 1$.

By Lemma 3, there always exists an optimal solution to (3) that is strongly sequentially revenue-ordered. In the appendix, we present an alternative argument for this property based on the fixed-point approach. Denote $R(A)$ as the revenue of a strongly sequentially revenue-ordered recommendation, given the lowest revenue at the k-th stage as the $k+1$-st element of A. By default, $A_1 = \infty$, More precisely,

$$R(A) \triangleq \sum_{k=1}^{K} \frac{\pi_k}{1 + \sum_{j:r_j \geq A_k} v_j} \frac{\sum_{i:A_k > r_i \geq A_{k+1}} v_i r_i}{1 + \sum_{j:r_j \geq A_{k+1}} v_j}.$$

Then the goal now is to find an optimal list of A to maximize $R(A)$.

Theorem 1. *Algorithm 1 finds the strongly sequentially revenue-ordered optimal solution to* (3) *with time complexity* $\mathcal{O}(n^3)$.

Algorithm 1 searches over the space of strongly sequentially revenue-ordered recommendation policies, which is of polynomial size and much smaller than the set of all possible recommendation policies. When $K = 1$, Theorem 1 reduces to the revenue-ordered property of the optimal assortment for the standard MNL model (see, e.g., Talluri and van Ryzin (2004); Gallego et al. (2004)).

Note that Algorithm 1 only requires the optimal solution to be strongly sequentially revenue-ordered and doesn't depend specifically on the assumption of Gumbel distribution. In Section 6, we study more general distributions that allow for multiple consumer types while still preserve this property. Our message is that for a wide range of consumer behavior, sequentially revenue-ordered recommendation is optimal and Algorithm 1 can be used to efficiently find the optimal policy.

3.2. Performance of One-Stage Recommendation

Given $K \geq 1$, we now study the performance if the seller only recommends his products at the first stage. To derive the optimal one among these simple strategies, the seller solves exactly the recommendation problem in the standard MNL model:

$$\tilde{r} \triangleq \max_{S \subseteq N} \frac{\sum_{i \in S} r_i v_i}{1 + \sum_{i \in S} v_i}. \tag{6}$$

It is well-known that the optimal solution to the above problem is a threshold policy: the products with revenues above $\tilde{r}$ will be recommended. Initialize the product indexes such that a product with a smaller index has a larger revenue. We want to compare the optimal revenue of problem (6) with that of problem (3). Clearly, the largest revenue of problem (3) is achieved when $K \to \infty$ and $\pi_1 = \cdots = \pi_k = \cdots = 1$. For such a setting, we can show that tne optimal solution satisfies $x_{ii} = 1$, $\forall i \in N$. That is, all products will be recommended and at each stage the seller only recommends one. As a result, the optimal revenue of problem (3) is as follows,

$$\tilde{R} \triangleq \sum_{i=1}^{n} \frac{1}{1 + \sum_{j<i} v_j} \frac{v_i r_i}{1 + \sum_{j \leq i} v_j}.$$

Then we are able to show the property of the ratio between the $\tilde{r}$ and $\tilde{R}$ as follows.

Proposition 1. *For problem* (3)*, if the seller optimally recommends products only at the first stage but does not recommend them at latter stages, then the performance will at least be half of the optimal sequential recommendation policy. More precisely,* $\tilde{r} \geq \frac{\tilde{R}}{2}$.

The optimal one-stage recommendation earns at least a constant bound compared to the optimal sequential recommendation. By the following example, we can show that this bound is actually tight when the more profitable products do not dominate others in the attractiveness.

Example 2. *Suppose that there are two products, and* $r_2 = \frac{v_1 r_1}{1+v_1}$. *Then* $\tilde{r} = r_2 = \frac{v_1 r_1}{1+v_1}$ *and* $\tilde{R} = \frac{v_1 r_1}{1+v_1} + \frac{1}{1+v_1}\frac{v_2 r_2}{1+v_1+v_2}$. *As a result, we have*

$$\frac{\tilde{R}}{\tilde{r}} = \frac{\frac{v_1 r_1}{1+v_1} + \frac{1}{1+v_1}\frac{v_2 r_2}{1+v_1+v_2}}{\tilde{r}} = \frac{\tilde{r} + \frac{1}{1+v_1}\frac{v_2 \tilde{r}}{1+v_1+v_2}}{\tilde{r}} = 1 + \frac{1}{1+v_1}\frac{v_2}{1+v_1+v_2}.$$

When $v_1 \to 0$ *and* $v_2 \to \infty$*, we have* $\frac{\tilde{R}}{\tilde{r}} \to 2$.

3.3. Efficient Recommendations

In Section 3.1, we analyse the seller's optimal recommendations to maximize his expected revenue. Now let the seller maximize a convex combination between the expected revenue and the sales volume. For example, when the seller is a new market entrant, he may not just want to make profits by sales but also intend to expand the consumer base by expanding the sales. In this situation, the objective function of the seller becomes as follows.

$$\max_{S_1,\cdots,S_K} \quad \sum_{k=1}^{K} \frac{\pi_k}{1+\sum_{j\in\cup_{m=1}^{k-1}S_m} v_j} \frac{\sum_{i\in S_k} v_i r_i}{1+\sum_{j\in\cup_{m=1}^{k}S_m} v_j} + z\sum_{k=1}^{K} \frac{\pi_k}{1+\sum_{j\in\cup_{m=1}^{k-1}S_m} v_j} \frac{\sum_{i\in S_k} v_i}{1+\sum_{j\in\cup_{m=1}^{k}S_m} v_j}, \tag{7}$$

subject to $\quad S_k \subseteq N, \forall k,$

$\quad S_m \cap S_n = \varnothing, \quad m \neq n.$

where in the objective function, the first term is the expected revenue and the second term is the seller's weigh on the sales volume, z, multiplied by the market share. We refer to the optimal solutions to the problem above as efficient recommendations, similar to the concept of "efficient sets" defined in Talluri and van Ryzin (2004). It is of interest to investigate the relationship between the efficient recommendations under different parameters. The change of the efficient recommendations will imply how a seller cares about the expansion of the consumer base.

Efficient recommendations are extremely useful also for the following two aspects when $z < 0$. First, we can treat $-z$ as a common production cost or the transaction fee for each sale charged by a third party. Then the change of the efficient recommendations will indicate how a seller responses to a production technology or a third party's contract. Second, it also relates to the single-leg dynamic capacity control problem[3], where $-z$ characterizes the marginal value of an additional capacity given any fixed remaining time and capacity. One can interpret $-z$ as the product's common opportunity cost. As shown in Gallego and van Ryzin (1994), when the remaining time increases or the remaining capacity decreases, the opportunity cost $-z$ will increase.

. We show the properties of the efficient recommendations in the following proposition.

Proposition 2. *1. For any z, there exists one efficient recommendation which is strongly sequentially revenue-ordered.*

2. For any real $z_1 < z_2$, there exist efficient recommendations for (7), satisfying $\cup_{i=1}^{k} S_i^*(z_1) \subseteq \cup_{i=1}^{k} S_i^*(z_2), \quad \forall k \in \{1, \cdots, K\}$.

The first part of Proposition 2 is straightforward, as problem (7) is similar to problem (3), and the only difference is that the revenue of each product is shifted by the same constant z. The second part implies that these efficient recommendations admit an inclusion relationship. Weighing more on the sales volume, or with a smaller production cost, or charged a smaller transaction fee, or having more remaining capacity or less remaining time, the seller will recommend more before each stage as z increases. In these situations, the seller prefers to sell his products quickly. Combined with the first part, this result also implies that the lowest revenue at each stage will decrease when z increases. A similar result is found in the standard MNL model by Talluri and van Ryzin (2004)

[3] A single-leg problem means that one unit consumption of any product requires the same quantity of the same resource. In the single-leg dynamic capacity control problem, the seller decides how to dynamically change his recommendations. This problem is widely studied in cargo revenue management (see, e.g., Slyke and Young (2000); Amaruchkul et al. (2007)) .

and Rusmevichientong et al. (2014). They show that when z increases, the revenue threshold, products with revenues above which are recommended, will decrease.

In Figure 2, we give a numerical example to illustrate the inclusion property. In the figure, we set $K = 3$, $\pi_1 = 1$, $\pi_2 = 0.8$, $\pi_3 = 0.5$. There are 30 products with randomly generated attractiveness and revenues. The result demonstrates that in the optimal recommendation strategy, as z increases, the total number of products recommended up to any stage gradually increases. As the optimal solutions are always strongly sequentially revenue-ordered, the result implies the inclusion property.

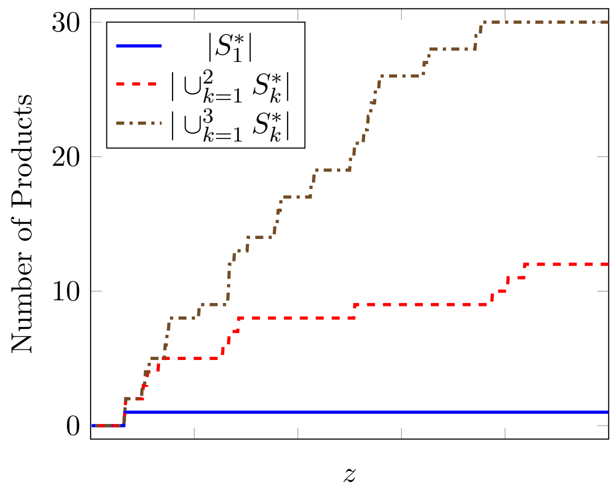


Figure 2: An illustration of the inclusion property of the efficient sequential sets with 30 products. The lines are the numbers of products recommended up to a specified stage.

4. Pure Pricing Problem

Besides the recommendation order decision, another leverage the seller can use is setting the prices of products, even when the order of recommendation sets of products is determined in advance and cannot be adjusted. There are two reasons studying the problem separately. First, the pricing problem sheds light on and serve as a building block for the joint recommendation order and pricing problem studied in Section 5. It is of independent interest technically as well. Second, there are situations in practice that the recommendation orders cannot be adjusted flexibly while the prices can be. Consider a consumer browsing products online and demanding a particular sorting of the display (in descending order of sales or consumer ratings, etc.). In this case, the recommendation order is chosen by the consumer and the seller can only hope to increase the revenue by setting prices.

In this section, we study the optimal pricing problem of the sequential selection model when the sequence of assortments $S_1, \ldots, S_K$ are given. The seller determines the price of each product, p_i for $i = 1, \ldots, n$. To incorporate the price sensitivity of consumers into the MNL model, we follow the standard in the literature (see, e.g., Gallego and Topaloglu (2019)) and modify the utility of product i of a random consumer to be $U_i = \mu_i - p_i + \epsilon_i$. With a little abuse of notations, we refer to $v_i = \exp(\mu_i)$ as the *intrinsic* attractiveness of product i. Note that in a standard MNL model, the choice probability of a product $i \in S$ becomes $\frac{v_i \exp(-p_i)}{1+\sum_{j \in S} v_j \exp(-p_j)}$, which depends explicitly on the intrinsic attractiveness multiplied by the pricing effect $\exp(-p_i)$.

The optimal pricing problem for given assortments can be formulated as below, similar to (3):

$$\max_{p_1,\ldots,p_n} \quad \sum_{k=1}^{K} \frac{\pi_k}{1 + \sum_{j \in \cup_{m=1}^{k-1} S_m} v_j e^{-p_j}} \times \frac{\sum_{i \in S_k} v_i e^{-p_i}(p_i - c_i)}{1 + \sum_{j \in \cup_{m=1}^{k} S_m} v_j e^{-p_j}}. \tag{8}$$

The next proposition shows that it is optimal for the seller to set the prices so that the products at the same stage have the same markup, $p_i - c_i$. In the case of zero cost $c_i \equiv 0$, the proposition implies that the products at the same stage have the same price. This is similar to the standard MNL models and nested logit models (Aydin and Ryan, 2000; Hopp and Xu, 2005; Gallego and Stefanescu, 2011; Li and Huh, 2011; Gallego and Wang, 2014).

Proposition 3. *The optimal prices to* (8) *satisfy the following condition: for all i such that $i \in S_k$, $p_i - c_i$ are equal.*

The unique question arising in our sequential model is s the relationship between prices and stages. The argument could go either way: setting higher prices for the first few stages will increase the revenue if the consumer finds at least one product satisfacing, while setting lower prices for the first few pages increases the sales volume as consumers are impatient and may leave after browsing a few stages if they find the products too expensive. From our next proposition, indeed, the optimal prices strike a balance in terms of the trade-off. By Proposition 3, we may denote $r^{(k)}$ as the optimal markup for the products at stage k.

Proposition 4. *The optimal prices to* (8) *satisfy* $\pi_1 r^{(1)} > \cdots > \pi_K r^{(K)}$.

To interpret the proposition, let's first investigate a few extreme cases. Recall that $\pi_k = \Pi_{m=1}^{k-1}(1-\alpha_m)$ is the probability that a consumer is patient enough to browse stage k if she finds none product satisfacing over the first $k-1$ stages. When $\pi_1 = \cdots = \pi_K = 1$, then all consumers are patient and always willing to browse the next stage if the current page does not include a satisficing product. In this case, Proposition 4 implies that the revenues of the products should be decreasing across stages. The seller always has incentives to recommend the profitable products first as he is not concerned about losing the consumers. In fact, even for a popular product appearing in later page, the seller does not set a relatively high price.

On the other hand, if only a fraction of consumers are willing to browse the next stage, i.e., $\pi_k > \pi_{k+1}$, then the optimal prices may satisfy $r^{(k)} < r^{(k+1)}$. To retain consumers, the seller may set lower prices at the first few stages. Due to the impatience of consumers, the structure of the optimal prices becomes complicated and can be non-monotone even products have the same production costs. This is consistent with the findings in Gao et al. (2018), which studies the price optimization under a general cascade click model. In the next example, we demonstrate such a case that even when the relatively attractive products are placed at the first stage, the price set for the first stage is lower than that of the second stage. Moreover, the optimal prices are non-monotone across stages.

Example 3. *Consider three products* ($n = 3$) *placed in order at three stages* ($K = 3$)*. The costs are zero* ($c_i \equiv 0$)*. Let* $\pi_1 = 1$, $\pi_2 = \pi_3 = 0.1$, $v_1 = 1 > v_2 = v_3 = 0.9$*. The optimal prices are* $p_1 = 1.33$, $p_2 = 1.50$ *and* $p_3 = 1.19$*. We have* $p_1 < p_2 > p_3$*, while* $\pi_1 p_1 = 1.33 > \pi_2 p_2 = 0.15 > \pi_3 p_3 = 0.12$.

Next in the following theorem, we show that the optimal prices will be unique and can be efficiently found by a binary search. We provide some intuition behind this theorem. Note that the prices are uniquely tied with the revenues $\{r^{(k)}\}$. By the first-order conditions of the objective over $\{r^{(k)}\}$, it will generate K equations. The equations have a special structure: if we are given $r^{(1)}$, then the first $K-1$ equations can be used to uniquely determine $r^{(2)}, \cdots, r^{(K)}$. The expression in the last equation, which turns out to be monotone in the choice of$r^{(1)}$, can be used to verify. Therefore, we can conduct a binary search to find $r^{(1)}$ so that the last equation holds.

Theorem 2. *In the pure pricing problem, the optimal pricing policy is unique, which can be efficiently found by a binary search.*

For the MNL model, Aydin and Porteus (2008) and Akçay et al. (2010) show that the profit function is unimodal and there exists a unique optimal pricing policy, which can also be found by solving the first-order conditions. Gallego and Wang (2014) study the pricing policy in the nested logit model with product-differentiated price sensitivities.

5. Joint Recommendation Order and Pricing Problem

In this section, we consider a seller deciding the order of products to recommend and their prices simultaneously. This is referred to as joint recommendation order and pricing problem. Similar to (3) and (8), the optimization problem can be formulated as (without loss of generality, we assume zero costs $c_i \equiv 0$):

$$\begin{aligned} \max_{r_1,\ldots,r_n,S_1,\cdots,S_K} \quad & \sum_{k=1}^{K} \frac{\pi_k}{1+\sum_{j\in\cup_{m=1}^{k-1}S_m} v_j e^{-r_j}} \times \frac{\sum_{i\in S_k} v_i e^{-r_i} r_i}{1+\sum_{j\in\cup_{m=1}^{k}S_m} v_j e^{-r_j}}, \\ \text{subject to} \quad & S_k \subseteq N, 1 \le k \le K, \\ & S_m \cap S_n = \varnothing, \quad m \neq n. \end{aligned} \tag{9}$$

As shown in Section 3 and Section 4, when one set of decisions is fixed (recommendation order or pricing), the optimization problem can be efficiently solved. However, the joint problem becomes significantly harder. In fact, we establish its NP-hardness in Theorem 3 below by reducing the well-known 2-PARTITION problem to a special case of our model. The result is quite different from the joint problem of the standard MNL model, which can still be efficiently solved.

Theorem 3. *Problem* (9) *is NP-hard.*

Next we provide an upper bound for the optimal value of (9). Consider the continuous relaxation introduced in Section 3.1. To optimize the continuous relaxation, by Proposition 3, the (potentially fractional) products at the same stage have the same revenue. Therefore, the objective function of the continuous relaxation can be expressed as

$$\sum_{k=1}^{K} \frac{\pi_k}{1+\sum_{m=1}^{k-1} V_m e^{-r^{(m)}}} \times \frac{V_k e^{-r^{(k)}} r^{(k)}}{1+\sum_{m=1}^{k} V_m e^{-r^{(k)}}}. \tag{10}$$

Here $r^{(k)}$ is the common price of the products at stage k; V_k is the total intrinsic attractiveness at stage k, i.e., $V_k \triangleq \sum_{i=1}^{n} x_{ik} v_i$ in (4). With the pricing power, the seller is always better off to recommend each product at least at the last stage. Then at optimality $\sum_k V_k$ will be same as $T \triangleq \sum_{i=1}^{n} v_i$, the total intrinsic attractiveness of all the seller's products.

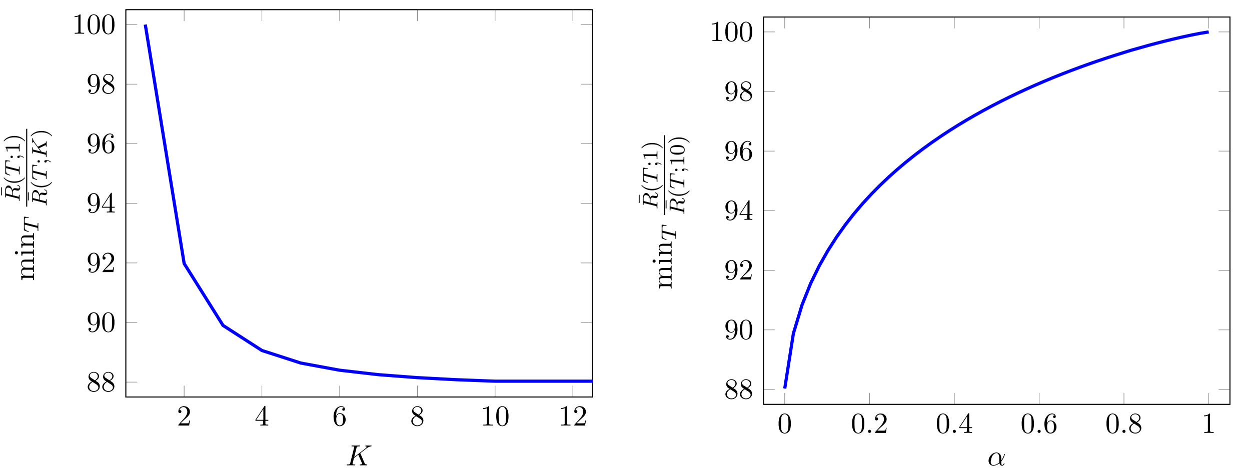
In addition, let $\alpha_k = 0$ for all $k < K$. Because the objective function of (9) is decreasing in α_k, this leads to an upper bound. The new optimization problem can be expressed as

$$\begin{aligned} \bar{R}(T;K) \triangleq \max_{r^{(1)},\ldots,r^{(K)},V_1,\ldots,V_K} \quad & \sum_{k=1}^{K} \frac{1}{1+\sum_{m=1}^{k-1} V_m e^{-r^{(m)}}} \times \frac{V_k e^{-r^{(k)}} r^{(k)}}{1+\sum_{m=1}^{k} V_m e^{-r^{(k)}}} \\ \text{subject to} \quad & \sum_{k=1}^{K} V_k = T, \\ & V_k \geq 0,\ 1 \leq k \leq K. \end{aligned} \tag{11}$$

According to the analysis above, $\bar{R}(T;K)$ serves as an upper bound of the optimal value of (9). Next we study the optimization of $\bar{R}(T;K)$, in order to derive useful insights and heuristics for (9). Define $\mathcal{L}(y)$ as the solution of x to $xe^x = y$ for any $y \geq 0$. We can show the following properties of $\bar{R}(T;K)$.

Lemma 4. *$\bar{R}(T;K)$ is increasing in K. Moreover,* $\bar{R}(T;\infty) = 2\ln(\frac{\sqrt{1+4T/e}+1}{2}) + \frac{1-\sqrt{1+4T/e}}{\sqrt{1+4T/e}+1}$ *and* $\bar{R}(T;1) = \mathcal{L}(T/e)$.

The objective function $\bar{R}(T;K)$ is increasing in K as the seller has more room to manipulate the order of the products when K increases. Although $\bar{R}(T;K)$ does not admit a close-form expression for all K, we can derive the two ends $K \to \infty$ and $K = 1$. Lemma 4 also leads to a simple heuristic: recommending all products at the first stage and optimizing their prices. Note that the revenue achieved by the heuristic is simply $\bar{R}(T;1)$. The following theorem provides a performance guarantee of the heuristic.



(a) $\alpha_k \equiv 0$.

(b) $K = 10$ and $\alpha_k \equiv \alpha$.

Figure 3: The percentage revenue of the heuristic relative to the optimal revenue under the continuous relaxtion.

Theorem 4. *For problem* (9)*, recommending all products at the first stage and optimizing their prices achieves a tight constant bound of the optimal revenue,* $\beta(T) = \frac{\bar{R}(T;1)}{\bar{R}(T;\infty)}$*. Moreover,* $\beta(T)$ *is first decreasing then increasing in* $T \in (0, \infty)$*, with* $\min_T \beta(T) = 87.8\%$.

When the seller can wield both operational weapons, the recommendation order and pricing, the simple heuristic that recommending all products at the first stage yields at least 87.8% of the optimal revenue. Besides, the bound becomes better when the total attractiveness of products are too large and small. In Figure 3a, we illustrate the performance of the heuristic against K, the total number of stages allowed. As we can see, when K increases, the performance becomes worse, but at around $K = 10$, the performance of the heuristic is stable and converges to 88%. The convergence has two-fold implications. First, it verifies the theorem. In practice, the heuristic can work much better because in the upper bound we make no-leave assumptions. Second, it implies $\bar{R}(T;K)$ converges almost at $K = 10$. That is, the seller does not benefit too much when the consumers are too patient to view more than 10 stages. In Figure 3b, we let $\alpha_k \equiv \alpha$, $K = 10$ and illustrate the performance of heuristic, compared to the optimal revenue in problem (10) under different α's. It can be seen that when consumers are more likely to leave, namely with a larger α, the heuristic works better. Therefore, if the seller cannot afford to solve the joint problem, either computationally or economically, then we suggest him adopt the heuristic and solve the one-stage optimal pricing using the binary search discussed in Section 4. On a side note, the good performance of the heuristic does not obliterate the value of our analysis in Section 3. If the prices cannot adjusted, then recommending all products at the first stage can be disastrous and leads to just half of the revenue at optimality.

6. Extensions

In this section, we provide some extensions on the sequential selection model, which allows multiple consumers types. We find some special cases that still admit the sequentially revenue-ordered recommendations as optimal, then by Theorem 1, which can be efficiently found by Algorithm 1.

6.1. Sequential Mixed MNL Model

We consider a mixture of consumer types where for each type of consumers, they behaves as in the model described in Section 3. The products may have different attractiveness for different types. We may refer to this choice model as a mixed sequential selection model. The seller's recommendation order problem can be formulated as follows,

$$
\begin{aligned}
\max_{\{S_1,\cdots,S_K\}} \quad & E_t[\sum_{k=1}^{K} \frac{\pi_k}{1+\sum_{j\in\cup_{m=1}^{k-1}S_m} v_j^t} \frac{\sum_{i\in S_k} v_i^t r_i}{1+\sum_{j\in\cup_{m=1}^{k}S_m} v_j^t}], \\
\text{subject to} \quad & S_k \subseteq N, 1 \le k \le K, \\
& S_m \cap S_n = \varnothing, \quad m \neq n,
\end{aligned}
$$

where t is the consumer's private and random type.

For the mixed sequential selection model, one can show that the recommendation order problem is NP-complete, as when $K = 1$, the seller's problem is reduced to that in the standard mixed MNL model (see, e.g., Rusmevichientong et al. (2014)). We will give several special cases that still admit sequentially revenue-ordered assortment as the optimal solution.

6.1.1. Random No-Purchase Mean Utilities

The first case is called the *Random No-Purchase Mean Utilities* form. In particular, we assume that the mean utility of product i is of the following form

$$\mu_i^t = \begin{cases} \mu_i, & i > 0, \\ \mu_0 + F(t), & i = 0, \end{cases}$$

where $F(t)$ is an arbitrary function of t. That is, only the mean utility of the outside option is shifted by the same quantity for any given type of consumers. The shift represents how picky a certain type of consumers are to this particular seller. Surprisingly, even we allow such randomness, the seller's optimal recommendation is still shown to be strongly sequentially revenue-ordered.

Lemma 5. *If the mean utilities of the products are of the Random No-Purchase Mean Utilities form, then the seller's optimal recommendation is strongly sequentially revenue-ordered.*

6.1.2. Value Conscious Mean Utilities

Next we provide another special case which still admits sequentially revenue-ordered recommendations as optimal. In particular, we assume that all products have different revenues and the realization of the mean utility of any two products $i, j > 0$ satisfies

$$r_i v_i^t \geq r_j v_j^t, \quad if \ r_j > r_i.$$

We call this as the *Value Conscious* form. Before proceeding to the recommendation order problem, we discuss the implications of this form. Let $c_i \equiv 0$. Notice that if $r_j > r_i$, this Value Conscious consumers condition requires that $v_i^t > v_j^t$. It corresponds to a situation where the consumer prefers less expensive (more precisely, more profitable for the seller) products, which is quite reasonable in many settings. We borrow the following example from Rusmevichientong et al. (2014) to help interpret this condition.

Example 4. *Consider the case where the mean utility of product i is of the form $v_i = Pr_i^{-B}$, where P and B are (possibly dependent) arbitrary random variables and B takes values in the interval* $[0, 1]$.

This corresponds to a situation where the price sensitivity of each consumer is sampled from the distribution of B and the consumers are not too price sensitive in the sense that B takes values in the interval $[0, 1]$. It is easy to check that this choice of mean utilities satisfies the Value Conscious *condition.*

The next lemma shows that sequentially revenue-ordered recommendations are optimal under the Value Conscious condition.

Lemma 6. *If the mean utilities of the products are of the Value Conscious form, then the seller's optimal recommendation is sequentially revenue-ordered.*

6.2. Mixed with Non-satisficing Choice Rule

Now Let's consider the case that fractions γ of consumers are following a sequential selection choice rule while the others will first view the first few stages and then make a MNL choice based on the products viewed. The consumer choice behavior of the latter kind is studied in Gallego et al. (2016). Let π'_k represent the fractions of consumers in the latter kind who first view $\hat{k}$ stages . Then the objective function of the seller can be written as follows,

$$\begin{aligned}
&\max_{\{S_1,\cdots,S_K\}} \quad (1-\gamma)\sum_{k=1}^{K}\frac{\pi'_k\sum_{i\in\cup_{m=1}^{k}S_m} v_i r_i}{1+\sum_{j\in\cup_{m=1}^{k}S_m} v_j} + \gamma\sum_{k=1}^{K}\frac{\pi_k}{1+\sum_{j\in\cup_{m=1}^{k-1}S_m} v_j}\frac{\sum_{i\in S_k} v_i r_i}{1+\sum_{j\in\cup_{m=1}^{k}S_m} v_j},\\
&(12) \quad \text{subject to} \quad S_k \subseteq N, 1\le k\le K,\\
&\qquad\qquad\qquad\quad S_m\cap S_n=\varnothing, \quad m\neq n.
\end{aligned}$$

The next lemma shows that sequentially revenue-ordered recommendations are still optimal.

Lemma 7. *For problem* (12)*, one optimal recommendation is strongly sequentially revenue-ordered.*

7. Application: Why Assortment Is Always Same on Different Date?

Consider that a seller provides one assortment $S_t \subseteq \mathcal{N} = \{1, \cdots, n\}$ on date t. Time is discrete, starts from 0, ends on date K, where K can be infinity. There are λ_t consumers arrive on date t, who are denoted as consumers of type t. Note that consumers of different types on the same date see the same assortment. The utilities of the no-purchase option (product i) for all consumers are $U_0 = \epsilon_0$ ($U_i = u_i + \epsilon_i$), where $\{u_i\}$ are deterministic and known and $\epsilon_0, \cdots, \epsilon_n$ are independent and follow a standard Gumbel distribution.

A consumer of type t makes decision in the following sequential way,

- If the best product on the current date (date k) is more attractive than the no-purchase option, $\max_{i \in S_k} U_i > U_0$, then she will buy the product and leave the system.

- If no product on the current date (date k) is more attractive than the no-purchase option (or no products are provided), then she leaves the system without purchasing with probability α_k^t, independent of everything else. Otherwise, with probability $1 - \alpha_k^t$, she moves on to date $k+1$ and repeats the process.

That is, all consumers are making decisions in a sequential satisficing rule. Note that there may be different number of consumers of different types, and the non-purchase leaving probability α_k^t may depend on the consumer's type, namely on which date a consumer arrives.

The seller maximizes the sum of revenues over all dates. We consider a robust optimization where both $\{\lambda_t\}$, and $\{\alpha_k^t\}$ are unknown. That is, the seller's objective is

$$
\begin{aligned}
(13) \qquad R^* = \max_{S_1, \cdots, S_K} \quad & \min_{\lambda_t \geq 0, 1 \geq \alpha_k^t \geq 0} \quad \sum_{t=0}^{K} \lambda_t \sum_{k=t}^{K} \frac{\pi_k^t}{1 + \sum_{j \in \cup_{m=t}^{k-1} S_m} v_j} \frac{\sum_{i \in S_k \setminus \cup_{m=t}^{k-1} S_m} v_i r_i}{1 + \sum_{j \in \cup_{m=t}^{k} S_m} v_j} \\
& \text{subject to} \quad S_k \subseteq N, 1 \leq k \leq K,
\end{aligned}
$$

where $\pi_k^t \triangleq \prod_{m=t}^{k-1}(1 - \alpha_m^t)$. The following theorem shows the performance of a static assortment policy,

Theorem 5. *For Problem (13), letting $S_k \equiv S$ be the optimal assortment in Problem (6), the seller will achieve at least half of the optimal revenue.*

Proof. We prove the claim by proving it is true for any $\{\lambda_t\}$, and $\{\alpha_k^t\}$. Denote

$$
R_t = \lambda_t \sum_{k=t}^{K} \frac{\pi_k^t}{1 + \sum_{j \in \cup_{m=t}^{k-1} S_m} v_j} \frac{\sum_{i \in S_k \setminus \cup_{m=t}^{k-1} S_m} v_i r_i}{1 + \sum_{j \in \cup_{m=t}^{k} S_m} v_j},
$$

which will just be the revenue for the seller earned from the consumers of type t. Denote the revenue earned from type t under the policy mentioned in the claim as $\tilde{R}_t$. Then by Proposition 1 $\tilde{R}_t \geq R_t^*/2$, where R_t^* is revenue earned from type t if the seller only maximizes R_t. Obviously, $R^* \leq \sum_t R_t^*$. Then we have the claim. $\square$

Similarly, if the seller can optimize the prices in a way as mentioned in Section 5, then by Theorem 4, the seller from an optimal static assortment will achieve at least 87.8% of the optimal revenue. The results above characterize why the assortment is always same in stores on different date.

8. Concluding Remark

We propose a tractable sequential recommendation-selection model. In our model, the seller sequentially recommend products to the consumers over multiple stages. Each consumer will at most view up to a certain stage. Before this stage, once the consumer finds a satisfactory product at the current stage, she will select the favorite one and leaves. We derive the optimal solution for the recommendation order problem. The relevant efficient recommendations, namely the optimal recommendations under different parameters, are also analysed, which admit an inclusion properties. The optimal pricing policy for a given recommendation order is shown to be unique, which can also efficiently solved. The optimal one-stage recommendation earns half of the revenue by the optimal sequential recommendation. In the optimal pricing policy, we observe some counter-intuitive results. For example, a product recommended earlier and with a larger intrinsic attractiveness may not be priced at a higher revenue for the seller. The joint problem is shown to be NP-hard, but based on a simple heuristic, we achieve a constant performance guarantee. We conclude this paper by pointing out some future directions for this work. Firstly, it is natural to consider a capacity constraint on the recommended products at each stage. Secondly, one may consider consumers recalls in the sequential decisions.

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A. Analyses and Proof

Proof of Lemma 1: Let $\mathcal{H}_{k-1}$ be the historical recommendations. We use $\mathcal{E}(\mathcal{H}_{k-1})$ to denote the event that a consumer has not found saitsificing products on the first $k-1$ pages and still moves on to the next page. Denote $P(U_0 = x|\mathcal{E}(\mathcal{H}_{k-1}))$ as the conditional distribution of consumer's non-purchase utility and $R_k(\mathcal{H}_{k-1})$ as the future expected revenue given the event of $\mathcal{E}(\mathcal{H}_{k-1})$. Then

$$\begin{aligned}
R_k(\mathcal{H}_{k-1}) = & \max_{\mathbf{p}, S_k \subseteq N \setminus H_{k-1}} \sum_{i \in S_k} r_i \sum_x P(U_0 = x|\mathcal{E}(\mathcal{H}_{k-1}))P(U_i \geq \max\{\max_{j \in S_k}\{U_j\}, x\}) \\
& \qquad + (1-\alpha_k) \sum_x P(U_0 = x|\mathcal{E}(\mathcal{H}_{k-1}))P(x \geq \max_{j \in S_k}\{U_j\})R_{k+1}(\mathcal{H}_k). \\
= & \max_{\mathbf{p}, S_k \subseteq N \setminus H_{k-1}} \sum_{i \in S_k} r_i P(U_i \geq \max_{j \in S_k}\{U_j\}, U_i > U_0|\mathcal{E}(\mathcal{H}_{k-1})) \\
& \qquad + (1-\alpha_k)P(U_0 \geq \max_{j \in S_k}\{U_j\}|\mathcal{E}(\mathcal{H}_{k-1}))R_{k+1}(\mathcal{H}_k).
\end{aligned}$$

After some algebra, we have

$$R_k(\mathcal{H}_{k-1}) = \max_{\mathbf{p}, S_k \cdots, S_K} \frac{1}{\pi_k} \sum_{m=k}^{K} \pi_m \sum_{i \in S_m} r_i P(U_i \geq \max_{j \in S_m}\{U_j\}, U_i > U_0|\mathcal{E}(\mathcal{H}_{m-1})).$$

If $\{\mathbf{p}, S_k^*, S_{k+1}^*, \cdots, S_K^*\}$ are the optimal recommendations for $R_k(\mathcal{H}_{k-1})$, then $\{\mathbf{p}_{-k}, S_{k+1}^*, \cdots, S_K^*\}$ are optimal for $R_{k+1}(\{\mathcal{H}_{k-1}, S_k^*\})$. Therefore we can only solve R_1 and a dynamic problem is equivalent to a static problem. ■

Derivation of Equation 2: Let $S \cap H = \varnothing$ and $i \in S$.

$$\begin{aligned}
& P(U_0 > \max_{j \in H} U_j,\, U_i > (U_0,\, \max_{j \in S,\, j \neq i} U_j)) \\
= & \int_{-\infty}^{\infty} P(U_0 > \max_{j \in H} U_j,\, U_i > (U_0,\, \max_{j \in S,\, j \neq i} U_j)|U_0 = s)f(s)ds \\
= & \int_{-\infty}^{\infty} P(s > \max_{j \in H} \mu_j + \epsilon_j)\, P(\mu_i + \epsilon_i > (s,\, \max_{j \in S,\, j \neq i} \mu_j + \epsilon_j))f(s)ds, \qquad (14)
\end{aligned}$$

where we have used the assumption that U_i are independent and $f(s)$ is the standard Gumbel density distribution. Now let us try to derive the probability of $P(\mu_i + \epsilon_i > (s, \max_{j \in S,\, j \neq i} \mu_j + \epsilon_j))$:

$$\begin{aligned}
P(\mu_i + \epsilon_i > (s, \max_{j \in S} \mu_j + \epsilon_j)) = & \int_{-\infty}^{\infty} P(\mu_i + \epsilon_i > (s,\, \max_{j \in S,\, j \neq i} \mu_j + \epsilon_j)|\epsilon_i = x)f(x)dx \\
= & \int_{s-\mu_i}^{\infty} P(\mu_i + x > \max_{j \in S,\, j \neq i} \mu_j + \epsilon_j)f(x)dx \\
= & \int_{s-\mu_i}^{\infty} \exp\Big(-\sum_{j \in S,\, j \neq i} e^{-(\mu_i - \mu_j + x)}\Big)\exp(-x)\exp\big(-e^{-x}\big)dx \\
= & \frac{1 - \exp\big(-\sum_{j \in S} e^{-s+\mu_j}\big)}{1 + \sum_{j \in S} \exp(\mu_j - \mu_i)}. \qquad (15)
\end{aligned}$$

Plug the equation (15) back into (14), we have

$$
\begin{aligned}
&P(U_0 > \max_{j\in H} U_j,\, U_i > (U_0,\, \max_{j\in S,\, j\neq i} U_j)) \\
&= \int_{-\infty}^{\infty} \exp\big(-\sum_{j\in H} e^{-s+\mu_j}\big) \frac{1-\exp\big(-\sum_{j\in S} e^{-s+\mu_j}\big)}{1+\sum_{j\in S_2}\exp(\mu_j-\mu_i)} exp(-s)\exp\big(-e^{-s}\big)ds \\
&= \Big(\frac{1}{1+\sum_{j\in H}\exp(\mu_j)} - \frac{1}{1+\sum_{j\in H}\exp(\mu_j)+\sum_{j\in S}\exp(\mu_j)}\Big)\frac{\exp(\mu_i)}{\sum_{j\in S_2}\exp(\mu_j)} \\
&= \frac{1}{1+\sum_{j\in S_1}\exp(\mu_j)}\frac{\exp(\mu_i)}{1+\sum_{j\in H}\exp(\mu_j)+\sum_{j\in S}\exp(\mu_j)} \\
&\equiv \frac{1}{1+\sum_{j\in H} v_j}\frac{v_i}{1+\sum_{j\in H} v_j+\sum_{j\in S} v_j}.
\end{aligned}
$$

As $P(U_i > U_0, U_i \geq \max_{j\in S_k} U_j, B_k) = \pi_k P(U_i > U_0, U_i \geq \max_{j\in H_k} \mu_j, U_0 > \max_{j\in H_{k-1}} U_j)$, we complete the proof. ■

Proof of Lemma 2: We first use a contradiction to prove part one. Suppose that at stage k there is a product with revenue r_1 and attractiveness δq_1, and at stage $k+1$, there is a product with revenue r_2 and attractiveness δq_2. Assume that $r_1 < r_2$. Denote $\delta q = \min\{\delta q_1, \delta q_2\}$. We will show that switching these two products with the same attractiveness δq will increase the revenue. After switching, the revenue extracted from products that are not switched will not change, therefore the revenue difference is as follows,

$$
\begin{aligned}
\Delta R =& \frac{\pi_k}{1+q_{k-1}}\frac{-r_1\delta q + r_2\delta q}{1+q_k} + \frac{\pi_{k+1}}{1+q_k}\frac{-r_2\delta q + r_1\delta q}{1+q_{k+1}} \\
=& (r_2-r_1)\delta q\Big[\frac{\pi_k}{1+q_{k-1}}\frac{1}{1+q_k} - \frac{\pi_{k+1}}{1+q_k}\frac{1}{1+q_{k+1}}\Big]
\end{aligned}
$$

As $\pi_k \geq \pi_{k+1}$ and $q_{k-1} \leq q_k \leq q_{k+1}$, $\Delta R \geq 0$. Therefore, we can always switch the products until that the total recommendation is sequentially revenue-ordered.

Next we show how to derive part two.

1) Q is a lattice on $\mathbb{R}^K$

Suppose $\mathbf{q}^1, \mathbf{q}^2 \in Q$. Then $q_i^1 \leq q_{i+1}^1$ and $q_i^2 \leq q_{i+1}^2$, from which we have $\max\{q_{i+1}^2, q_{i+1}^1\} \geq \max\{q_i^2, q_i^1\}$ and $\min\{q_{i+1}^2, q_{i+1}^1\} \geq \min\{q_i^2, q_i^1\}$. Accordingly $\mathbf{q}^1 \wedge \mathbf{q}^2 \in Q$ and $\mathbf{q}^1 \vee \mathbf{q}^2 \in Q$. Therefore Q is a lattice on $\mathbb{R}^K$.

2) $\frac{\partial^2 R}{\partial q_i \partial q_j} \geq 0$ for any $i \neq j$.

Denote the revenue earned from stage k as R^k, with

$$
R^k \triangleq \frac{\pi_k}{1+q_{k-1}}\frac{W(k,\mathbf{q})}{1+q_k},
$$

where $W(k,\mathbf{q}) = \sum_{i\in S_k} v_i r_i x_{ik}$. Due to part one , both R^k and $W(k,\mathbf{q})$ will only depend on q_{k-1} and q_k. Therefore, if $|k-k'| > 1$, $\frac{\partial^2 R^l}{\partial q_k \partial q_{k'}} = 0$ for all l and thus $\frac{\partial^2 R}{\partial q_k \partial q_{k'}} = 0$ where $R = \sum_k R^k$.

For ease of exposition, denote $W(q_{k-1}, q_k) \triangleq W(k,\mathbf{q})$. Let $\delta q_{k-1} \to 0$ and $\delta q_k \to 0$. Then

$W(q_{k-1}+\delta q_{k-1}, q_k+\delta q_k) = W(q_{k-1}, q_k) - r_a \delta q_{k-1} + r_b \delta q_k$, where due to Lemma **??**,

$$r_a = \begin{cases} \max_{i\in S_k} r_i, & \delta q_{k-1} \geq 0 \\ \min_{i\in S_{k-1}} r_i, & otherwise \end{cases} \text{ and } r_b = \begin{cases} \max_{i\in S_{k+1}} r_i, & \delta q_k \geq 0 \\ \min_{i\in S_k} r_i, & otherwise \end{cases}.$$

Moreover, due to part one,

$$r_a \geq \frac{W(q_{k-1}, q_k)}{q_k - q_{k-1}} \geq r_b.$$

As a result,

$$\begin{aligned} \frac{\partial^2 R}{\partial q_{k-1}\partial q_k} = \frac{\partial^2 R^k}{\partial q_{k-1}\partial q_k} &= \frac{\partial^2 R^k(q_{k-1}+\delta q_{k-1}, q_k+\delta q_k)}{\partial \delta q_{k-1}\partial \delta q_k}|_{\delta q_{k-1}=0,\delta q_k=0} \\ &= \frac{(1+q_{k-1})r_a - (1+q_k)r_b + W(q_{k-1},q_k)}{(1+q_{k-1})^2(1+q_k)^2} \\ &\geq \frac{(1+q_{k-1})\frac{W(q_{k-1},q_k)}{q_k-q_{k-1}} - (1+q_k)\frac{W(q_{k-1},q_k)}{q_k-q_{k-1}} + W(q_{k-1},q_k)}{(1+q_{k-1})^2(1+q_k)^2} = 0. \end{aligned}$$

As a result, $R(q_1, \cdots, q_K)$ is supermodular for $(q_1, \cdots, q_K) \in Q$. ■

Proof of Lemma 3: The revenue function can be written as

$$R(\mathbf{q}) = \cdots + \frac{\pi_k}{1+q_{k-1}} \frac{W(q_{k-1},q_k)}{1+q_k} + \frac{\pi_{k+1}}{1+q_k} \frac{W(q_k,q_{k+1})}{1+q_{k+1}} + \cdots.$$

Suppose that at optimality, products with the same revenue, r, are included in S_k and S_{k+1} and the optimal solution.is $\{q_1^*, \cdots, q_k^*, \cdots, q_K^*\}$. Now let $q_k = q_k^* + \delta q$, but keep other $q^*_{k'\neq k}$ unchanged. Let δq be small enough such that we only move the products with the revenue as r. Clearly, $W(q_{k-1}^*, q_k^*+\delta q) = W(q_{k-1}^*, q_k^*) + \delta q r$ and $W(q_k^*+\delta q, q_{k+1}^*) = W(q_k^*, q_{k+1}^*) - \delta q r$. Then

$$\begin{aligned} &\frac{\partial R(q_1^*, \cdots, q_k^*+\delta q, \cdots, q_K^*)}{\partial \delta q} \\ =&\frac{\pi_k(1+q_{k+1}^*)(r+rq_k^* - W(q_{k-1}^*, q_k^*)) - \pi_{k+1}(1+q_{k-1}^*)(r+rq_k^*+W(q_k^*,q_{k+1}^*))}{(1+q_{k-1}^*)(1+q_{k+1}^*)(1+q_k^*+\delta q)^2}. \end{aligned} \tag{16}$$

Since the optimal solution.is $\{q_1^*, \cdots, q_k^*, \cdots, q_K^*\}$, by first-order condition, we must have

$$\frac{\partial R(q_1^*, \cdots, q_k^*+\delta q, \cdots, q_K^*)}{\partial \delta q}|_{\delta q=0} = 0,$$

which implies the quantity in (16) is also 0 as its numerator does not depend on δq. Then $q_k^* + \delta q$ for small δq will still be optimal. Thus we have the lemma. ■

Alternative proof of the strongly sequentially revenue-ordered property. : Apart from the continuous relaxation, we provide the following fixed point approach to prove that: at optimality $\min\{i \in S_k\} r_i > \max\{i \in S_{k+1}\} r_i$, $\forall k \in \{1, \cdots, K\}$, where $S_{K+1} = N \setminus \cup_{k=1}^K S_k$. **I.** One stage

When there is only one stage, the seller's problem is identical to that in a standard MNL model, the optimal assortment is well-known to be revenue ordered.

II. Two stages

For each available product, the associated decision variables are x_{1i} and x_{2i}, both of which are binary denoting whether to be recommended in the first and second stage, respectively. Then the seller's problem is as follows

$$\begin{aligned} \max_x \quad & \frac{\sum_i r_i v_i x_{1i}}{1+\sum_i v_i x_{1i}} + \frac{\pi_2}{1+\sum_i v_i x_{1i}} \frac{\sum_i r_i v_i x_{2i}}{1+\sum_i v_i (x_{1i}+x_{2i})} \\ \text{subject to} \quad & x_{1i}+x_{2i} \le 1, \\ & x_{1i}, x_{2i} \in \{0,1\}, \end{aligned}$$

where we have assumed $\pi_1 = 1$.

Let the optimal objective to the seller's problem under two stages be z, then for all x,

$$z \ge \frac{\sum_i r_i v_i x_{1i}}{1+\sum_i v_i x_{1i}} + \frac{\pi_2}{1+\sum_i v_i x_{1i}} \frac{\sum_i r_i v_i x_{2i}}{1+\sum_i v_i (x_{1i}+x_{2i})}.$$

Multiplying both sides by $(1+\sum_i v_i x_{1i})(1+\sum_i v_i(x_{1i}+x_{2i}))$, and rearranging the terms yields

$$z = \max_x \sum_i \{r_i(1+\sum_i v_i(x_{1i}+x_{2i})) - z(2+\sum_i v_i(x_{1i}+x_{2i}))\} v_i x_{1i} + \sum_i (\pi_2 r_i - z) v_i x_{2i}.$$

The LHS is increasing in z, and RHS is decreasing in z. The optimal revenue z should be the unique fixed point of the equation above.

Step 1: At optimality $\min_{i\in S_1} r_i > \max_{i \in S_2} r_i$.

Suppose we already known the union of the optimal recommendation set as $S^* = S_1^* \cup S_2^*$, with total utility as $V \triangleq \sum_{i\in S^*} v_i$. Then

$$z = \max_x \sum_i [r_i(1+V) - z(2+V)] v_i x_{1i} + \sum_i (\pi_2 r_i - z) v_i x_{2i}.$$

As a result, product i in S^* should be put on the first stage when it satisfies

$$0 \le r_i(1+V) - z(2+V) - (\pi_2 r_i - z) = r_i(1-\pi_2+V) - (1+V)z.$$

The last expression is increasing in r_i. Therefore, the products with several largest price in S^* will be put on the first stage.

Step 2: At optimality $\min_{i\in S_2} r_i > \max_{i\in N\setminus(S_1\cup S_2)} r_i$.

It is not hard to prove it as when S_1^* is known, the remained assortment is just that of one-stage problem.

Then by these two claims, the property of optimal assortment for the two stages satisfies the revenue-ordered property.

II. K stages

Clearly, it is right for $K=1$ and $K=2$. Now consider the case that $K>2$.

Suppose that we have already known $S_1^*, \cdots. S_{m-1}^*, S_{m+2}^*, \cdots, S_K^*$ and $S_m^* \cup S_{m+1}^*$. We want to figure our how to partition $S_m^* \cup S_{m+1}^*$ for assortments at stage m and stage $m+1$. Note that no matter how we recommend the products at stage m and $m+1$, given $S_1^*, \cdots, S_{m-1}^*$, the revenue earned before stage m, $R_{<m}$, will be same. Let $a_0 = \sum_{i\in \cup_{k=1}^{m-1} S_k^*} v_i$ and $b_k = \sum_{i \in \cup_{l=m+2}^{k} S_l^*} v_i$ for $k > m+2$. If we recommend S_m at stage m and S_{m+1} at stage $m+1$ with $S_m \cap S_{m+1} = \varnothing$, the revenue earned from stage m and $m+1$ will be

$$R_{\{m,m+1\}}(S_m, S_{m+1}) = \frac{\pi_m}{1+a_0} \frac{\sum_{i\in S_m} r_i v_i}{1+a_0+\sum_{i\in S_m} v_i} + \frac{\pi_{m+1}}{1+a_0+\sum_{i\in S_m} v_i} \frac{\sum_{i\in S_{m+1}} r_i v_i}{1+a_0+\sum_{i\in S_m\cup S_{m+1}} v_i};$$

the revenue earned after $m+1$ will be

$$R_{>m+1}(S_m, S_{m+1}) = \sum_{k=m+2}^{K} \frac{\pi_k}{1+a_0+b_{k-1}+\sum_{j\in S_m\cup S_{m+1}} v_j} \frac{\sum_{i\in S_k^*} v_i r_i}{1+a_0+b_k+\sum_{j\in S_m\cup S_{m+1}} v_j}.$$

Consider the following problem,

$$\begin{aligned} \{\tilde{S}_m^*, \tilde{S}_{m+1}^*\} = & \operatorname*{argmax}_{S_m, S_{m+1}} R_{\{m,m+1\}}(S_m, S_{m+1}), \\ \text{subject to} \quad & S_m \cap S_{m+1} = \varnothing, \\ & S_m, S_{m+1} \subseteq S_m^* \cup S_{m+1}^*. \end{aligned}$$

Step 1: $\min_{i\in\tilde{S}_m^*} r_i > \max_{i\in\tilde{S}_{m+1}^*} r_i$.

It is true as shown in the two-stage problem.

Step 2: $\tilde{S}_m^* \cup \tilde{S}_{m+1}^* = S_m^* \cup S_{m+1}^*$.

Supposing it is not true, $R_{\{m,m+1\}}(\tilde{S}_m^*, \tilde{S}_{m+1}^*) > R_{\{m,m+1\}}(S_m^*, S_{m+1}^*)$ and $R_{\{>m+1\}}(\tilde{S}_m^*, \tilde{S}_{m+1}^*) > R_{\{>m+1\}}(S_m^*, S_{m+1}^*)$, where the first inequality is obvious and the second inequality is due to that the purchase probability of each product after stage $m+1$ increases. As a result, if the claim is not true, then

$$R(S_1^*, \cdots, S_{m-1}^*, \tilde{S}_m^*, \tilde{S}_{m+1}^*, S_{m+2}^*, \cdots, S_K^*) > R(S_1^*, \cdots, S_{m-1}^*, S_m^*, S_{m+1}^*, S_{m+2}^*, \cdots, S_K^*),$$

leading to a contradiction.

Step 3: $\tilde{S}_m^* = S_m^*$ and $\tilde{S}_{m+1}^* = S_{m+1}^*$.

Since when $S_m^* \cup S_{m+1}^*$ is known, $R_{<m}$ and $R_{>m+1}$ are fixed, the seller's problem is boiled down to maxmize $R_{\{m,m+1\}}(S_m, S_{m+1})$ subject to $S_m \cup S_{m+1} = S_m^* \cup S_{m+1}^*$. As $R(\tilde{S}_m^*, \tilde{S}_{m+1}^*) \geq R_{\{m,m+1\}}(S_m, S_{m+1})$ for all $S_m \cup S_{m+1} \subseteq S_m^* \cup S_{m+1}^*$, then we have the claim.

As a result, the revenues of products at stage m should be larger than that of any product at stage $m+1$. By induction, we have $\min_{i\in S_k} r_i > \max_{i\in S_{k+1}} r_i$ for any $k < K$. As shown in the one stage problem, $\min_{i\in S_K} r_i > \max_{i\in N\setminus\cup_{k=1}^K S_k} r_i$. As a result, we complete the proof. ■

Proof of Theorem 1: We prove a general claim for the algorithm: this algorithm always gives the best one among the strongly sequentially revenue-ordered assortment, if the choice probability satisfies: for any k and two assortments (not necessarily revenue ordered), S and $\tilde{S}$, if $\cup_{m=1}^{k-1} S_m = \cup_{m=1}^{k-1} \tilde{S}_m$ and $S_k = \tilde{S}_k$, then $P_i(S) = P_i(\tilde{S})$ for any $i \in S_k$ or $\tilde{S}_k$. This condition means the recommendation order of products recommended earlier does not affect the purchase probability of products recommended latter .

Different strongly sequentially revenue-ordered assortment can be characterized by the revenue threshold at different stage. Therefore, to find the best one, we only need to find a list of revenue thresholds. While, with the condition mentioned, if fixing the threshold of stage k, to find the best strongly sequentially revenue-ordered assortment can be divided into two problems: find the revenue thresholds of stages before k and these after k.

The algorithm uses a forward dynamic programming and the best strongly sequentially revenue-ordered assortment must be included in $\{A_K^j\}$. This is because, the best threshold of first stage must be included in $\{A_1^j\}$. Supposing the best threshold of second stage is r_i, then r_l must be the best threshold of first stage, where $l = \mathrm{argmax}_{1\le j\le i}\,\mathcal{R}(A_1^j \cup \underbrace{\{r_i,\cdots,r_i\}}_{\text{K-1 elements}})$. By induction, we have the best assortment among strongly sequentially revenue-ordered assortments. It is not hard to find that the time complexity is $\mathcal{O}(n^3)$. ■

Derivation of optimal revenue when $K\to\infty$ and $\pi_i = 1$ for all i. : We have shown that the optimal solution is revenue-ordered. Clearly, the seller will recommend all products, when $K\to\infty$ and $\pi_1 = \cdots = \pi_k = \cdots = 1$. Now suppose there are two products are included at the same stage, with revenue as r_m and r_{m+1}. The revenue extracted by these two products by dividing them into two stages subtracted by putting them at the same stage will be

$$\begin{aligned}\Delta R =&\frac{1}{1+V_0}\frac{v_m r_m}{1+V_0+v_m}+\frac{1}{1+V_0+v_m}\frac{v_{m+1}r_{m+1}}{1+V_0+v_m+v_{m+1}}-\frac{1}{1+V_0}\frac{v_m r_m+v_{m+1}r_{m+1}}{1+V_0+v_m+v_{m+1}}\\ =&\frac{v_m v_{m+1}(r_m-r_{m+1})}{(1+V_0+v_m+v_{m+1})(1+V_0)(1+V_0+v_m)(1+V_0+v_m+v_{m+1})},\end{aligned}$$

where $V_0=\sum_{i:r_i\le r_m} v_i$. Therefore, if $r_m \ge r_{m+1}$, by dividing them into two stages, the revenue earned from these two products will not decrease. As the revenue extracted by other products does not change, putting only one product on one stage is also optimal. If there are multiple products on the same stage, we treat them as two synsized products, and the result is same. Thus we prove the claim. ■

Proof of Proposition 1 : Let $\tilde n$ be the index of product with the lowest price in the optimal solution to problem (6).

Step 1: For all $m\le \tilde n$, $\frac{\sum_{i\le m} r_i v_i}{1+\sum_{i\le m} v_i} \ge \frac{1+\sum_{i\le m} v_i}{1+2\sum_{i\le m} v_i}\sum_{i\le m}\frac{1}{1+\sum_{j<i} v_j}\frac{v_i r_i}{1+\sum_{j\le i} v_j}$.
For $m=1$, it holds trivially.
Now suppose it holds for some m with $m+1\le\tilde n$. Then

$$\begin{aligned}&\frac{1+\sum_{i\le m+1} v_i}{1+2\sum_{i\le m+1} v_i}\sum_{i\le m+1}\frac{1}{1+\sum_{j<i} v_j}\frac{v_i r_i}{1+\sum_{j\le i} v_j}\\ =&\frac{1+\sum_{i\le m+1} v_i}{1+2\sum_{i\le m+1} v_i}[\sum_{i\le m}\frac{1}{1+\sum_{j<i} v_j}\frac{v_i r_i}{1+\sum_{j\le i} v_j}+\frac{1}{1+\sum_{j<m+1} v_i}\frac{v_{m+1}r_{m+1}}{1+\sum_{j\le m+1} v_j}]\\ \le&\frac{1+\sum_{i\le m+1} v_i}{1+2\sum_{i\le m+1} v_i}[\frac{1+2\sum_{i\le m} v_i}{1+\sum_{i\le m} v_i}\frac{\sum_{i\le m} r_i v_i}{1+\sum_{i\le m} v_i}+\frac{1}{1+\sum_{j<m+1} v_i}\frac{v_{m+1}r_{m+1}}{1+\sum_{j\le m+1} v_j}]\\ =&\frac{1+V_1+V_2}{1+2V_1+2V_2}[\frac{1+2V_1}{1+V_1}\frac{V_1 y_1}{1+V_1}+\frac{1}{1+V_1}\frac{V_2 y_2}{1+V_1+V_2}],\end{aligned}\tag{17}$$

where the inequality is due to that the the claim holds for m, $V_1=\sum_{i\le m} v_i$, $V_2 = v_{m+1}$, $y_1 = \frac{\sum_{i\le m} r_i v_i}{\sum_{i\le m} v_i}$ and $y_2 = r_{m+1}$. Note that As product $m+1$ is included in the assortment for problem

(6), then we have that $y_2 = r_{m+1} \geq \tilde{r} \geq \frac{\sum_{i\leq m} r_i v_i}{1+\sum_{i\leq m} v_i} = \frac{V_1 y_1}{1+V_1}$. Then

$$
\begin{aligned}
&\frac{\sum_{i\leq m+1} r_i v_i}{1+\sum_{i\leq m+1} v_i} - \frac{1+\sum_{i\leq m+1} v_i}{1+2\sum_{i\leq m+1} v_i} \sum_{i\leq m+1} \frac{1}{1+\sum_{j<i} v_j} \frac{v_i r_i}{1+\sum_{j\leq i} v_j} \\
\geq & \frac{V_1 y_1 + V_2 y_2}{1+V_1+V_2} - \frac{1+V_1+V_2}{1+2V_1+2V_2}\left[\frac{1+2V_1}{1+V_1}\frac{V_1 y_1}{1+V_1} + \frac{1}{1+V_1}\frac{V_2 y_2}{1+V_1+V_2}\right] \\
= & \frac{V_1 y_1}{1+V_1+V_2} - \frac{1+V_1+V_2}{1+2V_1+2V_2}\frac{1+2V_1}{1+V_1}\frac{V_1 y_1}{1+V_1} + \frac{V_2 y_2}{1+V_1+V_2} - \frac{V_2 y_2}{1+2V_1+2V_2}\frac{1}{1+V_1} \\
\geq & \frac{V_1 y_1}{1+V_1+V_2} - \frac{1+V_1+V_2}{1+2V_1+2V_2}\frac{1+2V_1}{1+V_1}\frac{V_1 y_1}{1+V_1} + \frac{V_2 \frac{V_1 y_1}{1+V_1}}{1+V_1+V_2} - \frac{V_2 \frac{V_1 y_1}{1+V_1}}{1+2V_1+2V_2}\frac{1}{1+V_1} \\
= & 0.
\end{aligned}
$$

where the first inequality is due to the inequality in (17) and the second inequality is due to $y_2 \geq \frac{V_1 y_1}{1+V_1}$. As a result, the claim holds.

Step 2: $\tilde{R} \leq 2\tilde{r}$.

Suppose $K \to \infty$ and $\pi_1 = \cdots = \pi_k = \cdots = 1$. Then the revenue of problem (3) after the $\tilde{n}$-th stage will be

$$R_{>\tilde{n}} = \sum_{i=\tilde{n}+1} \frac{1}{1+V+\sum_{j=\tilde{n}+1}^{i-1} v_j} \frac{v_i r_i}{1+V+\sum_{j=\tilde{n}+1}^{i} v_j},$$

where $V = \sum_{i\leq \tilde{n}} v_i$. Clearly $R_{\tilde{n}}$ is increasing in $r_{i\geq \tilde{n}}$. Note that $r_{i>\tilde{n}} < \tilde{r}$. If set $r_{i>\tilde{n}} \equiv \tilde{r}$ then $R_{>\tilde{n}}$ is maximized over the space $\{r_{i>\tilde{n}} < \tilde{r}\}$. In such a way, as proved in 3, at optimality, all products with index $i > \tilde{n}$ can be put at stage $\tilde{n}+1$, and

$$R_{>\tilde{n}} = \frac{1}{1+V} \frac{\tilde{r}\sum_{i>\tilde{n}} v_i}{1+V+\sum_{i>\tilde{n}} v_i}.$$

Let $\sum_{i>\tilde{n}} v_i \to \infty$, then the maximal value of $R_{>\tilde{n}}$ is achieved as $\frac{\tilde{r}}{1+V}$. Then the maximal revenue of problem (3) is as follows

$$\tilde{R} = \sum_{i\leq \tilde{n}} \frac{1}{1+\sum_{j=1}^{i-1} v_j} \frac{v_i r_i}{1+\sum_{j=1}^{i} v_j} + \frac{\tilde{r}}{1+\sum_{j=1}^{\tilde{n}} v_j}.$$

By Step 1,

$$\sum_{i\leq \tilde{n}} \frac{1}{1+\sum_{j=1}^{i-1} v_j} \frac{v_i r_i}{1+\sum_{j=1}^{i} v_j} + \frac{\tilde{r}}{1+\sum_{j=1}^{\tilde{n}} v_j} \leq \frac{(1+2\sum_{i\leq \tilde{n}} v_i)}{1+\sum_{i\leq \tilde{n}} v_i}\tilde{r} + \frac{\tilde{r}}{1+\sum_{j=1}^{\tilde{n}} v_j} = 2\tilde{r}.$$

Step 3: Tight bound example.

Suppose that there are two products and $r_2 = \frac{v_1 r_1}{1+v_1}$. Then $\tilde{r} = r_2 = \frac{v_1 r_1}{1+v_1}$ and $\tilde{R} = \frac{v_1 r_1}{1+v_1} + \frac{1}{1+v_1}\frac{v_2 r_2}{1+v_1+v_2}$. As a result, we have

$$\frac{\tilde{R}}{\tilde{r}} = \frac{\frac{v_1 r_1}{1+v_1} + \frac{1}{1+v_1}\frac{v_2 r_2}{1+v_1+v_2}}{\tilde{r}} = \frac{\tilde{r} + \frac{1}{1+v_1}\frac{v_2 \tilde{r}}{1+v_1+v_2}}{\tilde{r}} = 1 + \frac{1}{1+v_1}\frac{v_2}{1+v_1+v_2}.$$

When $v_1 \to 0$ and $v_2 \to \infty$, we have $\frac{\tilde{R}}{\tilde{r}} \to 2$. ∎

Proof of Proposition 2: First, introduce a notation as follows

$$
\begin{array}{rll}
R(\mathbf{q}; z) \triangleq & \max_{\mathbf{x}} & \sum_{k=1}^{K} \frac{\pi_k}{1+q_{k-1}} \frac{\sum_{i \in S_k} v_i (r_i + z) x_{ik}}{1+q_k} \\
& \text{subject to} & \sum_{i=1}^{k} \sum_{j \in S_i} x_{ji} v_j = q_k, \quad 1 \le k \le K, \\
& & \sum_{i=1}^{K} x_{ji} \le 1, j \in N, \\
& & x_{ji} \ge 0.
\end{array}
$$

It is easy to check that $R(\mathbf{q}; z_2) = R(\mathbf{q}; z_1) + U(\mathbf{q})(z_2 - z_1)$, where $U(\mathbf{q}) = \sum_{k=1}^{K} \frac{(\pi_k - \pi_{k+1}) q_k}{1+q_k} \ge 0$ is increasing in each component of $\mathbf{q}$. Define $\mathbf{q}^*(z) \triangleq \mathrm{argmax}_{\mathbf{q}} \{R(\mathbf{q}; z)\}$. Then for any $\mathbf{q} \in Q$, $0 \le U(\mathbf{q}) \le U(\mathbf{q}^*(z_1) \vee \mathbf{q})$ and due to Lemma 2,

$$R(\mathbf{q}; z_1) + R(\mathbf{q}^*(z_1); z_1) \le R(\mathbf{q}^*(z_1) \wedge \mathbf{q}; z_1) + R(\mathbf{q}^*(z_1) \vee \mathbf{q}; z_1).$$

As $R(\mathbf{q}^*(z_1) \wedge \mathbf{q}; z_1) \le R(\mathbf{q}^*(z_1); z_1)$, $R(\mathbf{q}; z_1) \le R(\mathbf{q}^*(z_1) \vee \mathbf{q}; z_1)$. Therefore, if $z_2 > z_1$, we have

$$R(\mathbf{q}; z_2) = R(\mathbf{q}; z_1) + U(\mathbf{q})(z_2 - z_1) \le R(\mathbf{q}^*(z_1) \vee \mathbf{q}; z_1) + U(\mathbf{q}^*(z_1) \vee \mathbf{q})(z_2 - z_1) = R(\mathbf{q}^*(z_1) \vee \mathbf{q}; z_2).$$

As a result, $\mathbf{q}_j^*(z_2) \ge \mathbf{q}_j^*(z_1)$, for any $j \in \{1, \cdots, K\}$. The corresponding recommendations of $\mathbf{q}_j^*(z_1)$ and $\mathbf{q}_j^*(z_2)$ may have products fractionally put in few sequential sets of $(S_1, \cdots, S_{K+1})$. By Lemma 3, we can put all these products in the last (first) set for $z_1(z_2)$ that containing them, which will still be optimal. As a result, there exist efficient sets, which satisfy $\sum_{k=1}^{j} \sum_{i \in S_i^*(z_2)} v_i = \mathbf{q}_j^*(z_2) \ge \mathbf{q}_j^*(z_1) = \sum_{k=1}^{j} \sum_{i \in S_i^*(z_1)} v_i$. As the optimal recommendation is strongly sequentially revenue-ordered, we complete the proof. ■

Proof of Proposition 3: Let r_i and r_j be the markups of any two products at the same stage. If fixing $v_i e^{-r_i} + v_j e^{-r_j}$, then the optimal markup for other products will keep as the same. Moreover, when $v_i e^{-r_i} + v_j e^{-r_j}$ is known, the revenue from these two products are proportional to $v_i e^{-r_i} r_i + v_j e^{-r_j} r_j$. Let $r_i = r - \delta$ and $r_j = r + \delta$, if $v_i e^{-r_i} + v_j e^{-r_j} = C$, then $r = \ln \frac{v_j e^{-\delta} + v_i e^{\delta}}{C}$. Therefore, fixing $v_i e^{-r_i} + v_j e^{-r_j} = C$, the pricing problem for product i and j is equivalent to maximize $\Pi(\delta) \triangleq v_i e^{-r+\delta}(r - \delta) + v_j e^{-r-\delta}(r + \delta)$ over d, with $r = \ln \frac{v_j e^{-\delta} + v_i e^{\delta}}{C}$. By the first order condition, namely $\frac{d\Pi(\delta)}{d\delta} = -\frac{4C v_i v_j e^{2\delta}}{(v_j + v_i e^{2\delta})^2} \delta = 0$, we have $\delta^* = 0$. That is, for any two products in the same stage, the optimal markups should be same. Then we have the claim. ■

Proof of Proposition 4: For ease of exposition, denote $V_k \triangleq \sum_{j \in S_k} v_j$, and $q_k \triangleq \sum_{i=1}^{k} V_i e^{-r^{(i)}}$ as the total attractiveness of products the first k stages when the revenue of stage i is $r^{(i)}$. By default, $q_0 = 0$. Note that from the definition of q, we will have

$$r^{(k)} = \ln V_k - \ln(q_k - q_{k-1}).$$

Therefore, there is one-to-one correspondence between q_k and $r^{(k)}$. Now we can reformulate the pure pricing problem as follows,

$$\max_{\mathbf{q}} \quad \{R(\mathbf{q}) = \sum_{k=1}^{K} \frac{\pi_k}{1 + q_{k-1}} \frac{(q_k - q_{k-1})(\ln V_k - \ln(q_k - q_{k-1}))}{1 + q_k}\}.$$

We denote $\Delta_k(\mathbf{q}) \triangleq (1+q_k)^2 \frac{\partial R(\mathbf{q})}{\partial q_k}$, then

$$\Delta_k(\mathbf{q}) = \begin{cases} \begin{array}{l} \pi_k(\ln V_k - \frac{1+q_k}{1+q_{k-1}} - \ln(q_k - q_{k-1})) \\ -\pi_{k+1}(\ln V_{k+1} - \frac{1+q_k}{1+q_{k+1}} - \ln(q_{k+1} - q_k)) \end{array}, & k < K \\ \pi_K(\ln V_K - \frac{1+q_K}{1+q_{K-1}} - \ln(q_K - q_{K-1})), & k = K \end{cases}. \tag{18}$$

From the first-order condition of optimization, $\Delta_k(\mathbf{q}) = 0$ for all k. As $r^{(k)} = \ln V_k - \ln(q_k - q_{k-1})$, the FOC could be written as,

$$r^{(K)} = \frac{1+q_K}{1+q_{K-1}} \text{ and } \pi_k r^{(k)} - \pi_k \frac{1+q_k}{1+q_{k-1}} = \pi_{k+1} r^{(k+1)} - \pi_{k+1} \frac{1+q_k}{1+q_{k+1}}, \forall k < K. \tag{19}$$

As $\frac{1+q_k}{1+q_{k-1}} > 1 > \frac{1+q_k}{1+q_{k+1}}$, we can easily prove the claim. ■

Proof of Theorem 2. We prove this claim by continuing the arguments in the proof of Proposition 4. To further simplify the expression, denote $t_k \triangleq (1+q_k)/(1+q_{k-1})$. Then $1+q_k = \Pi_{i=1}^k t_k$. Therefore, there is one-to-one correspondence between q_k and t_k. Then from (18), we have

$$\Delta_k(\mathbf{t}) \triangleq \begin{cases} \begin{array}{l} \pi_k(\ln V_k - t_k - \ln(t_k - 1) - \sum_{i=1}^{k-1} \ln t_i) \\ -\pi_{k+1}(\ln V_{k+1} - \frac{1}{t_{k+1}} - \ln(t_{k+1} - 1) - \sum_{i=1}^{k} \ln t_i) \end{array}, & k < K \\ \pi_K(\ln V_K - t_K - \ln(t_K - 1) - \sum_{i=1}^{K-1} \ln t_i), & k = K \end{cases}. \tag{20}$$

where we have used that $\ln(q_k - q_{k-1}) = \ln(1 + q_k - (1+q_{k-1})) = \ln(\frac{1+q_k}{1+q_{k-1}} - 1) + \ln(1+q_{k-1}) = \ln(t_k - 1) + \sum_{i=1}^{k-1} \ln t_i$. Note that as q_k is increasing in k, $t_k > 1$. Then as $t_k > 1$ and $\pi_k \geq \pi_{k+1}$, it is not hard to find that $\Delta_k(\mathbf{t})$ is decreasing in $t_{i \leq k}$, increasing in t_{k+1} and does not depend on $t_{i > k+1}$.

Note that at optimality, $\Delta_k(\mathbf{t}) = 0$ for all k. Then we can assign a number to t_1, then by $\Delta_1(\mathbf{t}) = 0$, we can solve t_2, which will be unique as $\Delta_1(\mathbf{t})$ is monotone in t_2. By induction, with $\{t_1, \cdots, t_k\}$, we can use $\Delta_k(\mathbf{t}) = 0$ to solve the unique t_{k+1}. At last, we can check whether $\Delta_K(\mathbf{t}) = 0$ to verify whether this is a solution.

From the monotonicity of $\Delta_k(\mathbf{t})$, when the assigned value to t_1 becomes larger, then all the other t_k will become larger by $\Delta_k(\mathbf{t}) = 0$ for $k < K$. As a result, $\Delta_K(t_1, t_2(t_1), \cdots, t_K(t_1))$ will be decreasing in t_1. Therefore, if exists, there will be a unique t_1 satisfies $\Delta_k(\mathbf{t}) = 0$ for all k. That is there is at most one solution satisfying the first-order condition. Since at optimality, r_i will be finite, it implies there is a global finite maximizer of q for $\mathrm{R}(\mathbf{q})$. As the first-order conditions have at most one solution, then it is global maximizer. Thus by a dichotomic search, we can find a unique $t_1 \geq 1$ satisfies $\Delta_k(\mathbf{t}) = 0$ for all k, by which we can derive the optimal pricing policy. ■

Proof of Theorem 3: Define

$$F(x) \triangleq \max_{r_1, r_2} \frac{r_1 e^{-r_1} x}{1 + e^{-r_1} x} + \frac{1}{1 + e^{-r_1} x} \frac{r_2 e^{-r_2}(1-x)}{1 + e^{-r_1} x + e^{-r_2}(1-x)},$$

and let $x^* \triangleq \mathrm{argmax}_{0 \leq x \leq 1} F(x)$.

Step 1: x^* is unique.

Introducing $t_1 = 1 + e^{-r_1}x$ and $t_2 = \frac{1+e^{-r_1}x+e^{-r_2}(1-x)}{1+e^{-r_1}x}$, then $t_1, t_2 > 1$ and $r_1 = \ln x - \ln(t_1 - 1)$ and $r_2 = \ln(1-x) - \ln t_1 - \ln(t_2 - 1)$. Therefore there is one-to-one correspondence between $\{t_1, t_2\}$ and $\{r_1, r_2\}$. Now we can express $F(x)$ as follows

$$F(x) = \max_{t_1,t_2} \frac{(t_1 - 1)[\ln x - \ln(t_1 - 1)]}{t_1} + \frac{(t_2 - 1)[\ln(1-x) - \ln t_1 - \ln(t_2 - 1)]}{t_1 t_2}.$$

At optimality, by the FOC, similar to (20), we have

$$\begin{aligned} 0 &= \ln \frac{x}{1-x} - t_1 - \ln(t_1 - 1) + \frac{1}{t_2} + \ln(t_2 - 1) + \ln t_1, \\ 0 &= \ln(1-x) - t_2 - \ln(t_2 - 1) - \ln t_1. \end{aligned}$$

Besides when $x = x^*$, due to the envelop theorem, we have

$$0 = \frac{dF(x)}{dx}|_{x=x^*} = \frac{1}{x^*}\frac{t_1 - 1}{t_1} - \frac{1}{1-x^*}\frac{t_2 - 1}{t_1 t_2}.$$

Combining these three conditions, we have

$$\begin{aligned} x^* &= \frac{t_2(t_1 - 1)}{t_1 t_2 - 1}, \\ 0 &= \ln t_1 - t_1 + \ln t_2 + \frac{1}{t_2}, \\ 0 &= t_2 + \ln t_1 + \ln(t_1 t_2 - 1). \end{aligned}$$

As $t_1, t_2 > 1$, it is not hard to prove that the RHS of the second equation is decreasing in t_1 and increasing in t_2. As a result, if we give an input to $t_1 > 1$, then a $t_2 > 1$ will be uniquely determined by the the second equation. Besides, when the input of t_1 increases, the derived t_2 will increase. However, the third equation should also be satisfied. As the RHS of the third equation is increasing both in t_1 and t_2, there are unique t_1 and t_2 satisfying both the second and third equations. As a result, from the first equation, x^* will be unique. By numerical calculation, we can find that $x^* \approx 0.520806$.

Step 2: NP-hardness by reduction from 2-PARTITION problem.

Now we prove the hardness of the joint problem by reducing the well-known 2-PARTITION problem to a special case of our model. The 2-PARTITION problem is defined as follows.

Definition 2. *Given as set of n non-negative rational numbers $w_1, \cdots, w_n$ with $\sum_i w_i = 1$, determine whether there is a set $S \subseteq \{1, \cdots, n\}$ such that $\sum_{i \in S} w_i = 1/2$.*

We now reduce the above 2-PARTITION problem into an instance of the joint problem.

Reduction 1. *Our reduction works as follows:*

- *Let $K = 2$, $\pi_1 = \pi_2 = 1$ and the attractiveness of the no-purchase option be 1.*
- *The seller has a universal set of products, $N = \{1, \cdots, n+2\}$. The intrinsic attractiveness of products are as follows: $v_i = 0.02 w_i$, for $i \leq n$, $v_{n+1} = x^* - 0.01$ and $v_{n+2} = 0.99 - x^*$.*

Under this setting, given the total attractiveness on the first stage as x, then the seller's revenue will just be same as $F(x)$. Therefore to determine whether the seller's revenue can be $F(x^*)$, is to find out whether there is a set $S \subseteq N$ such that $\sum_{i \in S} v_i = x^*$. First, notice that $\sum_{i \neq n+1} v_i < x^*$, then if $\sum_{i \in S} v_i = x_*$, product $n+1$ must be included in S. Second, notice that $v_{n+1} + v_{n+2} > x^*$, then if $\sum_{i \in S} v_i = x_*$, as product $n+1$ will be included, product $n+2$ can not lie in S. Now we have that $\max_x R(x) = F(x^*)$ if and only if there is a set $\tilde{S} \subseteq \{1, 2, \cdots, n\}$ such that $\sum_{i \in \tilde{S}} v_i = 0.01$. That is, $\max_x R(x) = F(x^*)$ if and only if the 2-PARTITION problem has a solution. It implies that the joint problem is NP-hard. ■

Proof of Lemma 4: Similar to the proof of Proposition 4, given $\{V_k\}$, the seller's revenue can be expressed as

$$R(\mathbf{V}) = \max_q \quad \sum_{k=1}^{K} \frac{[\ln V_k - \ln(q_k - q_{k-1})](q_k - q_{k-1})}{(1 + q_{k-1})(1 + q_k)}.$$

where $T = \sum_k V_k$ is the total attractiveness of products. Let the intrinsic attractivenesses of the first K-1 stages as the decision variables. When the context is clear, all the quantities below refer these at optimality.

Step 0: Optimal conditions.

At optimality, by the Envelope theorem, the first-order conditions for $k < K$ become as follows,

$$0 = \frac{\partial R}{\partial V_k} = \frac{1}{V_k} \frac{q_k - q_{k-1}}{(1 + q_{k-1})(1 + q_k)} - \frac{1}{V_K} \frac{q_K - q_{K-1}}{(1 + q_{K-1})(1 + q_K)} = \frac{e^{-r^{(k)}}}{(1 + q_{k-1})(1 + q_k)} - \frac{e^{-r^{(K)}}}{(1 + q_{K-1})(1 + q_K)}.$$

As a result, for all $k \neq k'$,

$$\frac{e^{-r^{(k)}}}{(1 + q_{k-1})(1 + q_k)} = \frac{e^{-r^{(k')}}}{(1 + q_{k'-1})(1 + q_{k'})}.$$

Let $k' = k + 1$, we have

$$\frac{1 + q_{k+1}}{1 + q_{k-1}} = e^{r^{(k)} - r^{(k+1)}}. \tag{21}$$

Besides, we have shown the optimal pricing policy satisfies Equation (19), namely

$$0 = \begin{cases} r^{(k)} - r^{(k+1)} - \frac{1+q_k}{1+q_{k-1}} + \frac{1+q_k}{1+q_{k+1}}, & k < K \\ r^{(K)} - \frac{1+q_K}{1+q_{K-1}}, & k = K. \end{cases} \tag{22}$$

where we have used $\pi_k = 1$ for all k in the relaxation problem. Then combining (21) and the first equation in (22), we have that

$$\frac{1 + q_k}{1 + q_{k-1}} - \ln \frac{1 + q_k}{1 + q_{k-1}} = \ln \frac{1 + q_{k+1}}{1 + q_k} + \frac{1 + q_k}{1 + q_{k+1}}.$$

Now we still introduce $t_k = \frac{1+q_k}{1+q_{k-1}} > 1$ to simplify the exposition. Then the above equation becomes

$$t_k - \ln t_k - \ln t_{k+1} - \frac{1}{t_{k+1}} = 0. \tag{23}$$

where the LHS is increasing in t_k as $t_k > 1$.

Step 1: At optimality $t_k \le t_{k+1}$ and $t_k \ge 2 - \frac{1}{t_{k+1}}$.

It is not hard to find that $t_{k+1} - 2\ln t_{k+1} - \frac{1}{t_{k+1}}$ is increasing in t_{k+1}, then $t_{k+1} - 2\ln t_{k+1} - \frac{1}{t_{k+1}} \ge [t_{k+1} - 2\ln t_{k+1} - \frac{1}{t_{k+1}}]_{t_{k+1}=1} = 0$. If $t_k > t_{k+1}$,

$$t_k - \ln t_k - \ln t_{k+1} - \frac{1}{t_{k+1}} > t_{k+1} - 2\ln t_{k+1} - \frac{1}{t_{k+1}} \ge 0,$$

leading to a contradiction with Equation 23. As a result, $t_k \le t_{k+1}$. Similarly, we will have $t_k \ge 2 - \frac{1}{t_{k+1}}$.

Step 2: At optimality $V_k \ge V_{k+1}$.

As $q_k \triangleq \sum_{i=1}^{k} V_i e^{-r^{(i)}}$, we have $V_k = (q_k - q_{k-1})e^{r^{(k)}}$. Then

$$(24)\quad \frac{V_k}{V_{k+1}} = \frac{(q_k - q_{k-1})e^{r^{(k)}}}{(q_{k+1} - q_k)e^{r^{(k+1)}}} = \frac{(q_k+1) - (q_{k-1}+1)}{(q_{k+1}+1) - (1+q_k)}\frac{1+q_{k+1}}{1+q_{k-1}} = \frac{t_{k+1}(t_k - 1)}{t_{k+1} - 1} \ge \frac{t_{k+1} - 1}{t_{k+1} - 1} = 1,$$

where the second equality is due to (21), the third equality is due to $1 + q_k = \Pi_{m=1}^{k} t_m$ and the last inequality is due to $t_k \ge 2 - \frac{1}{t_{k+1}}$.

Step 3: $\bar{R}(T;K)$ is increasing in K.

As any sequence of assortments with K stages is feasible for a problem with $K+1$ stages, we have that the revenue becomes larger as K increases.

Step 4: $\bar{R}(T;K=1) = \mathcal{L}(T/e)$.

When $K = 1$, the seller's revenue is as follows

$$\bar{R}(T;K=1) = \max_r \frac{re^{-r}T}{1+e^{-r}T}.$$

From the first-order condition, we will have $T - e^r(r-1) = 0$, from which we have $r = 1 + \mathcal{L}(T/e)$. Substituting into the expression of $\bar{R}(T;K=1)$ yields

$$\bar{R}(T;K=1) = \mathcal{L}(T/e).$$

Step 5: The simplification of $\bar{R}(T;K)$.

When $K \ge 2$, we first show how to simplify the expression of the revenue function $\bar{R}(T;K)$. Note that from the definition of q, the revenue on the k-th stage is $\frac{r^{(k)}(q_k - q_{k-1})}{(1+q_{k-1})(1+q_k)} = \frac{r^{(k)}}{1+q_{k-1}} - \frac{r^{(k)}}{1+q_k}$. Then,

$$\begin{aligned}
\bar{R}(T;K) = &\quad \textstyle\sum_k \frac{r^{(k)}}{1+q_{k-1}} - \frac{r^{(k)}}{1+q_k} \\
= &\quad r^{(1)} - \frac{r^{(1)}}{1+q_1} + \frac{r^{(2)}}{1+q_1} - \cdots + \frac{r^{(K-1)}}{1+q_{K-2}} - \frac{r^{(K-1)}}{1+q_{K-1}} + \frac{r^{(K)}}{1+q_{K-1}} - \frac{r^{(K)}}{1+q_K} \\
= &\quad r^{(1)} + (\textstyle\sum_{k=1}^{K-1} \frac{r^{(k+1)} - r^{(k)}}{1+q_k}) - \frac{r^{(K)}}{1+q_K} \\
= &\quad r^{(1)} - \frac{r^{(K)}}{1+q_K} + \textstyle\sum_{k=1}^{K-1}(\frac{1}{1+q_{k+1}} - \frac{1}{1+q_{k-1}}) \\
= &\quad r^{(1)} - 1 - \frac{1}{1+q_1} + \frac{1}{1+q_{K-1}} - \frac{r^{(K)}-1}{1+q_K}.
\end{aligned}$$

where we have used the fact $q_0 = 0$ in the second equality, the fourth equality is due to $r^{(k)} - r^{(k+1)} = \frac{1+q_k}{1+q_{k-1}} - \frac{1+q_k}{1+q_{k+1}}$ by (22).

Summing over all the equations in (22) yields

$$r^{(1)} = \frac{1+q_K}{1+q_{K-1}} + \sum_{k=1}^{K-1}(\frac{1+q_k}{1+q_{k-1}} - \frac{1+q_k}{1+q_{k+1}}) = t_K + \sum_{k=1}^{K-1}(t_k - \frac{1}{t_{k+1}}).$$

where in the last expression, we have used the definition of t_k. As a result,

$$\begin{aligned} \bar{R}(T;K) = & \ t_K + \textstyle\sum_{k=1}^{K-1}(t_k - \frac{1}{t_{k+1}}) - 1 - \frac{1}{1+q_1} + \frac{1}{1+q_{K-1}} - \frac{r^{(K)}-1}{1+q_K} \\ = & \ t_K + \textstyle\sum_{k=1}^{K-1}(t_k - \frac{1}{t_{k+1}}) - 1 - \frac{1}{t_1} + \frac{1}{1+q_{K-1}} - \frac{r^{(K)}-1}{1+q_K} \\ = & \ t_K + \textstyle\sum_{k=1}^{K-1}(\ln t_k + \ln t_{k+1}) - 1 - \frac{1}{t_1} + \frac{1}{1+q_{K-1}} - \frac{r^{(K)}-1}{1+q_K} \\ = & \ t_K + 2\textstyle\sum_{k=1}^{K-1}\ln t_k - \ln t_1 + \ln t_K - 1 - \frac{1}{t_1} + \frac{1}{1+q_{K-1}} - \frac{r^{(K)}-1}{1+q_K} \\ = & \ t_K + 2\ln(1+q_{K-1}) - \ln t_1 + \ln t_K - 1 - \frac{1}{t_1} + \frac{1}{1+q_{K-1}} - \frac{r^{(K)}-1}{1+q_K}, \end{aligned} \tag{25}$$

where we have used $1+q_1 = t_1$ in the second equality, the third equality is due to $t_k - \frac{1}{t_{k+1}} = \ln t_k + \ln t_{k+1}$ by (23) and the last equality is due to $1+q_{K-1} = \Pi_{k=1}^{K-1} t_k$ by the definition of t_k.

Step 6: At optimality, $T = e^{t_K}(q_{K-1}+1)q_K$.

From (24), $V_k = \frac{t_k - 1}{t_{k+1}-1} t_{k+1} V_{k+1}$. Then

$$\begin{aligned} V_k &= \frac{t_k - 1}{t_{k+1}-1} t_{k+1} V_{k+1} = \frac{t_k - 1}{t_{k+1}-1} t_{k+1} \frac{t_{k+1}-1}{t_{k+2}-1} t_{k+2} V_{k+2} \\ &= \cdots = \frac{(t_k - 1)\Pi_{i=k+1}^{K} t_i}{t_K - 1} V_K = \frac{V_K}{t_K - 1}(\Pi_{i=k}^{K} t_i - \Pi_{i=k+1}^{K} t_i). \end{aligned}$$

where $\Pi_{i=K+1}^{K} t_i = 1$ by default. Therefore

$$T = \sum_k V_k = \frac{V_K}{t_K - 1}\sum_k(\Pi_{i=k}^{K} t_i - \Pi_{i=k+1}^{K} t_i) = \frac{V_K}{t_K - 1}(\Pi_{i=1}^{K} t_i - 1) = \frac{V_K}{t_K - 1} q_K,$$

where in the last expression, we have used $1+q_K = \Pi_{i=1}^{K} t_i$ by the definition of t_k.

Note that from the definition of q_k and t_k, $r^{(K)} = \ln V_K - \ln(q_K - q_{K-1}) = \ln V_k - \ln[(t_K - 1)(1+q_{K-1})]$. Then from the last equation in (22), we have

$$0 = \ln V_K - \ln(t_K - 1) - \ln(1+q_{K-1}) - t_K,$$

namely,

$$V_K = e^{t_K}(t_K - 1)(q_{K-1}+1). \tag{26}$$

Therefore $T = e^{t_K}(q_{K-1}+1)q_K$.

Step 7: $K \to \infty$.

For ease of exposition, we add the superscript (K) to denote a quantity in the situation that the seller has K stages to recommend. For example, $V_K^{(K)}$ denote the intrinsic attractiveness on the last stage when the seller has K stages to recommend. Also, $V_k^{(K)}$ denote the intrinsic attractiveness on the k-th stage when the seller has K stages to recommend.

Step 7.1: $\lim_{K\to\infty} t_k^{(K)} = 1$ for all k.

By Step 2, $V_k^{(K)}$ is decreasing in k. As $\sum_k V_k^{(K)} = T$, $\lim_{K\to\infty} V_K^{(K)} = 0$. From (26), as $t_K^{(K)} \geq 1$ and $q_{K-1}^{(K)} \geq 0$, then $\lim_{K\to\infty} t_K^{(K)} = 1$. By Step 1, $t_k^{(K)}$ is increasing in k, and as $t_k^{(K)} \geq 1$ by its definition, we have $\lim_{K\to\infty} t_k^{(K)} = 1$ for all k.

Step 7.2: $\lim_{K\to\infty} r^{(K),(K)} = 1$.

From step 7.1, we cannot conclude $\lim_{K\to\infty} q_K^{(K)} = 1$, as $q_K^{(K)}$ is a product of infinite $t_k^{(K)}$. However, as $1 + q_K^{(K)} = t_K^{(K)}(1 + q_{K-1}^{(K)})$ from the definition of t_k, we have that $\lim_{K\to\infty} q_{K-1}^{(K)} = \lim_{K\to\infty} q_K^{(K)}$. Therefore from the last equation in 22, we have $\lim_{K\to\infty} r^{(K),(K)} = 1$.

Step 7.3: $\lim_{K\to\infty} q_K^{(K)} = \frac{\sqrt{1+4T/e}-1}{2}$.

From step 8, $T = e^{t_K}(q_{K-1}+1)q_K$, $\lim_{K\to\infty} t_K^{(K)} = 1$ and $\lim_{K\to\infty} q_{K-1}^{(K)} = \lim_{K\to\infty} q_K^{(K)}$, we have $\lim_{K\to\infty} q_K^{(K)} = \frac{\sqrt{1+4T/e}-1}{2}$.

Step 7.4: $\bar{R}(T;\infty) = 2\ln(\frac{\sqrt{1+4T/e}+1}{2}) + \frac{1-\sqrt{1+4T/e}}{\sqrt{1+4T/e}+1}$.

With these limitations, the revenue in (25) can be simplified as

$$
\begin{aligned}
\bar{R}(T;\infty) &= \lim_{K\to\infty} (t_K + 2\ln(1+q_{K-1}) - \ln t_1 + \ln t_K - 1 - \frac{1}{t_1} + \frac{1}{1+q_{K-1}} - \frac{r^{(K)}-1}{1+q_K}) \\
&= \lim_{K\to\infty} 2\ln(1+q_K^{(K)}) - 1 + \frac{1}{1+q_K^{(K)}} \\
&= 2\ln(\frac{\sqrt{1+4T/e}+1}{2}) + \frac{1-\sqrt{1+4T/e}}{\sqrt{1+4T/e}+1}.
\end{aligned}
$$

Now we complete all the claims. ■

Proof of Theorem 4: To prove the property of $\beta(T)$, let $T = ex(x+1)$. If $x \geq 0$, then $T \in [0,\infty)$. Therefore we alternatively prove the theorem by arguing that it is true for $x > 0$. By using x,

$$
\beta(x) = \frac{\bar{R}(T(x);1)}{\bar{R}(T(x);\infty)} = \frac{(1+x)\mathcal{L}(x+x^2)}{-x+2(1+x)\ln(1+x)}.
$$

Note that $\lim_{x\to 0}\beta(x) = \lim_{x\to\infty}\beta(x) = 1$, $\beta(x)$ is maximized at $x \to 0$ and $x \to \infty$. In the following, we prove that $\beta(x)$ is quasi-convex in $x \in (0,\infty)$. As a result, we can find the global minimizer of $\beta(x)$ by numerical calculation. The numerical result shows that $\min_x \beta(x) \approx 87.8\%$. By this, it leads to the theorem.

Due to the complexity of the function of $\mathcal{L}(\cdot)$, the proof of the quasi-convexity of $\beta(x)$ is quite involved. Note that if $x > 0$, then $\mathcal{L}(x) > 0$. We need to define several functions first:

$$
G(x) \triangleq -2 + 2(1/x+1)\ln(1+x) - \mathcal{L}(x+x^2),
$$

$$
H(x) \triangleq \frac{x^3}{2(1+x)} + x\ln(1+x) - (1+x)\ln^2(1+x),
$$

and

$$
K(x) \triangleq \frac{x(2+2x+x^2)}{(1+x)^2} - 2\ln(1+x).
$$

The relationships among these functions can be found as below,

- $\beta'(x) = \frac{(1+2x)\mathcal{L}(x+x^2)}{(x-2(1+x)\ln(1+x))^2(1+\mathcal{L}(x+x^2))} \times G(x)$. Note that the first term is always positive, thus $Sign[\beta'(x)] = Sign[G(x)]$ for all $x > 0$.

- $G'(x) = \frac{2(x-\ln(1+x))}{x^2} - \frac{(1+2x)\mathcal{L}(x+x^2)}{x(1+x)(1+\mathcal{L}(x+x^2))}$. If $G(x) = 0$, then $\mathcal{L}(x+x^2) = -2+2(1/x+1)\ln(1+x)$. Replacing $\mathcal{L}(x+x^2)$ by $-2+2(1/x+1)\ln(1+x)$ in the expression of $G'(x)$, then if $G(x) = 0$,

$$G'(x) = \frac{4}{x^2[2(1+x)\ln(1+x) - x]} \times H(x).$$

  Note that the first term is always positive, thus if $G(x) = 0$, $Sign[G'(x)] = Sign[H(x)]$ for all $x > 0$.

- $H''(x) = \frac{1}{1+x} \times K(x)$. Note that the first term is always positive, thus $Sign[H''(x)] = Sign[K(x)]$ for all $x > 0$.

Step 1: There exists a unique $x_* > 0$ such that $Sign[K(x)] = Sign[x - x_*]$ for all $x > 0$.

First, $K'(x) = \frac{x}{(1+x)^3} \times (x^2 + x - 2)$. Note that the first term is always positive, thus $K'(x) < 0$ for $0 < x < 1$ and $K'(x) > 0$ for $x > 1$. Then $K(x)$ is strictly quasi-convex in $x \in (0, \infty)$. As $K(0) = 0$, there exists a unique $x_* > 0$ such that $K(x_*) = 0$, then we have the claim.

Step 2: There exists a unique $x_{**} > 0$ such that $Sign[H(x)] = Sign[x - x_{**}]$ for all $x > 0$.

From the property of $K(x)$, then $H(x)$ is strictly concave in $x \in (0, x_*)$ and strictly convex in $x \in (x_*, \infty)$. Moreover, when $x \to 0^+$, $H(x) = -\frac{x^4}{12} + \mathcal{O}(x^4)$. Therefore $H(x)$ is decreasing in x, when $x \to 0^+$. As $H(x)$ is strictly concave in $x \in (0, x_*)$, $H(x)$ is decreasing in $x \in (0, x_*)$. Due to that $H(x)$ is strictly convex in $x \in (x_*, \infty)$ and $H(x)$ is decreasing in $x \in (0, x_*)$, $H(x)$ is quasi-convex in $x \in (0, \infty)$. As $H(0) = 0$, there exists a unique $x_{**} > 0$ such that $H(x_{**}) = 0$, then we have the claim.

Step 3: $\beta(x)$ is decreasing in $x \in (0, x_{**})$.

Suppose that there exists one point such that $\beta'(x) = 0$ for some $x \in (0, x_{**})$. Then at this point $G(x) = 0$ and $Sign[G'(x)] = Sign[H(x)] < 0$ (from Step 2). As a result, at this point

$$\begin{aligned}
\beta''(x) &= \frac{d}{dx} \frac{(1+2x)\mathcal{L}(x+x^2)}{(x-2(1+x)\ln(1+x))^2(1+\mathcal{L}(x+x^2))} \times G(x) \\
&\quad + \frac{(1+2x)\mathcal{L}(x+x^2)}{(x-2(1+x)\ln(1+x))^2(1+\mathcal{L}(x+x^2))} \times G'(x) \\
&= \frac{(1+2x)\mathcal{L}(x+x^2)}{(x-2(1+x)\ln(1+x))^2(1+\mathcal{L}(x+x^2))} \times G'(x) < 0,
\end{aligned}$$

where the second equality is due to $G(x) = 0$ at this point. That is, for $x \in (0, x_{**})$, if $\beta'(x) = 0$, we have $\beta''(x) < 0$. As a result, $\beta(x)$ is quasi-concave in $x \in (0, x_{**})$. As $\beta(x)$ is maximized at $x = 0$, then $\beta(x)$ is decreasing in $x \in (0, x_{**})$.

Step 4: $\beta(x)$ is quasi-convex in $x \in (x_{**}, \infty)$.

Suppose that there exists one point such that $\beta'(x) = 0$ for some $x \in (x_{**}, \infty)$. Similar to the previous step, we can show that at this point, $Sign[\beta''(x)] = Sign[G'(x)] > 0$. That is, for $x \in (x_{**}, \infty)$, if $\beta'(x) = 0$, we have $\beta''(x) > 0$. As a result, $\beta(x)$ is quasi-convex in $x \in (x_{**}, \infty)$.

Step 5: $\beta(x)$ is quasi-convex in $x \in (0, \infty)$.

By Step 3 and 4, this claim is obvious. Thus we prove the theorem. ■

Proof of Lemma 5: We use the continuous relaxation to prove the claim.

I. One stage

Let $g(v_0)$ be the distribution of $v_0 = exp(\mu_0) \geq 0$. Define the revenue of a continuous relaxation as follows,

$$R(q) = \max_{\sum_i x_i v_i = q, x_i \in [0,1]} \int g(v_0) \frac{\sum_i x_i r_i v_i}{q + v_0} dv_0.$$

Then fixing q, it is revenue-ordered. Now we can define $W(q) = \sum_i x_i r_i v_i$. Next we prove that $\max_q \{R(q)\}$ is maximizied at $x_i^* \in \{0, 1\}$. Suppose this is not true such that at optimality, product s is partially recommended. Let the total intrinsic attractiveness in recommendation as q^* . Then $W(q^* + \delta q) = W(q^*) + r_s \delta q$, with δq not large. Then,

$$0 = \frac{\partial R(q^* + \delta q)}{\partial \delta q}|_{\delta q = 0} = \int g(v_0) \frac{r_s q^* + r_s v_0 - W(q^*)}{(v_0 + q^*)^2} dv_0.$$

Let $\bar{v}_0$ satisfies $r_s q^* + r_s \bar{v}_0 - W(q^*) = 0$. Then

$$\begin{aligned} \frac{\partial^2 R(q^* + \delta q)}{\partial (\delta q)^2}|_{\delta q = 0} &= \int g(v_0) \frac{2(W(q^*) - r_s q^* - r_s v_0)}{(q^* + v_0)^3} dv_0 \\ &> \int_0^{\bar{v}_0} g(v_0) \frac{2(W(q^*) - r_s q^* - r_s v_0)}{(q^* + v_0)^2 (q^* + \bar{v}_0)} dv + \int_{\bar{v}_0}^{\infty} g(v_0) \frac{2(W(q^*) - r_s q^* - r_s v_0)}{(q^* + v_0)^2 (q^* + \bar{v}_0)} dv_0 \\ &= - \frac{2}{q^* + \bar{v}_0} \frac{\partial R(q^* + \delta q)}{\partial \delta q}|_{\delta q = 0} = 0, \end{aligned}$$

which leads to a contradiction.

II. Two stages

Define the revenue as follows,

$$R(q_1, q_2) = \max_x \int g(v_0) [\frac{\sum_i x_{i1} r_{i1} v_i}{q_1 + v_0} + \frac{\pi v_0}{q_1 + v_0} \frac{\sum_i x_{i2} r_{i2} v_i}{q_2 + v_0}] dv_0,$$

where $\sum_i x_{i1} v_i = q_1$ and $\sum_i (x_{i1} + x_{i2}) v_i = q_2$, Clearly, it is revenue-ordered by switching products. Now define $W_1(q_1) = \sum_i x_{i1} r_{i1} v_i$ and $W_2(q_1, q_2) = \sum_i x_{i2} r_{i2} v_i$. Suppose that for the optimal solution, (q_1^*, q_2^*), there is a product with the same revenue r_s is recommended on the first and second stage. Then with small δq, $W_1(q_1^* + \delta q) = W_1(q_1^*) + r_s \delta q$ and $W_2(q_1^* + \delta q, q_2^*) = W_2(q_1^*, q_2^*) - r_s \delta q$. Then

$$\frac{\partial R(q_1^* + \delta q, q_2^*)}{\partial \delta q} = \int g(v_0) \frac{K(v_0)}{(v_0 + q_1^* + \delta q)^2 (q_2^* + v_0)} dv_0,$$

where $K(v_0) = (q_1^* q_2^* r_s - q_2^* W_1(q_1^*)) + (q_1 r_s - \pi q_1^* r_s + q_2^* r_s - W_1(q_1^*) - \pi W_2(q_1^*, q_2^*)) v_0 + (r_s - \pi r_s) v_0^2$.

If $K(v_0) = 0$ for all v_0, then as (q_1^*, q_2^*) is optimal, $(q_1^* + \delta q, q_2^*)$ will also be optimal. Now assume $K(v_0) = 0$ does not hold for all v_0. As $r_s = \min_{i \in S_1} r_i$, $K(0) = q_1^* q_2^* r_s - q_2^* W_1(q_1^*) \leq 0$ and as $r_s \geq \pi r_s$, there exists one unique $\bar{v}_0 > 0$ satisfying $K(v_0) = 0$ to make $\frac{\partial R(q_1^* + \delta q, q_2^*)}{\partial \delta q}|_{\delta q = 0} = 0$ hold. Then for $v_0 < \bar{v}_0$, $K(v_0) \leq 0$ and otherwise, $K(v_0) \geq 0$.

$$
\begin{aligned}
\frac{\partial^2 R(q_1^*+\delta q, q_2^*)}{\partial(\delta q)^2}|_{\delta q=0} =& \int g(v_0)\frac{-2K(v_0)}{(q_1^*+v_0)^3(q_2+v_0)}dv_0 \\
&> \int_0^{\bar{v}_0} g(v_0)\frac{-2K(v_0)}{(q_1^*+\bar{v}_0)(q_1^*+v_0)^2(q_2+v_0)}dv_0 \\
&+ \int_{\bar{v}_0}^{\infty} g(v_0)\frac{-2K(v_0)}{(q_1^*+\bar{v}_0)(q_1^*+v_0)^2(q_2+v_0)}dv_0 \\
&= -\frac{2}{q_1^*+\bar{v}_0}\frac{\partial R(q_1^*+\delta q, q_2^*)}{\partial \delta q}|_{\delta q=0} = 0,
\end{aligned}
$$

which leads to a contradiction. Therefore, fixing q_2, $x_i^* \in \{0,1\}$ for all $i \in S_1$. When fixing q_1 with $x_i^* \in \{0,1\}$ for all $i \in S_1$, a similar approach as shown in **I.** leads to $x_i^* \in \{0,1\}$ for all $i \in S_2$

III. Multiple stages

From **II**, for any two stages, we can derive a similar result: the revenues on earlier stages will be larger than that on later stages, and products with the same revenue can be optimally put on one stage. From **I.**, we know the lowest revenue of the last stage will be larger than that not recommended. Thus we prove the claim. ∎

Proof of Lemma 6: It is not hard to prove that if it is not sequentially revenue-ordered, we can switch the products and the revenue for each consumer type will be improved, Thus we omit the details. ∎

Proof of Lemma 7: We use the continuous relaxation to prove this lemma.

$$
\begin{array}{lll}
R(q_1,\cdots,q_K) \triangleq & \mathbf{max_x} & \sum_{k=1}^K (1-\gamma)\pi_k' \frac{\sum_{m=1}^k \sum_{i\in S_m} v_i r_i x_{im}}{1+q_k} + \gamma \frac{\pi_k}{1+q_{k-1}} \frac{\sum_{i\in S_k} v_i r_i x_{ik}}{1+q_k} \\
& \text{subject to} & \sum_{i=1}^k \sum_{j=1}^n x_{ji} v_j = q_k, \quad 1 \le k \le K, \\
& & \sum_{i=1}^K x_{ji} \le 1, j \in N, \\
& & x_{ji} \ge 0,
\end{array}
$$

with x_{ji} denoting the fraction of product j recommended in S_i. By switching we can prove that the continuous relaxation admits sequentially revenue-ordered assortment as optimal. Let $(q_1^*,\cdots,q_K^*)$ be the optimal solution of $\max_{\mathbf{q}} R(q_1,\cdots,q_K)$ and assume that there is a product with revenue r_s which is included both in stage k and stage $k+1$. Then for small δq,

$$
\begin{aligned}
R(q_1^*,\cdots,q_k^*+\delta q,\cdots,q_K^*) =& \cdots + (1-\gamma)\pi_k' \frac{W_k'(q_k^*)+r_s\delta q}{1+q_k^*+\delta q} + \gamma\frac{\pi_k}{1+q_{k-1}^*}\frac{W_k(q_k^*)+r_s\delta q}{1+q_k^*+\delta q} \\
&+ (1-\gamma)\pi_{k+1}' \frac{W_{k+1}'(q_{k+1}^*)}{1+q_{k+1}^*} + \gamma\frac{\pi_{k+1}}{1+q_k^*+\delta q}\frac{W_k(q_{k+1}^*)-r_s\delta q}{1+q_{k+1}^*} + \cdots
\end{aligned}
$$

where $W_k'(q_k^*) = \sum_{m=1}^k \sum_{i\in S_m} v_i r_i x_{im}$ and $W_k(q_k^*) = \sum_{i\in S_k} v_i r_i x_{ik}$ given $\{x_{ik}\}$ as the solutions of $R(q_1^*,\cdots,q_K^*)$. Then

$$
R(q_1^*,\cdots,q_k^*+\delta q,\cdots,q_K^*) - R(q_1^*,\cdots,q_k^*+\delta q,\cdots,q_K^*) = \frac{\delta q}{1+q_k^*+\delta q}C,
$$

where C does not depend on δq. Since $\partial_{\delta q} R(q_1^*,\cdots,q_k^*+\delta q,\cdots,q_K^*)|_{\delta q=0} = 0$, then $C=0$. As a result, $(q_1^*, \cdots, q_k^* + \delta q, \cdots, q_K^*)$ is also optimal for $\max_{\mathbf{q}} R(q_1, \cdots, q_K)$. Then the optimal solution for $\max_{\mathbf{q}} R(q_1, \cdots, q_K)$ could be strongly revenue-ordered, which is feasible for problem (12), thus we complete the claim. ■