
Learning Deep ResNet Blocks Sequentially using Boosting Theory

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Abstract

Deep neural networks are known to be difficult to train due to the instability of back-propagation. A deep *residual network* (ResNet) with identity loops remedies this by stabilizing gradient computations. We prove a boosting theory for the ResNet architecture. We construct T weak module classifiers, each contains two of the T layers, such that the combined strong learner is a ResNet. Therefore, we introduce an alternative Deep ResNet training algorithm, *BoostResNet*, which is particularly suitable in non-differentiable architectures. Our proposed algorithm merely requires a sequential training of T “shallow ResNets” which are inexpensive. We prove that the training error decays exponentially with the depth T if the *weak module classifiers* that we train perform slightly better than some weak baseline. In other words, we propose a weak learning condition and prove a boosting theory for ResNet under the weak learning condition. Our results apply to general multi-class ResNets. A generalization error bound based on margin theory is proved and suggests ResNet’s resistant to overfitting under network with l_1 norm bounded weights.

1 Introduction

Deep neural networks have elicited breakthrough successes in machine learning, especially in image classification and object recognition (Krizhevsky *et al.*, 2012; Sermanet *et al.*, 2013; Simonyan & Zisserman, 2014; Zeiler & Fergus, 2014) in recent years. As the number of layers increases, the nonlinear network becomes more powerful, deriving richer features from input data. Empirical studies suggest that challenging tasks in image classification (He *et al.*, 2015; Ioffe & Szegedy, 2015; Simonyan & Zisserman, 2014; Szegedy *et al.*, 2015) and object recognition (Girshick, 2015; Girshick *et al.*, 2014; He *et al.*, 2014; Long *et al.*, 2015; Ren *et al.*, 2015) often require “deep” networks, consisting of tens or hundreds of layers. Theoretical analyses have further justified the power of deep networks (Mhaskar & Poggio, 2016) compared to shallow networks.

However deep neural networks are difficult to train despite their intrinsic viability. Stochastic gradient descent with back-propagation (BP) (LeCun *et al.*, 1989) and its variants are commonly used to solve the non-convex optimization problem. A major challenge that exists for training both shallow and deep networks is *vanishing/exploding gradients* (Bengio *et al.*, 1994; Glorot & Bengio, 2010). Recent works have proposed normalization techniques (Glorot & Bengio, 2010; LeCun *et al.*, 2012; Ioffe & Szegedy, 2015; Saxe *et al.*, 2013) to effectively ease the problem and achieve convergence. In training deep networks, however, a surprising *training performance degradation* is observed

(He & Sun, 2015; Srivastava *et al.*, 2015; He *et al.*, 2016): the training performance degrades rapidly along with the increased network depth after some saturation point. This *training performance degradation* is unexpected as one can easily construct a deep network identical to a shallow network by enforcing any part of the deep network to be the same as the shallow network and the rest layers to be identity maps. He *et al.* (He *et al.*, 2016) have presented a *residual network* (ResNet) learning framework to ease the training of networks that are substantially deeper than those used previously. And they explicitly reformulate the layers as learning residual functions with reference to the layer inputs by adding identity loops to the layers. It is shown in (Hardt & Ma, 2016) that identity loops ease the problem of spurious local optima in shallow networks. Srivastava *et al.* (Srivastava *et al.*, 2015) introduce a novel architecture that enables the optimization of networks with virtually arbitrary depth through the use of a learned gating mechanism for regulating information flow.

Although empirical evidence shows that these deep residual networks are easier to optimize than non-residual ones, there lacks a theoretical justification for this observation. For example, there have been no performance guarantees on the training or testing error for networks of this deep residual architecture. Furthermore, the entire training still relies on the unstable end-to-end back-propagation which are susceptible to suboptimal solutions (Ge *et al.*, 2015) in deep networks.

1.1 Summary of Results

We propose a novel framework, *multi-channel telescoping sum boosting* (defined in Section 4), to characterize a feed forward ResNet in Section 3. We show that the top level (final) output of a ResNet is a telescoping sum of its pairs of consecutive module differences. Theoretical analyses such as training error guarantees and generalization error bounds for telescoping sum boosting are provided.

To remedy the problem of training error degradation as depth increases, we introduce a guaranteed learning algorithm, called *BoostResNet*, to train modules of ResNet sequentially. BoostResNet adaptively selects training samples or changes the cost function (Section 4 Theorem 4.2). BoostResNet trains the ResNet with guarantees: **the training error decays exponentially with the depth of the network**. As discussed later in Section 4.4, the generalization error of BoostResNet is analyzed and advice to avoid over-fitting is provided. Our procedure trains each residual block sequentially, only requiring that each provides a better-than-random-guessing prediction of dataset labels.

Our *BoostResNet* algorithm enjoys several superiorities over the end-to-end back-propagation (*e2eBP*) training convention despite the theoretical guarantees. First, *BoostResNet* is substantially more memory efficient than *e2eBP* as the former requires only two modules of the network to be in the GPU whereas the latter inevitably keeps all modules in the GPU. The advantage is crucial for efficient training for deep networks. Also, *BoostResNet* is more computational efficient than *e2eBP* since each *e2eBP* step involves back propagating through the entire deep network.

Experimentally, we compare *BoostResNet* with *e2eBP* over two types of feed forward ResNets, *multilayer perceptron residual network* (MLP-ResNet) and *convolutional neural network residual network* (CNN-ResNet), on multiple datasets. *BoostResNet* shows drastic performance improvement under the MLP-ResNet architecture. Under *CNN-ResNet*, a slightly faster convergence for *BoostResNet* is observed. The tremendous advantages of *BoostResNet* on memory and computation efficiency are justified as well in the experiments. Our *multi-channel telescoping sum boosting* learning framework is not limited to ResNet and can be extended to other even non-differentiable nonlinear hypothesis units, such as decision trees and tensor decompositions.

1.2 Related Works

Training deep neural networks has been an active research area in the past few years. The main optimization challenge lies in the highly non-convex nature of the loss function. There are two main ways to address this optimization problem, one is to select a loss function and network architecture that have better geometric properties, and the other is to improve the network’s learning procedure.

Loss function and architecture selection In neural network optimization, there are many predefined loss functions and criteria that are commonly used. For instance, *mean squared error*, *negative log likelihood*, *margin criterion* and so forth. There are extensive works (Girshick, 2015; Rubinstein & Kroese, 2013; Tygert *et al.*, 2015) on selecting or modifying loss functions to prevent

empirical difficulties such as exploding/vanishing gradients or slow learning (Balduzzi *et al.*, 2017). However, there are no rigorous principles for selecting a loss function in general. Other works consider variations of the MLP or CNN by adding identity skip connections (He *et al.*, 2016), allowing information to bypass particular layers. However, no theoretical guarantees on the training error are provided despite breakthrough empirical successes. Hardt *et al.* (Hardt & Ma, 2016) have shown the advantage of identity loops in linear neural networks with theoretical justifications, however the linear setting is unrealistic in practice.

Learning algorithm design There have been extensive works on improving BP (LeCun *et al.*, 1989). For instance, *momentum* (Qian, 1999), *Nesterov accelerated gradient* (Nesterov, 1983), *Adagrad* (Duchi *et al.*, 2011) and its extension *Adadelta* (Zeiler, 2012). Most recently, *Adaptive Moment Estimation (Adam)* (Kingma & Ba, 2014), a combination of *momentum* and *adagrad*, has received substantial success in practice. All these methods are modifications of SGD, but our method only requires an arbitrary oracle, which does not necessarily need to be an SGD solver. Bengio *et al.* (Bengio *et al.*, 2006) introduce single hidden layer convex neural networks, and propose a gradient boosting algorithm to learn the weights of the linear classifier, however a generalization of their method to deep networks with more than one hidden layer is not possible. Shalev-Shwartz (Shalev-Shwartz, 2014) proposes a selfieBoost which boosts the accuracy of a single network. Our algorithm is different as we instead construct ensembles of classifiers. AdaBoost (Cortes *et al.*, 2016) also considers ensembles of classifiers, but they require connections between top layer and all other lower level layers, which are not amenable to a standard ResNet architecture.

2 Preliminaries

A *residual neural network* (ResNet) is composed of stacked entities referred to as modules. Each module consists of a multiple-layer neural network. Commonly used modules include MLP and CNN.

Throughout this paper, we consider training examples $(x, y) = \{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)\} \in \mathcal{D}$, where \mathcal{D} is the distribution of the data. We will use S to denote the samples.

A Module of ResNet Let each module map its input to $f_t(\cdot)$ where t denotes the level of the modules. Each module $f_t(x)$ is a nonlinear unit with n channels, i.e., $f_t(x) \in \mathbb{R}^n$. In *multilayer perceptron residual network* (MLP-ResNet), $f_t(x)$ is a shallow MLP, for instance, a connected nonlinear $f_t(x) = \tilde{V}_t^\top \sigma(\tilde{W}_t^\top x)$ where $\tilde{W}_t \in \mathbb{R}^{n \times k}$, $\tilde{V}_t \in \mathbb{R}^{k \times n}$ and σ is a nonlinear operator such as sigmoidal function or relu function. Similarly, in *convolutional neural network residual network* (CNN-ResNet), the function $f_t(x)$ represents the t -th convolutional module. Then the t -th module outputs $g_{t+1}(x)$

$$g_{t+1}(x) = f_t(g_t(x)) + g_t(x), \quad (1)$$

where x is the input fed to the ResNet. See Figure 1 for an illustration of a ResNet, which consists of stacked modules with identity loops.

Output of ResNet Due to the recursive relation specified in Equation (1), the output of the T -th module is equal to the summation over lower module outputs, i.e., $g_{T+1}(x) = f_T(g_T(x)) + g_T(x) = \sum_{t=1}^T f_t(g_t(x))$, where $g_1(x) = x$. For classification tasks, the final output of a ResNet given input x is rendered after a linear classifier $\mathbf{w} \in \mathbb{R}^n$ on representation $g_{T+1}(x)$:

$$\hat{y} = \tilde{\sigma}(F(x)) = \tilde{\sigma}(\mathbf{w}^\top g_{T+1}(x)) = \tilde{\sigma}\left(\mathbf{w}^\top \sum_{t=1}^T f_t(g_t(x))\right) \quad (2)$$

where $\tilde{\sigma}(\cdot)$ denotes a map from representation to labels $\tilde{\sigma}(z) : z \rightarrow \mathcal{Y}$. For instance $\tilde{\sigma}(z) = \text{sign}(z)$ for binary classification, or $\tilde{\sigma}(z) = \arg \max_i z_i$ for multi-class classification. The form of the output from a classification ResNet is thus as follows. The parameters of a depth- T ResNet are

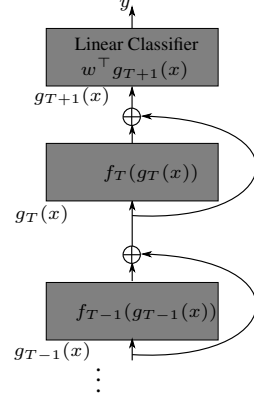


Figure 1: The architecture of a residual network (ResNet).

$\{\mathbf{w}, \{f_t(\cdot), \forall t \in T\}\}$. A ResNet training involves training the classifier \mathbf{w} and the weights of modules $f_t(\cdot) \forall t \in [T]$ when training samples $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$ are available.

Boosting Boosting (Freund & Schapire, 1995) assumes the availability of a *weak learning algorithm* which, given labeled training examples, produces a *weak classifier* (a.k.a. *base classifier*) $h_t(x)$. The goal of boosting is to improve the performance of the weak learning algorithm. The key idea behind boosting is to choose training sets for the weak classifier in such a fashion as to force it to infer something new about the data each time it is called. The weak learning algorithm will finally combine many weak classifiers $h_t(x)$ into a single combined *strong classifier* $\sum_{t=1}^T h_t(x)$ whose prediction power is strong.

From empirical experiences, ResNet remedies the problem of increased training error in deeper neural networks. We are curious about whether there is a theoretical justification that the training error of ResNet asymptotically converges to 0 as the depth T increases. More importantly, we are interested in proposing a new algorithm that avoids end-to-end back-propagation (*e2eBP*) through the deep network and thus is immune to the instability of the non-convex optimization.

3 ResNet in Telescoping Sum Boosting Framework

As we recall from Equation 2, ResNet indeed has a similar form as the strong classifier in boosting. The key difference is that boosting is an ensemble of estimated hypotheses $\sum_{t=1}^T h_t(x)$ whereas ResNet is an ensemble of estimated feature representations $F(x) = \sum_{t=1}^T f_t(g_t(x))$. To solve this problem, we introduce an auxiliary linear classifier \mathbf{w}_t on top of each ResNet module $f_t(g_t(x))$ to construct a *hypothesis module*. Formally, a *hypothesis module* is defined as

$$o_t(x) \stackrel{\text{def}}{=} \mathbf{w}_t^\top g_t(x) \in \mathbb{R} \quad (3)$$

in the binary classification setting. Therefore $o_{t+1}(x) = \mathbf{w}_{t+1}^\top [f_t(g_t(x)) + g_t(x)]$ as $g_{t+1}(x) = f_t(g_t(x)) + g_t(x)$. In the multi-class setting, let C be the number of classes, we define *hypothesis module* $o_t(x) \stackrel{\text{def}}{=} W_t^\top g_t(x) \in \mathbb{R}^C$ where linear classifier $W_t \in \mathbb{R}^{n \times C}$ is a matrix instead of a vector. Our analysis applies to both binary and multi-class, but we will focus on the binary class for simplicity in the main text and defer the multi-class analysis to the Appendix E.

Now the naive ensemble of the hypothesis module is not a ResNet unless all the auxiliary linear classifiers \mathbf{w}_t 's are equivalent to the ResNet's top classifier \mathbf{w} . However note that the input $g_{t+1}(x)$ of the $t+1$ -th module is the output $f_t(g_t(x)) + g_t(x)$ of the t -th module, one has to train the modules sequentially. Therefore the *common auxiliary linear classifier* assumption prevents us from training the T hypothesis module sequentially and is thus unrealistic. We design a **weak module classifier** using the idea of telescoping sum as follows.

Definition 3.1. A *weak module classifier* is defined as $h_t(x) \stackrel{\text{def}}{=} \alpha_{t+1} o_{t+1}(x) - \alpha_t o_t(x)$ where $o_t(x) \stackrel{\text{def}}{=} \mathbf{w}_t^\top g_t(x)$ is a *hypothesis module*. We call the boosting framework a “telescoping sum boosting” if the weak learners are restricted to the form of the weak module classifier.

ResNet: Ensemble of Weak Module Classifier Recall that the T -th module of a ResNet outputs $g_{T+1}(x)$, which is fed to the top/final linear classifier for the final classification. We show that an ensemble of the *weak module classifiers* is equivalent to a ResNet's final output. We state it formally in Lemma 3.2.

Lemma 3.2. Let the input $g_t(x)$ of the t -th module be the output of the previous module, i.e., $g_{t+1}(x) = f_t(g_t(x)) + g_t(x)$, then the summation of T weak module classifiers is identical to the output, $F(x)$ in Equation 2, of the depth- T ResNet,

$$F(x) \equiv \frac{1}{\alpha_{T+1}} \sum_{t=1}^T h_t(x), \quad (4)$$

where the weak module classifier is $h_t(x) \stackrel{\text{def}}{=} \alpha_{t+1} o_{t+1}(x) - \alpha_t o_t(x)$ and the hypothesis module is $o_t(x) \stackrel{\text{def}}{=} \mathbf{w}_t^\top g_t(x)$.

See Appendix A for the proof. Note that we will abusively call $F(x)$ the output of ResNet although a $\tilde{\sigma}$ function is applied on top of $F(x)$, mapping the output to the label space \mathcal{Y} . We now analyze the telescoping sum boosting framework in Section 4.

4 Telescoping Sum Boosting for Binary Classification

Recall that the weak module classifier is defined as $h_t(x) = \alpha_{t+1}o_{t+1}(x) - \alpha_t o_t(x)$. We restrict to bounded classifiers $|o_t(x)| \leq 1$. Throughout this paper, we will assume that the covariance between $\exp(-yo_{t+1}(x))$ and $\exp(yo_t(x))$ is non-positive. We propose a learning algorithm whose training error decays exponentially with the number of weak module classifiers T under a *weak learning condition* in the following.

4.1 Weak Learning Condition

Let $\tilde{\gamma}_t \stackrel{\text{def}}{=} \mathbb{E}_{i \sim D_{t-1}}[y_i o_t(x_i)] > 0$ be the edge of the hypothesis module $o_t(x)$, where D_{t-1} is the sample weight. A naive weak learning condition would be $\tilde{\gamma}_{t+1}^2 - \tilde{\gamma}_t^2 \geq \gamma'^2 > 0$. However this naive weak learning condition is too strong. Even when $\tilde{\gamma}_t$ is close to 1, we still seek weak learner which performances consistently better than γ' . Instead, we consider a much weaker weak learning condition as follows.

Definition 4.1 (γ -Weak Learning Condition). *Let $\tilde{\gamma}_t = \mathbb{E}_{i \sim D_{t-1}}[y_i o_t(x)] > 0$. A weak module classifier $h_t(x) = \alpha_{t+1}o_{t+1} - \alpha_t o_t$ satisfies the γ -weak learning condition with respect to a pair of distributions (D_t, D_{t-1}) if $\frac{\tilde{\gamma}_{t+1}^2 - \tilde{\gamma}_t^2}{1 - \tilde{\gamma}_t^2} \geq \gamma^2 > 0$.*

The weak learning condition is motivated by the learning theory and it is met in practice as shown in Figure 5a. For each weak module classifier, $\gamma_t \stackrel{\text{def}}{=} \sqrt{\frac{\tilde{\gamma}_{t+1}^2 - \tilde{\gamma}_t^2}{1 - \tilde{\gamma}_t^2}}$ is defined as the “edge” which characterizes the correlation between the true labels y and the weak module classifier $h_t(x)$ over the training samples. The condition specified in Definition 4.1 is extremely mild as it requires the weak module classifier $h_t(x)$ to perform only slightly better than random guessing. As the hypothesis module $o_t(x)$ is bounded by 1, we obtain that $|\tilde{\gamma}_t| \leq 1$.

4.2 BoostResNet

We now propose a novel training algorithm for telescoping sum boosting under the setting of binary-class classification as in Algorithm 1. In particular, we introduce a training procedure for deep ResNet in Algorithm 1 and 2, called **BoostResNet**, which only requires sequential training shallow ResNets. Each of the shallow ResNet $f_t(g_t(x)) + g_t(x)$ is combined with an auxiliary linear classifier w_{t+1} to form a *hypothesis module* $o_{t+1}(x)$. The weights of the ResNet are trained on these shallow ResNets and the auxiliary linear classifiers w_{t+1} are discarded (except for the top classifier). The training algorithm is therefore a *module-by-module* procedure following a *bottom-to-top* fashion as the outputs of the t -th module $g_{t+1}(x)$ are fed as the training examples to the next $t + 1$ -th module.

Algorithm 1 BoostResNet: telescoping sum boosting for binary-class classification

Input: m labeled samples $[(x_i, y_i)]_m$ where $y_i \in \{-1, +1\}$ and a threshold γ
Output: $\{f_t(\cdot), \forall t\}$ and \mathbf{w}_{T+1} ▷ Discard $\mathbf{w}_{t+1}, \forall t \neq T$
1: Initialize $t \leftarrow 0, \tilde{\gamma}_0 \leftarrow 0, \alpha_0 \leftarrow 0, o_0(x) \leftarrow 0$
2: Initialize sample weights at round 0: $D_0(i) \leftarrow 1/m, \forall i \in [m]$
3: **while** $\gamma_t > \gamma$ **do**
4: $f_t(\cdot), \alpha_{t+1}, \mathbf{w}_{t+1}, o_{t+1}(x) \leftarrow \text{Algorithm 2}(g_t(x), D_t, o_t(x), \alpha_t)$
5: Compute $\gamma_t \leftarrow \sqrt{\frac{\tilde{\gamma}_{t+1}^2 - \tilde{\gamma}_t^2}{1 - \tilde{\gamma}_t^2}}$ ▷ where $\tilde{\gamma}_{t+1} \leftarrow \mathbb{E}_{i \sim D_t}[y_i o_{t+1}(x_i)]$
6: Update $D_{t+1}(i) \leftarrow \frac{D_t(i) \exp(-y_i h_t(x_i))}{\sum_{i=1}^m D_t(i) \exp[-y_i h_t(x_i)]}$ ▷ where $h_t(x) = \alpha_{t+1}o_{t+1}(x) - \alpha_t o_t(x)$
7: $t \leftarrow t + 1$
8: **end while**
9: $T \leftarrow t - 1$

Theorem 4.2. [*Training error bound*] *The training error of a T -module telescoping sum boosting using Algorithm 1 and 2 decays exponentially with the number of modules T ,*

$$\Pr_{i \sim S} \left(\tilde{\sigma} \left(\sum_t h_t(x_i) \right) \neq y_i \right) \leq e^{-\frac{1}{2} T \gamma^2}$$

Algorithm 2 BoostResNet: oracle implementation for training a ResNet module

Input: $g_t(x), D_t, o_t(x)$ and α_t

Output: $f_t(\cdot), \alpha_{t+1}, \mathbf{w}_{t+1}$ and $o_{t+1}(x)$

- 1: $(f_t, \alpha_{t+1}, \mathbf{w}_{t+1}) \leftarrow \arg \min_{(f, \alpha, \mathbf{v})} \sum_{i=1}^m D_t(i) \exp(-y_i \alpha \mathbf{v}^\top [f(g_t(x_i)) + g_t(x_i)] + y_i \alpha_t o_t(x_i))$
 - 2: $o_{t+1}(x) \leftarrow \mathbf{w}_{t+1}^\top [f_t(g_t(x)) + g_t(x)]$
-

if $\forall t \in [T]$ the weak module classifier $h_t(x)$ satisfies the γ -weak learning condition defined in Definition 4.1 and the covariance between $\exp(-y o_{t+1}(x))$ and $\exp(y o_t(x))$ is non-positive $\forall t$.

The training error of Algorithm 1 and 2 is guaranteed to decay exponentially with the ResNet depth even when each weak learning module $h_t(x)$ performs only slightly better than random (i.e., $\gamma > 0$). The assumption of the covariance between $\exp(-y o_{t+1}(x))$ and $\exp(y o_t(x))$ being non-positive is suggesting that the weak module classifiers should not be adversarial, which is a reasonable assumption for ResNet. Refer to Appendix E for the algorithm and theoretical guarantees for multi-class classification.

4.3 Oracle Implementation for ResNet

In Algorithm 2, the implementation of the oracle is equivalent to

$$(f_t, \alpha_{t+1}, \mathbf{w}_{t+1}) = \arg \min_{(f, \alpha, \mathbf{v})} \frac{1}{m} \sum_{i=1}^m \exp(-y_i \alpha \mathbf{v}^\top [f(g_t(x_i)) + g_t(x_i)]) \quad (5)$$

In practice, there are various ways to implement Equation (5). For instance, Janzamin et al. (Janzamin *et al.*, 2015) propose a tensor decomposition technique which decomposes a tensor formed by some transformation of the features x combined with labels y and recovers the weights of a one-hidden layer neural network with guarantees. One can also use back-propagation as numerous works have shown that gradient based training are relatively stable on shallow networks with identity loops (Hardt & Ma, 2016; He *et al.*, 2016).

4.4 Generalization Error Analysis

In this section, we analyze the generalization error to understand the possibility of over-fitting under Algorithm 1. The strong classifier or the ResNet is $F(x) = \frac{\sum_t h_t(x)}{\alpha_{T+1}}$. Now we define the *margin* for example (x, y) as $yF(x)$.

For simplicity, we consider MLP-ResNet with multiple channels n and assume that the weight vector connecting a neuron at layer t with its precedent layer neurons is l_1 norm bounded by $\Lambda_{t,t-1}$. Recall that there exists a linear classifier \mathbf{w} on top, and we restrict to l_1 norm bounded classifiers, i.e., $\|\mathbf{w}\|_1 \leq C_0 < \infty$. The expected training examples are l_∞ norm bounded $r_\infty \stackrel{\text{def}}{=} \mathbb{E}_{S \sim \mathcal{D}} [\max_{i \in [m]} \|x_i\|_\infty] < \infty$. We introduce Lemma 4.3 according to Lemma 2 from (Cortes *et al.*, 2016).

Lemma 4.3. *Let \mathcal{D} be a distribution over $\mathcal{X} \times \mathcal{Y}$ and S be a sample of m examples chosen independently at random according to \mathcal{D} . With probability at least $1 - \delta$, for $\theta > 0$, the strong classifier $F(x)$ (ResNet) satisfies that*

$$\Pr_{\mathcal{D}}(yF(x) \leq 0) \leq \Pr_S(yF(x) \leq \theta) + \frac{4C_0 r_\infty}{\theta} \sqrt{\frac{\log(2n)}{2m}} \sum_{t=1}^T \Lambda_t + \frac{2}{\theta} \sqrt{\frac{\log T}{m}} + \beta(\theta, m, T, \delta) \quad (6)$$

where $\Lambda_t \stackrel{\text{def}}{=} \prod_{t'=1}^t 2\Lambda_{t',t'-1}$ and $\beta(\theta, m, T, \delta) \stackrel{\text{def}}{=} \sqrt{\left\lceil \frac{4}{\theta^2} \log\left(\frac{\theta^2 m}{\log T}\right) \right\rceil \frac{\log T}{m} + \frac{\log \frac{2}{\delta}}{2m}}$.

From Lemma 4.3, we obtain a generalization error bound in terms of margin bound $\Pr_S(yF(x) \leq \theta)$ and network complexity $\frac{4C_0 r_\infty}{\theta} \sqrt{\frac{\log(2n)}{2m}} \sum_{t=1}^T \Lambda_t + \frac{2}{\theta} \sqrt{\frac{\log T}{m}} + \beta(\theta, m, T, \delta)$. Larger margin bound (larger θ) contributes positively to generalization accuracy, and l_1 norm bounded weights (smaller

$\sum_{t=1}^T \Lambda_t$) are beneficial to control network complexity and to avoid overfitting. The dominant term in the network complexity is $\frac{4C_0 r_\infty}{\theta} \sqrt{\frac{\log(2n)}{2m}} \sum_{t=1}^T \Lambda_t$ which scales as least linearly with the depth T . See appendix C for the proof.

This lemma suggests that stronger weak module classifiers which produce higher accuracy predictions and larger edges, will yield larger margins and suffer less from over-fitting. The larger the value of θ , the smaller the term $\frac{4C_0 r_\infty}{\theta} \sqrt{\frac{\log(2n)}{2m}} \sum_{t=1}^T \Lambda_t + \frac{2}{\theta} \sqrt{\frac{\log T}{m}} + \beta(\theta, m, T, \delta)$ is. With larger edges on the training set and when $\hat{\gamma}_{T+1} < 1$, we are able to choose larger values of θ while keeping the error term zero or close to zero.

5 Experiments

We compare our proposed *BoostResNet* algorithm with *e2eBP* training a ResNet on the MNIST (LeCun *et al.*, 1998) and street view house numbers (SVHN) (Netzer *et al.*, 2011) benchmark datasets. Two different types of architectures are tested: *multilayer perceptron residual network* (MLP-ResNet) and *convolutional neural network residual network* (CNN-ResNet). In each experiment the architecture of both algorithms is identical, and they are both initialized with the same random seed. Our experiments were programmed in the Torch deep learning framework for Lua and executed on NVIDIA Tesla P100 GPUs.

In our training, a mini-batch size 100 is used for both *BoostResNet* and *e2eBP*. The learning rate is initialized at 1e-2 with a decaying rate of 1e-4. The optimization method used in *e2eBP* is the state-of-the-art *Adaptive Moment Estimation (Adam)* Kingma & Ba (2014) variant of SGD introduced earlier. The oracle we used in *BoostResNet* to solve the *weak module classifier* is Adam as well, but could be extended to tensor methods Janzamin *et al.* (2015), decision trees Safavian & Landgrebe (1991) or other nonlinear classifiers for non-differentiable data.

ResNet-MLP on MNIST The MNIST database (LeCun *et al.*, 1998) of handwritten digits has a training set of 60,000 examples, and a test set of 10,000 examples. The data contains 10 number of classes. We test the performance of *BoostResNet* on MLP-ResNet using MNIST dataset, and compare it with *e2eBP* baseline. Each residual block is composed of an MLP with a single, 1024-dimensional hidden layer. The training and testing error between *BoostResNet* and *e2eBP* is in Figure 2 as a function of depth. Surprisingly, we observe a training error degradation for *e2eBP* although the ResNet’s identity loop is supposed to alleviate this problem. Despite the presence of identity loops, the *e2eBP* eventually is susceptible to spurious local optima. Our proposed sequential training procedure, *BoostResNet*, relieves gradient instability issues, and continues to perform well as depth increases.

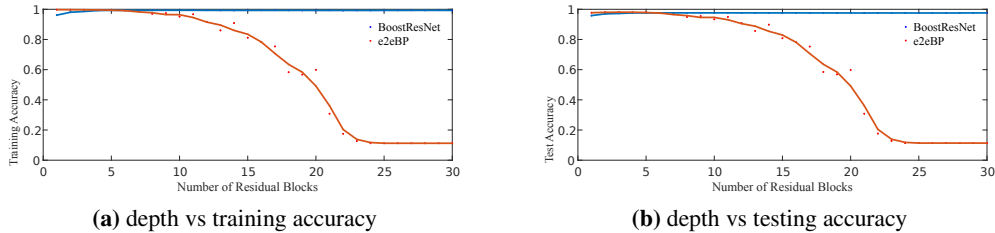


Figure 2: Comparison of *e2eBP* (ours, blue) and *BoostResNet* (baseline, red) on multilayer perceptron residual network on MNIST dataset.

ResNet-CNN on SVHN SVHN (Netzer *et al.*, 2011) is a real-world image dataset, obtained from house numbers in Google Street View images, for recognizing digits and numbers in natural scene images and is therefore significantly harder than MNIST. Many of the images contain some distractors at the sides. The dataset contains over 600,000 digit images, an order of magnitude more labeled data. There are 604,388 digits for training, 26,032 digits for testing. We test the performance of *BoostResNet* on CNN-ResNet using SVHN dataset, and compare it with *e2eBP*. Each residual block is composed of a CNN using $15 \times 3 \times 3$ filters. The training and testing error between *BoostResNet* and *e2eBP* is in Figure 3. The training error degradation of *e2eBP* is alleviated on CNN-ResNet, but the learning is relatively slow compare to our *BoostResNet*.

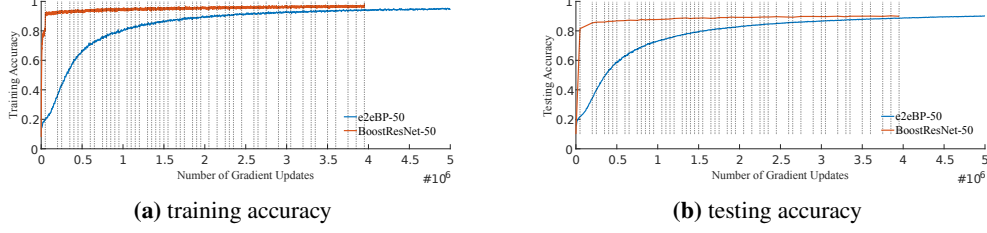


Figure 3: Convergence performance comparison between *e2eBP* (ours, red) and *BoostResNet* (baseline, blue) using convolutional neural network residual network of 50 residual blocks on SVHN dataset.

We further investigate performance of *e2eBP* and *BoostResNet* on a shallower fine tuned network of 25 residual blocks and 50 layers in total ¹. Each residual block contains convolution(5x5), batch normalization, ReLU, convolution(5x5), batch normalization, addition with identity loop and ReLU. The learning rate is fixed to be $10e-4$ with no decay as suggested. The accuracy result is shown in Figure 4. Both algorithms are comparable using *e2eBP*'s fine tuned network architecture and hyper parameters although the parameters of *BoostResNet* is not fine tuned.

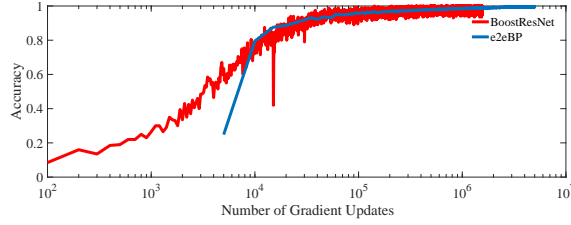


Figure 4: Accuracy of SVHN on a fine tuned ResNet with 25 residual blocks. The rapid convergence *BoostResNet* enjoys is less than on deeper networks.

Weak Learning Condition Check The weak learning condition (Definition 4.1) inspired by learning theory is checked in Figure 5. The required better than random guessing edge γ_t is depicted in Figure 5a, it is always greater than 0 and our weak learning condition is thus non-vacuous. In Figure 5b, the representations we learned using *BoostResNet* is increasingly better (for this classification task) as the depth increases.

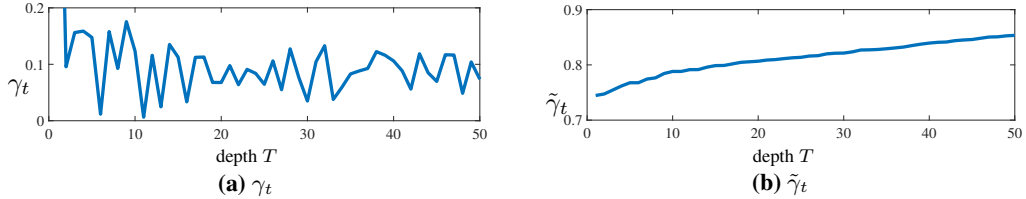


Figure 5: Visualization of required larger than 0 (better than random guessing) edge γ_t and edge for each residual block $\tilde{\gamma}_t$. The x-axis represents depth, and the y-axis represents γ_t or $\tilde{\gamma}_t$ values. The plots are for convolutional neural network residual network of depth 50 on SVHN dataset.

Computational and Memory Efficiency It is worth noting that *BoostResNet* training is memory efficient as the training process only requires parameters of two consecutive residual blocks to be in memory. Given that the limited GPU memory being one of the main bottlenecks for computational efficiency, *BoostResNet* requires significantly less training time than *e2eBP* in deep networks as a result of reduced communication overhead and the speed-up in shallow gradient forwarding and back-propagation. Let M_1 be the memory required for one module, and M_2 be the memory required for one linear classifier, the memory consumption is $M_1 + M_2$ by *BoostResNet* and $M_1T + M_2$ by *e2eBP*. Let the flops needed for gradient update over one module and one linear classifier be C_1 and C_2 respectively, the computation cost is $C_1 + C_2$ by *BoostResNet* and $C_1T + C_2$ by *BoostResNet*.

¹The network is from <https://github.com/facebook/fb.resnet.torch>.

In practice, a 50 layer CNN-ResNet requires 3 days to train using *e2eBP* but 1 day to train using *BoostResNet*.

6 Conclusions and Future Works

Our proposed BoostResNet algorithm achieves exponentially decaying (with the depth T) training error under the weak learning condition. BoostResNet is much more computationally efficient compared to end-to-end back-propagation in deep ResNet. More importantly, the memory required by BoostResNet is trivial compared to end-to-end back-propagation. It is particularly beneficial given the limited GPU memory and large network depth. Our learning framework is natural for non-differentiable data. For instance, our learning framework is amendable to take weak learning oracles using tensor decomposition techniques. Tensor decomposition, a spectral learning framework with theoretical guarantees, is applied to learning one layer MLP in (Janzamin *et al.*, 2015). We plan to extend our learning framework to non-differentiable data using general weak learning oracles.

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Appendix: Learning Deep ResNet Blocks Sequentially using Boosting Theory

A Proof for Lemma 3.2: the strong learner is a ResNet

Proof. In our algorithm, the input of the next module is the output of the current module

$$g_{t+1}(x) = f_t(g_t(x)) + g_t(x), \quad (7)$$

we thus obtain that each weak learning module is

$$h_t(x) = \alpha_{t+1} \mathbf{w}_{t+1}^\top (f_t(g_t(x)) + g_t(x)) - \alpha_t \mathbf{w}_t^\top g_t(x) \quad (8)$$

$$= \alpha_{t+1} \mathbf{w}_{t+1}^\top g_{t+1}(x) - \alpha_t \mathbf{w}_t^\top g_t(x), \quad (9)$$

and similarly

$$h_{t+1} = \alpha_{t+2} \mathbf{w}_{t+2}^\top g_{t+2}(x) - \alpha_{t+1} \mathbf{w}_{t+1}^\top g_{t+1}(x). \quad (10)$$

Therefore the sum over $h_t(x)$ and $h_{t+1}(x)$ is

$$h_t(x) + h_{t+1}(x) = \alpha_{t+2} \mathbf{w}_{t+2}^\top g_{t+2}(x) - \alpha_t \mathbf{w}_t^\top g_t(x) \quad (11)$$

And we further see that the weighted summation over all $h_t(x)$ is a telescoping sum

$$\sum_{t=1}^T h_t(x) = \alpha_{T+1} \mathbf{w}_{T+1}^\top g_{T+1}(x) - \alpha_1 \mathbf{w}_1^\top g_1(x) = \alpha_{T+1} \mathbf{w}_{T+1}^\top g_{T+1}(x). \quad (12)$$

□

B Proof for Theorem 4.2: binary class telescoping sum boosting theory

Proof. We will use a 0-1 loss to measure the training error. In our analysis, the 0-1 loss is bounded by exponential loss.

The training error is therefore bounded by

$$\Pr_{i \sim D_1} (p(\alpha_{T+1} \mathbf{w}_{T+1}^\top g_{T+1}(x_i)) \neq y_i) \quad (13)$$

$$= \sum_{i=1}^m D_1(i) \mathbf{1}\{\tilde{\sigma}(\alpha_{T+1} \mathbf{w}_{T+1}^\top g_{T+1}(x_i)) \neq y_i\} \quad (14)$$

$$= \sum_{i=1}^m D_1(i) \mathbf{1}\left\{\tilde{\sigma}\left(\sum_{t=1}^T h_t(x_i)\right) \neq y_i\right\} \quad (15)$$

$$\leq \sum_{i=1}^m D_1(i) \exp\left\{-y_i \sum_{t=1}^T h_t(x_i)\right\} \quad (16)$$

$$= \sum_{i=1}^m D_{T+1}(i) \prod_{t=1}^T Z_t \quad (17)$$

$$= \prod_{t=1}^T Z_t \quad (18)$$

where $Z_t = \sum_{i=1}^m D_t(i) \exp(-y_i h_t(x_i))$.

We choose α_{t+1} to minimize Z_t .

$$\frac{\partial Z_t}{\partial \alpha_{t+1}} = - \sum_{i=1}^m D_t(i) y_i o_{t+1} \exp(-y_i h_t(x_i)) \quad (19)$$

$$= -Z_t \sum_{i=1}^m D_{t+1}(i) y_i o_{t+1}(i) = 0 \quad (20)$$

Furthermore each learning module is bounded as we see in the following analysis. We obtain

$$Z_t = \sum_{i=1}^m D_t(i) e^{-y_i h_t(x_i)} \quad (21)$$

$$= \sum_{i=1}^m D_t(i) e^{-\alpha_{t+1} y_i o_{t+1}(x_i) + \alpha_t y_i o_t(x_i)} \quad (22)$$

$$\leq \sum_{i=1}^m D_t(i) e^{-\alpha_{t+1} y_i o_{t+1}(x_i)} \sum_{i=1}^m D_t(i) e^{\alpha_t y_i o_t(x_i)} \quad (23)$$

$$= \sum_{i=1}^m D_t(i) e^{-\alpha_{t+1} \frac{1+y_i o_{t+1}(x_i)}{2} + \alpha_{t+1} \frac{1-y_i o_{t+1}(x_i)}{2}} \sum_{i=1}^m D_t(i) e^{\alpha_t \frac{1+y_i o_t(x_i)}{2} - \alpha_t \frac{1-y_i o_t(x_i)}{2}} \quad (24)$$

$$\leq \sum_{i=1}^m D_t(i) \left(\frac{1+y_i o_{t+1}(x_i)}{2} e^{-\alpha_{t+1}} + \frac{1-y_i o_{t+1}(x_i)}{2} e^{\alpha_{t+1}} \right). \quad (25)$$

$$\sum_{i=1}^m D_t(i) \left(\frac{1+y_i o_t(x_i)}{2} e^{\alpha_t} + \frac{1-y_i o_t(x_i)}{2} e^{-\alpha_t} \right) \quad (26)$$

$$= \sum_{i=1}^m D_t(i) \left(\frac{1+y_i o_{t+1}(x_i)}{2} e^{-\alpha_{t+1}} + \frac{1-y_i o_{t+1}(x_i)}{2} e^{\alpha_{t+1}} \right) \frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \quad (27)$$

$$= \sum_{i=1}^m D_t(i) \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}}}{2} + \frac{e^{-\alpha_{t+1}} - e^{\alpha_{t+1}}}{2} y_i o_{t+1}(x_i) \right) \frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \quad (28)$$

$$= \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}}}{2} + \frac{e^{-\alpha_{t+1}} - e^{\alpha_{t+1}}}{2} \tilde{\gamma}_t \right) \frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \quad (29)$$

Equation (23) is due to the non-positive correlation between $\exp(-y o_{t+1}(x))$ and $\exp(y o_t(x))$. Jensen's inequality in Equation (26) holds only when $|y_i o_{t+1}(x_i)| \leq 1$ which is satisfied by the definition of the weak learning module.

The algorithm chooses α_{t+1} to minimize Z_t . We achieve an upper bound on Z_t , $\sqrt{\frac{1-\tilde{\gamma}_t^2}{1-\tilde{\gamma}_{t-1}^2}}$ by minimizing the bound in Equation (29)

$$Z_t \leq \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}}}{2} + \frac{e^{-\alpha_{t+1}} - e^{\alpha_{t+1}}}{2} \tilde{\gamma}_t \right) \frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \Big|_{\alpha_{t+1} = \frac{1}{2} \ln \left(\frac{1+\tilde{\gamma}_t}{1-\tilde{\gamma}_t} \right)} \quad (30)$$

$$= \sqrt{\frac{1-\tilde{\gamma}_t^2}{1-\tilde{\gamma}_{t-1}^2}} = \sqrt{1-\gamma_t^2} \quad (31)$$

Therefore over the T modules, the training error is upper bounded as follows

$$\Pr_{i \sim D} (p(\alpha_{T+1} w_{T+1}^\top g_{T+1}(x_i))) \neq y_i \leq \prod_{t=1}^T \sqrt{1-\gamma_t^2} \leq \prod_{t=1}^T \sqrt{1-\gamma^2} = \exp \left(-\frac{1}{2} T \gamma^2 \right) \quad (32)$$

Overall, Algorithm 1 leads us to consistent learning of ResNet. \square

C Proof for Lemma 4.3: Generalization Bound

Rademacher complexity technique is powerful for measuring the complexity of \mathcal{H} any family of functions $h : \mathcal{X} \rightarrow \mathbb{R}$, based on easiness of fitting any dataset using classifiers in \mathcal{H} (where \mathcal{X} is any space). Let $S = \langle x_1, \dots, x_m \rangle$ be a sample of m points in \mathcal{X} . The empirical Rademacher complexity of \mathcal{H} with respect to S is defined to be

$$\mathcal{R}_S(\mathcal{H}) \stackrel{\text{def}}{=} \mathbb{E}_\sigma \left[\sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m \sigma_i h(x_i) \right] \quad (33)$$

where σ is the Rademacher variable. The Rademacher complexity on m data points drawn from distribution \mathcal{D} is defined by

$$\mathcal{R}_m(\mathcal{H}) = \mathbb{E}_{S \sim \mathcal{D}} [\mathcal{R}_S(\mathcal{H})]. \quad (34)$$

Proposition C.1. (Theorem 1 Cortes et al. (2014)) Let \mathcal{H} be a hypothesis set admitting a decomposition $\mathcal{H} = \cup_{i=1}^l \mathcal{H}_i$ for some $l > 1$. \mathcal{H}_i are distinct hypothesis sets. Let S be a random sequence of m points chosen independently from \mathcal{X} according to some distribution \mathcal{D} . For $\theta > 0$ and any $H = \sum_{t=1}^T h_t$, with probability at least $1 - \delta$,

$$\begin{aligned} \Pr_{\mathcal{D}}(yH(x) \leq 0) &\leq \Pr_S(yH(x) \leq \theta) + \frac{4}{\theta} \sum_{t=1}^T \mathcal{R}_m(\mathcal{H}_{k_t}) + \frac{2}{\theta} \sqrt{\frac{\log l}{m}} \\ &\quad + \sqrt{\left\lceil \frac{4}{\theta^2} \log \left(\frac{\theta^2 m}{\log l} \right) \right\rceil \frac{\log l}{m} + \frac{\log \frac{2}{\delta}}{2m}} \end{aligned} \quad (35)$$

for all $h_t \in \mathcal{H}_{k_t}$.

Lemma C.2. Let $\tilde{h} = \tilde{\mathbf{w}}^\top \tilde{\mathbf{f}}$, where $\tilde{\mathbf{w}} \in \mathbb{R}^n$, $\tilde{\mathbf{f}} \in \mathbb{R}^n$. Let $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{F}}$ be two hypothesis sets, and $\tilde{h} \in \tilde{\mathcal{H}}$, $\tilde{\mathbf{f}}_j \in \tilde{\mathcal{F}}$, $\forall j \in [n]$. The Rademacher complexity of $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{F}}$ with respect to m points from \mathcal{D} are related as follows

$$\mathcal{R}_m(\tilde{\mathcal{H}}) = \|\tilde{\mathbf{w}}\|_1 \mathcal{R}_m(\tilde{\mathcal{F}}). \quad (36)$$

C.1 ResNet Module Hypothesis Space

Let n be the number of channels in ResNet, i.e., the number of input or output neurons in a module $\mathbf{f}_t(\mathbf{g}_t(x))$. We have proved that ResNet is equivalent as

$$F(x) = \mathbf{w}^\top \sum_{t=1}^T \mathbf{f}(\mathbf{g}_t(x)) \quad (37)$$

We define the family of functions that each neuron $f_{t,j}$, $\forall j \in [n]$ belong to as

$$\mathcal{F}_t = \{x \rightarrow \mathbf{u}_{t-1,j}(\sigma \circ \mathbf{f}_{t-1})(x) : \mathbf{u}_{t-1,j} \in \mathbb{R}^n, \|\mathbf{u}_{t-1,j}\|_1 \leq \Lambda_{t,t-1}, \mathbf{f}_{t-1,i} \in \mathcal{F}_{t-1}\} \quad (38)$$

where $\mathbf{u}_{t-1,j}$ denotes the vector of weights for connections from unit j to a lower layer $t-1$, $\sigma \circ \mathbf{f}_{t-1}$ denotes element-wise nonlinear transformation on \mathbf{f}_{t-1} . The output layer of each module is connected to the output layer of previous module. We consider 1-layer modules for convenience of analysis.

Therefore in ResNet with probability at least $1 - \delta$,

$$\begin{aligned} \Pr_{\mathcal{D}}(yF(x) \leq 0) &\leq \Pr_S(yF(x) \leq \theta) + \frac{4}{\theta} \sum_{t=1}^T \|\mathbf{w}\|_1 \mathcal{R}_m(\mathcal{F}_t) + \frac{2}{\theta} \sqrt{\frac{\log T}{m}} \\ &\quad + \sqrt{\left\lceil \frac{4}{\theta^2} \log \left(\frac{\theta^2 m}{\log T} \right) \right\rceil \frac{\log T}{m} + \frac{\log \frac{2}{\delta}}{2m}} \end{aligned} \quad (39)$$

for all $f_t \in \mathcal{F}_t$.

Define the maximum infinity norm over samples as $r_\infty \stackrel{\text{def}}{=} \mathbb{E}_{S \sim \mathcal{D}} [\max_{i \in [m]} \|x_i\|_\infty]$ and the product of l_1 norm bound on weights as $\Lambda_t \stackrel{\text{def}}{=} \prod_{t'=1}^t 2\Lambda_{t',t'-1}$. According to lemma 2 of Cortes et al. (2016), the empirical Rademacher complexity is bounded as a function of r_∞ , Λ_t and n :

$$\mathcal{R}_m(\mathcal{F}_t) \leq r_\infty \Lambda_t \sqrt{\frac{\log(2n)}{2m}} \quad (40)$$

Overall, with probability at least $1 - \delta$,

$$\begin{aligned} \Pr_{\mathcal{D}}(yF(x) \leq 0) &\leq \Pr_S(yF(x) \leq \theta) + \frac{4\|\mathbf{w}\|_1 r_\infty \sqrt{\frac{\log(2n)}{2m}}}{\theta} \sum_{t=1}^T \Lambda_t \\ &\quad + \frac{2}{\theta} \sqrt{\frac{\log T}{m}} + \sqrt{\left\lceil \frac{4}{\theta^2} \log \left(\frac{\theta^2 m}{\log T} \right) \right\rceil \frac{\log T}{m} + \frac{\log \frac{2}{\delta}}{2m}} \end{aligned} \quad (41)$$

for all $f_t \in \mathcal{F}_t$.

D Proof for Theorem D: Margin and Generalization Bound

Theorem D.1. [*Generalization error bound*] Given algorithm 1, the fraction of training examples with margin at most θ is at most $(1 + \frac{2}{\frac{1}{\tilde{\gamma}_{T+1}} - 1})^{\frac{\theta}{2}} \exp(-\frac{1}{2}\gamma^2 T)$. And the generalization error $\Pr_D(yF(x) \leq 0)$ satisfies

$$\begin{aligned} \Pr_D(yF(x) \leq 0) &\leq (1 + \frac{2}{\frac{1}{\tilde{\gamma}_{T+1}} - 1})^{\frac{\theta}{2}} \exp(-\frac{1}{2}\gamma^2 T) \\ &\quad + \frac{4C_0 r_\infty}{\theta} \sqrt{\frac{\log(2n)}{2m}} \sum_{t=1}^T \Lambda_t + \frac{2}{\theta} \sqrt{\frac{\log T}{m}} + \beta(\theta, m, T, \delta) \end{aligned} \quad (42)$$

with probability at least $1 - \delta$ for $\beta(\theta, m, T, \delta) \stackrel{\text{def}}{=} \sqrt{\left\lceil \frac{4}{\theta^2} \log \left(\frac{\theta^2 m}{\log T} \right) \right\rceil \frac{\log T}{m} + \frac{\log \frac{2}{\delta}}{2m}}$.

Now the proof for Theorem D is the following.

Proof. The fraction of examples in sample set S being smaller than θ is bounded

$$\Pr_S(yF(x) \leq \theta) \leq \frac{1}{m} \sum_{i=1}^m \mathbf{1}\{y_i F(x_i) \leq \theta\} \quad (43)$$

$$= \frac{1}{m} \sum_{i=1}^m \mathbf{1}\{y_i \sum_{t=1}^T h_t(x_i) \leq \theta \alpha_{T+1}\} \quad (44)$$

$$\leq \frac{1}{m} \sum_{i=1}^m \exp(-y_i \sum_{t=1}^T h_t(x_i) + \theta \alpha_{T+1}) \quad (45)$$

$$= \exp(\theta \alpha_{T+1}) \frac{1}{m} \sum_{i=1}^m \exp(-y_i \sum_{t=1}^T h_t(x_i)) \quad (46)$$

$$= \exp(\theta \alpha_{T+1}) \prod_{t=1}^T Z_t \quad (47)$$

To bound $\exp(\theta \alpha_{T+1}) = \sqrt{(\frac{1+\tilde{\gamma}_{T+1}}{1-\tilde{\gamma}_{T+1}})^\theta}$, we first bound $\tilde{\gamma}_{T+1}$: We know that $\sum_{t=1}^T \prod_{t'=t+1}^T (1 - \gamma_{t'}^2) \gamma_t^2 \leq (1 - \gamma^2)^{T-t} \gamma^2$ for all $\forall \gamma_t \geq \gamma^2 + \epsilon$ if $\gamma^2 \geq \frac{1-\epsilon}{2}$. Therefore $\forall \gamma_t \geq \gamma^2 + \epsilon$ and $\gamma^2 \geq \frac{1-\epsilon}{2}$

$$\tilde{\gamma}_{T+1}^2 = (1 - \gamma_T^2) \tilde{\gamma}_T^2 + \gamma_T^2 \quad (48)$$

$$= \sum_{t=1}^T \prod_{t'=t+1}^T (1 - \gamma_{t'}^2) \gamma_t^2 + \prod_{t=1}^T (1 - \gamma_t^2) \tilde{\gamma}_1^2 \quad (49)$$

$$\leq \sum_{t=1}^T (1 - \gamma^2)^{T-t} \gamma^2 + (1 - \gamma^2)^T \tilde{\gamma}_1^2 \quad (50)$$

$$= \sum_{t=0}^{T-1} (1 - \gamma^2)^t \gamma^2 + (1 - \gamma^2)^T \tilde{\gamma}_1^2 \quad (51)$$

$$= 1 - (1 - \gamma^2)^T + (1 - \gamma^2)^T \tilde{\gamma}_1^2 \quad (52)$$

$$= 1 - (1 - \tilde{\gamma}_1^2)(1 - \gamma^2)^T \quad (53)$$

Therefore

$$\Pr_S(yF(x) \leq \theta) \leq \exp(\theta\alpha_{T+1}) \prod_{t=1}^T Z_t \quad (54)$$

$$= \left(\frac{1 + \tilde{\gamma}_{T+1}}{1 - \tilde{\gamma}_{T+1}}\right)^{\frac{\theta}{2}} \prod_{t=1}^T Z_t \quad (55)$$

$$= \left(\frac{1 + \tilde{\gamma}_{T+1}}{1 - \tilde{\gamma}_{T+1}}\right)^{\frac{\theta}{2}} \prod_{t=1}^T \sqrt{1 - \gamma_t^2} \quad (56)$$

$$= \left(1 + \frac{2}{\frac{1}{\tilde{\gamma}_{T+1}} - 1}\right)^{\frac{\theta}{2}} \exp\left(-\frac{1}{2}\gamma^2 T\right) \quad (57)$$

$$\leq \left(1 + \frac{2}{\frac{1}{\sqrt{1-(1-\tilde{\gamma}_1^2)(1-\gamma^2)^T}} - 1}\right)^{\frac{\theta}{2}} \exp\left(-\frac{1}{2}\gamma^2 T\right) \quad (58)$$

As $T \rightarrow \infty$, $\Pr_S(yF(x) \leq \theta) \leq 0$ as $\exp(-\frac{1}{2}\gamma^2 T)$ decays faster than $\left(1 + \frac{2}{\frac{1}{\sqrt{1-(1-\tilde{\gamma}_1^2)(1-\gamma^2)^T}} - 1}\right)^{\frac{\theta}{2}}$. \square

E Telescoping Sum Boosting for Multi-class Classification

Recall that the weak module classifier is defined as

$$h_t(x) = \alpha_{t+1}o_{t+1}(x) - \alpha_t o_t(x) \in \mathbb{R}^C, \quad (59)$$

where $o_t(x) \in \Delta^{C-1}$.

The weak learning condition for multi-class classification is different from the binary classification stated in the previous section, although minimal demands placed on the weak module classifier require prediction better than random on any distribution over the training set intuitively.

We now define the weak learning condition. It is again inspired by the slightly better than random idea, but requires a more sophisticated analysis in the multi-class setting.

E.1 Cost Matrix

In order to characterize the training error, we introduce the cost matrix $\mathbf{C} \in \mathbb{R}^{m \times C}$ where each row denote the cost incurred by classifying that example into one of the C categories. We will bound the training error using exponential loss, and under the exponential loss function defined as in Definition F.1, the optimal cost function used for best possible training error is therefore determined.

Lemma E.1. *The optimal cost function under the exponential loss is*

$$\mathbf{C}_t(i, l) = \begin{cases} \exp(s_t(x_i, l) - s_t(x_i, y_i)) & \text{if } l \neq y_i \\ -\sum_{l' \neq y_i} \exp(s_t(x_i, l') - s_t(x_i, y_i)) & \text{if } l = y_i \end{cases} \quad (60)$$

where $s_t(x) = \sum_{\tau=1}^t h_\tau(x)$.

E.2 Weak Learning Condition

Definition E.2. Let $\tilde{\gamma}_{t+1} = \frac{-\sum_{i=1}^m \langle \mathbf{C}_t(i, :), o_{t+1}(x_i) \rangle}{\sum_{i=1}^m \sum_{l \neq y_i} \mathbf{C}_t(i, l)}$ and $\tilde{\gamma}_t = \frac{-\sum_{i=1}^m \langle \mathbf{C}_{t-1}(i, :), o_t(x_i) \rangle}{\sum_{i=1}^m \sum_{l \neq y_i} \mathbf{C}_{t-1}(i, l)}$. A multi-class

weak module classifier $h_t(x) = \alpha_{t+1}o_{t+1}(x) - \alpha_t o_t(x)$ satisfies the γ -weak learning condition if $\frac{\tilde{\gamma}_{t+1}^2 - \tilde{\gamma}_t^2}{1 - \tilde{\gamma}_t^2} \geq \gamma^2 > 0$.

We propose a novel learning algorithm using the optimal edge-over-random cost function for training ResNet under multi-class classification task as in Algorithm 3.

Algorithm 3 BoostResNet: telescoping sum boosting for multi-class classification

Input: Given $(x_1, y_1), \dots, (x_m, y_m)$ where $y_i \in \mathcal{Y} = \{1, \dots, C\}$ and a threshold γ
Output: $\{f_t(\cdot), \forall t\}$ and W_{T+1} \triangleright Discard $\mathbf{w}_{t+1}, \forall t \neq T$

- 1: Initialize $t \leftarrow 0, \tilde{\gamma}_0 \leftarrow 1, \alpha_0 \leftarrow 0, o_0 \leftarrow \mathbf{0} \in \mathbb{R}^C, s_0(x_i, l) = 0, \forall i \in [m], l \in \mathcal{Y}$
- 2: Initialize cost function $\mathbf{C}_0(i, l) \leftarrow \begin{cases} 1 & \text{if } l \neq y_i \\ 1 - C & \text{if } l = y_i \end{cases}$
- 3: **while** $\gamma_t > \gamma$ **do**
- 4: $f_t(\cdot), \alpha_{t+1}, W_{t+1}, o_{t+1}(x) \leftarrow \text{Algorithm 4}(g_t(x), \mathbf{C}_t, o_t(x), \alpha_t)$
- 5: Compute $\gamma_t \leftarrow \sqrt{\frac{\tilde{\gamma}_{t+1}^2 - \tilde{\gamma}_t^2}{1 - \tilde{\gamma}_t^2}}$ \triangleright where $\tilde{\gamma}_{t+1} \leftarrow \frac{-\sum_{i=1}^m \mathbf{C}_t(i, \cdot) \cdot o_{t+1}(x_i)}{\sum_{i=1}^m \sum_{l \neq y_i} \mathbf{C}_t(i, l)}$
- 6: Update $s_{t+1}(x_i, l) \leftarrow s_t(x_i, l) + h_t(x_i, l) \triangleright$ where $h_t(x_i, l) = \alpha_{t+1} o_{t+1}(x_i, l) - \alpha_t o_t(x_i, l)$
- 7: Update cost function $\mathbf{C}_{t+1}(i, l) \leftarrow \begin{cases} e^{s_{t+1}(x_i, l) - s_{t+1}(x_i, y_i)} & \text{if } l \neq y_i \\ -\sum_{l' \neq y_i} e^{s_{t+1}(x_i, l') - s_{t+1}(x_i, y_i)} & \text{if } l = y_i \end{cases}$
- 8: $t \leftarrow t + 1$
- 9: **end while**
- 10: $T \leftarrow t - 1$

Algorithm 4 BoostResNet: oracle implementation for training a ResNet module (multi-class)

Input: $g_t(x), s_t, o_t(x)$ and α_t
Output: $f_t(\cdot), \alpha_{t+1}, W_{t+1}$ and $o_{t+1}(x)$

- 1: $(f_t, \alpha_{t+1}, W_{t+1}) \leftarrow \arg \min_{(f, \alpha, V)} \sum_{i=1}^m \sum_{l \neq y_i} e^{\alpha V^\top [f(g_t(x_i), l) - f(g_t(x_i), y_i) + g_t(x_i, l) - g_t(x_i, y_i)]}$
- 2: $o_{t+1}(x) \leftarrow W_{t+1}^\top [f_t(g_t(x)) + g_t(x)]$

Theorem E.3. *The training error of a T -module ResNet using Algorithm 3 and 4 decays exponentially with the depth of the ResNet T ,*

$$\frac{C-1}{m} \sum_{i=1}^m L_\eta^{\text{exp}}(s_T(x_i)) \leq (C-1)e^{-\frac{1}{2}T\gamma^2} \quad (61)$$

if the weak module classifier $h_t(x)$ satisfies the γ -weak learning condition $\forall t \in [T]$.

The exponential loss function defined as in Definition F.1

E.3 Oracle Implementation

We implement an oracle to minimize $Z_t \stackrel{\text{def}}{=} \sum_{i=1}^m \sum_{l \neq y_i} e^{s_t(x_i, l) - s_t(x_i, y_i)} e^{h_t(x_i, l) - h_t(x_i, y_i)}$ given current state s_t and hypothesis module $o_t(x)$. Therefore minimizing Z_t is equivalent to the following.

$$\min_{(f, \alpha, V)} \sum_{i=1}^m \sum_{l \neq y_i} e^{s_t(x_i, l) - s_t(x_i, y_i)} e^{-\alpha_t(o_t(x_i, l) - o_t(x_i, y_i))} e^{\alpha V^\top [f(g_t(x_i), l) - f(g_t(x_i), y_i) + g_t(x_i, l) - g_t(x_i, y_i)]} \quad (62)$$

$$\equiv \min_{(f, \alpha, V)} \sum_{i=1}^m \sum_{l \neq y_i} e^{\alpha V^\top [f(g_t(x_i), l) - f(g_t(x_i), y_i) + g_t(x_i, l) - g_t(x_i, y_i)]} \quad (63)$$

$$\equiv \min_{\alpha, f, v} \sum_{i=1}^m e^{-\alpha v^\top [f(x_i, y_i) + g_t(x_i, y_i)]} \sum_{l \neq y_i} e^{\alpha v^\top [f(x_i, l) + g_t(x_i, l)]} \quad (64)$$

F Proof for Theorem E.3 multi-class boosting theory

Proof. To characterize the training error, we use the exponential loss function

Definition F.1. Define loss function for a multi-class hypothesis $H(x_i)$ on a sample (x_i, y_i) as

$$L_\eta^{exp}(H(x_i), y_i) = \sum_{l \neq y_i} \exp((H(x_i, l) - H(x_i, y_i))). \quad (65)$$

Define the accumulated weak learner $s_t(x_i, l) = \sum_{t'=1}^t h_{t'}(x_i, l)$ and the loss $Z_t = \sum_{i=1}^m \sum_{l \neq y_i} \exp(s_t(x_i, l) - s_t(x_i, y_i)) \exp(h_t(x_i, l) - h_t(x_i, y_i))$.

Recall that $s_t(x_i, l) = \sum_{t'=1}^t h_{t'}(x_i, l) = \alpha_{t+1} W_{t+1}^\top g_{t+1}(x_i)$, the loss for a T -module multiclass resnet is thus

$$\Pr_{i \sim D_1} (p(\alpha_{T+1} W_{T+1}^\top g_{T+1}(x_i)) \neq y_i) \leq \frac{1}{m} \sum_{i=1}^m L_\eta^{exp}(s_T(x_i)) \quad (66)$$

$$\leq \frac{1}{m} \sum_{i=1}^m \sum_{l \neq y_i} \exp(\eta(s_T(x_i, l) - s_T(x_i, y_i))) \quad (67)$$

$$\leq \frac{1}{m} Z_T \quad (68)$$

$$= \prod_{t=1}^T \frac{Z_t}{Z_{t-1}} \quad (69)$$

Note that $Z_0 = \frac{1}{m}$ as the initial accumulated weak learners $s_0(x_i, l) = 0$.

The loss fraction between module t and $t-1$, $\frac{Z_t}{Z_{t-1}}$, is related to $Z_t - Z_{t-1}$ as $\frac{Z_t}{Z_{t-1}} = \frac{Z_t - Z_{t-1}}{Z_{t-1}} + 1$.

The Z_t is bounded

$$Z_t = \sum_{i=1}^m \sum_{l \neq y_i} \exp(s_t(x_i, l) - s_t(x_i, y_i) + h_t(x_i, l) - h_t(x_i, y_i)) \quad (70)$$

$$\leq \sum_{i=1}^m \sum_{l \neq y_i} e^{s_t(x_i, l) - s_t(x_i, y_i)} e^{\alpha_{t+1} o_{t+1}(x_i, l) - \alpha_{t+1} o_{t+1}(x_i, y_i)} \sum_{i=1}^m \sum_{l \neq y_i} e^{s_t(x_i, l) - s_t(x_i, y_i)} e^{-\alpha_t o_t(x_i, l) + \alpha_t o_t(x_i, y_i)} \quad (71)$$

$$\leq \sum_{i=1}^m \sum_{l \neq y_i} e^{s_t(x_i, l) - s_t(x_i, y_i)} \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}}}{2} + \frac{e^{-\alpha_{t+1}} - e^{\alpha_{t+1}}}{2} (o_{t+1}(x_i, y_i) - o_{t+1}(x_i, l)) \right) \sum_{i=1}^m \sum_{l \neq y_i} e^{s_{t-1}(x_i, l) - s_{t-1}(x_i, y_i)} \left(\frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \right) \quad (72)$$

$$= \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}} - 2}{2} Z_{t-1} + \frac{e^{\alpha_{t+1}} - e^{-\alpha_{t+1}}}{2} \sum_{i=1}^m \langle C_t(x_i, :), o_{t+1}(x_i, :) \rangle \right) \left(\frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \right) \quad (73)$$

$$\leq \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}} - 2}{2} Z_{t-1} + \frac{e^{\alpha_{t+1}} - e^{-\alpha_{t+1}}}{2} \sum_{i=1}^m \langle C_t(x_i, :), U_{\tilde{\gamma}_t}(x_i, :) \rangle \right) \left(\frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \right) \quad (74)$$

$$= \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}} - 2}{2} Z_{t-1} + \frac{e^{\alpha_{t+1}} - e^{-\alpha_{t+1}}}{2} (-\tilde{\gamma}_t) Z_{t-1} \right) \left(\frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \right) \quad (75)$$

Therefore

$$\frac{Z_t}{Z_{t-1}} \leq \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}}}{2} + \frac{e^{-\alpha_{t+1}} - e^{\alpha_{t+1}}}{2} \tilde{\gamma}_t \right) \left(\frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \right) \quad (76)$$

The algorithm chooses α_{t+1} to minimize Z_t . We achieve an upper bound on Z_t , $\sqrt{\frac{1-\tilde{\gamma}_t^2}{1-\tilde{\gamma}_{t-1}^2}}$

by minimizing the bound in Equation (76)

$$Z_t \leq \left(\frac{e^{-\alpha_{t+1}} + e^{\alpha_{t+1}}}{2} + \frac{e^{-\alpha_{t+1}} - e^{\alpha_{t+1}}}{2} \tilde{\gamma}_t \right) \frac{e^{\alpha_t} + e^{-\alpha_t}}{2} \Big|_{\alpha_{t+1} = \frac{1}{2} \ln \left(\frac{1+\tilde{\gamma}_t}{1-\tilde{\gamma}_t} \right)} \quad (77)$$

$$= \sqrt{\frac{1-\tilde{\gamma}_t^2}{1-\tilde{\gamma}_{t-1}^2}} = \sqrt{1-\gamma_t^2} \quad (78)$$

Therefore over the T modules, the training error is upper bounded as follows

$$\Pr_{i \sim D} (p(\alpha_{T+1} w_{T+1}^\top g_{T+1}(x_i)) \neq y_i) \leq \prod_{t=1}^T \sqrt{1-\gamma_t^2} \leq \prod_{t=1}^T \sqrt{1-\gamma^2} = \exp \left(-\frac{1}{2} T \gamma^2 \right) \quad (79)$$

Overall, Algorithm 3 and 4 leads us to consistent learning of ResNet. \square

G Minimal Weak Learning Condition

Mukherjee and Schapire Mukherjee & Schapire (2013) introduced a sufficient and necessary weak learning condition based on a concept called *edge over random*.

It is intuitive to restrict the cost matrices to a space of edge-over-random cost-matrices \mathcal{C}^{eor} who put the least cost on the correct label.

Definition G.1. *The space of edge-over-random cost matrices $\mathcal{C}^{eor} \subseteq \mathbb{R}^{m \times C}$ is the space of cost matrices $C \in \mathcal{C}^{eor}$ that satisfies*

$$\mathbf{C}(i, y_i) \leq \mathbf{C}(i, l), \quad \forall l \neq y_i, \forall i \in \{1, \dots, m\}. \quad (80)$$

Let us define a random classifier as a baseline predictor $\mathbf{B} \in \mathbb{R}^{m \times C}$, where each row lies on a simplex $\mathbf{B}(i, :) \in \Delta\{1, \dots, C\}$. We consider the space of baselines, called edge-over-random, who have a faint clue about the correct answer.

Definition G.2. *The space of edge-over-random baselines Mukherjee & Schapire (2013) $\mathcal{B}_\gamma^{eor} \subseteq \mathbb{R}^{m \times C}$ is the space of baselines $\mathbf{B} \in \mathcal{B}_\gamma^{eor}$ that satisfy*

$$B(i, y_i) \geq B(i, l) + \gamma, \quad \forall l \neq y_i, \forall i \in \{1, \dots, m\}. \quad (81)$$

Note that cost function for sample x_i is $\mathbf{C}(i, :) \in \mathbb{R}^{1 \times C}$, weak module classifier vector $h_t(x_i) \in \mathbb{R}^C$ and baseline predictor $\mathbf{B}(i, :) \in \mathbb{R}^{1 \times C}$.

Definition G.3. *A multi-class weak module classifier $h_t(x) = \alpha_{t+1} o_{t+1} - \alpha_t o_t$ satisfies the γ -weak learning condition if $\forall \mathbf{C} \in \mathcal{C}^{eor}, \exists h_t^{V, f(\cdot), \alpha}$ such that*

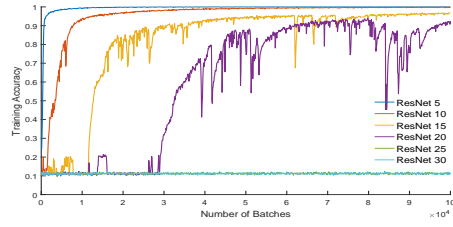
$$\sum_{i=1}^m \mathbf{C}(i, :) h_t^{V, f(\cdot), W}(x_i) \leq \max_{\mathbf{B} \in \mathcal{B}_\gamma^{eor}} \sum_{i=1}^m \mathbf{C}(i, :) \mathbf{B}(i, :)^T. \quad (82)$$

Lemma G.4. *A weak classifier space \mathcal{H} is boostable if and only if \mathcal{H} satisfies the weak-learning condition defined in Definition G.3.*

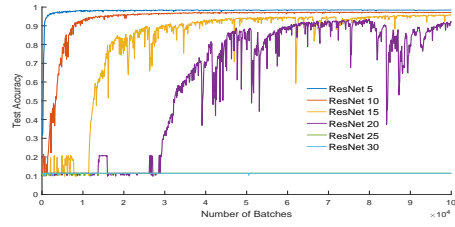
Refer to Mukherjee and Schapire Mukherjee & Schapire (2013) for the proof.

H Experiments

We investigate e2eBP training performance on various depth ResNet. Surprisingly, we observe a training error degradation for *e2eBP* although the ResNet's identity loop is supposed to alleviate this problem. Despite the presence of identity loops, the *e2eBP* eventually is susceptible to spurious local optima. This phenomenon is explored further in Figures 6a and 6b, which respectively show how training and test accuracies vary throughout the fitting process. Our proposed sequential training procedure, *BoostResNet*, relieves gradient instability issues, and continues to perform well as depth increases.



(a) e2ebp training accuracy



(b) e2ebp testing accuracy

Figure 6: Convergence of *e2eBP* (baseline) on multilayer perceptron residual network (of various depths) on MNIST dataset.