Python Programming in Finance

Futures & Options

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Online Course

Introduction

- A derivative is a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying assets.
- Four major types of derivatives are forwards[1], futures[2], options[3], and swaps[4].

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[1] See https://en.wikipedia.org/wiki/Forward_contract.

[2] See https://en.wikipedia.org/wiki/Futures_contract and https://www.cmegroup.com/education/courses/introduction-to-futures.html.

[3] See https://www.cmegroup.com/education/courses/introduction-to-options.html.

[4] See https://en.wikipedia.org/wiki/Swap_(finance).

Futures Contracts

- A futures contract is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price.
- For example, index futures (say TX[5] and S&P 500 E-Mini[6]), equity futures[7], commodity futures[8] (say crude oil, gold, corn, and cocoa), weather futures[9], and cryptocurrency futures (say, Bitcoin[10]).

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[5] See `https://www.taifex.com.tw/cht/2/tX`.

[6] See `https://www.investing.com/indices/us-spx-500-futures`.

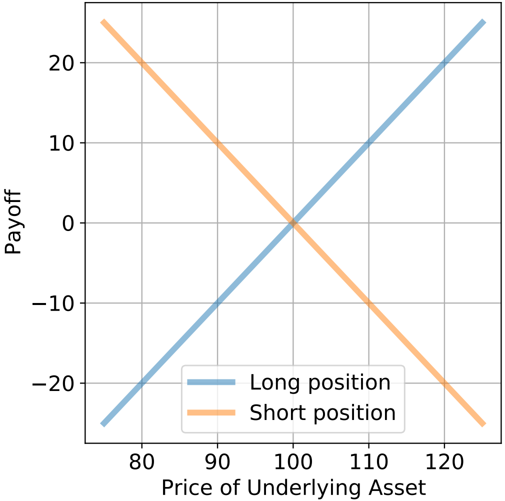
[7] See `https://pchome.megatime.com.tw/group/mkt5/cidE008.html`.

[8] See `https://www.cnbc.com/futures-and-commodities/` and `http://www.stockq.org/market/commodity.php`.

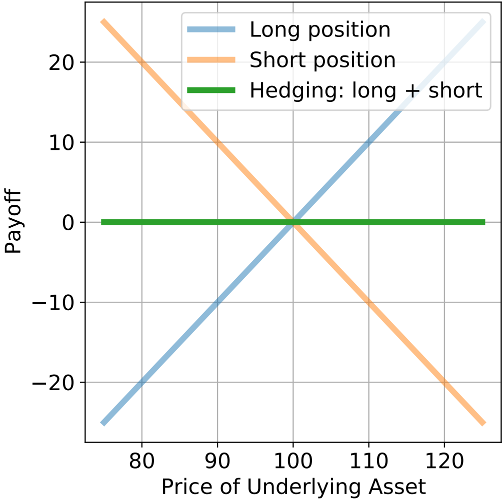
[9] See `https://www.cmegroup.com/trading/weather/files/weather-futures-and-options-fact-card.pdf`.

[10] See `https://www.cmegroup.com/markets/cryptocurrencies/bitcoin/bitcoin.html`.

Payoff of Futures



One of Applications: Hedging



Futures Contracts (Cont'd)

- A contract's expiration date is the last day you can trade that contract.
  - For example, the expiration date of TX contracts is the 3rd Wednesday each month.
- Futures margin is an amount of capital one needs to deposit to control a futures contract.
  - Be responsible for assuring that adequate funds are on deposit with the brokerage firm for margin purposes.[11]
  - If not, you cannot trade any futures.
  - It is called the margin trading.

[11] See `https://www.taifex.com.tw/cht/5/indexMarging`.

Futures Contracts (Concluded)

- An index futures is a cash-settled futures contract while a commodity futures may be settled by physical delivery.
  - For example, WTI crude oil futures.[12]
- Further information about futures could be found in the footnote.[13]
  - For example, futures markets provide a price discovery mechanism, especially when trading in the off-market hours.[14]

[12] See https://www.oilandgas360.com/wti-crude-oil-futures-prices-fell-below-zero-because-of-low-liquidity-

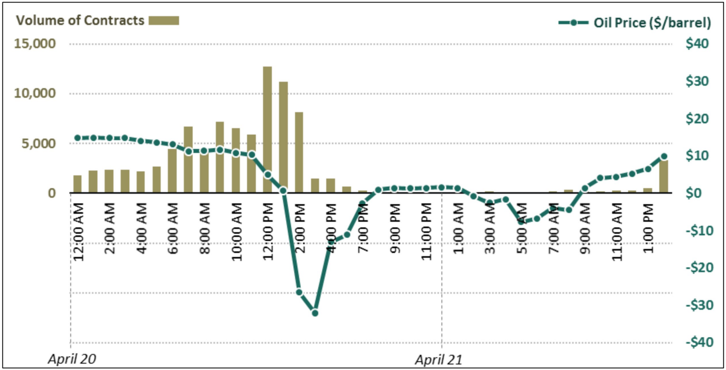
[13] See https://www.yuantafutures.com.tw/futureinvest_01, https://rich01.com/futures-least-margin-100/, and https://zerodha.com/varsity/module/futures-trading/.

[14] See https://www.cmegroup.com/education/courses/introduction-to-futures/price-discovery.html.

News: Crude Oil Futures Prices Turn Negative[15]

Figure 1. Hourly May Contract Volumes and WTI Futures Prices

April 20-21, 2020



Source: NYMEX (New York Mercantile Exchange) via ICE (the Intercontinental Exchange).

Notes: After 2:30 PM on April 21, the May futures contract stopped trading, and the June contract became the front month. Trading does not usually occur daily between 5 PM and 6 PM, so that the market can reset for the next day.

[15] See `https://crsreports.congress.gov/product/pdf/IN/IN11354`.

Futures Pricing

- Let S_0 be the spot price, and F_0 be the futures price at time 0.
- The (theoretical) futures price is

$$F_0 = S_0 e^{cT}, \tag{1}$$

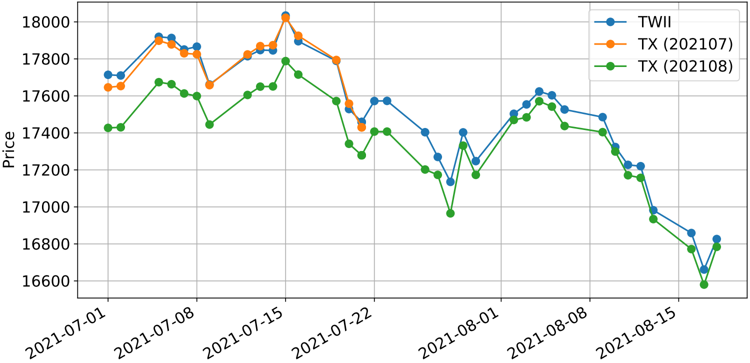
  where T is the time to maturity, and c is the cost of carry for the futures contract.
  - For a non-dividend-paying stock, $c = r$ where r is the risk-free interest rate.
  - For a dividend-paying stock, $c = r - q$ where q is the dividend yield rate.
  - For a commodity requiring storage costs at rate u, $c = r + u$.
- The situation of $F_0 > S_0$ is called contango; backwardation when $F_0 < S_0$.

Example

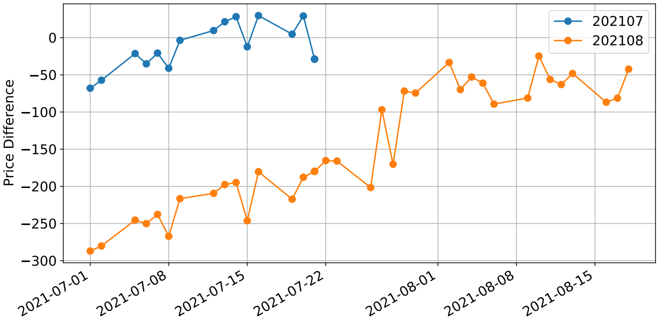
- Assume that the current market index is 17000 with $r = 3\%$ and $q = 5\%$.
- Then the resulting price of the near-month $(T = 1\,/\,12)$ futures is
$$17000 \times e^{-0.02/12} = 16971.69.$$
- For the next-month $(T = 1\,/\,6)$ futures, its price is 16943.43.

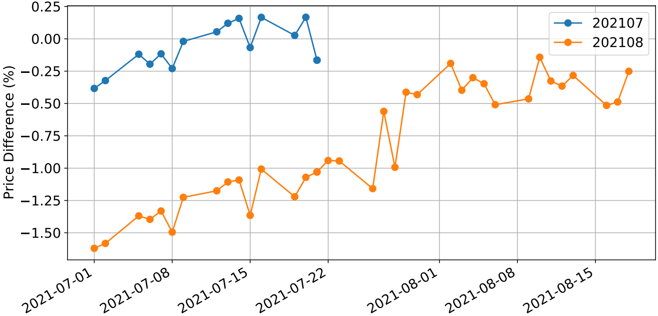
Convergence between Spot Price and Futures Price[16]

- By equation (1), the futures price always converges upon the spot price on the day of the expiry.



[16] You can find the historical futures prices [here](#).


202107
202108
Price Difference
0
−50
−100
−150
−200
−250
−300
2021-07-01
2021-07-08
2021-07-15
2021-07-22
2021-08-01
2021-08-08
2021-08-15


202107
202108
Price Difference (%)
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.25
−1.50
2021-07-01
2021-07-08
2021-07-15
2021-07-22
2021-08-01
2021-08-08
2021-08-15

Option Contracts[18]

- A call/put option gives the holder to the right to buy/sell an asset by the expiration (or maturity) date, denoted by T, for the exercise (or strike) price, denoted by X.
- European options can be exercised only on the expiration date itself, whereas American options can be exercised at any time up to the expiration date.
- For example, index options (say TXO[17]) and foreign exchange options, options on futures.

[17] See https://www.taifex.com.tw/cht/2/tXO.

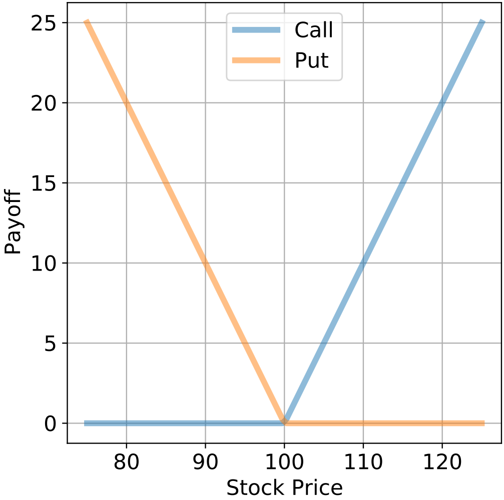
[18] You may also hear about the warrant contracts. See http://www.differencebetween.net/business/finance-business-2/difference-between-warrants-and-options.

到期年月 202105 | 商品 台指選TXO | 期貨 TXFE1 | 期 台指期貨05 | 預設欄位 | 欄位設定 | ◉ 同步模式 ○ 對稱模式

加權指數 17,285.00 ▲ 290.64 1.71% 4525.67 億 | 期貨 17333 ▲ 50 37,082 | 賣出價(量) 17333 4 | 買進價(量) 17329 8

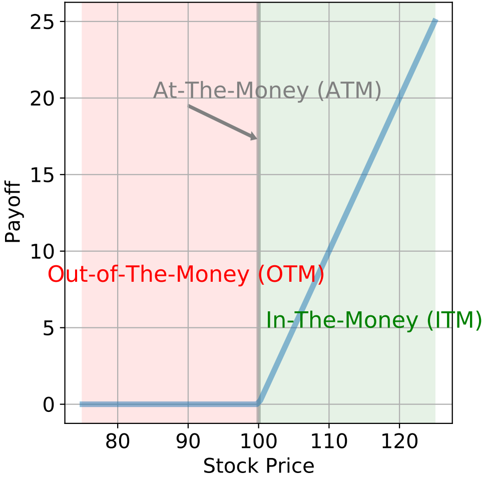
買權							賣權					
買進價	賣出價	成交價	漲跌	漲跌幅	單量	履約價	買進價	賣出價	成交價	漲跌	漲跌幅	單量
515.00	-	-	-	0.00	-	15500	4.70	6.00	6.00	▲ 1.00	20.00	1
1.10	-	-	-	0.00	-	15600	5.00	7.00	7.90	▲ 2.10	36.20	1
1.10	-	-	-	0.00	-	15700	6.00	9.00	7.10	▲ 0.50	7.57	1
67.00	-	-	-	0.00	-	15800	6.60	8.90	7.20	▼ 0.40	-5.26	20
63.00	-	-	▲ 270.00	19.42	1	15900	7.40	10.00	10.00	▲ 0.70	7.52	1
1310.00	-	1320.00	▲ 30.00	2.32	1	16000	10.50	11.50	10.50	-	0.00	1
237.00	-	-	▲ 295.00	24.58	1	16100	13.00	15.00	12.50	▼ 1.00	-7.40	1
197.00	-	-	▲ 300.00	27.27	1	16200	15.00	16.50	15.00	▼ 1.00	-6.25	2
955.00	1580.00	1030.00	▲ 30.00	3.00	1	16300	18.50	20.00	20.00	-	0.00	1
750.00	1500.00	-	▲ 210.00	23.07	1	16400	23.00	25.50	26.50	▲ 0.50	1.92	1
775.00	1400.00	845.00	▲ 30.00	3.68	1	16500	28.00	29.50	30.00	▼ 3.50	-10.44	2
740.00	830.00	750.00	▲ 25.00	3.44	1	16600	35.00	38.00	35.50	▼ 7.50	-17.44	1
610.00	695.00	690.00	▲ 55.00	8.66	1	16700	43.50	47.00	43.00	▼ 12.00	-21.81	2
570.00	635.00	595.00	▲ 40.00	7.20	1	16800	51.00	58.00	52.00	▼ 16.00	-23.52	1
494.00	525.00	496.00	▲ 23.00	4.86	12	16900	65.00	76.00	69.00	▼ 19.00	-21.59	4
420.00	450.00	413.00	▲ 16.00	4.03	1	17000	88.00	89.00	89.00	▼ 23.00	-20.53	1
336.00	372.00	345.00	▲ 19.00	5.82	2	17100	109.00	115.00	118.00	▼ 23.00	-16.31	1
276.00	285.00	276.00	▲ 16.00	6.15	1	17200	135.00	148.00	147.00	▼ 28.00	-16.00	2
223.00	231.00	223.00	▲ 23.00	11.50	1	17300	178.00	187.00	185.00	▼ 32.00	-14.74	1
159.00	168.00	161.00	▲ 9.00	5.92	4	17400	221.00	236.00	218.00	▼ 50.00	-18.65	4
119.00	125.00	120.00	▲ 7.00	6.19	1	17500	268.00	293.00	275.00	▼ 55.00	-16.66	4
83.00	88.00	83.00	▲ 4.00	5.06	1	17600	322.00	400.00	311.00	▼ 85.00	-21.46	2
56.00	60.00	56.00	▲ 1.00	1.81	1	17700	374.00	910.00	430.00	▼ 44.00	-9.28	1
40.50	41.50	40.50	▲ 5.00	14.08	1	17800	385.00	615.00	475.00	▼ 75.00	-13.63	1
26.00	29.00	28.00	▲ 3.50	14.28	1	17900	134.00	-	555.00	▼ 85.00	-13.28	1
18.50	19.50	19.50	▲ 3.00	18.18	6	18000	545.00	880.00	710.00	▼ 25.00	-3.40	4
10.50	12.50	12.50	▲ 1.00	8.69	1	18100	183.00	-	-	-	0.00	-
8.00	8.80	8.00	▲ 0.10	1.26	1	18200	214.00	-	-	-	0.00	-
4.90	7.80	7.80	▲ 1.80	30.00	1	18300	261.00	2730.00	-	-	0.00	-
4.70	9.00	4.70	▲ 0.10	2.17	1	18400	62.00	-	-	-	0.00	-
2.60	3.80	3.20	▼ 0.60	-15.78	2	18500	800.00	-	-	-	0.00	-
1.00	6.00	3.00	▼ 0.10	-3.22	1	18600	73.00	1390.00	-	-	0.00	-
2.50	5.80	2.50	▼ 0.30	-10.71	2	18700	78.00	-	-	-	0.00	-
1.70	7.00	1.60	▼ 0.60	-27.27	50	18800	62.00	-	-	-	0.00	-
0.10	3.80	1.70	-	0.00	10	18900	66.00	-	-	-	0.00	-
0.10	9.90	-	-	0.00	-	19000	68.00	-	-	-	0.00	-
0.10	25.00	-	-	0.00	-	19100	100.00	-	-	-	0.00	-
0.20	5.20	-	-	0.00	-	19200	-	-	-	-	0.00	-
-	19.00	-	-	0.00	-	19300	-	-	-	-	0.00	-
-	14.50	-	▼ 0.10	-9.09	1	19400	-	-	-	-	0.00	-
0.10	29.00	-	▲ 0.70	63.63	2	19500	-	-	-	-	0.00	-
0.10	19.50	-	-	0.00	-	19600	-	-	-	-	0.00	-
-	16.00	-	-	0.00	-	19700	-	-	-	-	0.00	-
-	20.00	-	▲ 0.20	22.22	2	19800	-	-	-	-	0.00	-
-	16.50	-	-	0.00	-	19900	-	-	-	-	0.00	-
0.60	3.00	0.60	▼ 0.10	-14.28	20	20000	-	-	-	-	0.00	-
-	11.00	-	▲ 0.10	14.28	1	20100	-	-	-	-	0.00	-
-	11.00	-	-	0.00	-	20200	-	-	-	-	0.00	-
0.20	1.00	-	-	0.00	1	20300	-	-	-	-	0.00	-

Payoff for Call & Put

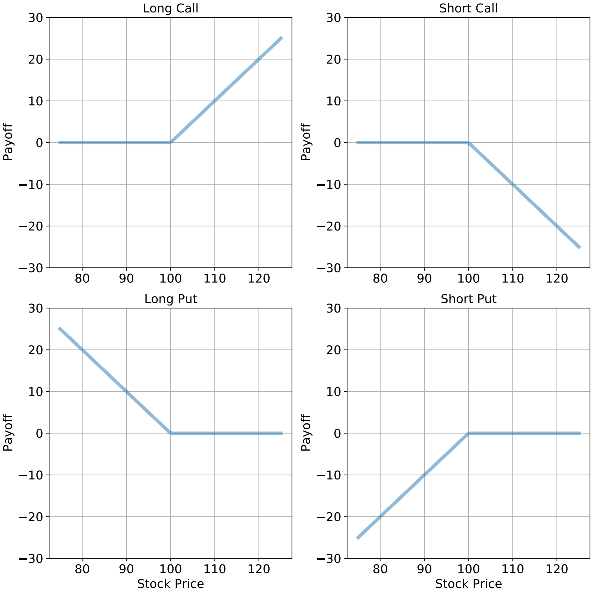


Moneyness

- Take a call option as an example:



Long/Short Position in Call/Put



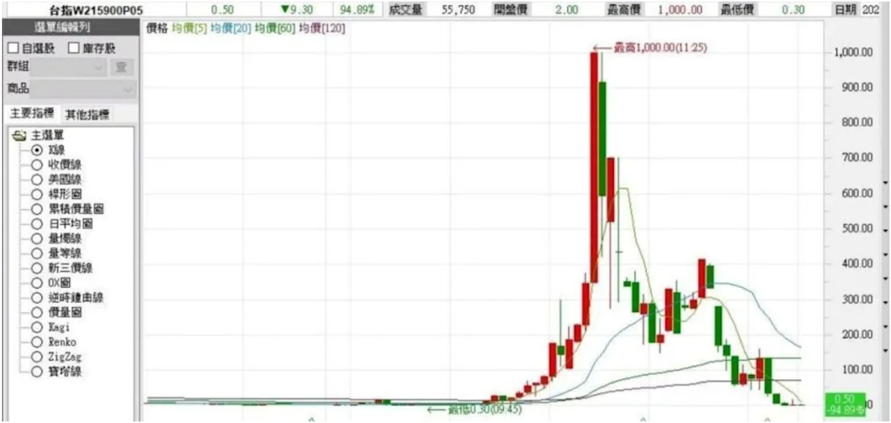
!!! WARNING !!!

- Buying a call/put option offers limited risk and unlimited reward.[19]
- Selling a naked call/put option, however, has limited reward, and unlimited risk![20]

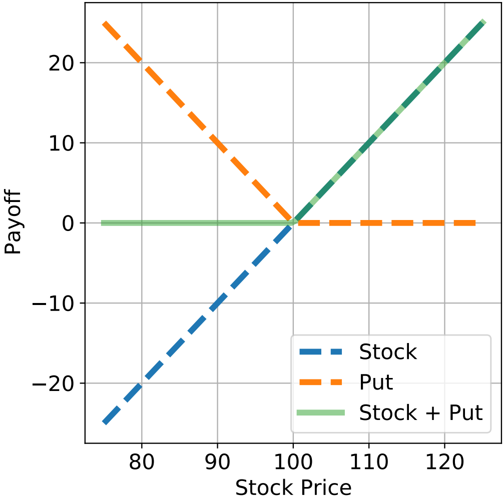
[19] See https://finance.ettoday.net/news/1999215.

[20] See https://www.peopo.org/news/379679.

Example: 3300% Rate of Return?



Hedging: Protective Put



Put-Call Parity

	futures	call	put
long position	$S - X$	$(S - X)^+$	$(X - S)^+$
short position	$-(S - X)$	$-(S - X)^+$	$-(X - S)^+$

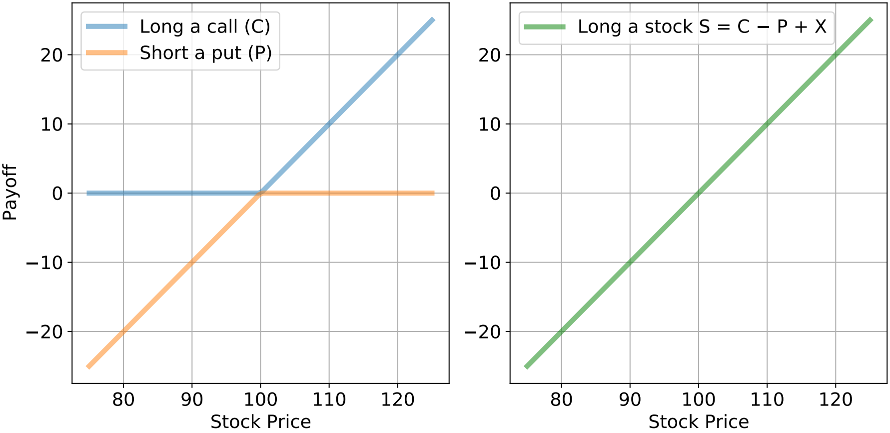
- Note that $(\cdot)^+$ denotes $\max(0, \cdot)$.
- It is clear that

$$\boxed{\text{long futures} = \text{long call} + \text{short put}}$$

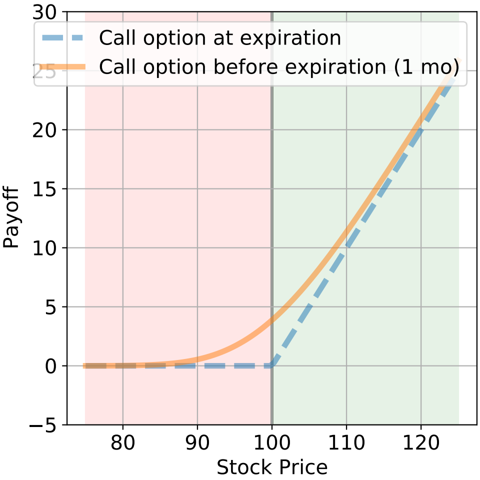
because $S - X = (S - X)^+ - (X - S)^+$.[21]

[21] More precisely, the best bid of futures could be the best ask of put minus the best bid of call while the best ask of futures could be the best ask of call minus the best bid of put.

Illustration

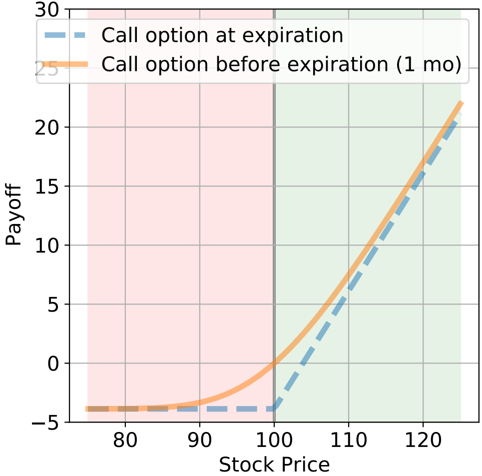


Time Value



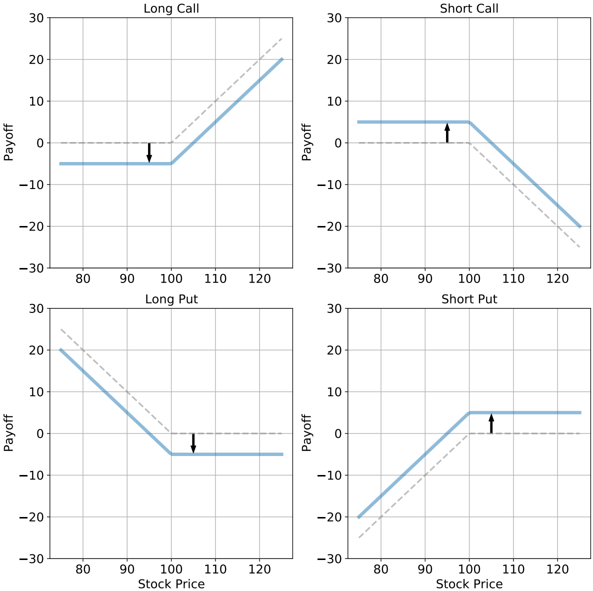
$\boxed{\text{Option value} = \text{intrinsic value} + \text{time value}}$

Premium: There is No Free Lunch

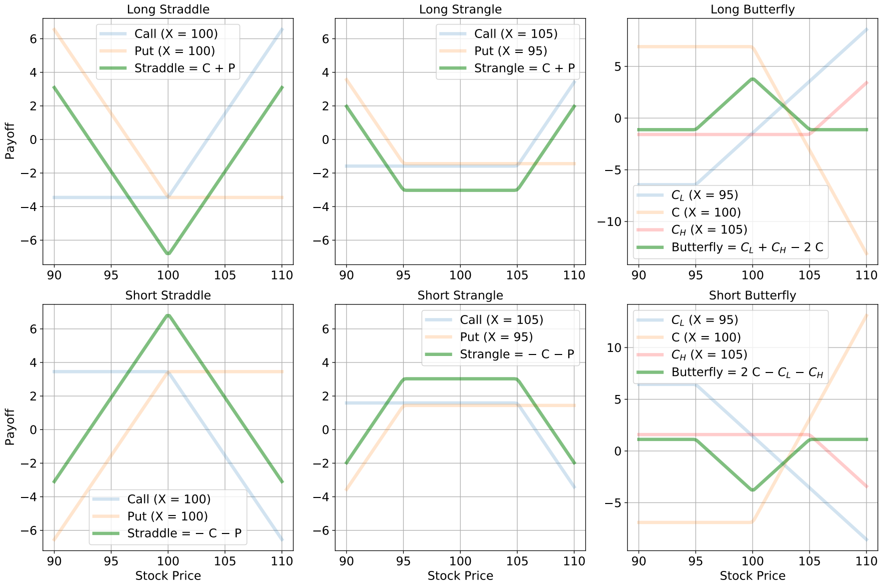


- In this case, the option holder (buyer) pays \$3.87 to the option writer (seller).

Long/Short Position in Call/Put (Revisited)



Selected Trading Strategies with Options: Illustration



Arbitrage

- Let $V_\Pi^0 = 0$ be the initial value of the portfolio Π at time 0.
- Then Π is arbitrageous if $\mathbf{Pr}(V_\Pi^1 \geq 0) = 1$.
- Simply put, Π is a sure win.
- This is called arbitrage, a process that you make a profit without bearing any risk.
- The arbitrage-free principle is the central concept of option pricing.[22]

[22] Also see the arbitrage pricing theory.

Example 1: Protective Put (Revisited)

- To hedge your long position in stock XYZ, you decide to buy its put option with the strike price \$100.
- For simplicity, one option contract covers one stock share.
- Now you have 1000 shares of XYZ.
- Assume that the stock price will be \$150 or \$50.
- How many contracts of this put option you need to buy?

- Let x be the number of option contracts to protect one share.
- Then the final value of the portfolio (XYZ + protective put) should be

$$150 + 0x = 50 + 50x,$$

- It is clear that $x = 2$ and the final value is \$150,000.
- So we need 2000 contracts of this put option.

Example 2: Pricing

- Assume that the initial stock price is \$100 and the risk-free rate for one period is 5%.
- Let P be the put price.
- The initial portfolio value is

$$(100 + 2P) \times 1000.$$

- By the arbitrage-free principle,

$$100 + 2P = \frac{150}{1.05} = 142.86.$$

- So it is clear that $P = 21.43$.
- You need \$42,860 ($= P \times 2000$) to buy protective puts.[23]

[23] You need to pay (insurance) premium to acquire protection.

Example 3: Mispricing

- If the put price is overpriced, say $P' = 25$, then you could now sell the portfolio and buy to close the short position in the end:

$$(100 + 2 \times 25) \times 1.05 - 150 > 0.$$

- If the put price is underpriced, say $P'' = 20$, then you could play the previous strategy in opposite direction:

$$150 - (100 + 2 \times 20) \times 1.05 > 0.$$

- The above two cases show the presence of arbitrage.
- Hence $P = 21.43$ is the fair price of this put option.

The 1st and 2nd Theorem of Asset Pricing

- **Fundamental Theorem of Arbitrage Pricing**: there exists a risk-neutral probability measure if and only if arbitrages do not exist.
- An arbitrage-free market may admit more than one risk-neutral probability measure.
- Economists call such markets incomplete.[24]
- **Completeness Theorem**: a complete market is one that has a unique equilibrium measure.
- In a complete market, any derivative security can be hedged by replicating the portfolio from the market.

[24] It could be said that there are necessarily derivative securities that cannot be hedged.

Pricing $\leftrightarrow$ Replication $\leftrightarrow$ Hedging

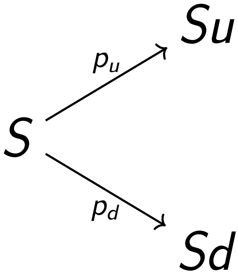
Binomial Option Pricing Model (BOPM)

One-Period Case

- Let X be the strike price and r be the risk-free interest rate.
- Let S be the price at time 0.
- Let u and d be the jump size for the up and down movement at time T.[25]
- Let p_u and p_d denote the transitional probability for Su and Sd, respectively.

[25] It could be proved that $0 < d \leq u$ is equivalent to the arbitrage-free principle.

Binomial Tree



- For a fair bet, solve the following simultaneous equations

$$\begin{cases} p_u + p_d & = 1, \\ p_u Su + p_d Sd & = Se^{rT}. \end{cases}$$

- It is easy to see that

$$p_u = \frac{e^{rT} - d}{u - d}.$$

- If $0 \leq p_u, p_d \leq 1$, then the call price C is

$$C = e^{-rT} \left[p_u(Su - X)^+ + p_d(Sd - X)^+ \right].$$

- Note that p_u and p_d are so-called the risk-neutral probabilities.
- Under this risk-neutral probability measure,

$$C = e^{-rT} \mathbf{E}^{\mathbb{Q}} \left[(S_T - X)^+ \right]. \tag{2}$$

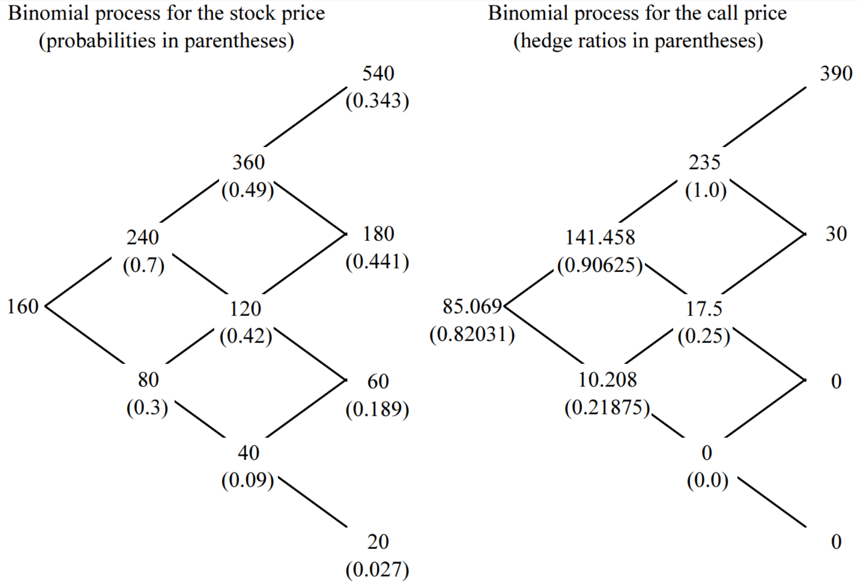
- This equality also holds for continuous-time cases.

BOPM: A 3-Period Case

- A non-dividend-paying stock is selling for \$160.
- Assume that $u = 1.5, d = 0.5$, and $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
- Hence the probability of an up move is

$$p = \frac{R - d}{u - d} = 0.7.$$

- Consider a European call on this stock with $X = 150$.
- The call value is \$85.069 by backward induction.


Binomial process for the stock price
(probabilities in parentheses)
540
(0.343)
360
(0.49)
240
(0.7)
180
(0.441)
160
120
(0.42)
80
(0.3)
60
(0.189)
40
(0.09)
20
(0.027)
Binomial process for the call price
(hedge ratios in parentheses)
390
235
(1.0)
141.458
(0.90625)
30
85.069
(0.82031)
17.5
(0.25)
10.208
(0.21875)
0
0
(0.0)
0

```python
import numpy as np

s0 = 160 # spot price
X = 150  # strike price
n = 3    # periods
R = 1.2  # gross return for each period
u = 1.5  # size of up move
d = 0.5  # size of down move

p = (R - d) / (u - d) # probability of up move

# generate the forward price tree
tree = np.zeros([n + 1, n + 1])
tree[0, 0] = s0
for period in range(1, n + 1):
    for level in range(period  + 1):
        tree[level, period] = s0 * u ** (period - level) * d ** 
            level

# see the next page
```

```python
# calculate the payoff at expiration
for level in range(n + 1):
    payoff = tree[level, n] - X
    tree[level, n] = payoff if payoff > 0 else 0

# calculate the option price at time 0 by backward induction
for period in range(n - 1, -1, -1):
    for level in range(period + 1):
        tree[level, period] = (p * tree[level, period + 1] + (1
            - p) * tree[level + 1, period + 1]) / R

print(tree)
print("c =", round(tree[0, 0], 2)) # output 85.07
```

Risk-Neutral Valuation[26] for Call Options

履約價

折現因子

風險中立測度

$$C = \boxed{e^{-rT}}\,\mathbf{E}^{\boxed{\mathbb{Q}}}\left[\boxed{(S_T - X)^+}\right]$$

買權價格

報酬函數

期望值

到期日的標的物價格

[26] Also called martingale pricing. See
https://en.wikipedia.org/wiki/Martingale_pricing.

Remarks

- The arbitrage-free principle determines the fair price of options.
- The fair price is the discounted payoff of the contingency claim under the risk-neutral probability measure.
- A further question is,

How does the price evolve over time?

Geometric Brownian Motion[29]

- Let μ be the expected return rate, σ be the annual volatility, and T be the time to maturity.
- Then we assume that the return rate of S_t follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{3}$$

where $W_t \sim N(0, T)$ is the Wiener process.[27]

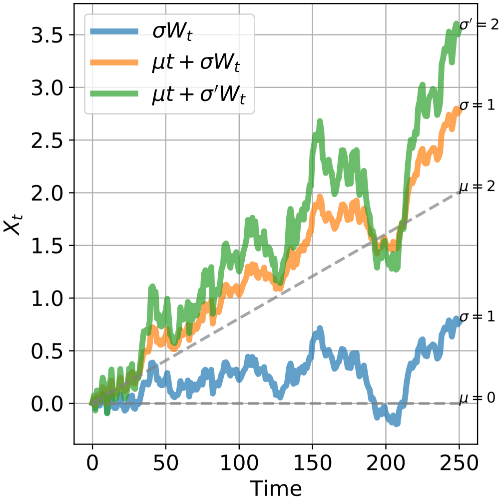
- By Itô's lemma[28], the stock price at time T is

$$S_T = S_0 e^{(\mu - \sigma^2/2)T + \sigma W_T},$$

where S_0 is the spot price.

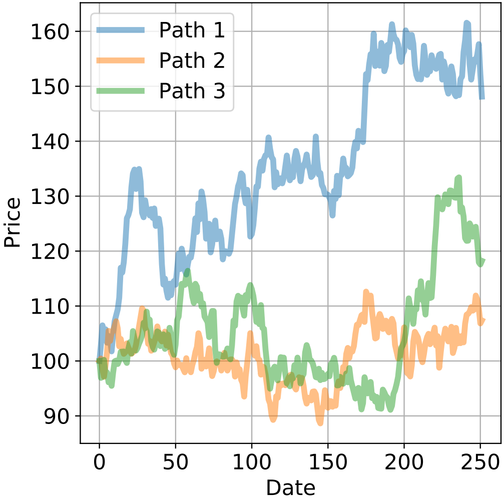
[27] See https://en.wikipedia.org/wiki/Wiener_process.
[28] Itô (1944). See https://en.wikipedia.org/wiki/Ito's_lemma.
[29] See https://en.wikipedia.org/wiki/Brownian_motion.

Generalized Wiener Process: Illustration[30]



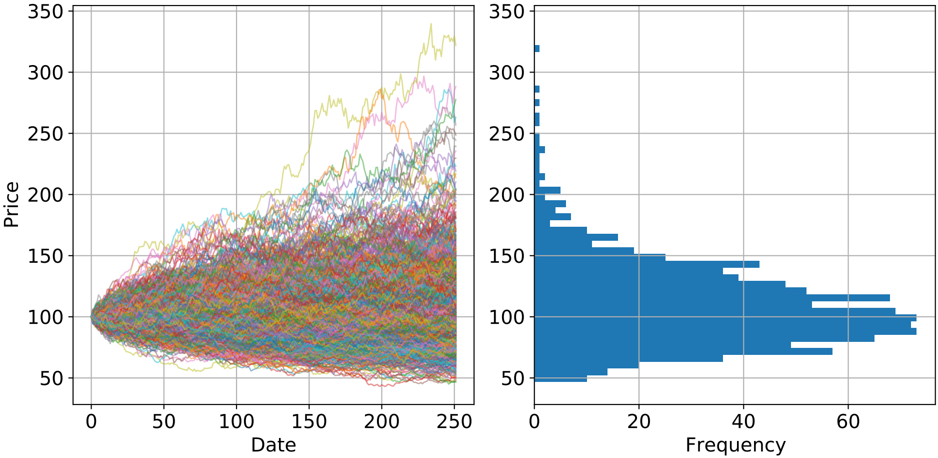
30

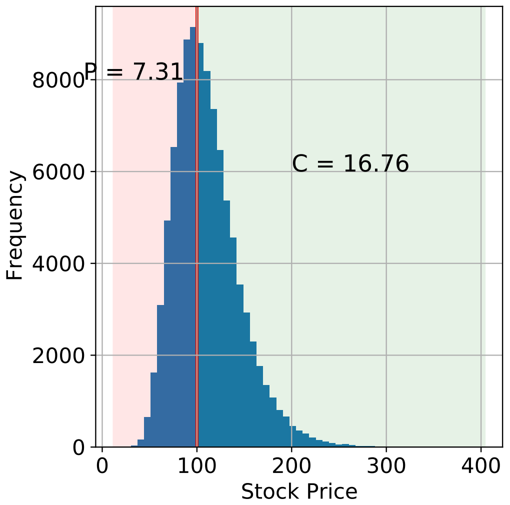
Example: Price Path Simulation[31]



[31] $S_0 = 100, \mu = 0.1, \sigma = 0.3, T = 1, N = 250.$

Example: Monte Carlo Method for European Options





- The number of simulation paths can be determined by its standard error of the estimator.

```python
import numpy as np

S0 = 100
X = 100
r = 0.1
v = 0.3
T = 1

N = 100000 # for example
Z = np.random.normal(size = (N, 1))
S = S0 * np.exp((r - 0.5 * v ** 2) * T + v * np.sqrt(T) * Z)
C = np.sum(S[S > X] - X) / N / np.exp(r * T)
P = np.sum(X - S[X > S]) / N / np.exp(r * T)
```

From BOPM to Black-Scholes Formula

- Consider the log price $\ln S_t$.
- By Itô's lemma, it can be shown that

$$d \ln S_t = rdt + \sigma dW_t,$$

where $r = \mu - \dfrac{\sigma^2}{2}$.

- To match the mean and the variance of the above equation,

$$\begin{cases} q_u \ln u + (1 - q_u) \ln d & = r\Delta t, \\ q_u \ln^2 u + (1 - q_u) \ln^2 d & = r^2 \Delta t^2 + \sigma^2 \Delta t. \end{cases}$$

- Choosing $ud = 1$, it can be shown that

$$u = e^{\sigma\sqrt{\Delta t}}, q_u = \frac{1}{2} + \frac{r\sqrt{\Delta t}}{2\sigma}.$$

- It is worth to mention that

$$p_u = \frac{e^{r\Delta t} - d}{u - d} \to q_u$$

as $\Delta t \to 0$.

- This is well known as the CRR binomial model[32], which is proved to converge to the Black-Scholes formula.

[32] Cox, Ross and Rubinstein (1979).

Black-Scholes Formula for European Options[34]

- Insert equation (3) into equation (2).
- Then we could drive the famous option pricing formula

$$C(S, X, r, \sigma, T) = SN(d_1) - Xe^{-rT}N(d_2), \tag{4}$$

where $N(\cdot)$ is the normal cdf, $d_1 = \frac{(\ln(S/X)+(r+\sigma^2/2)T)}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.[33]

- By the put-call parity,

$$P = C - S + Xe^{-rT}.$$

[33] Try `https://www.taifex.com.tw/cht/9/calOptPrice`.

[34] Black and Scholes (1973), Merton (1973). Scholes and Merton are awarded with the Nobel Prize in Economic Sciences in 1997 for a new method of determining the value of derivatives. See `https://www.nobelprize.org/prizes/economic-sciences/1997/summary/`.

Example

```
import numpy as np
import scipy.stats

s0 = 100
X = 100
r = 0.1
v = 0.3
T = 1

d1 = (np.log(s0 / X) + (r + 0.5 * v ** 2)) / np.sqrt(T) / v
d2 = d1 - np.sqrt(T) * v
C = s0 * scipy.stats.norm.cdf(d1) - X * np.exp(-r * T) * scipy.
    stats.norm.cdf(d2)
print(C)
```

Trading Volatility

- Options are a popular vehicle for speculation.
- Speculative trades would arguably be more inclined to buy a call than to buy its stock if the stock is bullish and expected high future volatility.
- Buying options is buying volatility and selling options is selling volatility.

Historical/Implied Volatility

- Historical volatility (HV) is the annualized standard deviation of daily returns.
- Implied volatility (IV) is the volatility input in a pricing model that, in conjunction with the other four inputs, returns the theoretical value of an option matching the market price.
- What is the difference between HV and IV?
  - HV: what has happened.
  - IV: derived from the market's expectation for future volatility.

How to Use IV?

- You could deannualize IV to the daily IV by

$$\frac{\text{IV}}{\sqrt{256}} = \text{1-day expected } \sigma.$$

- For example, assume that the period in question is one month and there are 22 business days remaining in that month.
- The ATM call for the \$100 stock is traded at 32% IV.
- Then it has a one-month volatility

$$\frac{0.32}{\sqrt{256}} \times \sqrt{22} = 9.38\%.$$

- It is equivalent to say that there is 68% chance of the stock's closing between \$90.62 and \$109.38.

How to Calculate IV?

- We know the spot price S, the time to maturity T, the risk-free rate r, and also the option price C with the strike price X.
- The only unobservable parameter is the volatility σ.
- We invert equation (4) to calculate (more precisely, estimate) σ by the Newton's method.[35]
- This resulting volatility is called the implied volatility because the volatility is implied by the Black-Scholes formula with the market prices.

[35] You may also use the bisection method in this problem.

Newton's Method[36]

- Let $f(x)$ be the target function and x^* be the root of f.
- Then x^* could be found by using the iterative formula

$$x^{k+1} \leftarrow x^k - \frac{f}{f'},$$

where $f' = \dfrac{df}{dx}$ at x^k.

- The loop halts when the convergence criteria is satisfied, say

$$|\, x^{k+1} - x^k \,| < \varepsilon$$

for a constant ε small enough.

[36] See https://en.wikipedia.org/wiki/Newton's_method.

How to Calculate IV? (Concluded)

- Now consider the European call $C = f(S, X, r, \sigma, T)$.
- Then

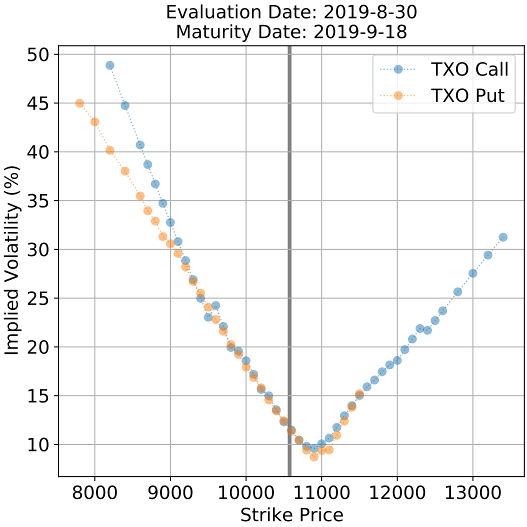
$$\sigma_{k+1} \leftarrow \sigma_k + \frac{C - f(S, X, r, \sigma_k, T)}{\nu},$$

where $\nu = \dfrac{\partial f}{\partial \sigma}$ is called the vega.
- Note that the vega of European options is

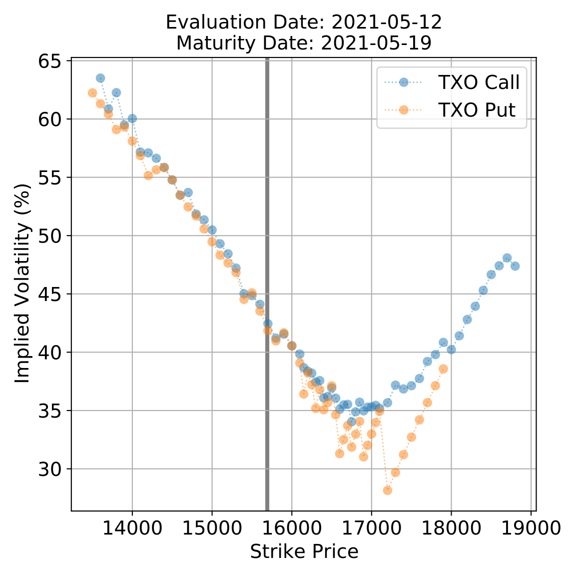
$$\nu = SN(d_1)\sqrt{T} > 0.$$

- The higher the volatility input, the higher the theoretical value, holding all other variables constant.

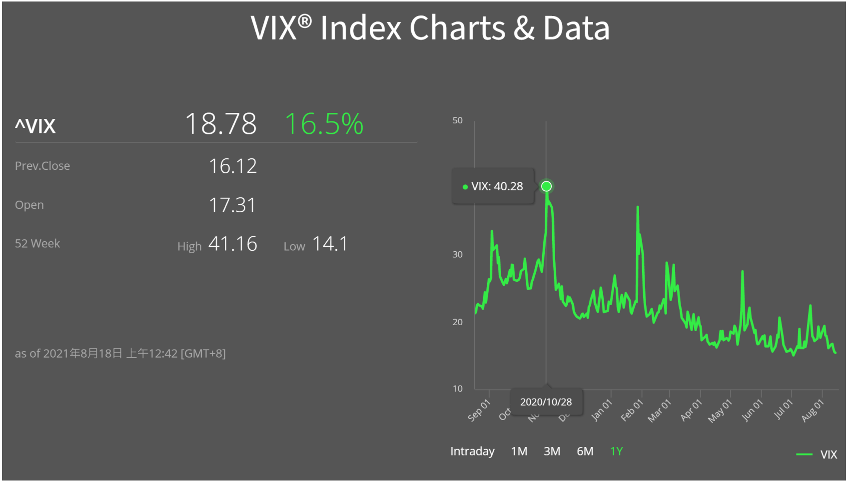
Examples



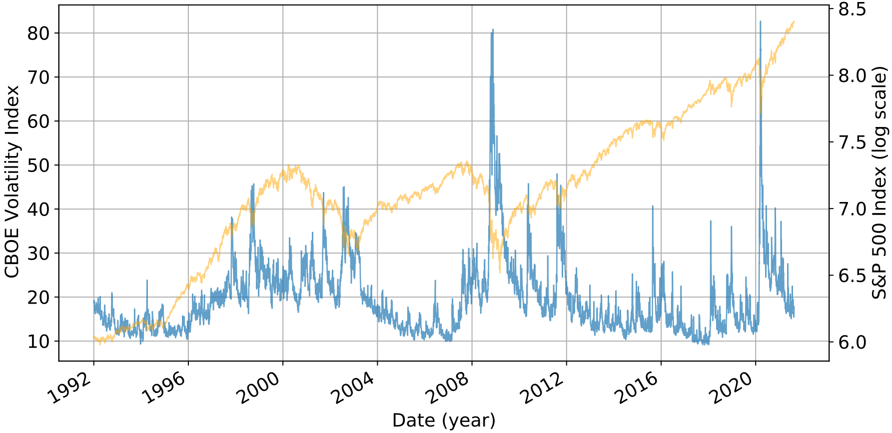
Evaluation Date: 2019-8-30
Maturity Date: 2019-9-18
TXO Call
TXO Put
Implied Volatility (%)
Strike Price
10
15
20
25
30
35
40
45
50
8000
9000
10000
11000
12000
13000


Evaluation Date: 2021-05-12
Maturity Date: 2021-05-19
TXO Call
TXO Put
Implied Volatility (%)
Strike Price
65
60
55
50
45
40
35
30
14000
15000
16000
17000
18000
19000

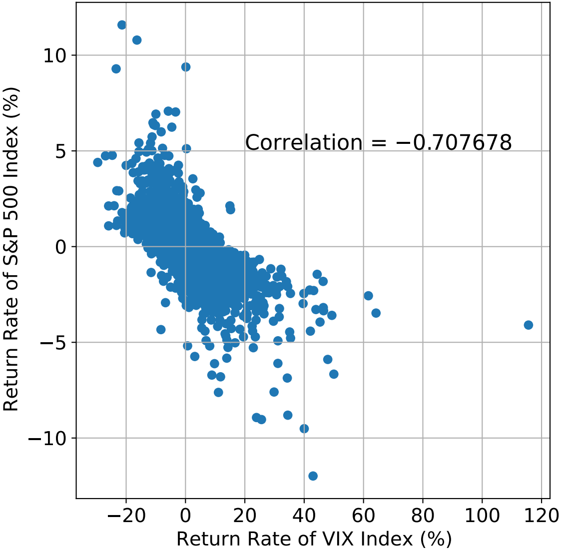
Case Study: CBOE VIX Index[37]



[37] See https://www.cboe.com/tradable_products/vix/.



CBOE Volatility Index
S&P 500 Index (log scale)
Date (year)
80
70
60
50
40
30
20
10
8.5
8.0
7.5
7.0
6.5
6.0
1992
1996
2000
2004
2008
2012
2016
2020



Correlation = −0.707678
Return Rate of S&P 500 Index (%)
Return Rate of VIX Index (%)
10
5
0
−5
−10
−20
0
20
40
60
80
100
120

Beyond the Black-Scholes Formula

- As we can see, the IV curve violets the assumption of constant volatility.[38]
- Empirically, stock returns tend to have fat tails, inconsistent with the Black-Scholes model's assumptions.
- Later, the stochastic volatility models and the jump-diffusion models are proposed to tackle with this issue.
- In the time series analysis, the ARCH model and its variants are also used to estimate the volatility, especially in risk management.[39]

[38] This is called the volatility smile (smirk). See https://en.wikipedia.org/wiki/Volatility_smile. Also see https://en.wikipedia.org/wiki/Volatility_clustering and https://www.investopedia.com/terms/a/assymetricvolatility.asp.

[39] See https://en.wikipedia.org/wiki/Autoregressive_conditional_heteroskedasticity.

Cox-Ingersoll-Ross (CIR) Model[40]

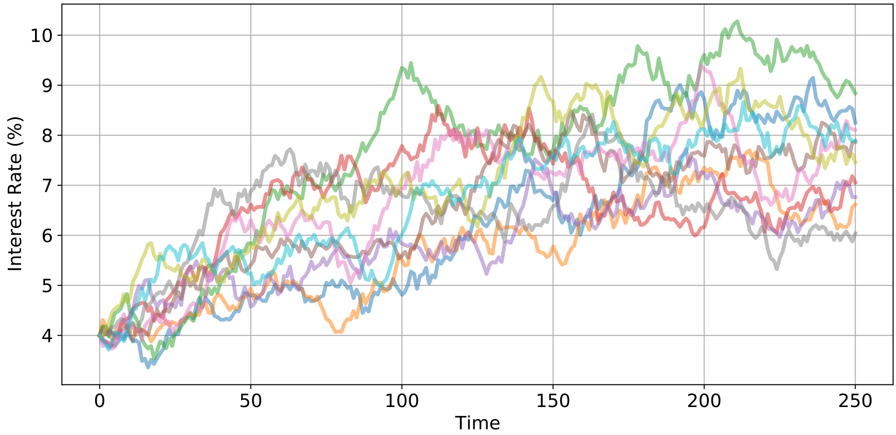
- The CIR Model is a square-root mean reverting process following

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t,$$

  where r_t is the interest rate at time t, κ is the mean-reverting rate, θ is the long-term mean of r_t, σ is the volatility of r_t, and dW_t is a Wiener process.
- The CIR model is widely used in modeling interest rate, commoditiy price, and price volatility (or variance).

[40] Cox, Ingersoll, Ross (1985).

Illustration[41]



[41] $r_0 = 4\%, \theta = 8\%, \kappa = 2, \sigma = 10\%, T = 1, N = 250.$

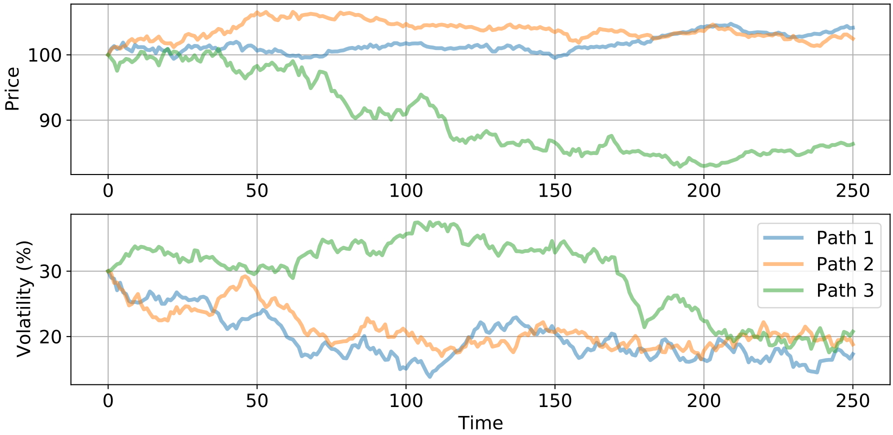
Heston Model

- Heston (1993) exploits the CIR model to capture the dynamic of price volatility:

$$\frac{dS}{S} = \mu dt + \sqrt{v_t} dW_{1,t},$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t},$$

  where v_r is the price variance of S_t and $dW_{1,t} \times dW_{2,t} = \rho$.
- In particular, ρ is typically negative in the stock market.

Illustration[42]



[42] $S_0 = 100, r = 1\%, v_0 = 0.09, \theta = 0.04, \kappa = 3, \sigma = 0.3, \rho = -0.7, N = 250.$

Merton's Jump-Diffusion Model

- Merton (1976) superimposes a jump component on a diffusion component.
- The jump component is composed of lognormal jumps driven by a Poisson process.[43]
- It models the sudden changes in the stock price because of the arrival of important new information.
- The risk-neutral jump-diffusion process follows

$$\frac{dS_t}{S_t} = (r - \lambda\overline{\kappa})dt + \sigma dW_t + \kappa dq_t,$$

where q_t is compound Poisson process with intensity λ, where κ denotes the magnitude of the random jump.

[43] See https://en.wikipedia.org/wiki/Poisson_point_process.

Illustration