Newton's Method

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目录

1 一般的方法

(I) 当 $F(x)$ 是标量方程的时候，我们通过迭代的式子:

$$x = x - \frac{F(x)}{F'(x)}$$

实际计算时用到的迭代式子为

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}$$

(II) 当 $\boldsymbol{F}(\boldsymbol{x})$ 是形如下面的向量方程时,

$$\boldsymbol{F}(\boldsymbol{x}) \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Jacobian 矩阵 $J(\boldsymbol{x})$:

$$J(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

求解向量用到的公式为

$$\boldsymbol{x} = \boldsymbol{x} - [J(\boldsymbol{x})]^{-1}\boldsymbol{F}(\boldsymbol{x})$$

2 块中心差分方法 Jacobin 矩阵

$Darcy - Forchheimer$ 的离散形式

$$(\frac{\mu}{k} + \beta\rho[Qu]_i) \cdot u_i + \frac{p_r - p_i}{h^x_{i+1/2}} = f_{1,i}$$

$$(\frac{\mu}{k} + \beta\rho[Qv]_j) \cdot v_j + \frac{p_a - p_i}{h^y_{j+1/2}} = f_{2,i}$$

$$\frac{u_{i,1} - u_{i,3}}{h^x_i} + \frac{v_{i,2} - v_{i,0}}{h^y_i} = g_i$$

其中

$$[Qu]_i = \frac{1}{4h^x_{i+1/2}}\{h^x_i(R(u_i, v_j) + R(u_i, v_{j-1})) + h^x_{i+1}(R(u_i, v_{j+ny+1}) + R(u_i, v_{j+ny}))\}$$

$$[Qv]_j = \frac{1}{4h^y_{j+1/2}}\{h^y_i(R(u_i, v_j) + R(u_{i+1}, v_j)) + h^y_{j+1}(R(u_{i+ny}, v_j) + R(u_{i+ny+1}, v_j))\}$$

则有

$$F_{1,i} = (\frac{\mu}{k} + \beta\rho[Qu]_i) \cdot u_i + \frac{p_r - p_i}{h^x_{i+1/2}} - f_{1,i}$$

$$F_{2,i} = (\frac{\mu}{k} + \beta\rho[Qv]_j) \cdot v_j + \frac{p_a - p_i}{h^y_{j+1/2}} - f_{2,i}$$

$$F_{3,i} = \frac{u_{i,1} - u_{i,3}}{h^x_i} + \frac{v_{i,2} - v_{i,0}}{h^y_i} - g_i$$

由于 F 的规模是 $(NE + NC) \times (NE + NC)$, 因此 $J(\boldsymbol{x})$ 的规模也是 $(NE + NC) \times (NE + NC)$ 的。

对 F 求偏导可得

$$\frac{\partial F_{1,i}}{\partial u_i} = \beta\rho\frac{1}{4h^x_{i+1/2}}\{h^x_i(\frac{2u_i}{R(u_i, v_j)} + \frac{2u_i}{R(u_i, v_{j-1})}) + h^x_{i+1}(\frac{2u_i}{R(u_i, v_{j+ny+1})} + \frac{2u_i}{R(u_i, v_{j+ny})})\}u_i + (\frac{\mu}{k} + \beta\rho[Qu]_i)$$

$$\frac{\partial F_{1,i}}{\partial v_j} = \beta\rho\frac{1}{4h^x_{i+1/2}}\{h^x_i(\frac{2v_j}{R(u_i, v_j)})\}u_i$$

$$\frac{\partial F_{1,i}}{\partial v_{j-1}} = \beta\rho\frac{1}{4h^x_{i+1/2}}\{h^x_i(\frac{2v_{j-1}}{R(u_i, v_{j-1})})\}u_i$$

$$\frac{\partial F_{1,i}}{\partial v_{j+ny+1}} = \beta\rho\frac{1}{4h^x_{i+1/2}}\{h^x_{j+1}(\frac{2v_{j+ny+1}}{R(u_i, v_{j+ny+1})})\}u_i$$

$$\frac{\partial F_{1,i}}{\partial v_{j+ny}} = \beta\rho\frac{1}{4h^x_{i+1/2}}\{h^x_{j+1}(\frac{2v_{j+ny}}{R(u_i, v_{j+ny})})\}u_i$$

$$\frac{\partial F_{1,i}}{\partial p_r} = \frac{1}{h^x_{i+1/2}}$$

$$\frac{\partial F_{1,i}}{\partial p_i} = -\frac{1}{h^x_{i+1/2}}$$

$$\frac{\partial F_{2,j}}{\partial v_j} = \beta\rho\frac{1}{4h^y_{j+1/2}}\{h^y_i(\frac{2v_j}{R(u_i, v_j)} + \frac{2v_j}{R(u_{i+1}, v_j)}) + h^y_{j+1}(\frac{2v_j}{R(u_{i+ny}, v_j)} + \frac{2v_j}{R(u_{i+ny+1}, v_j)})\}v_j + (\frac{\mu}{k} + \beta\rho[Qv]_j)$$

$$\frac{\partial F_{2,j}}{\partial u_i} = \beta\rho\frac{1}{4h^y_{j+1/2}}\{h^y_j(\frac{2u_i}{R(u_i, v_j)})\}v_j$$

$$\frac{\partial F_{2,j}}{\partial u_{i+1}} = \beta\rho\frac{1}{4h^y_{j+1/2}}\{h^y_j(\frac{2u_{i+1}}{R(u_{i+1}, v_j)})\}v_j$$

$$\frac{\partial F_{2,j}}{\partial u_{i+ny}} = \beta\rho\frac{1}{4h^y_{j+1/2}}\{h^y_{j+1}(\frac{2u_{i+ny}}{R(u_{i+ny}, v_j)})\}v_j$$

$$\frac{\partial F_{2,j}}{\partial u_{i+ny+1}} = \beta\rho\frac{1}{4h^y_{j+1/2}}\{h^y_{j+1}(\frac{2u_{i+ny+1}}{R(u_{i+ny+1}, v_j)})\}v_j$$

$$\frac{\partial F_{2,j}}{\partial p_a} = \frac{1}{h^y_{j+1/2}}$$

$$\frac{\partial F_{2,j}}{\partial p_i} = -\frac{1}{h^y_{j+1/2}}$$

$$\frac{\partial F_{3,i}}{\partial u_{i,3}} = -\frac{1}{h^x_i}$$

$$\frac{\partial F_{3,i}}{\partial u_{i,1}} = \frac{1}{h^x_i}$$

$$\frac{\partial F_{3,i}}{\partial v_{i,2}} = \frac{1}{h^y_j}$$

$$\frac{\partial F_{3,i}}{\partial v_{i,0}} = -\frac{1}{h^y_j}$$

因此 Jacobian 矩阵为

$$\begin{bmatrix} \frac{\partial F_{1,i}}{\partial u_i} & \frac{\partial F_{1,i}}{\partial v_i} & \frac{\partial F_{1,i}}{\partial p_i} \\ \frac{\partial F_{2,i}}{\partial u_i} & \frac{\partial F_{2,i}}{\partial v_i} & \frac{\partial F_{2,i}}{\partial p_i} \\ \frac{\partial F_{3,i}}{\partial u_i} & \frac{\partial F_{3,i}}{\partial v_i} & \frac{\partial F_{3,i}}{\partial p_i} \end{bmatrix}$$

是 3×3 的分块矩阵。

其中

$$\begin{aligned} F_{1,i} &= F(u_1, \cdots, u_{Nu}, v_1, \cdots, v_{Nv}, p_1, \cdots, p_{Np}) \\ F_{2,i} &= F(u_1, \cdots, u_{Nu}, v_1, \cdots, v_{Nv}, p_1, \cdots, p_{Np}) \\ F_{3,i} &= F(u_1, \cdots, u_{Nu}, v_1, \cdots, v_{Nv}, p_1, \cdots, p_{Np}) \end{aligned}$$

Nu 表示 u 的个数,Nv 表示 v 的个数,Np 表示 p 的个数。

参考文献