# Homework/Programming 5. (Due Oct. 28)

Math. 609-600

**Problem 1.** Do Exercise 3 on p.5 of the 639 lecture 7 notes. You will have to use the (positive) square operator defined on p.4-5 of those notes.

The next problem that we consider here comes from a finite element approximation to a one dimensional second order elliptic boundary value problem. The problem that we consider is to approximate the solution $y : [0,1] \to \mathbb{R}$. of

$$\begin{aligned} -y''(x) &= f(x), \quad x \in (0,1), \\ y(0) &= y(1) = 0. \end{aligned} \tag{1.1}$$

The approximation involves a mesh of equally spaced points on $[0,1]$, namely, $\{x_i := ih\}$, where $h = 1/N$ and $N > 1$ is an integer. We define the finite element approximation space, $V_h$ to be the set of functions on $[0,1]$ which are continuous and piecewise linear with respect to the above mesh and vanish on the endpoints, i.e., at 0 and 1. This space and the resulting approximation will be discussed in more detail in recitation, in particular:

(a) The functions in $V_h$ are uniquely determined by their values at the nodes, $x_i$, $i = 1, 2, \ldots, N-1$.

(b) There is a natural basis for $V_h$ (called the finite element basis), namely, $\xi_i$, $i = 1, \ldots, N-1$ where

$$\xi_i(x_j) = \delta_{i,j}$$

with $\delta_{i,j}$ denoting the Kronecker delta function.

(c) To define the weak formulation of (1.1), we multiply (1.1) by a "test function" $\phi$ (which vanishes at the endpoints), integrate the product over $[0,1]$ and integrate by parts. The weak form of (1.1) is then to find $y \in V$ satisfying

$$A(y, \phi) = (f, \phi), \qquad \text{for all } \phi \in V \tag{1.2}$$

where

$$A(v, w) := \int_0^1 v'(x) w'(x)\, dx \qquad \text{and} \qquad (v, w) := \int_0^1 v(x) w(x)\, dx.$$

Here $V$ is a space of functions defined on $[0, 1]$ (the precise definition of $V$ is beyond the scope of this class). This is a weak formulation as the functions in $V$ need only have integrable first derivatives while the strong form (1.1) requires the solution $y \in C^2$.

(d) The Galerkin approximation to $y$ solving (1.2) is the unique function $y_h \in V_h$ satisfying

$$A(y_h, \xi_h) = (f, \xi_h), \qquad \text{for all } \xi_h \in V_h. \tag{1.3}$$

(e) Expanding $y_h = \sum_{i=1}^{N-1} c_i \xi_i$ in the finite element basis we are led to the linear system:

$$\widetilde{A}c = \widetilde{B}F \tag{1.4}$$

with $\widetilde{A}$ and $\widetilde{B}$ being tri-diagonal $(N-1) \times (N-1)$ matrices with entries

$$\widetilde{A}_{ij} = h^{-1} \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\widetilde{B}_{ij} = \frac{h}{6} \begin{cases} 4 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and, finally, $F_j = (f, \xi_j)$, for $j = 1, 2, \ldots, N-1$. Both $\widetilde{A}$ and $\widetilde{B}$ are symmetric and positive definite.

(f) We rewrite (1.4) as

$$\mathcal{A}y := \widetilde{B}^{-1}\widetilde{A}y = F. \tag{1.5}$$

**Problem 2.** (a) Using the fact that both $\widetilde{A}$ and $\widetilde{B}$ are symmetric and positive definite, show that the matrix $\mathcal{A}$ is self adjoint with respect to the $\widetilde{B}$ inner product even in the case of non-uniformly spaced meshs although the entries of $\widetilde{A}$ and $\widetilde{B}$ are a little more complicated in that case.

(b) It is known that there are positive constants $c_0$ and $c_1$ not depending on $h$ and satisfying

$$c_0 \le \lambda_1 \le \lambda_{N-1} \le c_1 h^{-2} \tag{2.1}$$

with $\lambda_1$ and $\lambda_{N-1}$ denoting the smallest and largest eigenvalue of $\mathcal{A}$. Let $e^j$ denote the error in the $j$'th iterate of Richardson's method with $\tau = 1/\lambda_{N-1}$ and expand

$$e^j = \sum_{i=1}^{N-1} c_{i,j}\phi_i$$

with $\phi_i$ as in Theorem 2 of 639 lecture 7 (with $A$ and $B$ of that theorem replaced by $\mathcal{A}$ and $\widetilde{B}$, resp.). Show that

$$|c_{i,j}| = |(1 - \lambda_i/\lambda_{N-1})^j c_{i,0}| \le (1 - 1/K)^j |c_{i,0}|.$$

Here $K := \lambda_{N-1}/\lambda_1$ is the spectral condition number.

(c) Use linear Maclauren expansion of $\ln(1 + x)$ to show the expected number of steps required to reduce the error by $\epsilon$ is $j \approx K \ln(\epsilon^{-1})$.

(d) Now switch to the method with optimal parameter, i.e., $\tau = 2/(\lambda_1 + \lambda_{N-1})$. Show that in this case the expected number of steps requred to reduce the error is $j \approx K \ln(\epsilon^{-1})/2$. Hint: Write $\rho(G)$ on p. 7 of the 639 lecture notes 7 in terms of $K$ and, also, use the linear Maclauren approximation to $(1 - x)/(1 + x)$.

**Problem 3.** This is a programming problem using the two iteration methods considered in the above problem. In this case, we have

$$\lambda_j = 6h^{-2}(2 - 2\cos(j\pi/N))/(4 + 2\cos(j\pi/N)), \quad \text{for } j = 1, \ldots, N - 1.$$

For simplicity, we consider iterating for the solution of the somewhat trivial problem

$$\mathcal{A}x = 0 \tag{3.1}$$

with initial vector $x^0$ being the vector in $\mathbb{R}^{N-1}$ with all entries set to 1. The advantages of solving this problem is that the error $e^j = x - x^j = -x^j$ is trivial to compute.

(a) Run your code with $\tau = 1/\lambda_{N-1}$ and report the number of iterations required to reduce the maximum norm of error to $\epsilon = 10^{-2}$ and $\epsilon = 10^{-4}$ as a function of $N$ over the range $4, 8, \ldots .128$. Also, for $N > 4$, report the iteration increase factor, i.e., the number of iterations at $N$ divided by the number of iterations at $N/2$ (for the same $\epsilon$). The asymtotic iteration increase factor (as $N \to \infty$)

should follow (2.1) and is related to the answers of Parts (c) and (d) of the previous problem.

(b) Switch to the optimal $\tau$ and make the runs and report erros analogus to those of Part (a) above.