MATH 609, FINAL EXAM

The following questions are to be answered independently without working with others or getting help from other students, faculty or the TA. You will have until Wednesday November 25 at 1:00 PM to work on it. Please organize you work into one PDF file which is to be emailed to me (pasciak@math.tamu.edu) by that date and time. Good luck.

Note that the definition of A-stablity and the region of absolute stability are given in Lecture 27 and discussed there and also in Lecture 28.

Problem 1. *Consider the ODE*

$$x'(t) = f(t, x(t)), \qquad t > 0, \quad x(0) = x_0. \tag{1.1}$$

Consider the numerical ODE schemes (with parameter $\theta \in [0, 1]$):

$$X_{k+1} - X_k = h[\theta f(t_{k+1}, X_{k+1}) + (1 - \theta) f(t_k, X_k)].$$

(a) Show that the scheme is A-Stable if $\theta \in [1/2, 1]$.
(b) Show that the scheme is not A-Stable if $\theta \in [0, 1/2)$.

Problem 2. *Let A be a nonsymmetric, nonsingular $n \times n$ real matrix and consider the problem $Ax = b$ with b given in $\mathbb{R}^n$. Since A^tA is symmetric and positive definite,*

$$\langle x, y\rangle := (Ax) \cdot (Ay)$$

is an innerproduct on $\mathbb{R}^n$. Consider the iteration

$$x_{i+1} = x_i + \alpha_i r_i$$

with a given $x_0 \in \mathbb{R}^n$ and $r_i = b - Ax_i$. Set $e_i = x - x_i$.

(a) Find the value of α_i which makes $\langle e_{i+1}, r_i\rangle = 0$.
(b) Is the resulting scheme computable without knowing e_i? Explain your answer.
(c) Let $|||\cdot|||$ denote the norm corresponding to the $\langle\cdot,\cdot\rangle$ inner product. Show that if α_i is chosen as in Part (a),

$$|||e_{i+1}||| = \min_{\alpha\in\mathbb{R}} |||e_i - \alpha r_i|||.$$

(d) Write down the steepest descent algorithm applied to the problem

$$\mathcal{A}x = \mathcal{B}$$

with $\mathcal{A} := A^tA$, $\mathcal{B} := A^tb$ and initial iterate x_0. Does this algorithm give the same sequence of iterates as that of Parts (a) and (b) above? Explain your answer.

Problem 3. *Consider the multistep method for approximating solutions to (1.1) defined for $k = 1, 2, 3, \ldots$ by*

$$x_{k+1} = x_{k-1} + 2hf(t_k, x_k). \tag{3.1}$$

Here $t_k = kh$, $h = 1/M$ for $M \geq 2$ and $x_k \approx x(t_k)$. Of course, the startup approximations x_0 and x_1 are assumed given. The scheme is obviously explicit.

(a) Give an expression for the local truncation error in terms of h and derivatives of $x(t)$ which leads to the highest order in h.
(b) Show that the scheme is stable (see, Definition 25.1).
(c) Show that the region of absolute stability is the empty set.

Problem 4. This problem involves some simple programming, i.e., developing a code implementing the scheme of the previous problem. Write a code implementing the scheme (3.1) applied to the ODE,

$$x'(t) = \lambda x(t), \qquad t > 0, \quad x(0) = x_0. \tag{4.1}$$

Take $x_0 = 1$ and use one step of the forward Euler method to set x_1.

(a) Even though (c) of the previous problem holds, Theorem 25.1 of the 639d lecture notes still applies. Now for $N = 8, 16, 32, 64, 128$ and $h = 1/N$, run you code approximating the solution of (4.1) at $t = 1$ with $\lambda = -1$. Report the errors in the solution at each N. Your errors should reflect the convergence order of Part (a) of Problem 3 above.
(b) Now again using $\lambda = -1$ but $h = 10/N$, run your code for

$$N = 10, 100, 1000, \ldots, 1000000$$

to approximate the solution of (4.1) at $t = 10$ Report the errors in the solution at each N. (You will only have a couple of significant digits of accuracy at the largest N.)

The following is a motivation for the next problem and depends on the part of Lecture 28 assigned this week.

The analysis of the backward difference numerical ODE approximations depend on stability of the numerical solutions approximating the ODE

$$x' = 0, \qquad x(0) = 1.$$

(mentioned in Remark 28.2 of the 609d notes). The roots of the polynomial S_0 play an important role as the backward difference (approximate) solutions can be expanded using them (see (52)–(54) of Lecture 28).

Definition: S_0 satisfies the **root condition** if:

(a) All roots of S_0 are of absolute value at most 1.
(b) If r_j is a root of S_0 of absolute value 1, then r_j is simple (i.e., has multiplicity 1).

It is known that S_0 satisfies the root condition for $m = 2, 3, \ldots, 6$. The result of the following problem can be used to prove a theorem similar to Theorem 26.1 of the 609d notes but applied to the backward difference ODE schemes for the above m.

Problem 5. *Let $P(x) \in \mathcal{P}^m$ be the characteristic polynomial of an $m \times m$ matrix M. Show that if $P(x)$ satisfies the root condition and $\rho(M) = 1$, then there is a norm $\|\cdot\|_*$ on $\mathbb{C}^m$ such that the induced norm matrix norm satisfies*

$$\|M\|_* = 1.$$

(Hint: Modify the proof given in the 639d class notes 4 of Theorem 2 of the 639d class notes 3. You will have to use the observation that if r_j is a root on the unit circle then its the Jordan Block is 1×1).