Homework/Programming 4. (Due Sept. 30)

Math. 609-600

Problem 1. Use Taylor's series and undetermined coefficients to derive a numerical approximation to the derivative of the form

$$f'(x) = Af(x) + Bf(x-h) + Cf(x-3h)$$

and use the Taylor series error formula to show that it is second order.

Problem 2. Using Taylor series, show that the approximation

$$(a(x)u'(x))' \approx \frac{a(x-h/2)(u(x-h)-u(x)) + a(x+h/2)(u(x+h)-u(x))}{h^2}$$

is a second order approximation. You may assume that $a(x)$ is in $C^2[x-h, x+h]$. Make sure that you explain why the order of the scheme is two.

Problem 3. (1) Use exactness conditions to determine the coefficients for a quadrature approximation of the form

$$I(g) := \int_0^1 g(x)\, dx \approx a_1 g(1) + b_1 g(0) + c_1 g(-1). \tag{3.1}$$

(2) Use exactness conditions to determine the coefficients for quadrature approximations of the form

$$I(g) := \int_0^1 g(x)\, dx \approx a_2 g(0) + b_2 g(-1) + c_2 g(-2). \tag{3.2}$$

Problem 4. Let $k > 0$ be a parameter and set $t_j = jk$. Translate the two quadrature schemes in the previous problem to the interval $[t_{j-1}, t_j]$ to obtain two quadrature approximations to

$$I_j(g) := \int_{t_{j-1}}^{t_j} g(x)\, dx.$$

The resulting quadrature schemes are the basis for implicit and explicit multistep methods for approximating solutions of ODE's (ordinary differential equations) as we shall see later in this course.

Problem 5. Find the monic polynomial whose roots $\{x_0, x_1\}$ give the two point Gaussian quadrature approximating

$$\int_0^{\pi/2} \sin(x) f(x)\, dx \approx \sum_{i=0}^{1} w_i f(x_i).$$

Hint:

$$\int x \sin(x)\, dx = \sin(x) - x\cos(x),$$
$$\int x^2 \sin(x)\, dx = 2x\sin(x) - (x^2-2)\cos(x),$$
$$\int x^3 \sin(x)\, dx = (3x^2-6)\sin(x) - (x^3-6x)\cos(x).$$

You will need to apply the error formula for Gaussian quadrature for Part c of the next problem. This result and its proof were not originally included in the 609d lecture notes. This was somewhat of an oversight and the result has now been added (see the paragraph on the bottom of P.55 and the theorem and proof at the top of P.56 of the 609d lecture notes). (make sure that you refresh 609d lecture notes). It will also be gone over in recitation.

Problem 6. The polynomial

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

satisfies

$$\int_{-1}^{1} P_4(x) q(x)\, dx = 0, \qquad \text{for all } q \in \mathbb{P}^3.$$

(a) What are the nodes for the 4 point Gaussian quadrature

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=0}^{3} w_i f(x_i).$$

The corresponding weights are $w_s = (18+\sqrt{30})/36$ and $w_l = (18-\sqrt{30})/36$ with w_s being the weights for the two nodes of smaller absolute value and w_l being the weights for the two nodes of larger absolute value.

(b) Write down the corresponding composite quadrature based on a partition $0 = y_0 < y_1 < \cdots < y_m = 1$ for approximating

$$\int_0^1 f(x)\, dx.$$

Make sure that you give expressions for the local weights and quadrature nodes on each sub-interval $[y_{j-1}, y_j]$.

(c) Use the error formula for Gaussian quadrature and derive an error bound for the composite quadrature above by first deriving local bounds and combining them into a global bound. You may assume the appropriate regularity needed on the function appearing in the integrand. What is the global order of approximation?