Homework/Programming 2. (Due Sept. 6)

Math. 609-600

The first problem involves the Lagrange polynomials for the Hermite cubic interpolation problem. A linear functional is a linear transformation of a vector space (on $\mathbb{R}$) into $\mathbb{R}$. We consider the 4 linear functionals defined by

$$\eta_1(f) = f(0), \qquad \eta_2(f) = f(1),$$
$$\eta_3(f) = f'(0), \qquad \eta_4(f) = f'(1).$$

You may need to review linear tranformations of vector spaces from your course on linear algebra. With this notation, the Hermite cubic interpolation problem on $(0, 1)$ becomes: Given $f \in C^1[0, 1]$, find $p \in \mathbb{P}^3$ satisfying

$$\eta_j(p) = \eta_j(f), \qquad \text{for } j = 1, 2, 3, 4. \tag{0.1}$$

The Lagrange polynomials ℓ_j, for $j = 1, 2, 3, 4$ are defined by

$$\eta_i(\ell_j) = \delta_{i,j} := \begin{cases} 1: & \text{if } i = j, \\ 0: & \text{otherwise,} \end{cases} \qquad \text{for } i = 1, 2, 3, 4.$$

It is not hard to see that the solution $p \in \mathbb{P}^3$ to (0.1) is given by

$$p = \sum_{i=1}^{4} \eta_i(f)\ell_i.$$

Problem 1. Explicitly compute the Lagrange polynomials for the Hermite cubic interpolation problem above.

For the remainder of this homework set, we consider continuous piecewise linear approximations to a function $f(x)$ on $[0, 1]$ based on the partition

$$0 = y_0 < y_1 < \cdots < y_m = 1 \tag{1.1}$$

and set $J_j = [y_{j-1}, y_j]$, for $j = 1, 2, \ldots, m$. Of course, this means that we use $\{y_{j-1}, y_j\}$ as the interpolation nodes on J_j.

We first consider the case when f is smooth. Recall Theorem 6.1 of the 609d lecture notes which shows that if $f(x)$ is in $C^2[0, 1]$, then piecewise linear approximations $f_h(x)$ to $f(x)$ satisfy

$$\|f(x) - f_h(x)\|_{L^\infty(0,1)} \le \frac{\|f''\|_{L^\infty(0,1)}}{8} h^2.$$

We set $h = 1/m$ and define the partition $y_j = jh$ leading to

$$\|f(x) - f_h(x)\|_{L^\infty(0,1)} \le Cm^{-2} \tag{1.2}$$

with C depending on f but not m. The goal of the remainder of this homework is to use non-uniform partitions for singular f which preserve the estimate (1.2).

In the remainder of this assignment, we consider piecewise linear approximations to the function

$$f(x) = x^s, \qquad \text{for } x \in [0,1].$$

Here $s \in (0,1)$ is fixed. Note that f is continuous on $[0,1]$ but is singular since all of its derivatives blow up at 0. As f is continuous, the continuous piecewise linear approximation using the end points of the subintervals as the interpolation points is well defined for any partition of the form of (1.1). We set $\alpha = 2/s$ and $y_i = (i/m)^\alpha$. Note that this partitioning (or mesh) is refined near 0 as the spacing near 0 is much less than near 1.

The first observation is that the interpolation error formula (Theorem 6.1) is useless on J_1 as the second derivative of $f(x)$ is unbounded on that interval. Let P_1 be the linear interpolant of f using the end points as interpolation nodes on J_1. We note that as $s \in (0,1)$, $f(x)$ is increasing and is greater than or equal to P_1 on J_1 and hence

$$|f(x) - f_h(x)| = |f(x) - P_1(x)| \le f(y_1) \text{ for } x \in [0, y_1]. \tag{1.3}$$

We use (1.3) for J_1 and

$$\|f(x) - P_j(x)\|_{L^\infty(y_{j-1},y_j)} \le \frac{\|f''\|_{L^\infty(y_{j-1},y_j)}}{8}(y_j - y_{j-1})^2. \tag{1.4}$$

for the remaining intervals.

In the Problems 2 and 3 below, you will need to bound monotonic functions on J_j for $j > 1$. These will be bounded by their values at one of the two end points.

Problem 2. For $j > 1$, derive a bound for

$$\|f''\|_{L^\infty(y_{j-1},y_j)}$$

in terms of t_{j-1} or t_j.

Problem 3. Setting $Y(t) = t^\alpha$ and $t_j = (j/m)$, we note that $y_j = Y(t_j)$. The mean value theorem shows that for $j > 1$,

$$y_j - y_{j-1} = Y(t_j) - Y(t_{j-1}) = Y'(\zeta)(t_j - t_{j-1}) = Y'(\zeta)/m$$

where ζ is in (t_{j-1}, t_j). Derive a bound for $y_j - y_{j-1}$ in terms of t_{j-1} or t_j by bounding

$$\|Y'\|_{L^\infty(t_{j-1},t_j)}.$$

For the following problem, you will need to use the inequality: For $j > 1$ and $\beta > 0$,

$$t_j^\beta = \left(\frac{j}{j-1}\right)^\beta t_{j-1}^\beta \leq 2^\beta t_{j-1}^\beta. \tag{3.1}$$

Problem 4. Choose β so that combining (1.3), (1.4), (3.1) and the bounds of Problem 2 and 3 implies

$$\|f - f_h\|_{L^\infty(0,1)} \leq C/m^2 \tag{4.1}$$

with C only depending on s. Be explicit with the values of β used in (3.1) and C.