MATH 609

Oct. 7, 2018

Student (print) ______________________________

Last First Middle

Student (sign) ______________________________

Student ID ______________________________

This is a take home test and is due at 3:00PM, Oct. 8. You are to work on this test independently without seeking help from faculty, other students or TA's. Please submit two documents, the first being a PDF file with your work on Problems 1 and 3 and the results requested on Problem 2. The second being a text file containing your the code which you developed for Problem 2 which can be compiled and run. E-mail both files to

pasciak@math.tamu.edu.

1	/40	2	/30
3	/30		
total			

Problem 1. This problem involves a quadrature rule based on values of the function and of the derivative, namely,

$$\int_0^1 f(x)\,dx \approx w_1 f(0) + w_2 f(1) + w_3 f'(0) + w_4 f'(1). \tag{1.1}$$

(a) (15 pts.) Use undetermined coefficients to compute the weights w_i, $i = 1, 2, 3, 4$, which makes the above quadrature exact on cubics. Is this scheme exact on $\mathcal{P}^4$?

(b) (5 pts.) Recall the Lagrange polynomials for Hermite cubics, ℓ_i, $i = 1, 2, 3, 4$ computed in Problem 1 of Homework 2. If you did Part (a) correctly, you should find that

$$w_i = \int_0^1 \ell_i(x)\,dx. \quad \text{for } i = 1, 2, 3, 4. \tag{1.2}$$

Now consider the interval $[0, h]$. On this interval, we set

$$\begin{aligned} \tilde\eta_1(f) &= f(0), & \tilde\eta_2(f) &= f(h), \\ \tilde\eta_3(f) &= f'(0), & \tilde\eta_4(f) &= f'(h). \end{aligned}$$

so that the Lagrange polynomials $\{\tilde\ell_i\}$ for Hermite cubics on $[0, h]$ satisfy

$$\tilde\eta_i(\tilde\ell_j) = \delta_{i,j} := \begin{cases} 1: & \text{if } i = j, \\ 0: & \text{otherwise,} \end{cases} \qquad \text{for } i = 1, 2, 3, 4.$$

For each i, express $\tilde\ell_i$ in terms of ℓ_i.

(c) (10 pts.) Use Part (b) to compute

$$\tilde w_i = \int_0^h \tilde\ell_i(t)\,dt$$

in terms of w_j (using (1.2)) and then compute $\tilde w_j$ from Part (a). The scheme

$$\int_\alpha^{\alpha+h} f(x)\,dx \approx \tilde w_1 f(\alpha) + \tilde w_2 f(\alpha+h) + \tilde w_3 f'(\alpha) + \tilde w_4 f'(\alpha+h)$$

is the scheme (1.1) translated to the interval $[\alpha, \alpha+h]$ and is exact on cubics.

(d) (10 pts.) Use the error identity for Hermite cubic interpolation to derive an identity for the error of the scheme of Part (c) on the interval $[\alpha, \alpha+h]$. You may assume without proof that the function $f^{(4)}(\zeta_x)$ appearing in the cubic interpolation error identity is a continuous function of x. What is the (local) order of approximation on this interval?

Problem 2. This problem involves the algorithm discussed in Lecture 12 for Choleski factorization.

(a) (10 pts.) Write a routine (in Python or Matlab) which given a $k \times k$ lower triangular matrix L and a vector $y \in \mathbf{R}^k$, forward solves for the solution $s = L^{-1}y$.

(b) (20 pts.) Write a routine which computes the Choleski factorization of an $n \times n$ symmetric and positive definite matrix, calling the routine of Part (a) when appropriate.

Debug and test your code by using it to compute the Choleski factorization of the $n \times n$ Hilbert matrix A_n[1] for $n = 2, 3, 4, 5$. Report the resulting L in the Choleski factorization of A_n for each n above.

Problem 3. *Let $\{x_i\}$, $i = 1, \ldots, n$ be the nodes of the Gaussian quadrature approximating*

$$\int_{-1}^{1} (1-x^2) f(x)\, dx \approx \sum_{i=1}^{n} a_i f(x_i). \tag{3.1}$$

(a) (5 pts.) Set $x_0 = -1$, $x_{n+1} = 1$ and $\{x_i\}$, $i = 1, \ldots, n$ as above. Consider the quadrature

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=0}^{n+1} A_i f(x_i). \tag{3.2}$$

The coefficients $\{A_i\}$ are chosen so that the above scheme is exact for $\mathcal{P}^{n+1}$. Write down an expression for the coefficient A_0.

[1]The Hilbert matrix is defined by $(A_n)_{i,j} = (i+j-1)^{-1}$, for $i, j = 1, \ldots, n$.

(b) *(15 pts.) Derive a simple relationship between the coefficients* a_i *and* A_i *for* $i = 1, \dots, n$.

(c) *(10 pts.) Show that the scheme* (3.2) *is, in fact, exact for* $\mathcal{P}^{2n+1}$. *(Hint: use parts (a) and (b)).*