January 2022.

Problem 1:

(a). Let $L: \mathcal{H} \to \mathcal{H}$ be a bounded linear operator whose range, $R(L)$, is closed. Then the equation $Lf = g$ can be solved if and only if $\langle g, v \rangle = 0$ for all $v \in N(L^*)$. Equivalently.

$R(L) = N(L^*)^\perp$

proof: Let $g \in R(L)$, so that there is an $h \in \mathcal{H}$ s.t. $Lh = g$.

If $v \in N(L^*)$, then

$\langle g, v \rangle = \langle Lh, v \rangle = \langle h, L^* v \rangle = \langle h, 0 \rangle = 0$.

$\therefore g \in N(L^*)^\perp \Rightarrow R(L) \in N(L^*)^\perp$.

Let $g \in N(L^*)^\perp$ Since $R(L)$ is closed, the projection theorem implies that we can define $\underbrace{P: \mathcal{H} \to R(L)}_{\text{orthogonal projection}}$. such that.

$Pg \in R(L)$ and $g - Pg \in R(L)^\perp$

Since $g \in N(L^*)^\perp$, $Pg \in R(L) \subseteq N(L^*)^\perp$. then $g - Pg \in N(L^*)^\perp$

So we have $g - Pg \in R(L)^\perp \cap N(L^*)^\perp$.

Hence for all $h \in \mathcal{H}$.

$$0 = \langle Lh, g-Pg \rangle = \langle h, L^*(g-Pg) \rangle$$

Choose $h = L^*(g-Pg) \Rightarrow g-Pg \in N(L^*)$.

Thus $g-Pg \in N(L^*) \cap N(L^*)^\perp = \{0\}$.

$\therefore g = Pg \in R(L)$.

Thus $N(L^*)^\perp \subseteq R(L)$.

Hence $R(L) = N(L^*)^\perp$

(b). If $K \in C(\mathcal{H})$, then $L = I - \lambda K$ has closed range $\lambda \in \mathbb{C}$.

(c). We try to find the $N(L^*)$.

Let $w \in N(L^*)$

$$L^* w = w(x) - \bar{\lambda} \int_0^1 \overline{k(y,x)}\, w(y)\, dy.$$

$$= w(x) - \bar{\lambda} \int_0^1 \left(\sum_{j=1}^{n} \overline{\psi_j(y)}\, \phi_j(x) \right) w(y)\, dy = 0$$

$$\Rightarrow \quad w(x) = \sum_{j=1}^{n} \bar{\lambda} \int_0^1 \overline{\psi_j(y)}\, w(y)\, dy \;\; \phi_j(x) = 0.$$

$$\therefore \quad w(x) \in \operatorname{span}\{\phi_j(x)\}_{j=1}^{n}.$$

$$\cancel{\therefore \quad N(L^*) = \operatorname{span}\{\phi_j(x)\}_{j=1}^{n}}$$

write $w(x) = \sum_{k=1}^{n} c_k \phi_k(x)$. substitute back.

$$\sum_{k=1}^{n} c_k \phi_k(x) - \sum_{j=1}^{n} \bar{\lambda} \int_0^1 \overline{\psi_j(y)} \left(\sum_{k=1}^{n} c_k \phi_k(y)\right) dy \;\; \phi_j(x) = 0.$$

$$\sum_{k=1}^{n} c_k \phi_k(x) - \sum_{j=1}^{n} \bar{\lambda} \sum_{k=1}^{n} c_k \int_0^1 \overline{\psi_j(y)}\, \phi_k(y)\, dy \;\; \phi_j(x) = 0.$$

$$\sum_{k=1}^{n} c_k \phi_k(x) - \sum_{j=1}^{n} \bar{\lambda} \left(\sum_{k=1}^{n} c_k \cdot A_{kj}\right) \phi_j(x) = 0.$$

Compare coefficients. Let $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$$c - \bar{\lambda} \cdot A^T c = 0.$$

Since K is compact. Fredholm Alternative applies.

$Lu = f$ has an solution for all f $\iff$ $N(L^*) = \{0\}$. $\iff$ $c = 0$.

$\left(\frac{1}{\lambda} I - A\right) C = 0$ has solution $C = \{0\}$.

$\therefore \frac{1}{\lambda}$ is not an eigenvalue of A. $\blacksquare$

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problem 2:

(a). From Cauchy-Schwartz, we have

$$|\langle Lu, u\rangle| \leq \|Lu\|\cdot\|u\| \leq \|L\|\cdot\|u\|^2$$

taking supremum over all $\|u\|=1$.

$$\sup_{\|u\|=1} |\langle Lu, u\rangle| \leq \|L\|.$$

Then we set $\alpha := \sup_{\|u\|=1} |\langle Lu, u\rangle|$, we need to prove $\|L\| \leq \alpha$.

Claim: For any $y \in \mathcal{H}$, we have $\|y\| = \sup_{\|z\|=1} |\langle y, z\rangle|$.

proof: Given $y \in \mathcal{H}$, define $\phi_y : \mathcal{H} \to \mathbb{C}$ by $\phi_y(z) := \langle y, z\rangle$

From Cauchy-Schwartz.

$$|\phi_y(z)| = |\langle y, z\rangle| \leq \|y\|\cdot\|z\| \Rightarrow \|\phi_y\| \leq \|y\|.$$

On the other hand,

$$\phi_y(y) = \|y\|^2 \leq \|\phi_y\|\cdot\|y\| \Rightarrow \|y\| \leq \|\phi_y\|.$$

$$\therefore \|y\| = \|\phi_y\| = \sup_{\|z\|=1} |\langle y, z\rangle| \quad \square$$

Then we apply our claim to write

$$\|L\| = \sup_{\|u\|=1} \|Lu\| = \sup_{\|u\|=1} \sup_{\|v\|=1} |\langle Lu, v\rangle| = \sup_{\substack{\|u\|=1 \\ \|v\|=1}} |\langle Lu, v\rangle| .$$

Let $u, v \in \mathcal{H}$ and $\|u\| = \|v\| = 1$, we use the hint to get.

$$\langle L(u+v), u+v\rangle - \langle L(u-v), u-v\rangle = 2\langle Lu, v\rangle + 2\langle Lv, u\rangle$$

$$\overset{\text{self-adjoint}}{=} 2\langle u, Lv\rangle + 2\langle v, Lu\rangle$$

Recall $\langle f, g\rangle = \overline{\langle g, f\rangle}$, we obtain.

$$\langle L(u+v), u+v\rangle - \langle L(u-v), u-v\rangle = \underline{4\,\mathrm{Re}(\langle Lu, v\rangle)}.$$

Next, we let $\tilde{v} = e^{i\theta} v$, where θ is the argument of $\langle Lu, v\rangle \in \mathbb{C}$.

We see that $\qquad \langle Lu, v\rangle = |\langle Lu, v\rangle| \cdot e^{i\theta}$.

$$\langle L(u+\tilde{v}), u+\tilde{v}\rangle - \langle L(u-\tilde{v}), u-\tilde{v}\rangle = 4\,|\langle Lu, v\rangle|$$

From there, we let

$$x := \frac{u+\tilde{v}}{\|u+\tilde{v}\|} \qquad y := \frac{u-\tilde{v}}{\|u-\tilde{v}\|}.$$

By definition of $\lambda := \sup_{\|u\|=1} |\langle Lu, u\rangle|$, we get

$$4\,|\langle Lu, v\rangle| \le \|u+\tilde{v}\|^2 |\langle Lx, x\rangle| + \|u-\tilde{v}\|^2 |\langle Ly, y\rangle| \le \lambda\left(\|u+\tilde{v}\|^2 + \|u-\tilde{v}\|^2\right)$$

From the polarization identity, we have

$$\|u+\tilde{v}\|^2 + \|u-\tilde{v}\|^2 = 2\|u\|^2 + 2\|\tilde{v}\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Therefore, for any $u, v \in \mathcal{H}$, $\|u\| = \|v\| = 1$, we get.

$4|\langle Lu, v\rangle| \leq 4\alpha$., taking supremum and using $\|L\| = \sup\limits_{\substack{\|u\|=1 \\ \|v\|=1}} |\langle Lu, v\rangle|$.

so $\|L\| \leq \alpha$. $\boxed{///}$

(b). By part (a) $\|K\| = \sup\limits_{\|u\|=1} |\langle Ku, u\rangle|$.

We can choose a sequence $\{u_n\}_{n=1}^{\infty}$, $\|u_n\| = 1$, such that.

$$\|K\| = \lim_{n\to\infty} |\langle Ku_n, u_n\rangle|.$$

Taking away the absolute values, we see that the sequence $\langle Ku_n, u_n\rangle$ will converge to $\|K\|$ or $-\|K\|$, or may have subsequences that convergen to either of these.

We assume that $\langle Ku_n, u_n\rangle \to \|K\|$ as $n \to \infty$.

$$\|Ku_n - \|K\| u_n\|^2 = \|Ku_n\|^2 - 2\|K\| \langle Ku_n, u_n\rangle + \|K\|^2$$

$$\leq \|K\|^2 - 2\|K\| \langle Ku_n, u_n \rangle + \|K\|^2$$

$$\leq 2\|K\| \left(\|K\| - \langle Ku_n, u_n \rangle \right).$$

Because $\|K\| = \lim_{n\to\infty} \langle Ku_n, u_n \rangle$, we have

$$\lim_{n\to\infty} \| Ku_n - \|K\| u_n \| = 0$$

Let $z_n = Ku_n - \|K\| u_n$.

Since K is compact, $\{u_n\}_{n=1}^{\infty}$ is bounded.

then there is a convergent subsequence $\{u_{n_k}\}_{k=1}^{\infty}$. 强.

It follows that

$$\|K\| \lim_{k\to\infty} u_{n_k} = \lim_{k\to\infty} K u_{n_k} - \lim_{k\to\infty} z_{n_k}.$$

Since $z_{n_k} \to 0$ and the $\lim_{k\to\infty} K u_{n_k}$ exists, we have that

$u := \lim_{k\to\infty} u_{n_k}$ exists as well. and consequently.

$$\lim_{k\to\infty} K u_{n_k} = K u.$$

Since $z_{n_k} \to 0$. $\quad \|K\| u = K u \qquad$ Also $\|u\| = \| \lim_{k\to\infty} u_{n_k} \| = 1$.

$\therefore$ $\|K\|$ is an eigenvalue of K.

(c).

$$\|M\|_{op} = \sup_{\|u\|=1} |\langle Mu, u\rangle| = \sup_{\|u\|=1} \left| \int_0^1 x\, u^2(x)\, dx \right|$$

$$\le \sup_{\|u\|=1} \int_0^1 u^2(x)\, dx = \|u\|_{L_2}^2 = 1.$$

Also, choose $u_n(x) = \sqrt{2n+1}\, x^n$.

$$\|u_n(x)\|_{L_2}^2 = \int_0^1 (2n+1) x^{2n}\, dx = x^{2n+1}\Big|_0^1 = 1.$$

$$\frac{|\langle Mu_n, u_n\rangle|}{\cancel{\|Mu_n\|_{L_2}^2}} = \int_0^1 x(2n+1)x^{2n}\, dx = \frac{2n+1}{2n+2} x^{2n+2}\Big|_0^1 = \frac{2n+1}{2n+2}.$$

$$\therefore \|M\|_{op} \ge \cancel{\|Mu_n\|_2}\ |\langle Mu_n, u_n\rangle| = \frac{2n+1}{2n+2} \quad \text{for all } n.$$

as $n \to \infty$. $\|M\|_{op} \ge 1$.

Thus $\|M\|_{op} = 1$.

<u>Not Compact!</u>

Let $\{f_n\}$ be an orthonormal set in $L^2[1/2, 1]$, and extend the function to be 0 on $[0, 1/2]$. Thus

$\{\tilde{f}_n\}_{n=1}^{\infty}$ is an orthonormal set in $L^2[0,1]$. (bounded).

But.

$$\|T\tilde{f}_n - T\tilde{f}_m\|_{L^2}^2 = \|x\tilde{f}_n - x\tilde{f}_m\|_{L^2}^2$$

$$= \int_{1/2}^{1} x^2 (f_n(x) - f_m(x))^2 \, dx$$

$$\geq 1/4 \int_{1/2}^{1} |f_n(x) - f_m(x)|^2 \, dx$$

$$= 1/4 \left(\|f_n\|_{L^2[1/2,1]}^2 + \|f_m\|_{L^2[1/2,1]}^2 \right) = 1/2.$$

No convergent subsequence.

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Problem 3:

Homogeneous Solution $e^{i\sqrt{\lambda}x}$ $e^{-i\sqrt{\lambda}x}$

$$G(x,y) = \begin{cases} \alpha_1(y) e^{i\sqrt{\lambda}x} + \alpha_2(y) e^{-i\sqrt{\lambda}x}, & -\infty \le x \le y \le +\infty. \\ \beta_1(y) e^{i\sqrt{\lambda}x} + \beta_2(y) e^{-i\sqrt{\lambda}x}, & -\infty \le y \le x \le +\infty. \end{cases}$$

① Boundary condition.

$G(x,y) \in L^2((-\infty,+\infty) \times (-\infty,+\infty))$

when $x \to -\infty$. sine $\operatorname{Im}\sqrt{\lambda} > 0$, then $i\sqrt{\lambda}x = -\operatorname{Im}(\sqrt{\lambda}) \cdot x \to +\infty$.

which is not in L^2

Thus $\alpha_1(y) = 0$, simplarly when $x \to +\infty$. $\beta_2(y) = 0$.

$$G(x,y) = \begin{cases} \alpha_2(y) e^{-i\sqrt{\lambda}x}, & -\infty \le x \le y \le +\infty \\ \beta_1(y) e^{i\sqrt{\lambda}x}, & -\infty \le y \le x \le +\infty. \end{cases}$$

② Continuity

$G(y^+, y) = G(y^-, y) \Rightarrow \alpha_2(y) e^{-i\sqrt{\lambda}y} = \beta_1(y) e^{i\sqrt{\lambda}y}$.

③ Jump Condition:

$\partial_x G(y^+, y) - \partial_x G(y^-, y) = 1 \Rightarrow \beta_1(y) i\sqrt{\lambda} e^{i\sqrt{\lambda}y} - \alpha_2(y)(-i\sqrt{\lambda}) e^{-i\sqrt{\lambda}y} = 1$

Solve
$$\beta_1(y) = \frac{1}{2i\sqrt{\lambda}} e^{-i\sqrt{\lambda} y} \qquad \alpha_2(y) = \frac{1}{2i\sqrt{\lambda}} e^{i\sqrt{\lambda} y}$$

So
$$G(x,y) = \begin{cases} \frac{1}{2i\sqrt{\lambda}} e^{-i\sqrt{\lambda} x} e^{i\sqrt{\lambda} y}, & -\infty \le x \le y \le +\infty. \\ \frac{1}{2i\sqrt{\lambda}} e^{-i\sqrt{\lambda} y} e^{i\sqrt{\lambda} x}, & -\infty \le y \le x \le +\infty. \end{cases}$$

$$= \begin{cases} \frac{-i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda} (\overbrace{y-x}^{>0\ |x-y|})}, & -\infty \le x \le y \le +\infty. \\ \frac{-i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda} (\overbrace{x-y}^{>0\ |x-y|})}, & -\infty \le y \le x \le +\infty. \end{cases}$$

$$= \frac{-i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda} |x-y|}.$$

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problem 4

(a) symmetry and linearity are easy to check.

$\langle u,u\rangle = \int_0^1 (u'')^2 \, dx \geq 0.$

If $\langle u,u\rangle = 0 \Leftrightarrow \int_0^1 (u'')^2 \, dx = 0 \Leftrightarrow u''(x) = 0. \Leftrightarrow u(x) = ax+b.$

Also $u(0) = u(1) = 0 \Leftrightarrow u(x) = 0.$

inner product.

$\{\phi_j\}_{j=1}^{n-1} \cup \{\psi_j\}_{j=0}^{n}$ is linearly independent.

To show: $\underbrace{\text{span}\{\phi_j\}_{j=1}^{n-1} \cup \{\psi_j\}_{j=0}^{n}}_{V} = S_0^{1/n}(3,1).$

If $s \in V$, then $s \in S^{1/n}(3,1)$ and $s(0) = 0 \cdot s(1) = 0$ by properties of ϕ_j, ψ_j.

So $s \in S_0^{1/n}(3,1). \Longrightarrow V \subseteq S_0^{1/n}(3,1)$

If $s \in S_0^{1/n}(3,1)$

$s \in S^{1/n}(3,1) \Rightarrow s \in \text{span}\{\phi_j, \psi_j\}_{j=0}^{n}$

but $s(0) = 0 \Rightarrow a_0\phi_0(0) = a_0 = 0$

$s(1) = 0 \Rightarrow a_n\phi_n(1) = a_n = 0$

$\Rightarrow s \in \underbrace{\text{span}\{\phi_j\}_{j=1}^{n-1} \cup \{\psi_j\}_{j=0}^{n}}_{V}.$

(b). $\langle \psi_j, \psi_k \rangle = \int_0^1 \psi_j''(x)\, \psi_k''(x)\, dx$

$= \int_0^1 n\, \psi''(nx-j) \cdot n\, \psi''(nx-k)\, dx$.

Notice that $\psi''(x) \neq 0$ if $|x| \le 1$.

~~Since $|j-k|>1$.~~ The support of $\psi''(nx-j)$. is $|nx-j| \le 1$

$\Rightarrow \quad x \in [\frac{j-1}{n}, \frac{j+1}{n}]$.

support of $\psi''(nx-k)$ is $[\frac{k-1}{n}, \frac{k+1}{n}]$.

Since $|j-k|>1$, then $[\frac{j-1}{n}, \frac{j+1}{n}] \cap [\frac{k-1}{n}, \frac{k+1}{n}] = \{0\}$.

$= 0$.

(c). The solution $s^*(x) = \sum_{j=1}^{n-1} a_j \phi_j(x) - \sum_{j=0}^{n} \alpha_j \psi_j(x)$

since $\{\phi_j(x)\}_{j=1}^{n-1} \cup \{\psi_j(x)\}_{j=0}^{n}$ is a basis for $S_0^{1/n}(3,1)$.

From condition $s(k/n) = f_k$, $k = 1, \cdots, n-1$ and condition

$\phi_j(\frac{k}{n}) = \delta_{j,k}$ $\psi_j(\frac{k}{n}) = 0$. we have

$s^*(k/n) = \sum_{j=1}^{n-1} a_j \phi_j(k/n) - \sum_{j=0}^{n} \alpha_j \psi_j(k/n) = a_k = f_k \qquad k = 1, \cdots, n-1$

~~$\Rightarrow S^*$~~ Next, we show d_j's satisfy a tridiagonal system.

Let $s^*(x) = \arg\min \{ \|s\| : s \in S_0^{1/n},\ s(j/n) = f_j,\ j=1,\cdots,n-1 \}$.

and $v \in \text{span}\{\psi_j\}_{j=0}^{n}$ $\forall t \in \mathbb{R}$.

$(s^* + tv)(\frac{j}{n}) = s^*(\frac{j}{n}) + t\, v(j/n) = s^*(j/n)$.

$\|s^* + tv\|^2 = \|s^*\|^2 + 2t\langle s^*, v\rangle + t^2\|v\|^2$ is minimized

when $t=0$.

$\langle s^*, v\rangle = 0$ for all $v \in \text{span}\{\psi_j\}_{j=0}^{n}$

$\therefore s^* \perp \text{span}\{\psi_j\}_{j=0}^{n}$

$$0 = \langle s^*, \psi_k \rangle = \langle \sum_{j=1}^{n-1} f_j \phi_j(x) - \sum_{j=0}^{n} d_j \psi_j(x), \psi_k(x) \rangle$$

$$\Rightarrow \sum_{j=0}^{n} \langle \psi_j(x), \psi_k(x) \rangle d_j = \langle \sum_{j=1}^{n-1} f_j \phi_j(x), \psi_k(x) \rangle \quad \forall k = 0, 1, \cdots, n.$$

$$\underbrace{\begin{bmatrix} \ & & \\ & & \\ & & \end{bmatrix}}_{A} \begin{bmatrix} d_0 \\ \vdots \\ d_n \end{bmatrix} = \underbrace{\begin{bmatrix} \ \\ \\ \ \end{bmatrix}}_{b}$$

$A_{k,j} = \langle \psi_j(x), \psi_k(x) \rangle$

$b_k = \langle \sum_{j=1}^{n-1} f_j \phi_j(x), \psi_k(x) \rangle$.

by (b) if $|j-k|>1$ $\langle \psi_j, \psi_k \rangle = 0$.

so $A_{k,j} = 0$ if $|j-k|>1$.

so A is tridiagonal. (symmetric).

Conclusion:
$$\begin{bmatrix} a & b & & \\ c & a & b & \\ & \ddots & \ddots & b \\ & & c & a \end{bmatrix}$$

eigenvalue

$$\lambda_k = a + 2\sqrt{bc}\cos\left(\frac{k\pi}{n+1}\right) \qquad k = 0, \cdots, n \text{ 是}.$$

In our case, we can compute $a = 4n$. $b = c = 2n$.

$$\lambda_k = 4n + 4n\cos\left(\frac{k\pi}{n+1}\right) \neq 0 \quad \text{for all } k = 0, \cdots, n.$$

是 invertible!