August 2020

problem 1:

This is not closed range theorem.

$L = I - \lambda K$.

Here $L = T - \lambda K$. but we can use the proof. of Closed Range Theorem.

Goal: Show that if there is a sequence $\{g_n\} \subset R(L)$ s.t.

$g_n \to g$, then $g \in R(L)$.

Step 1:

When $N(L) \neq \{0\}$, the solution to $Lf = g$ is not unique. To make it unique, we simply project out the null space.

We will show that if $f \in N(L)^{\perp}$, then there is a constant $C > 0$, independent of f, such that

$$\|Lf\| \geq C\|f\|.$$

If not, then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset N(L)^{\perp}$ s.t $\|f_n\| = 1$. and $\|Lf_n\| \to 0$ as $n \to \infty$.

Note that $Lf_n = Tf_n - \lambda K f_n$. so

$$f_n = T^{-1}(\lambda K f_n + L f_n)$$

Since f_n's are bounded and K is compact, we may choose a subsequence $\{f_{n_k}\}$ s.t. $\{K f_{n_k}\}$ is convergent.

Thus

$$\tilde{f} := \lim_{k\to\infty} f_{n_k} = \lambda \lim_{k\to\infty} T^{-1} K f_{n_k} + \lim_{k\to\infty} T^{-1} L f_n.$$

~~Since $\|L f_n\| \to 0$, then $\lim_{k\to\infty} \|T^{-1} L f_n\| \le \lim_{n\to\infty} \|T^{-1}\| \|L f_n\| \to 0$.~~

∴

T^{-1} is bounded, since both terms on the right are convergent.

$L\tilde{f} = \lim_{k\to\infty} L f_{n_k} = 0$ and $\tilde{f} \in N(L)$.

$N(L)^{\perp}$ is closed and $\{f_n\}_{n=1}^{\infty} \subset N(L)^{\perp}$.

$\therefore \tilde{f} \in N(L) \cap N(L)^{\perp} \Rightarrow \tilde{f} = 0$. $\quad 1 = \lim_{n\to\infty} \|f_n\| = \lim_{n\to\infty} \|f_{n_k}\| = \|\tilde{f}\| = 0$

Contradiction!

Step 2: Notice that $g_n \in R(L)$, there exist $f_n \in \mathcal{H}$ s.t

$g_n = Lf_n$. Fix $f_n \in N(L)^{\perp}$.

By step 1, we have

$$\|g_n - g_m\| = \|Lf_n - Lf_m\| \geq C\|f_n - f_m\|$$

Because $\{g_n\}_{n=1}^{\infty}$ ~~is~~ convergent, is Cauchy.

this implies $\{f_n\}_{n=1}^{\infty}$ is Cauchy. Thus $f_n \to f \in \mathcal{H}$.

$g = \lim_{n\to\infty} Lf_n = Lf$, so $g \in R(L)$.

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problem 2:

(a). Fix $k \in \{0, 1, \cdots n-1\}$.

we can write $x^k = \sum_{j=0}^{k} a_j \phi_j(x)$.

so $\langle x^k, \phi_n(x) \rangle = \langle \sum_{j=0}^{k} a_j \phi_j(x), \phi_n(x) \rangle$

$= \sum_{j=0}^{k} a_j \langle \phi_j(x), \phi_n(x) \rangle = 0$. since $\langle \phi_j, \phi_n \rangle = 0$ for all $j = 0, \cdots, n-1$.

(b). Let $\{\tilde{\phi}_n(x)\}$ be a set of polynomials ~~orthog~~ gotten by using the Gram-schmidt process, and that coefficient of x^n in $\tilde{\phi}_n(x)$ is $b_n > 0$.

$\frac{1}{k_n}\phi_n(x) - \frac{1}{b_n}\tilde{\phi}_n(x)$ polynomial of degree $n-1$

① $\langle \frac{1}{k_n}\phi_n(x) - \frac{1}{b_n}\tilde{\phi}_n(x), \frac{1}{k_n}\phi_n(x) \rangle = 0$ by part (a).

② $\langle \frac{1}{k_n}\phi_n(x) - \frac{1}{b_n}\tilde{\phi}_n(x), \frac{1}{b_n}\tilde{\phi}_n(x) \rangle = 0$ by Gram-Schmidt process

① - ② $\Rightarrow \| \frac{1}{k_n}\phi_n(x) - \frac{1}{b_n}\tilde{\phi}_n(x) \|^2 = 0 \Rightarrow \phi_n(x) = \frac{k_n}{b_n}\tilde{\phi}_n(x)$ ☒

we can write

(c). $x\phi_n(x) = \sum_{k=0}^{n+1} \beta_k \phi_k(x)$. Let's determine β_k.

Notice that $\langle x\phi_n(x), \phi_\ell(x)\rangle = \langle \phi_n(x), x\phi_\ell(x)\rangle = 0$.

for all $\ell \in \{0, \cdots, n-2\}$. by part (a).

~~so. $\beta_\ell = 0$~~ so $\beta_\ell = 0$ for all $\ell \in \{0, \cdots, n-2\}$.

so $x\phi_n(x) = \beta_{n+1}\phi_{n+1}(x) + \beta_n \phi_n(x) + \beta_{n-1}\phi_{n-1}(x)$.

If $\ell = n-1$, $x\phi_{n-1}(x) = k_{n-1}x^n + \underbrace{q(x)}_{\in P_{n-1}} = \frac{k_{n-1}}{k_n}\phi_n(x) + \underbrace{\tilde{q}(x)}_{\in P_{n-1}}$

so $\langle x\phi_n(x), \phi_{n-1}(x)\rangle = \langle \phi_n(x), x\phi_{n-1}(x)\rangle$

$= \langle \phi_n(x), \frac{k_{n-1}}{k_n}\phi_n(x) + \tilde{q}(x)\rangle = \frac{k_{n-1}}{k_n}\|\phi_n\|^2$.

Also $\langle x\phi_n(x), \phi_{n-1}(x)\rangle = \beta_{n-1}\|\phi_{n-1}\|^2$

$$\beta_{n-1} = \frac{k_{n-1}\|\phi_n\|^2}{k_n\|\phi_{n-1}\|^2}$$

If $\ell = n$. $\langle x\phi_n(x), \phi_n(x)\rangle = \langle \beta_n\phi_n(x), \phi_n(x)\rangle$

$$\Rightarrow \beta_n = \frac{\langle x\phi_n(x), \phi_n(x)\rangle}{\|\phi_n(x)\|^2}$$

对 $l = n+1$, $x\phi_n(x) = k_n x^{n+1} + q(x) = \frac{k_n}{k_{n+1}} \phi_{n+1}(x) + \underbrace{\tilde{q}(x)}_{\in P_n}$

$$\langle x\phi_n(x), \phi_{n+1}(x)\rangle = \langle \frac{k_n}{k_{n+1}} \phi_{n+1}(x) + \tilde{q}(x), \phi_{n+1}(x)\rangle$$

$$= \frac{k_n}{k_{n+1}} \|\phi_{n+1}\|^2$$

$$\langle x\phi_n(x), \phi_{n+1}(x)\rangle = \langle \beta_{n+1}\phi_{n+1}, \phi_{n+1}\rangle = \beta_{n+1}\|\phi_{n+1}\|^2.$$

$$\Rightarrow \beta_{n+1} = \frac{k_n}{k_{n+1}}$$

So. $$x\phi_n(x) = \frac{k_n}{k_{n+1}} \phi_{n+1}(x) + \frac{\langle x\phi_n(x), \phi_n(x)\rangle}{\|\phi_n\|^2} \phi_n(x)$$

$$+ \frac{k_{n-1}}{k_n} \frac{\|\phi_n\|^2}{\|\phi_{n-1}\|^2} \phi_{n-1}(x).$$

$$\Rightarrow \phi_{n+1}(x) = \left(\frac{k_{n+1}}{k_n} x - \frac{k_{n+1}}{k_n} \frac{\langle x\phi_n, \phi_n\rangle}{\|\phi_n\|^2} \right) \phi_n(x).$$

$$+ \frac{k_{n-1}\, k_{n+1}}{k_n^2} \frac{\|\phi_n\|^2}{\|\phi_{n-1}\|^2} \phi_{n-1}(x).$$

$$\therefore A_n = \frac{k_{n+1}}{k_n}.$$

▯.

August 2020

Problem 3:

(a). $\langle Lu, u \rangle = \int_{-\infty}^{-\infty} -u''(x)u(x)dx = -u'(x)u(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -u'(x)u'(x)\,dx$

$= \int_{-\infty}^{+\infty} u'(x)^2\,dx \geq 0.$

If $\langle Lu, u \rangle = 0 \Rightarrow \int_{-\infty}^{+\infty} u'(x)^2\,dx = 0$

$\Rightarrow u'(x) = 0. \Rightarrow u(x) = c$ but $u \in L^2_{\mathbb{R}}$

$\therefore c = 0!$ $u(x) = 0.$

$\langle Lu, v \rangle = \int_{-\infty}^{+\infty} -u''(x)v(x)\,dx = -u'(x)v(x)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} u'(x)v'(x)\,dx$

$= -u'(x)v(x)\Big|_{-\infty}^{+\infty} + u(x)v'(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u(x)\underbrace{v''(x)}_{Lv}\,dx.$

$\therefore Lv = -v''$ $v \in L^2$, $Lv \in L^2$.

(b).

(b). Homogeneous

$\cancel{G(x,y) =}$

$$G(x,y) = \begin{cases} \alpha_1(y)\, e^{i\sqrt{\lambda}x} + \alpha_2(y)\, e^{-i\sqrt{\lambda}x}, & -\infty \le x \le y \le +\infty. \\ \beta_1(y)\, e^{i\sqrt{\lambda}x} + \beta_2(y)\, e^{-i\sqrt{\lambda}x}, & -\infty \le y \le x < +\infty. \end{cases}$$

Boundary Condition. $(\mathrm{Im}\sqrt{\lambda} > 0)$.

when $x \to -\infty$ $\qquad \alpha_1(y) = 0.$

when $x \to +\infty$ $\qquad \beta_2(y) = 0.$

Continuity:

$$\alpha_2(y)\, e^{-i\sqrt{\lambda}y} = \beta_1(y)\, e^{i\sqrt{\lambda}y}.$$

Jump Condition:

$$\beta_1(y) \cdot i\sqrt{\lambda}\, e^{i\sqrt{\lambda}y} - \alpha_2(y)\,(-i\sqrt{\lambda})\, e^{-i\sqrt{\lambda}y} = -1$$

$$\beta_1(y) = \frac{-1}{2i\sqrt{\lambda}} e^{-i\sqrt{\lambda} y}$$

$$\alpha_1(y) = \frac{-1}{2i\sqrt{\lambda}} e^{i\sqrt{\lambda} y}$$

$$G(x,y) = \begin{cases} \dfrac{-1}{2i\sqrt{\lambda}} e^{i\sqrt{\lambda} y} \cdot e^{-i\sqrt{\lambda} x}, & -\infty \le x \le y \le +\infty \\ \dfrac{-1}{2i\sqrt{\lambda}} e^{-i\sqrt{\lambda} y} \cdot e^{i\sqrt{\lambda} x}, & -\infty \le x \le y \le +\infty . \end{cases}$$

$$G(x,y) = \frac{i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda}|x-y|}.$$

(c).

$$\int_{[-\infty,+\infty]\times[-\infty,+\infty]} G(x,y)\, dx\, dy . < +\infty .$$

Not compact!

Aug 2020

problem 4:

(a) For any real symmetric matrix A.

$$\lambda_k = \min_{C \in R^{(k-1)\times n}} \max_{\substack{\|x\|=1 \\ Cx=0}} x^T A x.$$

proof: Textbook 1

(b). Choose $C = (1.1.1)$.

this reads $x_1 + x_2 + x_3 = 0$

$$x^T A x = (x_1, x_2, \overset{x_3}{\cancel{x_1 = x_2}}) \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \underset{x_3}{\cancel{x_1 x_2}} \end{pmatrix}.$$

$$= 2x_1x_2 + 6x_1x_3 + 4x_2x_3.$$

$$= 2x_1x_2 + 6x_1(-x_1 - x_2) + 4x_2(-x_1 - x_2)$$

$$= -6x_1^2 - 4x_2^2 - 8x_1x_2$$

$$\leq -4x_1^2 - 4x_2^2 - 8x_1x_2$$

$$= -4(x_1 + x_2)^2 \leq 0. \qquad \text{for all } Cx = 0.$$

taking supremum over x with $\|x\| = 1$.

$$\therefore \quad \lambda_k \leq \max_{\substack{\|x\|=1 \\ (1 \cdot 1 \cdot 1)^T x = 0}} x^T A x \leq 0. \qquad \square$$