January 2021.

Problem 1:

(a) Let S_n be the set of periodic sequences of complex numbers with period n. If $y, z \in S_n$ and $\underline{x = y * z}$, then $\hat{x}_k = \hat{y}_k \hat{z}_k$.

$$x_j = \sum_{m=0}^{n-1} y_m z_{j-m}$$

proof: $\hat{x}_k = \sum_{j=0}^{n-1} x_j \overline{w}^{jk} = \sum_{j=1}^{n-1} \left(\sum_{m=0}^{n-1} y_m z_{j-m} \right) \overline{w}^{jk}$

$$= \sum_{m=0}^{n-1} y_m \sum_{j=0}^{n-1} z_{j-m} \overline{w}^{jk} \overset{i=j-m}{=\!=} \sum_{m=0}^{n-1} y_m \sum_{i=-m}^{n-1-m} z_i \overline{w}^{(i+m)k}$$

$$= \sum_{m=0}^{n-1} y_m \overline{w}^{mk} \sum_{i=-m}^{n-1-m} z_i \overline{w}^{ik}.$$

If we can prove $\sum_{i=-m}^{n-1-m} z_i \overline{w}^{ik} = \sum_{i=0}^{n-1} z_i \overline{w}^{ik}$ (1), then we have by definition

$$= \hat{y}_k \cdot \hat{z}_k$$

To prove (1). we use induction on m

$$\overline{w}^{(n-m)k} = \overline{w}^{nk} \, \overline{w}^{-mk} = e^{-2k\pi i} \cdot \overline{w}^{-km} = \overline{w}^{-mk} \quad \forall m \in \mathbb{Z}.$$

~~when~~ if $m=1$

$$\sum_{i=-1}^{n-2} z_i \overline{w}^{ik} = z_{-1} \overline{w}^{-k} + \sum_{i=0}^{n-2} z_i \overline{w}^{ik} \overset{n\text{-periodic}}{=\!=\!=} z_{n-1} \overline{w}^{(n-1)k} + \sum_{i=0}^{n-2} z_i \overline{w}^{ik} = \sum_{i=0}^{n-1} z_i \overline{w}^{ik}$$

Now suppose (+) is true for $m-1$. then for m.

$$\sum_{i=-m}^{n-1-m} z_i \bar{w}^{ik} = z_{-m}\bar{w}^{-mk} + \sum_{i=-(m-1)}^{n-1-m} z_i \bar{w}^{ik}$$

$$= z_{n-m}\bar{w}^{(n-m)k} + \sum_{i=-(m-1)}^{n-1-m} z_i \bar{w}^{ik}$$

$$= \sum_{i=-(m-1)}^{(n-1)-(m-1)} z_i \bar{w}^{ik} \underset{\substack{\uparrow \\ \text{inductive} \\ \text{hypothesis}}}{=} \sum_{i=0}^{n-1} z_i \bar{w}^{ik}. \qquad \boxed{}$$

(b). The j-th entry of Ax.

$$(Ax)_j = \sum_{\ell=0}^{n-1} x_\ell a_{n-\ell+j} \qquad j=0,1,\cdots,n-1$$

$$\overset{n\text{-periodic}}{=} \sum_{\ell=0}^{n-1} x_\ell a_{j-\ell}. \underset{\substack{\uparrow \\ \text{communitativity}}}{=} [a*x]_j$$

$\therefore \quad y_j = (Ax)_j = [a*x]_j \qquad j=0,1,\cdots,n-1$

So $y = a*x$.

(c) $Ax = y \iff \eta = a * \xi$.

$Ax = \lambda x \iff \lambda \xi = a * \xi$.

by part (a). $(\widehat{\lambda\xi})_j = \hat{a}_j \hat{\xi}_j$, for all $j = \{0, \cdots, n-1\}$.

$\Downarrow$

$\lambda \hat{\xi}_j = \hat{a}_j \hat{\xi}_j$

$\Downarrow$

$\lambda = \hat{a}_j$.

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Problem 2:

(a): Let $u \in D_L$ and $v \in D_{L^*}$

$$\langle Lu, v\rangle = \int_0^1 -u''(x)\,v(x)\,dx \overset{\text{IBP}}{=} -u'(1)\,v(1) + u'(0)\,v(0) + \int_0^1 u'(x)\,v'(x)\,dx$$

$$\overset{BC}{=} u'(0)\left(v(1)+v(0)\right) + \int_0^1 u'(x)\,v'(x)\,dx$$

$$\overset{\text{IBP}}{=} u'(0)\left(v(1)+v(0)\right) + u(1)\,v'(1) - u(0)\,v'(0) + \int_0^1 -v''(x)\,u(x)\,dx.$$

$$\overset{BC}{=} u'(0)\underbrace{\left(v(1)+v(0)\right)}_{=0} + u(1)\underbrace{\left(v'(1)+v'(0)\right)}_{=0.} + \int_0^1 \underbrace{-v''(x)}_{L^*v}\,u(x)\,dx$$

$\therefore$ $L^*v = -v''(x)$, $0 \le x \le 1$ with domain of L^* given by

$D_{L^*} = \{ v \in L^2[0,1] : v'' \in L^2[0,1],\ v(0) = -v(1),\ v'(0) = -v'(1) \}.$

$\therefore$ $L = L^*$. $D_L = D_{L^*}$.

So L is self-adjoint.

(b). $G(x,y) = \begin{cases} \alpha_1(y) + \alpha_2(y)x, & 0 \le x < y \le 1 \\ \beta_1(y) + \beta_2(y)x, & 0 \le y < x \le 1. \end{cases}$

Boundary Condition:

$\alpha_1(y) = -(\beta_1(y) + \beta_2(y)).$ ①

$\alpha_2(y) = -\beta_2(y).$ ②

Continuity Condition:

$\alpha_1(y) + \alpha_2(y) \cdot y = \beta_1(y) + \beta_2(y) \cdot y.$ ③.

Jump Condition:

$\partial_x G(y^+, y) - \partial_x G(y^-, y) = 1/-1$

$\beta_2(y) - \alpha_2(y) = -1$ ④.

②, ④ $\Rightarrow$ $\beta_2(y) = -1/2$, $\alpha_2(y) = 1/2$

$\alpha_1(y) = \frac{1-2y}{4}$ $\beta_1(y) = \frac{1+2y}{4}.$

$$G(x,y) = \begin{cases} \dfrac{1-2y}{4} + \dfrac{1}{2}x, & 0 \le x \le y \le 1 \\[2ex] \dfrac{1+2y}{4} + \left(-\dfrac{1}{2}x\right), & 0 \le y \le x \le 1 \end{cases}$$

$$= \begin{cases} \dfrac{1-2y+2x}{4}, & 0 \le x \le y \le 1 \\[2ex] \dfrac{1-2x+2y}{4}, & 0 \le y \le x \le 1. \end{cases}$$

(c). $K^*u(x) = \int_0^1 G(y,x)\, u(y)\, dy.$

Since $G(x,y) = \overline{G(y,x)}$, then $K = K^*$. self-adjoint.

$G(x,y) \in L^2([0,1]\times[0,1])$ since G is continuous.

Then K is Hilbert-Schmidt operator, and hence compact.

(d). Since K is a compact, self-adjoint operator. Then from. among the eigenfunctions of K, including those for $\lambda = 0$, we may select an orthonormal basis for $\mathcal{H}$.

$K = L^{-1}$ is invertible. So have no zero eigenvalues.

Then we ~~set~~ let $\{\phi_n\}_{n=1}^{\infty}$ be the selected orthonormal basis for $\mathcal{H}$ and $\{\lambda_n\}_{n=1}^{\infty}$ be the corresponding eigenvalues, we can define $\mu_n = 1/\lambda_n$.

Since $\lambda_n \phi_n = K \phi_n$. we applies L to both sides.

$$\lambda_n L \phi_n = \phi_n \Rightarrow \lambda_n \phi_n'' = \phi_n .$$

$$\Rightarrow \phi_n'' = 1/\lambda_n \, \phi_n \Rightarrow L\phi_n = \mu_n \phi_n .$$

So $\{\phi_n\}_{n=1}^{\infty}$ are eigenfunctions of L and contain an orthonormal set that is complete in $L^2[0,1]$.

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problem 3:

(a). K is compact operator and $\lambda \in \mathbb{C}$. then $L = \cancel{I} - \lambda K$ has closed range.

(b) $k(x,y) = x^3 y^2 \in L^2([0,1]\times[0,1])$

so K is Hilbert-Schmidt operator, and hence compact.

(c). Since L is compact (b) and has closed range (a), then FA applies.

Find $N(L^*)$

$$L^* w(x) = w(x) - \bar{\lambda} \int_0^1 y^3 x^2 \, w(y)\,dy = 0.$$

$$\Rightarrow w(x) = Cx^2. \quad \cancel{N(L^*) = \text{span}\{x^2\}}.$$

$$Cx^2 - \bar{\lambda} \int_0^1 y^3 x^2 \cdot Cy^2\,dy = 0 \Rightarrow Cx^2 - C\bar{\lambda} x^2 \cdot \frac{1}{6} = 0.$$

$$\Rightarrow (C - C\bar{\lambda}/6)\, x^2 = 0 \quad \Rightarrow \quad C - C\bar{\lambda}/6 = 0.$$

If $1 - \bar{\lambda}/6 \neq 0$, $(\lambda \neq 6)$ then $C = 0 \Rightarrow N(L^*) = \{0\}$.

Then by FA, $Lu = f$ can be solved for all $f \in L^2[0,1]$.

To find $u(x)$, we need to determined $\int_0^1 y^2 u(y)\,dy$.

we multiply x^2 to $Lu=f$ and integrate over $[0,1]$.

$$\int_0^1 x^2 u(x)\,dx - \int_0^1 x^2\lambda \int_0^1 x^3y^2 u(y)\,dy\,dx = \int_0^1 x^2 f(x)\,dx.$$

$$\int_0^1 y^2 u(y)\,dy - \lambda\cdot\int_0^1 y^2 u(y)\,dy\cdot 1/6 = \int_0^1 y^2 f(y)\,dy.$$

$$\therefore \quad \int_0^1 y^2 u(y)\,dy = \frac{1}{1-\lambda/6}\int_0^1 y^2 f(y)\,dy.$$

$$\begin{aligned}\therefore \quad u(x) &= f(x) + \lambda x^3\int_0^1 y^2 u(y)\,dy.\\ &= f(x) + \lambda\cdot x^3\frac{1}{1-\lambda/6}\int_0^1 y^2 f(y)\,dy.\\ &= f(x) + \frac{6\lambda}{1-\lambda}\int_0^1 k(x,y)f(y)\,dy\\ &= f(x) + \frac{6\lambda}{1-\lambda}kf\end{aligned}$$

In the operator sense.

$$(I-\lambda k)^{-1} = I + \frac{6\lambda}{1-\lambda}k.$$

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Problem 4:

(a). $\phi_j(\frac{k}{n}) = \phi(n\cdot\frac{k}{n} - j) = \phi(k-j)$.

If $k=j$, $\phi_j(\frac{k}{n}) = \phi(0) = (0-1)^2\cdot(2\cdot0+1) = 1$.

If $k\neq j$, $|k-j|=1$, $\phi_j(\frac{k}{n}) = \begin{cases} \phi(1) = (1-1)^2\cdot(2\times1+1)=0, & k=j+1 \\ \phi(-1) = (1-1)^2\cdot(2\times1+1)=0, & k=j-1 \end{cases}$

If $k\neq j$, $|k-j|>1$, $\phi_j(\frac{k}{n}) = \phi(k-j) = 0$.

$\phi_j'(\cancel{\frac{k}{n}}\, x) = n\,\phi'(nx-j)$.

$\phi_j'(\frac{k}{n}) = n\cdot\phi'(k-j)$.

$$\phi'(x) = \begin{cases} 0, & |x|>1 \\ 2(x-1)(2x+1)+2(x-1)^2, & 0\le x\le 1. \\ 2(x+1)(-2x+1) - 2(x+1)^2, & -1\le x\le 0. \end{cases}$$

If $k=j$, $\phi'(0) = -2+2=0$.

If $k\neq j$, $|k-j|>1$ $\phi'(k-j)=0$.

If $k\neq j$, $|k-j|=1$ $\phi'(1) = 0+0=0$

$\phi'(-1) = 0+0=0$.

(a) $\psi_j(k/n) = \frac{1}{n}\psi(k-j)$.

$\psi(0)=0$. $\psi(1) = 1\cdot(1-1)^2=0$ $\psi(k-j)=0$ if $|k-j|>1$.

$$\psi_j'(x) = \begin{cases} 0 & |x|>1. \\ (x-1)^2 + x\cdot 2(x-1) & 0\le x\le 1 \\ (-x-1)^2 + x\cdot 2(-x-1) & -1\le x\le 0. \end{cases}$$

$\psi_j'(k/n) = \psi'(nx-j) = \psi'(k-j)$

if $k=j$, $\psi'(0)=1$

if $k\neq j$. $|k-j|=1$ $\psi_j'(1)=0$ $\psi_j'(-1)=0$

if $k\neq j$. $|k-j|>1$, $\psi_j'(x)=0$.

(b). $\{\phi_j, \psi_j\}_{j=0}^{n}$ $2n+2$ elements.

we need to show $\{\phi_j, \psi_j\}_{j=0}^{n}$ linearly independent.

$s(x) = \sum_{j=0}^{n} \alpha_j \phi_j(x) + \sum_{j=0}^{n} \beta_j \psi_j(x)$

Using (a). we see that

$$s(k/n) = \alpha_k$$

$$s'(k/n) = \beta_k$$

if $s(x) \equiv 0$, $s'(x) \equiv 0$ $\Rightarrow$ $\alpha_k = 0$, $\beta_k = 0$. for all k.

so $\{\phi_j, \psi_j\}_{j=1}^{n}$ are linearly independent. hence basis.

Define

(c). $Pf(x) = \sum_{j=0}^{N} f(j/N)\,\phi_j(x) + \sum_{j=0}^{N} f'(\frac{j}{N})\,\psi_j(x)$.

show it's a projection ~~PP f~~ $P^2 f = Pf$

$$P^2 f(x) = P(Pf(x)) = \sum_{k=0}^{N} Pf(\tfrac{k}{N})\,\phi_k(x) + \sum_{k=0}^{N} (Pf)'(\tfrac{k}{N})\,\psi_k(x)$$

$$= \sum_{k=0}^{N} f(\tfrac{k}{N})\,\phi_k(x) + \sum_{k=0}^{N} f'(\tfrac{k}{N})\,\psi_k(x)$$

$$Pf(\tfrac{k}{N}) = \sum_{j=0}^{N} f(\tfrac{j}{N}) \underbrace{\phi_j(\tfrac{k}{N})}_{\delta_{j,k}} + \sum_{j=0}^{N} f'(\tfrac{j}{N}) \underbrace{\psi_j(\tfrac{k}{N})}_{0} = f(\tfrac{k}{N})$$

$$Pf'(\tfrac{k}{N}) = \sum_{j=0}^{N} f(\tfrac{j}{N}) \underbrace{\phi_j'(\tfrac{k}{N})}_{0} + \sum_{j=0}^{N} f'(\tfrac{j}{N}) \underbrace{\psi_j'(\tfrac{k}{N})}_{\delta_{j,k}} = f'(\tfrac{k}{N}).$$

$\square$