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problem 1:

(a).

D: A sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{D}$ converges to some $\phi \in \mathcal{D}$ iff.

$\|\phi_n^{(m)} - \phi^{(m)}\|_{L^\infty} \to 0$ as $n \to \infty$ for every fixed $m \in \mathbb{N}$ and

$\bigcup_{n=1}^{\infty} \text{supp}(\phi_n - \phi) \subset [-k, k]$. some $k > 0$.

$\mathcal{D}'$: A sequence of distribution $\{T_n\}_{n=1}^{\infty} \subset \mathcal{D}'$ converges to some distribution T in $\mathcal{D}'$ if $\langle T_n, \phi\rangle \to \langle T, \phi\rangle$ as $n \to \infty$ for every $\phi \in \mathcal{D}$.

(b).

$$\phi(x) := \begin{cases} e^{-(1-|x|^2)^{-1}}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

(c). "$\Rightarrow$" $\psi(0) = 0 \cdot \phi(0) = 0$ ~~$\psi'(0)$ trivial~~

~~$\psi'(0) =$~~

$\psi'(x) = x^2 \phi'(x) + 2x\phi(x)$ $\psi'(0) = 0$.

"$\Leftarrow$" we may define $\phi(x) = \frac{1}{x^2}\psi(x)$.

since $\psi \in \mathcal{D}$.

the support of $\phi(x)$ is the same as the support of $\psi(x)$.

$\phi(x)$ is well defined

$$\lim_{x\to 0}\phi(x) = \lim_{x\to 0}\frac{\psi(x)}{x^2} \overset{L'H}{=} \lim_{x\to 0}\frac{\psi'(x)}{2x} \overset{L'H}{=} \lim_{x\to 0}\frac{\psi''(x)}{2} = \frac{\psi''(0)}{2}.$$

Similarly $\lim_{x\to 0}\phi^{(n)}(x) = \dfrac{\psi^{(n+2)}(0)}{(n+2)}$.

So ϕ is smooth C^∞ since $\psi \in C^\infty$.

$$= C_1 \langle \delta, \phi \rangle + C_2 \langle \delta, \phi' \rangle$$

$$= C_1 \langle \delta, \phi \rangle - C_2 \langle \delta', \phi \rangle$$

$$= \langle C_1 \delta - C_2 \delta', \phi \rangle \quad \text{for any } \phi.$$

$\therefore T = C_1 \delta - C_2 \delta'$ where $C_1 = \langle T, \phi_0(x) \rangle$

$C_2 = \langle T, x \phi_0(x) \rangle$

δ is the Dirac-Delta distribution.

(d). Choose test function $\phi_0 \in \mathcal{D}$.

$\phi_0(x) = 1$ on $[-1,1]$ and 0 otherwise.

For any test function $\phi \in \mathcal{D}$, we decompose it as.

$$\phi(x) = \phi(0)\, \phi_0(x) + x\, \phi'(0)\, \phi_0(x) + \psi(x)$$

where $\psi(x) = \phi(x) - \phi(0)\, \phi_0(x) - x\, \phi'(0)\, \phi_0(x)$.

Check

$$\psi(0) = \phi(0) - \phi(0) \cdot 1 - 0 = 0.$$

$$\psi'(0) = \phi'(0) - \phi(0) \cdot 0 - \phi'(0) \cdot \underbrace{\phi_0(0)}_{1} - 0 \cdot \phi'(0) \cdot 0 = 0.$$

So $\psi(x) = x^2 \varphi(x)$ for some $\varphi(x) \in \mathcal{D}$.

Since $x^2 T = 0$, then $0 = \langle x^2 T, \varphi \rangle = \langle T, x^2 \varphi \rangle$.

It means. For any ϕ

$$\langle T, \psi \rangle = 0 \Rightarrow \langle T, \phi \rangle = \phi(0) \underbrace{\langle T, \phi_0(x) \rangle}_{c_1} - \phi'(0) \underbrace{\langle T, x\phi_0(x) \rangle}_{c_2}.$$

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problem 2:

(a) Let $f \in C[0,1]$. Then, for every $\varepsilon > 0$, we can find a polynomial p such that $\|f-p\|_{C[0,1]} < \varepsilon$.

(b). For any $f \in L^2_w[0,1]$

$$\|f\|^2_{L^2_w} = \int_0^1 |f(x)|^2 w(x)\,dx \geq \min_{x\in[0,1]} w(x) \int_0^1 |f(x)|^2\,dx.$$

$$\text{so } \|f\|^2_{L^2[0,1]} \leq \frac{1}{\min\limits_{x\in[0,1]} w(x)} \cdot \|f\|^2_{L^2_w[0,1]} < \infty.$$

Since $C[0,1]$ is dense in $L^2[0,1]$, then for any $f \in L^2_w[0,1] \subseteq L^2[0,1]$.

$\varepsilon > 0$, one can find a continuous function $g \in C[0,1]$ s.t.

$$\|f-g\|_{L^2} < (4\|w\|_\infty)^{-1/2}\varepsilon$$

Since polynomial is dense in $C[0,1]$ (Stone-Weirstrass).

$$\|g-p\|_{C[0,1]} \leq (4\|w\|_\infty)^{-1/2}\varepsilon$$

$$\text{So } \|f-p\|^2_{L^2_w[0,1]} = \int_0^1 |f(x)-p(x)|^2 w(x)\,dx \leq \|w\|_{C[0,1]}\left(2\|f-g\|^2_{L^2} + 2\|g-p\|^2_{L^2}\right)$$

$$\leq \|w\|_{L^\infty} \cdot 2 \cdot \left(\|f-g\|_{L^2}^2 + \|g-p\|_{C[0,1]}^2 \right).$$

$$\leq \|w\|_\infty \cdot 2 \cdot \left(\frac{1}{4\|w\|_\infty} \varepsilon^2 + \frac{1}{4\|w\|_\infty} \varepsilon^2 \right)$$

$$= 2\|w\|_\infty \cdot \frac{1}{2\|w\|_\infty} \varepsilon^2 = \varepsilon^2$$

$$\therefore \|f-p\|_{L^2_w[0,1]} < \varepsilon.$$

So $\mathcal{P}$ is dense in $L^2_w[0,1]$.

(c) Completeness: Lecture Notes: <u>Orthonormal Sets and expansions.</u>

Proposition:

Let D be a dense subset of $\mathcal{H}$. An o.n set $\mathcal{U}$ is complete if and only if $D \subset \mathcal{H}_u$, where $\mathcal{H}_u = \{ g \in \mathcal{H} : g = \sum_{j=1}^{\infty} \alpha_j u_j \}$.

D: $\mathcal{P}$ the set of polynomials.

$\mathcal{H}$: $L^2_w[0,1]$

$\mathcal{U} := \{p_n\}_{n=0}^{\infty}$

It's clear that $\mathcal{P} \subset \operatorname{span}\{p_n\}_{n=1}^{\infty}$ and $\mathcal{P}$ is dense in $L^2_w[0,1]$

Hence, $\mathcal{U}$ is complete.

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problem 3.

(a). suppose not. Then we may select a subsequence $\{\phi_{n_j}\}_{j=1}^{\infty}$ of $\{\phi_n\}_{n=1}^{\infty}$ for which $\|K\phi_{n_j}\| \geq \alpha > 0$ for all j.

Because K is compact, we can also select a subsequence $\{\phi_{n_{j_k}}\}_{k=1}^{\infty}$ such that $K\phi_{n_{j_k}} \to \psi \in \mathcal{H}$.

So we have $\|K\phi_{n_{j_k}}\| \to \|\psi\| \geq \alpha > 0$.

Next, note that

$$\lim_{k\to\infty} \langle K\phi_{n_{j_k}}, \psi \rangle = \|\psi\|^2$$

However

$$\lim_{k\to\infty} \langle K\phi_{n_{j_k}}, \psi \rangle = \lim_{k\to\infty} \langle \phi_{n_{j_k}}, K^*\psi \rangle = 0 \text{ by Bessel's inequality.}$$

Thus $\|\psi\|^2 = 0$ Contradiction!

Review:

Bessel's inequality. If $\{\phi_n\}_{n=1}^{\infty}$ is any orthonormal sequence, then for any $v \in \mathcal{H}$. we have $\sum_{n=1}^{\infty} |\langle \phi_n, v \rangle|^2 \leq \|v\|^2 < \infty$.

So it implies $\lim_{n\to\infty} |\langle \phi_n, v \rangle|^2 = 0 \Rightarrow \lim_{n\to\infty} |\langle \phi_n, v \rangle| = 0 \Rightarrow \lim_{n\to\infty} \langle \phi_n, v \rangle = 0$.

(b). We first compute

$$\int_{-\infty}^{+\infty} e^{-|x-y|^2} u(y)\,dy = \int_{-\infty}^{+\infty} e^{-|x-y|^2/2} u(y) \; e^{-|x-y|^2/2}\,dy$$

$$\overset{CS}{\leq} \left(\int_{-\infty}^{+\infty} e^{-|x-y|^2} u(y)^2\,dy\right)^{1/2} \underbrace{\left(\int_{-\infty}^{+\infty} e^{-|x-y|^2}\,dy\right)^{1/2}}_{\text{compute.}}$$

$$= \left(\int_{-\infty}^{+\infty} e^{-|x-y|^2} u(y)^2\,dy\right)^{1/2} \cdot (\sqrt{\pi})^{1/2}$$

$$\|Tu\|_{L^2}^2 = \int_{-\infty}^{+\infty} Tu(x)^2\,dx \leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-|x-y|^2} u(y)^2\,dy \cdot \sqrt{\pi}\,dx$$

$$\overset{\text{Fubini}}{=} \sqrt{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-|x-y|^2}\,dx \; u(y)^2\,dy$$

$$= \pi \|u\|_{L^2}^2$$ T is Bounded.

(c) $T\phi_j = \int_j^{j+1} e^{-|x-y|^2}\,dy$.

$$\|T\phi_j\|_{L^2}^2 = \int_{-\infty}^{+\infty} \left(\int_j^{j+1} e^{-|x-y|^2}\,dy \right)^2 dx$$

$$\overset{\text{Let } t=y-j}{=} \int_{-\infty}^{+\infty} \left(\int_0^1 e^{-|x-t-j|^2}\,dt \right)^2 dx$$

$$\overset{u=x-j}{=} \int_{-\infty}^{+\infty} \left(\int_0^1 e^{-|u-t|^2}\,dt \right)^2 du$$

$$= \int_{-\infty}^{+\infty} (T\phi_0)^2\,dx = \|T\phi_0\|^2$$

(d). Suppose T is compact, by part (a)

$\lim_{j\to\infty} T\phi_j = 0$. $\quad \therefore \lim_{j\to\infty} \|T\phi_j\| = 0$.

But $\lim_{j\to\infty} \|T\phi_j\| = \|T\phi_0\| \neq 0$. Contradiction!

So T is not compact.

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Problem 4

(a). For any $u \in D_L$ and $v \in D_{L^*}$.

$$\langle Lu, v\rangle = \int_1^2 \underbrace{x^2 u''(x)\, v(x)}_{①} + \underbrace{2x\, u'(x)\, v(x)}_{②} - 2u(x)\, v(x)\, dx.$$

$$= x^2 v(x)\, u'(x)\Big|_1^2 - \underbrace{\int_1^2 u'(x)\, 2x\, v(x)}_{②} + u'(x)\, x^2 v'(x)\, dx + \int_1^2 2x\, u'(x)\, v(x) - 2u(x)\, v(x)\, dx.$$

$$= 4v(2)\, u'(2) - \underbrace{v(1)\, u'(1)}_{0} - \underbrace{\int_1^2 u'(x)\, x^2 v'(x)\, dx}_{③} - \int_1^2 2u(x)\, v(x)\, dx$$

$$= 4v(2)\, u'(2) - x^2 v'(x)\, u(x)\Big|_1^2 + \int_1^2 2x v'(x)\, u(x) + x^2 v''(x)\, u(x) - 2u(x)\, v(x)\, dx$$

$$= \underbrace{4v(2)}_{=0}\, u'(2) - \cancel{4v'(2)\underbrace{u(2)}_{0}} + \underbrace{v'(1)}_{=0}\, u(1) + \int_1^2 \underbrace{[x^2 v''(x) + 2x v'(x) - 2v(x)]}_{L^* v}\, u(x)\, dx$$

~~Let $L^* v =$~~

$D_{L^*} = \{v \in L^2[1,2] : v'(1) = 0,\ v(2) = 0\}.$

$L^* v = x^2 v''(x) + 2x v'(x) - 2v(x)$

$L = L^*$.

(b). $G(x,y) = \begin{cases} \alpha_1(y)\, x + \alpha_2(y)\, x^{-2}, & 1 \le x \le y \le 2 \\ \beta_1(y)\, x + \beta_2(y)\, x^{-2}, & 1 \le y \le x \le 2. \end{cases}$

Boundary Condition:

$$\partial_x G(1,y) = \alpha_1(y) + (-2)\cdot \alpha_2(y) = 0 \implies \boxed{\alpha_1(y) = 2\alpha_2(y)}$$

$$G(2,y) = 2\beta_1(y) + \frac{1}{4}\beta_2(y) = 0 \implies \boxed{\beta_1(y) = -\frac{1}{8}\beta_2(y)}$$

Continuity Condition:

$$\alpha_1(y)\cdot y + \alpha_2(y)\cdot y^{-2} = \beta_1(y)\cdot y + \beta_2(y)\cdot y^{-2}$$

$$\boxed{\alpha_2(y)\,(2y + y^{-2}) = \beta_2(y)\left(-\frac{1}{8}y + y^{-2}\right)}$$

Jump condition:

$$\partial_x G(y^+,y) - \partial_x G(y^-,y) = \frac{1}{a_2(y)} = \frac{1}{y^2}$$

$$\beta_1(y) + (-2)\beta_2(y)\, y^{-3} - \alpha_1(y) - (-2)\,\alpha_2(y)\, y^{-3} = \frac{1}{y^2}$$

$$\beta_1(y)\, y^3 - 2\beta_2(y) - \alpha_1(y)\, y^3 + 2\alpha_2(y) = y.$$

$$\boxed{-\frac{1}{8}\beta_2(y)\, y^3 - 2\beta_2(y) - 2\alpha_2(y)\, y^3 + 2\alpha_2(y) = y.}$$

$$\beta_2(y) = \frac{y(2y+y^{-2})}{(-\frac{1}{8}y^3-2)(2y+y^{-2}) - (-\frac{1}{8}y+y^{-2})(2y^3-2)}$$

$$\alpha_2(y) = \frac{y(-\frac{1}{8}y+y^{-2})}{(-\frac{1}{8}y^3-2)(2y+y^{-2}) - (-\frac{1}{8}y+y^{-2})(2y^3-2)}$$

$$G(x,y) = \begin{cases} \alpha_2(y)\,(2x+x^{-2}), & \cancel{0} \le x \le y \le 2 \\ \beta_2(y)\,(-\frac{1}{8}x+x^{-2}), & 1 \le y \le x \le 2. \end{cases}$$

$G(x,y)$ is continuous and hence in $L^2([1,2]\times[1,2])$

so K is Hilbert-Schmidt kernel and hence compact.

$G(x,y) = G(y,x)$ $\therefore$ K is self-adjoint.

(c). Spectral Theorem:

Let K be a compact, self-adjoint operator. Then, from among the eigenvectors of K, including those for $\lambda=0$, we may select an orthonormal basis for $\mathcal{H}$.

$K = L^{-1}$ is compact and self-adjoint, then by spectral theorem, one may select an orthonormal basis for $\mathcal{H}$, denoted by $\{\phi_n\}_{n=1}^{\infty}$ and

$\{\lambda_n\}_{n=1}^{\infty}$ are corresponding eigenvalues.

② $K = L^{-1}$ is invertible, so K has no zero eigenvalues.

Then we set $\mu_n = -1/\lambda_n$.

since $\lambda_n \phi_n = K \phi_n$. then we apply L to both sides.

$$\lambda_n L\phi_n = \phi_n \Rightarrow L\phi_n = -1/\lambda_n \phi_n = \mu_n \phi_n.$$

$\therefore \{\phi_n\}_{n=1}^{\infty}$ are eigenvectors of L, also forms a complete orthonormal set.