Optimization Theory and Algorithms

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Outline

- Syllabus
- Background information
- Mathematical optimization
- Optimization problems
- Examples

Syllabus

- Reference textbook: *Convex Optimization* by Stephen Boyd and Lieven Vandenberghe
  Available at https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf

- Grading scheme: 40% homework (4 times) + 60% exam

Course Information

- Application driven: manufacturing; transportation; scheduling; investment…
- Decision problem
- Mathematical modeling
- Topics: introduction to optimization; convex optimization; duality theory; convex optimization algorithms.

Mathematical Optimization

$$
\begin{aligned}
\mathit{minimize} \quad & f(x) \\
\mathit{subject\ to} \quad & g_i(x) \leq 0, i = 1, \dots, m \\
& h_i(x) = 0, i = 1, \dots, l \\
& x \in X
\end{aligned}
$$

- $x = (x_1, \dots, x_n)$: optimization/decision variables
- $f(\cdot)\colon \mathbb{R}^n \longrightarrow \mathbb{R}$: objective function
- $g_i(x)$, $i = 1, \dots, m$: inequality constrain functions
- $h_i(x)$, $i = 1, \dots, l$: equality constrain functions
- $X \subseteq \mathbb{R}^n$: feasible region

Global minimizer x^*: has the smallest value of f for all $x \in X$, i.e.,$f(x^*) \leq f(x)$.

Local minimizer x': there exist an ε such that $f(x') \leq f(x)$ for all $x \in X \cap B(x', \varepsilon)$.

$\{\boldsymbol{x} \in \mathbb{R}^n \colon ||x - x'||_2 \leq \varepsilon\}$

Euclidean ball of radius $\varepsilon > 0$ centered at x'

Linear Programming

$$\begin{aligned} & \mathit{minimize} && c^T x \\ & \mathit{subject\ to} && Ax = b \\ & && Gx \boxed{\preccurlyeq} h \end{aligned}$$

Component-wise inequality

- $x \in \mathbb{R}^n, c \in \mathbb{R}^n$
- $A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p, G \in \mathbb{R}^{m \times n}, h \in \mathbb{R}^m$
- Linear objective function, linear equality constraint, and linear inequality constraint.

Example of Linear Programming

Diet problem: To find the cheapest combination of foods that satisfies some nutritional requirements.

$$
\begin{aligned}
&\textit{minimize} \quad && c^T x \\
&\textit{subject to} \quad && Gx \succcurlyeq h \\
& && x \succcurlyeq 0
\end{aligned}
$$

- $x \in \mathbb{R}^n, c \in \mathbb{R}^n, G \in \mathbb{R}^{m \times n}, h \in \mathbb{R}^m$
- x_j: units of food j; c_j: per-unit price of food j
- G_{ij}: content of nutrient i in per unit of food j
- h_i: minimum required intake of nutrient i

Example of Linear Programming

Air traffic control problem

- There are n airplanes: airplane i arrival interval $[a_i, b_i]$.
- Assume airplane arrive in the order 1,2,…,n.
- Control objective: maximize the minimum over all inter-arrival times between two consecutive planes.

$$
\begin{array}{lll}
maximize & \min_{1 \le j \le n-1} (t_{j+1} - t_j) & \\
subject\ to & a_i \le t_i \le b_i, & i = 1, \dots, n \\
 & t_i \le t_{i+1}, & i = 1, \dots, n-1
\end{array}
$$

Introduce a new decision variable

$$
\begin{array}{lll}
maximize & z & \\
subject\ to & t_{j+1} - t_j \ge z, & i = 1, \dots, n-1 \\
 & a_i \le t_i \le b_i, & i = 1, \dots, n \\
 & t_i \le t_{i+1}, & i = 1, \dots, n-1
\end{array}
$$

decision variables: z, t_i, $i = 1, \dots, n$,

Quadratic Programming

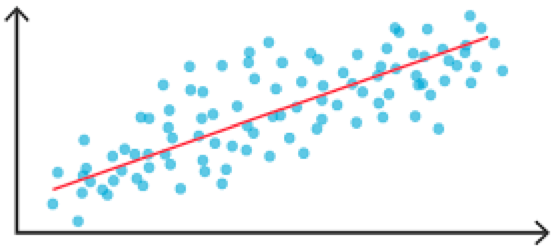
$$\begin{aligned} &minimize \;\; \frac{1}{2}x^TQx + q^Tx \\ &subject\ to \;\; Ax = b \\ &\qquad\qquad\quad\; Gx \preccurlyeq h \end{aligned}$$

- $Q = [Q_{ij}] \in \mathbb{R}^{n\times n}$, $x^TQx \equiv \sum_{i=1}^{n}\sum_{i=1}^{n} Q_{ij}x_ix_j$
- Assume Q is symmetric.

$$x^TQx = x^T\left(\frac{Q+Q^T}{2}\right)x$$

Example of Quadratic Programming

Data fitting problem



$$\underset{x}{minimize} \, ||Ax - b||_2^2 = (Ax - b)^T(Ax - b) = x^T A^T A x - 2b^T A x + b^T b$$

- $A \in \mathbb{R}^{m \times n}$: each row i represents a data point with n features.
- $b \in \mathbb{R}^m$: each component i represents a prediction.
- To find the parameter x that minimized the squared L2-norm of the error.

p-norm of a vector x ($p \geq 1$):

$$\|x\|_p \triangleq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

Example of Quadratic Programming

Portfolio optimization:

- There n assets or stocks. Let x_i denote the amount of asset i invested.
- Constraint: a minimum return, budget feasibility, and non-negative investment.
- Objective: minimizing overall risk.

$$\begin{aligned} &minimize && x^T \Sigma x \\ &subject\ to && R^T x \geq r_{min} \\ & && \mathbf{1}^T x = B \\ & && x \succcurlyeq 0 \end{aligned}$$

- $R \in \mathbb{R}^n$: expected return for each invested asset; $r_{min} \in \mathbb{R}$: minimum return.
- $\mathbf{1} \in \mathbb{R}^n$: every component is 1; B: budget.
- $\Sigma \in \mathbb{R}^{n \times n}$: covariance matrix for the prices of all assets; indicates investment risk.

Semidefinite Programming

Positive semidefinite matrix ($Q \succcurlyeq \mathbf{0}$ or $Q \in \mathbf{S}_+^n$):
An $n \times n$ symmetric matrix Q is positive semidefinite if

$$x^T Qx \geq 0 \text{ for all } x \in \mathbb{R}^n.$$

positive definite ($Q \succ 0$ or $Q \in \mathbf{S}_{++}^n$)⇒
$x^T Qx > 0$ for all $x \neq \mathbf{0}$

$$\begin{array}{ll} minimize & C \bullet X \\ subject\ to & A_i \bullet X = b_i, \quad \text{for } i = 1, \dots, m \\ & X \succcurlyeq \mathbf{0} \end{array}$$

- $C \in \mathbf{S}_+^n$: $n \times n$ symmetric matrix.
- $C \bullet X \triangleq \sum_{i=1}^n \sum_{j=1}^n C_{ij} Z_{ij}$.

tr(Z): sum of matrix Z diagonal elements
tr(CX)= $\sum_{i=1}^n \sum_{j=1}^n C_{ij} Z_{ij}$

Example of Semidefinite Programming

Minimize the largest eigenvalue:

- Consider k $n \times n$ symmetric matrix $A_1, \dots, A_k$.
- Consider a function A: $R^k \to R^{n \times n}$

$$A(x) = \sum_{i=1}^{k} x_i A_i$$

eigenvalue

$$minimize \ \max_i \boxed{\lambda_i(A(x))}$$

Introduce an auxiliary valuable t

$$\begin{aligned} &minimize \ \ t \\ &subject\ to\ tI - A(x) \succcurlyeq \mathbf{0} \end{aligned}$$

Proposition

Let Z be an arbitrary $n \times n$ symmetric matrix, and let $\lambda_{max}(Z)$ be the largest eigenvalue of Z. Then, we have $t \geq \lambda_{max}(Z)$ if and only if $tI - Z \succcurlyeq \mathbf{0}$.

What's Next

- Basic concepts of convexity
- Convex optimization
- Optimality condition and duality theory
- Optimization algorithms
- Applications