

# Optimization Theory and Algorithms

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# Outline

- General optimization problem
- Convex optimization problem
- Optimality condition
- Equivalent transformation

# Optimization problem in standard form

$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p\end{array}$$

- $x = (x_1, \dots, x_n)$  : optimization/decision variables
- $f_0(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ : objective function
- $f_i(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, m$ : inequality constrain functions
- $h_i(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, p$ : equality constrain functions
- Domain:  $\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$
- If a point  $x \in \mathcal{D}$  satisfies all the constraints  $f_i(x) \leq 0, i = 1, \dots, m$ , and  $h_i(x) = 0, i = 1, \dots, p$ , it is a *feasible point*. The set of all feasible point is *feasible set*.
- The problem is *feasible* if there exists at least one feasible point, and *infeasible* otherwise.

# Notations

$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p\end{array}$$

- Optimal value

$$p^* = \inf \{f_0(x) \mid f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$

- $p^* = -\infty$ : if there are feasible points  $x_k$  with  $f_0(x_k) \rightarrow -\infty$  as  $k \rightarrow \infty$ , i.e., the problem is unbounded below.

- Optimal points:  $x^*$  is feasible and  $f_0(x^*) = p^*$ , i.e.,  $x^*$  solves the problem.

- Optimal set

$$X_{\text{opt}} = \{x \mid f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p, f_0(x) = p^*\}$$

- Example (unconstrained optimization problem):

- $f_0(x) = \frac{1}{x}$ ,  $\mathcal{D} = \mathbb{R}_{++}$ ,  $p^* = 0$ , no optimal point

- $f_0(x) = -\log x$ ,  $\mathcal{D} = \mathbb{R}_{++}$ ,  $p^* = -\infty$ , unbounded below

- $f_0(x) = x \log x$ ,  $\mathcal{D} = \mathbb{R}_{++}$ ,  $p^* = -1/e$ ,  $x = 1/e$  is optimal point

# Notations

$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p\end{array}$$

- $\varepsilon$ -suboptimal points: feasible  $x$  with  $f_0(x) = p^* + \varepsilon$
- $\varepsilon$ -suboptimal set: the set of all  $\varepsilon$ -suboptimal points.
- Locally optimal points solves the following problem for an  $R > 0$ :

$$\begin{array}{ll}\min & f_0(z) \\ \text{s.t.} & f_i(z) \leq 0, i = 1, \dots, m \\ & h_i(z) = 0, i = 1, \dots, p \\ & \|z - x\|_2 < R\end{array}$$

# Convex optimization problem

$$\min f_0(x)$$

$$\text{s.t. } f_i(x) \leq 0, i = 1, \dots, m$$

$$a_i^T x = b_i, i = 1, \dots, p \longrightarrow Ax = b, A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p$$

- Convex optimization problem:
  - Objective function  $f_0$  is convex function
  - Inequality constraint functions  $f_1, \dots, f_m$  are convex functions
  - Equality constraint functions are affine ( $a_i^T x = b_i$ ).
- Property: feasible set of a convex optimization problem is convex

# Abstract form convex optimization problem

$$\begin{aligned} \min \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(x) = x_1/(1 + x_2^2) \leq 0, \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{aligned}$$

- The problem is not a convex problem  
inequality constraint function is not convex;  
equality constraint function is not affine
- Equivalent convex problem

$$\begin{aligned} \min \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(x) = x_1 \leq 0, \\ & h_1(x) = x_1 + x_2 = 0 \end{aligned}$$

# Local and global optima

Any locally optimal point of a convex problem is (globally) optimal

Proof by contradiction:

- Let  $x$  be locally optimal.
- Suppose  $x$  is not globally optimal, i.e., there exists  $y$  such that  $f_0(y) < f_0(x)$ .
- Consider  $\theta \in [0,1]$ :

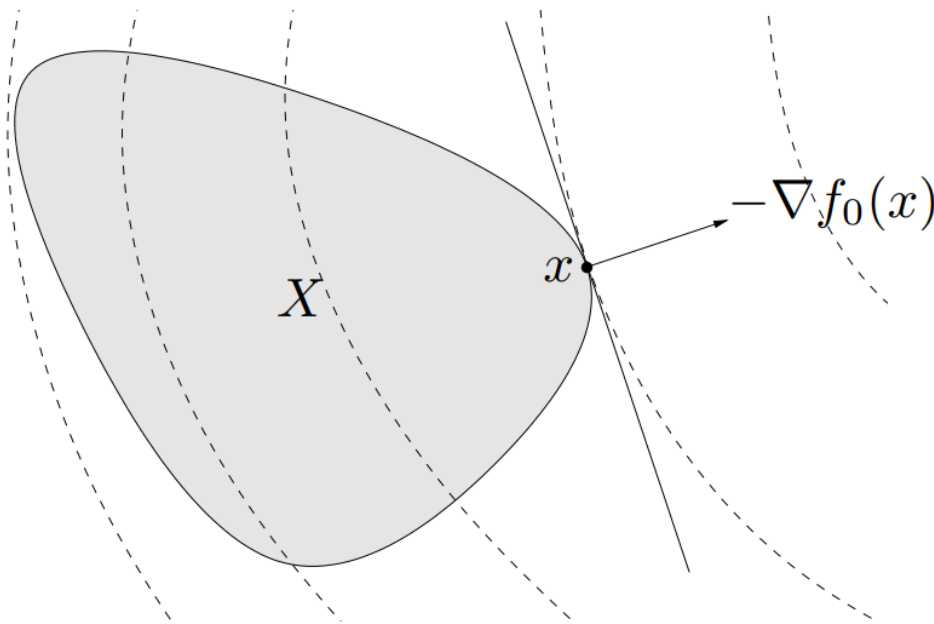
$$f_0(\theta y + (1 - \theta)x) \leq \theta f_0(y) + (1 - \theta)f_0(x) < \theta f_0(x) + (1 - \theta)f_0(x) = f_0(x),$$

i.e.,  $f_0(\theta x + (1 - \theta)y) < f_0(x)$ , which is a contradiction when  $\theta = 1$ .



# Optimality criterion for differentiable $f_0$

First-order condition:  $x$  is optimal if and only if it is feasible and  
 $\nabla f_0(x)^T (y - x) \geq 0$  for all feasible  $y$



If  $\nabla f_0(x)$  is nonzero,  $-\nabla f_0(x)$  defines a supporting hyperplane to the feasible set at  $x$

# Unconstrained problems as a special case

$$\min f_0(x)$$

$x$  is optimal if and only if it is feasible and  $\nabla f_0(x)^T(y - x) \geq 0$  for all feasible  $y$

reduce

$x$  is optimal if and only if  $\nabla f_0(x) = 0$

Example:

$$\min f_0(x) = \frac{1}{2}x^T Qx + b^T x$$

The first-order condition:  $\nabla f_0(x) = Qx + b = 0$

- Case 1:  $Qx = -b$  has unique solution.
- Case 2:  $Qx = -b$  has infinitely many solutions.
- Case 3:  $Qx = -b$  has no solution, i.e.,  $\min f_0(x) = -\infty$ .

# Equivalent transformation

## Eliminating equality constraints

$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, i = 1, \dots, m \\ & Ax = b\end{array}$$



$$\begin{array}{ll}\min_z & f_0(Fz + x_0) \\ \text{s.t.} & f_i(Fz + x_0) \leq 0, i = 1, \dots, m\end{array}$$

$F$  and  $x_0$  are such that  
 $Ax = b \Leftrightarrow x = Fz + x_0$  for some  $z$

## Introducing equality constraints

$$\begin{array}{ll}\min & f_0(A_0x + b_0) \\ \text{s.t.} & f_i(A_ix + b_i) \leq 0, i = 1, \dots, m\end{array}$$



$$\begin{array}{ll}\min_{x, y_i} & f_0(y_0) \\ \text{s.t.} & f_i(y_i) \leq 0, i = 1, \dots, m \\ & y_i = A_ix + b_i, i = 1, \dots, m\end{array}$$

# Equivalent transformation

Introducing slack variables for *linear inequalities*

$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & a_i^T x \leq b_i, i = 1, \dots, m\end{array}$$



$$\begin{array}{ll}\min_{x, s_i} & f_0(x) \\ \text{s.t.} & a_i^T x + s_i = b_i, i = 1, \dots, m \\ & s_i \geq 0, i = 1, \dots, m\end{array}$$

Epigraph form

$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, i = 1, \dots, m \\ & Ax = b\end{array}$$



$$\begin{array}{ll}\min_{x, t} & t \\ \text{s.t.} & f_i(x) \leq 0, i = 1, \dots, m \\ & Ax = b \\ & f_i(x) - t \leq 0\end{array}$$