

Optimization Theory and Algorithms

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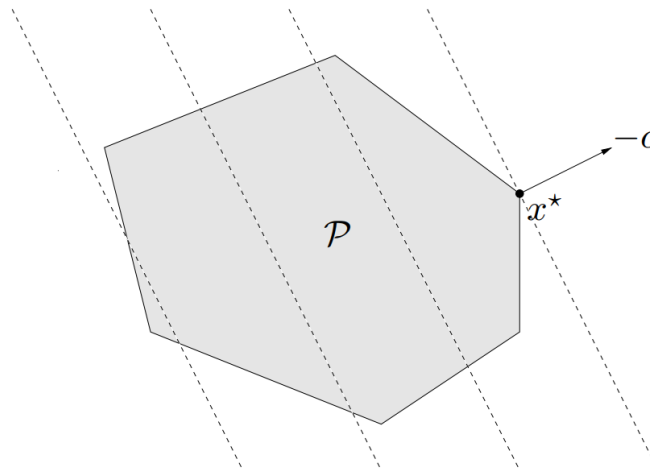
Outline

- Linear programming
- Quadratic programming
- Quadratically constrained quadratic programming
- Second-order cone programming
- Semidefinite programming
- Conic programming

Linear Programming (LP)

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

- Affine objective and constraint functions
- minimize an affine function over a polyhedron



- Solution: (i) $-\infty$; (ii) at a vertex

Linear Programming: standard form

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

- The only inequalities are $x \geq 0$
- Converting general form to standard form:

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Gx \leq h \\ & Ax = b\end{array} \quad \longrightarrow \quad \begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0\end{array}$$

- Introducing **slack variables** s for the inequalities:

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Gx \leq h \\ & Ax = b\end{array} \quad \longrightarrow \quad \begin{array}{ll}\min & c^T x \\ \text{s.t.} & Gx + s = h \\ & Ax = b \\ & s \geq 0\end{array}$$

- Decompose the variable x as the difference of two non-negative variables

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Gx + s = h \\ & Ax = b \\ & s \geq 0\end{array} \quad \longrightarrow \quad \begin{array}{ll}\min & c^T x^+ - c^T x^- \\ \text{s.t.} & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & s \geq 0, x^+ \geq 0, x^- \geq 0\end{array}$$

LP: examples

Diet problem: To find the cheapest combination of foods that satisfies some nutritional requirements.

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0\end{array}$$

- x_j : units of food j ; c_j : per-unit price of food j
- A_{ij} : content of nutrient i in per unit of food j
- b_i : minimum required intake of nutrient i

LP: examples

Transportation: Ship commodities from given sources to destinations at minimum cost

$$\begin{aligned} \min_x \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^n x_{ij} \leq s_i, \quad i = 1, \dots, m \\ & \sum_{i=1}^m x_{ij} \geq d_j, \quad j = 1, \dots, n, \quad x \geq 0 \end{aligned}$$

- x_{ij} : units shipped from i to j
- c_{ij} : per-unit shipping cost from i to j
- s_i : supply at source $i, i = 1, \dots, m$
- d_j : demand at destination $j, j = 1, \dots, n$

LP: examples

Piecewise-linear minimization

$$\min \max_{i=1,\dots,m} a_i^T x + b_i$$

Equivalent LP:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & a_i^T x + b_i \leq t, \quad i = 1, \dots, m \end{array}$$

Absolute value minimization

$$\begin{array}{ll} \min & |c^T x + d| \\ \text{s.t.} & Ax = b \end{array}$$

Equivalent LP:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & c^T x + d \leq t \\ & -c^T x - d \leq t \\ & Ax = b \end{array}$$

LP: examples

L_∞ -norm minimization


$$\begin{array}{ll}\min & \|x\|_\infty \\ \text{subject to} & Gx \leq h \\ & Ax = b\end{array}$$

$$\|x\|_\infty \triangleq \max_i |x_i|$$

Equivalent LP:

$$\begin{array}{ll}\min_{t \in \mathbb{R}, x \in \mathbb{R}^n} & t \\ \text{subject to} & Gx \leq h \\ & Ax = b \\ & x \leq t \mathbf{1} \\ & -t \mathbf{1} \leq x\end{array}$$

$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$



L_1 -norm minimization

$$\begin{array}{ll}\min & \|x\|_1 \\ \text{subject to} & Gx \leq h \\ & Ax = b\end{array}$$

$$\|x\|_1 \triangleq \sum_i |x_i|$$

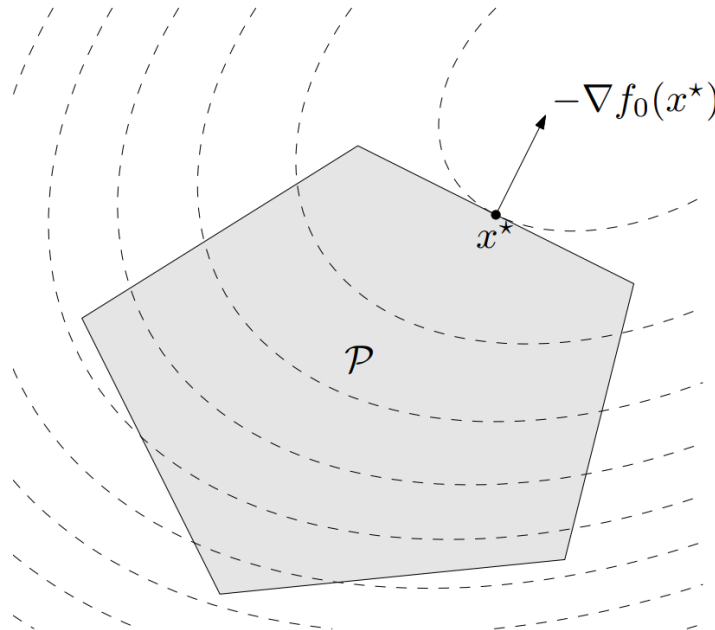
Equivalent LP:

$$\begin{array}{ll}\min_{t \in \mathbb{R}^n, x \in \mathbb{R}^n} & \mathbf{1}^T t \\ \text{subject to} & Gx \leq h \\ & Ax = b \\ & x \leq t \\ & -t \leq x\end{array}$$

Quadratic Programming (QP)

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Px + q^T x + r \\ \text{s.t.} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

- $P \in \mathbb{S}_+^n$, so the objective is convex function
- Minimize a convex quadratic function over a polyhedron



QP: examples

Least-squares and regression

$$\begin{aligned} \min_x \quad & ||Ax - b||_2^2 = x^T A^T A x - 2b^T A x + b^T b \\ \text{subject to} \quad & l_i \leq x_i \leq u_i, i = 1, \dots, n \end{aligned}$$

Linear programming with random cost

Deterministic

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

Non-deterministic

$$\begin{aligned} \min \quad & \mathbf{E}[c^T x] + \gamma \mathbf{var}[c^T x] = \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

- c is random vector with mean \bar{c} and covariance Σ
- $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$

QP: examples

Portfolio optimization

$$\begin{aligned} & \text{minimize} && x^T \Sigma x \\ & \text{subject to} && R^T x \geq r_{\min} \\ & && \mathbf{1}^T x = B \\ & && x \geq 0 \end{aligned}$$

- Price changes of all invested assets has mean $R \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n \times n}$
- r_{\min} : minimum return.
- $\mathbf{1} \in \mathbb{R}^n$: every component is 1;
- B : budget.

Quadratically constrained quadratic programming (QCQP)

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- $P \in \mathbb{S}_+^n$, so the objective and constraint functions are convex
- Minimize a convex quadratic function over a intersection of m ellipsoids and an affine set

QCQP: example

Portfolio optimization

$$\begin{aligned} &\text{minimize} && x^T \Sigma_0 x \\ &\text{subject to} && x^T \Sigma_i x \leq d_i, i = 1, \dots, m \\ & && R^T x \geq r_{\min} \\ & && \mathbf{1}^T x = B \\ & && x \geq 0 \end{aligned}$$

- There are a few estimation of the covariance of the price changes, $\Sigma_i, i = 0, \dots, m$

Second-order cone programming (SOCP)

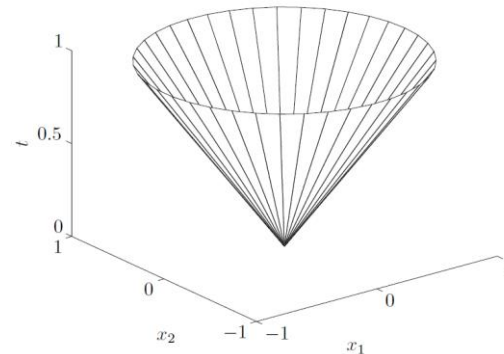
$$\begin{array}{ll}\text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \dots, m \\ & Fx = g,\end{array}$$

- $A_i \in \mathbb{R}^{n_i \times n}$
- Inequalities are second-order cone constraints

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbb{R}^{n_i+1}$$

- If $A_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP.

$\{(x, t) \mid \|x\|_2 \leq t\}$ is *second-order cone*,
also called *ice cream cone*.



SOCP: examples

Robust linear programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m,\end{array}$$

- a_i is inaccurate, but are known in ellipsoids: $a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$
 $\bar{a}_i \in \mathbb{R}^n, P_i \in \mathbb{R}^{n \times n}$



$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \dots, m\end{array}$$

Equivalent SOCP:

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \dots, m\end{array}$$

- $\sup_{\|u\|_2 \leq 1} \bar{a}_i^T x + (P_i u)^T x = \sup_{\|u\|_2 \leq 1} \bar{a}_i^T x + u^T (P_i^T x) = \bar{a}_i^T x + \|P_i^T x\|_2$

SOCP: examples

Robust linear programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m,\end{array}$$

- a_i is Gaussian with mean \bar{a}_i , and covariance Σ_i
- $a_i^T x$ is Gaussian with mean $\bar{a}_i^T x$, and variance $x^T \Sigma_i x = \left\| \Sigma_i^{1/2} x \right\|_2$



$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \dots, m,\end{array}$$

Equivalent SOCP:

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \left\| \Sigma_i^{1/2} x \right\|_2 \leq b_i, \quad i = 1, \dots, m\end{array}$$

$\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^x e^{-t^2/2} dt$ is CDF of $\mathcal{N}(0, 1)$

$$\Pr(a_i^T x \leq b_i) = \Pr\left(\frac{a_i^T x - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2} \leq \frac{b_i - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2}\right) \geq \eta \Leftrightarrow \frac{b_i - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2} \geq \Phi^{-1}(\eta)$$

Semidefinite programming (SDP)

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Ax = b\end{array}$$

$$x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0$$

linear matrix inequality (LMI)

$$Y \preceq 0 \Leftrightarrow -Y \succeq 0$$

- $F_i, G \in \mathbb{S}^k$
- Multiple LMI is equivalent to single LMI:

$$x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0$$

$$x_1 H_1 + x_2 H_2 + \cdots + x_n H_n + L \preceq 0$$



$$x_1 \begin{bmatrix} F_1 & 0 \\ 0 & H_1 \end{bmatrix} + x_2 \begin{bmatrix} F_2 & 0 \\ 0 & H_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} F_n & 0 \\ 0 & H_n \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & L \end{bmatrix} \preceq 0$$

Semidefinite programming (SDP): standard form

$$\begin{aligned} \min \quad & \text{Tr}(CX) \\ \text{subject to} \quad & \text{Tr}(A_i X) = b_i, \quad \text{for } i = 1, \dots, p \\ & X \succcurlyeq \mathbf{0} \end{aligned}$$

- $\text{tr}(Z)$: sum of matrix Z diagonal elements; $\text{tr}(CX) = \sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij}$
- Converting general form to standard form:

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x_1 F_1 + x_2 F_2 + \dots + x_n F_n + G \preceq 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \min \quad & \text{Tr}(CX) \\ \text{subject to} \quad & \text{Tr}(A_i X) = b_i, \quad \text{for } i = 1, \dots, p \\ & X \succcurlyeq \mathbf{0} \end{aligned}$$

- Introducing **slack variable** $S \succcurlyeq \mathbf{0}$ for the inequalities
- Decompose the variable x as the difference of two non-negative variables

$$x = x^+ - x^-, \quad x^+ \geq 0, \quad x^- \geq 0$$

- Constructing block matrix out of x^+, x^-, S as semidefinite matrix.

SDP: example

Eigenvalue minimization

$$\min \lambda_{\max}(F(x))$$

- $F(x) = \sum_{i=1}^k x_i F_i, F_i \in \mathbb{S}^k$
- $t \geq \lambda_{\max}(Z)$ if and only if $tI - Z \succcurlyeq \mathbf{0}$.

Equivalent SDP:

$$\begin{aligned} & \min t \\ & \text{subject to } tI - F(x) \succcurlyeq \mathbf{0} \end{aligned}$$

Matrix norm minimization

$$\min \|F(x)\|_2 = (\lambda_{\max}(F(x)^T F(x)))^{1/2}$$

- 2-norm of a matrix is its maximum singular value: $\|F\|_2 \triangleq \sigma_{\max}(F) = \lambda_{\max}(F^T F)$
- Schur complement theorem: $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succcurlyeq 0 \Leftrightarrow A - BC^{-1}B^T \succcurlyeq 0$

Equivalent SDP:

$$\begin{aligned} & \min t \\ & \text{subject to } t^2 I - F(x)^T F(x) \succcurlyeq \mathbf{0} \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} & \min t \\ & \text{subject to } \begin{bmatrix} tI & F(x) \\ F(x)^T & tI \end{bmatrix} \succcurlyeq \mathbf{0} \end{aligned}$$

Connection

$$\text{LP} \subseteq \text{QP} \subseteq \text{SOCP} \subseteq \text{SDP}$$

LP

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

QP

$$\begin{array}{ll}\min & \frac{1}{2} x^T P x + q^T x + r \\ \text{s.t.} & Gx \preceq h \\ & Ax = b\end{array}$$

SOCP

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & \|F_i x + e_i\|_2 \leq g_i^T x + d_i, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

SDP

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Ax = b \\ & x_1 F_1 + x_2 F_2 + \dots + x_n F_n + G \preceq 0\end{array}$$

LP and QP

$$\text{LP} \subseteq \text{QP} \subseteq \text{SOCP} \subseteq \text{SDP}$$

LP

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

$$P = 0$$



QP

$$\begin{array}{ll} \min & \frac{1}{2} x^T P x + q^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

$$\text{LP} \subseteq \text{QP}$$

QP and SOCP

$$\text{LP} \subseteq \text{QP} \subseteq \text{SOCP} \subseteq \text{SDP}$$

QP

$$\begin{array}{ll} \min & \frac{1}{2}x^T Px + q^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$



SOCP

$$\begin{array}{ll} \min & q^T x + t \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

$$\boxed{\frac{1}{2}x^T Px \leq t}$$



$$\left\| \left(\frac{1}{\sqrt{2}} P^{1/2} x, \frac{1}{2} - \frac{t}{2} \right) \right\|_2 \leq \frac{1}{2} + \frac{t}{2}$$

$$\text{QP} \subseteq \text{SOCP}$$

SOCP and SDP

$$\text{LP} \subseteq \text{QP} \subseteq \text{SOCP} \subseteq \text{SDP} \subseteq ?$$

SOCP

$$\|x\|_2 \leq t$$



SDP

$$\begin{bmatrix} t & x^T \\ x & tI \end{bmatrix} \succcurlyeq \mathbf{0}$$

$$\text{SOCP} \subseteq \text{SDP}$$

Conic programming

LP \subseteq QP \subseteq SOCP \subseteq SDP \subseteq Conic programming

General conic programming

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Fx + g \preccurlyeq_K 0 \\ & Ax = b\end{array}$$

- \preccurlyeq_K : generalized inequality

Proper cone

A convex cone $K \subseteq \mathbb{R}^n$ is a proper cone if

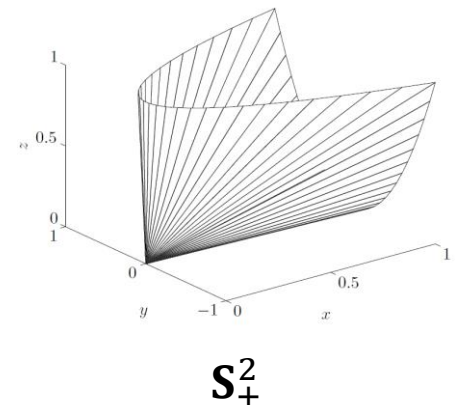
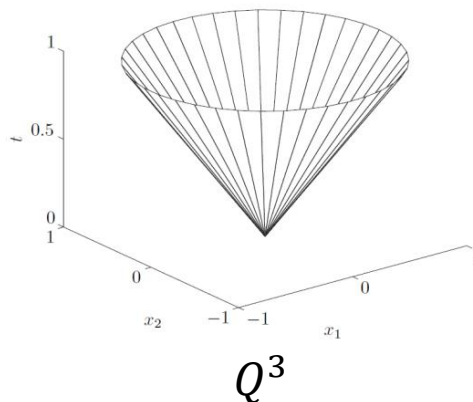
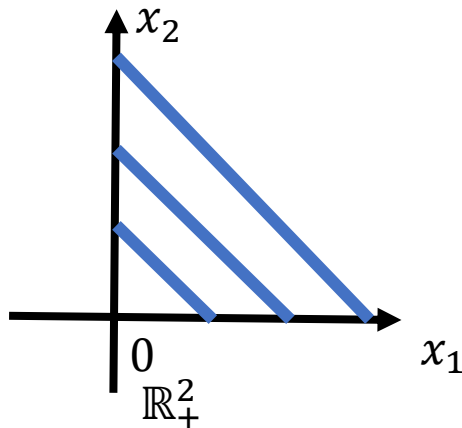
- K is closed, i.e., contains its boundary
- K is solid, i.e., has nonempty interior
- K is pointed, i.e., contains no line, or $x \in K$, and $-x \in K \Rightarrow x = 0$

Convex cone:

for any $x_1, x_2 \in K$, $\theta_1, \theta_2 \geq 0$,
 $\theta_1 x_1 + \theta_2 x_2 \in K$

Examples:

- Nonnegative orthant $K = \mathbb{R}_+^n \triangleq \{x | x_i \geq 0\}$
- Second-order cone $K = Q^{n+1} \triangleq \{(x, t) | \|x\|_2 \leq t\}$
- Positive semidefinite cone $K = \mathbf{S}_+^n \triangleq \{X \in \mathbf{S}^n | z^T X z \geq 0 \text{ for all } z \in \mathbb{R}^n\}$.



Generalized inequality

Generalized inequality is defined in a proper cone K to denote a partial ordering

$$x \preceq_K y \iff y - x \in K$$

$$x \prec_K y \iff y - x \in \mathbf{int} K$$

Examples:

- Nonnegative orthant $K = \mathbb{R}_+^n \triangleq \{x | x_i \geq 0\}$: component wise inequality

$$x \preceq_{\mathbb{R}_+^n} y \iff y - x \in \mathbb{R}_+^n, \text{ or } y_i - x_i \geq 0$$

- Second-order cone $K = Q^{n+1} \triangleq \{(x, t) | \|x\|_2 \leq t\}$

$$y \preceq_{Q^{n+1}} z \iff z - y \in Q^{n+1}, \text{ or } \|z_{1:n} - y_{1:n}\|_2 \leq z_{n+1} - y_{n+1}$$

- Positive semidefinite cone $K = \mathbf{S}_+^n \triangleq \{X \in \mathbf{S}^n | z^T X z \geq 0 \text{ for all } z \in \mathbb{R}^n\}$.

$$X \preceq_{\mathbf{S}_+^n} Y \iff Y - X \in \mathbf{S}_+^n, \text{ or } Y - X \succcurlyeq 0 \text{ is positive semidefinite matrix}$$

Properties of generalized inequality

Generalized inequality also preserves \leq in \mathbb{R}

\leq in \mathbb{R}

- Additivity: $x \preceq_K y$ and $u \preceq_K v \implies x + u \preceq_K y + v$ ➤ $x \leq y$ and $u \leq v \implies x + u \leq y + v$
- Transitivity: $x \preceq_K y$ and $y \preceq_K z \implies x \preceq_K z$ ➤ $x \leq y$ and $y \leq z \implies x \leq z$
- Anti-symmetric: $x \preceq_K y$ and $y \preceq_K x \implies x = y$ ➤ $x \leq y$ and $y \leq x \implies x = y$
- Homogeneity: $x \preceq_K y$ and $a \geq 0 \implies ax \preceq_K ay$ ➤ $x \leq y$ and $a \geq 0 \implies ax \leq ay$

Conic programming

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Fx + g \preccurlyeq_K 0 \\ & Ax = b\end{array}$$

- $K = \mathbb{R}_+^n \Rightarrow$ linear programming (LP)

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Fx + g \preccurlyeq 0 \\ & Ax = b\end{array}$$

- $K = Q^{n+1} \Rightarrow$ second-order cone programming (SOCP)

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & \|F_i x + e_i\|_2 \leq g_i^T x + d_i, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- $K = \mathbf{S}_+^n \Rightarrow$ semidefinite programming (SDP)

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \dots + x_n F_n + G \preccurlyeq 0 \\ & Ax = b\end{array}$$

Conic programming: standard form

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Ax = b \\ & x \succcurlyeq_K 0\end{array}$$

- LP: $K = \mathbb{R}_+^n$

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \text{ (or, } x \in \mathbb{R}_+^n\text{)}\end{array}$$

- SOCP: $K = Q^{n+1}$

$$\begin{array}{ll}\min & c^T x \\ \text{subject to} & Ax = b \\ & x \succcurlyeq_{Q^{n+1}} 0 \text{ (or, } x \in Q^{n+1}\text{)}\end{array}$$

- SDP: $K = \mathbf{S}_+^n$

$$\begin{array}{ll}\min & \text{Tr}(CX) \\ \text{subject to} & \text{Tr}(A_i X) = b_i, \quad \text{for } i = 1, \dots, p \\ & X \succcurlyeq_{\mathbf{S}_+^n} \mathbf{0} \text{ (or, } x \in \mathbf{S}_+^n\text{)}\end{array}$$