SSE5107 Optimization Theory and Algorithms
Homework 1 solutions

Problem 1

Explain whether the following sets are convex.

1. A *slab*, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$.
2. A *rectangle*, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, ..., n\}$.
3. A *wedge*, i.e., $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$.
4. The set of points closer to a given point than a given set, i.e.,
$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\},$$
where $S \subseteq \mathbb{R}^n$.
5. The set of points closer to one set than another, i.e.,
$$\{x \mid \text{dist}(x, S) \leq \text{dist}(x, T)\},$$
where $S, T \subseteq \mathbb{R}^n$, and
$$\text{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$
6. The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.
7. The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$, where $a \neq b$ and $0 \leq \theta \leq 1$.

Problem 1 solutions

1. A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).
2. A rectangle is a convex set and a polyhedron because it is a finite intersections of halfspaces.
3. A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.
4. The set is convex because it can be expressed as an intersection of halfspaces, i.e.,
$$\bigcap_{y \in \mathcal{S}} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}.$$
Notice that for fixed y, the set $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$ is a halfspace:
$$\|x - x_0\|_2 \leq \|x - y\|_2 \Leftrightarrow (x - x_0)^T (x - x_0) \leq (x - y)^T (x - y) \Leftrightarrow 2(x_0 - y)^T x \leq x_0^T x_0 - y^T y$$

5. In general this set is not convex. For example, consider $\mathcal{S} = \{-1, 1\}$ and $\mathcal{T} = \{0\}$. We have

$$\{x \mid \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\} = \{x \in \mathbf{R} \mid x \leq -1/2 \;\text{ or }\; x \geq 1/2\},$$

   which is not convex.

6. This set is convex. $x + \mathcal{S}_2 \subseteq \mathcal{S}_1$ if $x + y \in \mathcal{S}_1$ for all $y \in \mathcal{S}_2$. Therefore,

$$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in \mathcal{S}_2} \{x \mid x + y \in \mathcal{S}_1\} = \bigcap_{y \in \mathcal{S}_2} (\mathcal{S}_1 - y),$$

   i.e., the interaction of convex sets $\mathcal{S}_1 - y$ is convex.

7. The set is convex.

$$\begin{aligned}\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\} &= \{x \mid \|x - a\|_2^2 \leq \theta \|x - b\|_2^2\} \\ &= \{x \mid (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x + (a^T a - \theta^2 b^T b) \leq 0\}\end{aligned}$$

   If $\theta = 1$, this is a halfspace. If $\theta \leq 1$, it is a ball

$$\{x \mid (x - x_0)^T (x - x_0) \leq R\},$$

   with center x_0 and radius R given by

$$x_0 = \frac{a - \theta^2 b}{1 - \theta^2}, \qquad R = \left(\frac{\theta^2 \|b\|_2^2 - \|a\|_2^2}{1 - \theta^2} - \|x_0\|_2^2 \right)^{1/2}.$$

Problem 2

Let $P = \left\{x \in \mathbb{R}^n \mid a_i^T x \leq b_i \;\text{ for }\; i = 1, \ldots, m\right\}$, where $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$ are given. Recall that a ball with center $\bar{x} \in \mathbb{R}^n$ and radius $r > 0$ is defined as the set $B(\bar{x}, r) = \{x \in \mathbb{R}^n \mid \|x - \bar{x}\|_2 \leq r\} = \{\bar{x} + x \in \mathbb{R}^n \mid \|x\|_2 \leq r\}$. We are interested in finding a ball with the largest possible radius, subject to the condition that it is entirely contained within the set P (also known as the largest inscribed ball in P). Give a linear programming formulation of this problem.

Problem 2 solutions

Notice that a ball can be represented as $B(\bar{x}, r) = \{\bar{x} + x \in \mathbb{R}^n \mid \|x\|_2 \leq r\}$. Observe that for $i \in \{1, ..., m\}$, $B(\bar{x}, r) \subset H_i = \left\{x \in \mathbb{R}^n \mid a_i^T x \leq b_i\right\}$ iff $a_i^T(\bar{x} + x) \leq b_i$, where $\|x\|_2 \leq r$. By the Cauchy-Schwarz inequality, we have

$$\sup_{u \in \mathbb{R}^n \mid \|x\|_2 \leq r} \left\{a_i^T x\right\} = a_i^T \left(r \cdot \frac{a_i}{\|a_i\|_2} \right) = r \|a_i\|_2$$

It follows that $B(\bar{x}, r) \subset H_i$ iff

$$a_i^T \bar{x} + r \|a_i\|_2 \leq b_i, \tag{1}$$

which is a linear inequality in $\bar{x} \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$. In particular, we have $B(\bar{x}, r) \subset P = \big\{x \in \mathbb{R}^n : a_i^T x \leq b_i$ for $i = 1, \ldots, m\big\}$ iff (1) holds for $i = 1, \ldots, m$. Hence, the problem of finding the largest inscribed ball in P can be formulated as the following LP:

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T \bar{x} + r \|a_i\|_2 \leq b_i \quad \text{for } i = 1, \ldots, m, \\ & \bar{x} \in \mathbb{R}^n, r \geq 0 \end{array}$$

Problem 3

Let $S = \left\{x \in \mathbb{R}^n \mid x^T A x + b^T x + c \leq 0\right\}$, where $A \in \mathcal{S}^n, b \in \mathbb{R}^n$, and $c \in \mathbb{R}$ are given.

1. Show that S is convex if $A \succeq \mathbf{0}$. Is the converse true? Explain.
2. Let $H = \left\{x \in \mathbb{R}^n \mid g^T x + h = 0\right\}$, where $g \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $h \in \mathbb{R}$. Show that $S \cap H$ is convex if $A + \lambda g g^T \succeq \mathbf{0}$ for some $\lambda \in \mathbb{R}$.

Problem 3 solutions

1. Let $x_1, x_2 \in S$, and let $\alpha \in (0,1)$. Then, we have

$$x_1^T A x_1 + b^T x_1 + c \leq 0 \tag{2}$$
$$x_2^T A x_2 + b^T x_2 + c \leq 0 \tag{3}$$

Now, we compute

$$\begin{aligned}
&(\alpha x_1 + (1-\alpha)x_2)^T A (\alpha x_1 + (1-\alpha)x_2) + b^T (\alpha x_1 + (1-\alpha)x_2) + c \\
=&(\alpha x_1 + (1-\alpha)x_2)^T A (\alpha x_1 + (1-\alpha)x_2) + \alpha \left(b^T x_1 + c\right) + (1-\alpha)\left(b^T x_2 + c\right) \\
\leq&(\alpha x_1 + (1-\alpha)x_2)^T A (\alpha x_1 + (1-\alpha)x_2) - \alpha x_1^T A x_1 - (1-\alpha) x_2^T A x_2 \qquad (4)\\
=&-\alpha(1-\alpha) x_1^T A x_1 - (1-\alpha)(1-(1-\alpha)) x_2^T A x_2 + 2\alpha(1-\alpha) x_1^T A x_2 \\
=&-\alpha(1-\alpha)\left(x_1^T A x_1 - 2 x_1^T A x_2 + x_2^T A x_2\right) \\
=&-\alpha(1-\alpha) \cdot (x_1 - x_2)^T A (x_1 - x_2) \\
\leq&0 \qquad (5)
\end{aligned}$$

where (4) follows from the fact that $b^T x_i + c \leq -x_i^T A x_i$ for $i = 1, 2$ (by (2) and (3)), and (5) follows from the assumption that $A \succeq \mathbf{0}$. This proves that S is convex if $A \succeq \mathbf{0}$.

Note that the converse of the claim need not be true. Indeed, let $n = 1$, and let $A = -1$, $b = c = 0$. Then, we have $S = \left\{x \in \mathbb{R} : -x^2 \leq 0\right\} = \mathbb{R}$, which is trivially convex.

2. Let $x_1, x_2 \in S \cap H$, and let $\alpha \in (0,1)$. From the calculations in part 1, we have

$$\begin{aligned}
&(\alpha x_1 + (1-\alpha)x_2)^T A (\alpha x_1 + (1-\alpha)x_2) + b^T (\alpha x_1 + (1-\alpha)x_2) + c \\
\leq&-\alpha(1-\alpha) \cdot (x_1 - x_2)^T A (x_1 - x_2) \qquad (6)
\end{aligned}$$

Since $A + \gamma g g^T \succeq \mathbf{0}$, we have

$$0 \leq (x_1 - x_2)^T \left(A + \gamma g g^T\right)(x_1 - x_2) = (x_1 - x_2)^T A (x_1 - x_2) + \gamma \left(g^T (x_1 - x_2)\right)^2,$$

i.e.,

$$-(x_1 - x_2)^T A (x_1 - x_2) \leq \gamma \left(g^T (x_1 - x_2)\right)^2.$$

It follows from (6) that

$$\begin{aligned}
&(\alpha x_1 + (1-\alpha)x_2)^T A (\alpha x_1 + (1-\alpha)x_2) + b^T (\alpha x_1 + (1-\alpha)x_2) + c \\
\leq&-\alpha(1-\alpha) \cdot (x_1 - x_2)^T A (x_1 - x_2) \\
\leq&\alpha(1-\alpha) \cdot \gamma \left(g^T (x_1 - x_2)\right)^2 \\
=&0.
\end{aligned}$$

where the last equality follows from the fact that $g^T x_1 + h = g^T x_2 + h = 0$. So $\alpha x_1 + (1-\alpha)x_2 \in S \cap H$.