Optimization Theory and Algorithms

Instructor: Prof. LIAO, Guocheng （廖国成）

Email: liaogch6@mail.sysu.edu.cn

School of Software Engineering
Sun Yat-sen University

Outline

- General optimization problem
- Convex optimization problem
- Optimality condition
- Equivalent transformation

Optimization problem in standard form

$$
\begin{aligned}
\text{min} \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \leq 0, i = 1, \dots, m \\
& h_i(x) = 0, i = 1, \dots, p
\end{aligned}
$$

- $x = (x_1, \dots, x_n)$: optimization/decision variables
- $f_0(\cdot): \mathbb{R}^n \longrightarrow \mathbb{R}$: objective function
- $f_i(\cdot): \mathbb{R}^n \longrightarrow \mathbb{R}, i = 1, \dots, m$: inequality constrain functions
- $h_i(\cdot): \mathbb{R}^n \longrightarrow \mathbb{R}, i = 1, \dots, p$: equality constrain functions
- Domain: $\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \ \cap \ \bigcap_{i=1}^{p} \mathbf{dom}\, h_i$
- If a point $x \in \mathcal{D}$ satisfies all the constraints $f_i(x) \leq 0, i = 1, \dots, m,$ and $h_i(x) = 0,$ $i = 1, \dots, p,$ it is a *feasible point*. The set of all feasible point is *feasible set*.
- The problem is *feasible* if there exists at least one feasible point, and *infeasible* otherwise.

Notations

$$\begin{aligned} &\text{min } f_0(x) \\ &\text{s.t. } f_i(x) \le 0, i = 1, \ldots, m \\ &\quad\;\; h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

- Optimal value

$$p^\star = \inf\{f_0(x) \mid f_i(x) \le 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$$

  - $p^\star = -\infty$: if there are feasible points x_k with $f_0(x_k) \to -\infty$ as $k \to \infty$, i.e., the problem is unbounded below.

- Optimal points: $x^\star$ is feasible and $f_0(x^\star) = p^\star$, i.e., $x^\star$ solves the problem.

- Optimal set

$$X_{\text{opt}} = \{x \mid f_i(x) \le 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p,\ f_0(x) = p^\star\}$$

- Example (unconstrained optimization problem):

  - $f_0(x) = \frac{1}{x}, \mathcal{D} = \mathbb{R}_{++}, p^\star = 0$, no optimal point
  - $f_0(x) = -\log x, \mathcal{D} = \mathbb{R}_{++}, p^\star = -\infty$, unbounded below
  - $f_0(x) = x \log x, \mathcal{D} = \mathbb{R}_{++}, p^\star = -1/e, x = 1/e$ is optimal point

Notations

$$\begin{aligned} \text{min } & f_0(x) \\ \text{s.t. } & f_i(x) \le 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p \end{aligned}$$

- ε-suboptimal points: feasible x with $f_0(x) = p^\star + \varepsilon$
- ε-suboptimal set: the set of all ε-suboptimal points.
- Locally optimal points solves the following problem for an $R > 0$:

$$\begin{aligned} \text{min } & f_0(z) \\ \text{s.t. } & f_i(z) \le 0, i = 1, \dots, m \\ & h_i(z) = 0, i = 1, \dots, p \\ & \|z - x\|_2 \le R \end{aligned}$$

Convex optimization problem

$$\begin{aligned} \min\ & f_0(x) \\ \text{s.t.}\ & f_i(x) \le 0, i = 1, \dots, m \\ & \boxed{a_i^T x = b_i, i = 1, \dots, p} \longrightarrow Ax = b, A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p \end{aligned}$$

- Convex optimization problem:
  - Objective function f_0 is convex function
  - Inequality constraint functions $f_1, \dots, f_m$ are convex functions
  - Equality constraint functions are affine ($a_i^T x = b_i$).

- Property: feasible set of a convex optimization problem is convex

Abstract form convex optimization problem

$$\begin{aligned}
\min\ & f_0(x) = x_1^2 + x_2^2 \\
\text{s.t.}\ & f_1(x) = x_1/(1 + x_2^2) \leq 0, \\
& h_1(x) = (x_1 + x_2)^2 = 0
\end{aligned}$$

- The problem is not a convex problem
  inequality constraint function is not convex;
  equality constraint function is not affine

- Equivalent convex problem

$$\begin{aligned}
\min\ & f_0(x) = x_1^2 + x_2^2 \\
\text{s.t.}\ & f_1(x) = x_1 \leq 0, \\
& h_1(x) = x_1 + x_2 = 0
\end{aligned}$$

Local and global optima

Any locally optimal point of a convex problem is (globally) optimal

Proof by contradiction:

- Let x be locally optimal, i.e., $f_0(x) = \inf\{f_0(z) | z \text{ is feasible}, \|z - x\|_2 \leq R\}$ for some $R > 0$.
- Suppose x is not globally optimal, i.e., there exists y ($\|y - x\|_2 > R$) such that $f_0(y) < f_0(x)$.
- Consider a point $z = \theta y + (1 - \theta)x$, $\theta = \frac{R}{2\|y-x\|_2} < \frac{1}{2}$. The distance between z and x is

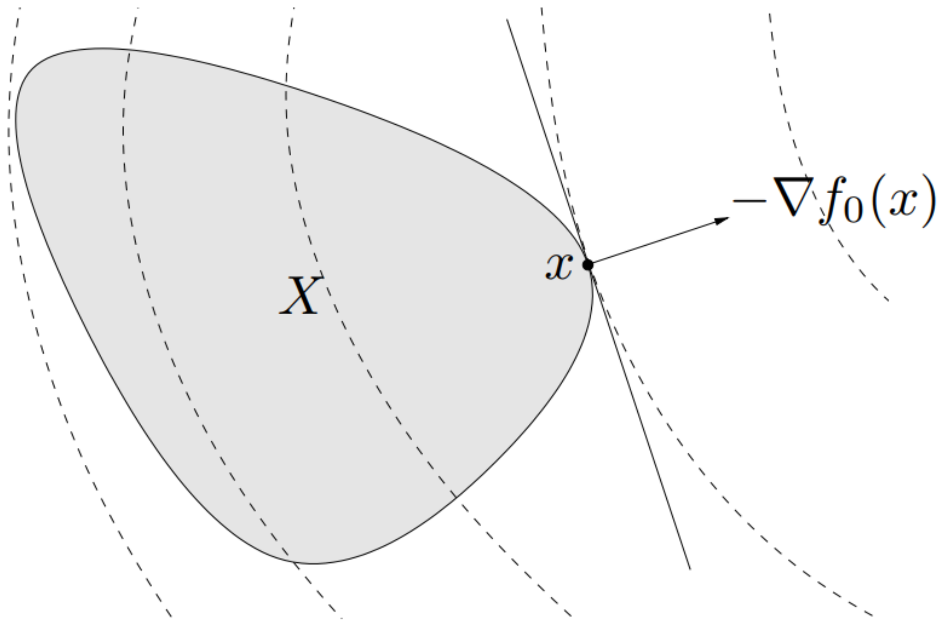
  $\|z - x\|_2 = \frac{R}{2} < R$. By local optimality of x, we have $f_0(x) < f_0(z)$ -- (1)
- Meanwhile,

$$
\begin{aligned}
f_0(z) &= f_0(\theta y + (1 - \theta)x) \\
&\leq \theta f_0(y) + (1 - \theta) f_0(x) && \text{convexity of } f_0 \\
&< \theta f_0(x) + (1 - \theta) f_0(x) && f_0(y) < f_0(x) \\
&= f_0(x),
\end{aligned}
$$

i.e., $f_0(z) < f_0(x)$, which contradicts equation (1).

Optimality criterion for differentiable f_0

First-order condition: x is optimal if and only if it is feasible and
$\nabla f_0(x)^T(y - x) \geq 0$ for all feasible y



If $\nabla f_0(x)$ is nonzero, $-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at x

Unconstrained problems as a special case

$$\min \ f_0(x)$$

$\boxed{x \text{ is optimal if and only if it is feasible and } \nabla f_0(x)^T(y-x) \geq 0 \text{ for all feasible } y}$ $\xrightarrow{\text{reduce}}$ $\boxed{x \text{ is optimal if and only if } \nabla f_0(x) = 0}$

Example: $\min \ f_0(x) = \frac{1}{2}x^TQx + b^Tx$

The first-order condition: $\nabla f_0(x) = Qx + b = 0$

- Case 1: $Qx = -b$ has unique solution.
- Case 2: $Qx = -b$ has infinitely many solutions.
- Case 3: $Qx = -b$ has no solution, i.e., $\min \ f_0(x) = -\infty$.

Equivalent transformation

Eliminating equality constraints

$$\begin{aligned} \min \ & f_0(x) \\ \text{s.t.} \ & f_i(x) \le 0, i = 1, \dots, m \\ & Ax = b \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min_z \ & f_0(Fz + x_0) \\ \text{s.t.} \ & f_i(Fz + x_0) \le 0, i = 1, \dots, m \end{aligned}$$

F and x_0 are such that
$Ax = b \Leftrightarrow x = Fz + x_0$ for some z

Introducing equality constraints

$$\begin{aligned} \min \ & f_0(A_0 x + b_0) \\ \text{s.t.} \ & f_i(A_i x + b_i) \le 0, i = 1, \dots, m \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min_{x, y_i} \ & f_0(y_0) \\ \text{s.t.} \ & f_i(y_i) \le 0, i = 1, \dots, m \\ & y_i = A_i x + b_i, i = 1, \dots, m \end{aligned}$$

Equivalent transformation

Introducing slack variables for *linear inequalities*

$$\begin{aligned} \min \;& f_0(x) \\ \text{s.t.} \;& a_i^T x \le b_i, i = 1, \dots, m \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min_{x, s_i} \;& f_0(x) \\ \text{s.t.} \;& a_i^T x + s_i = b_i, i = 1, \dots, m \\ & s_i \ge 0, i = 1, \dots, m \end{aligned}$$

Epigraph form

$$\begin{aligned} \min \;& f_0(x) \\ \text{s.t.} \;& f_i(x) \le 0, i = 1, \dots, m \\ & Ax = b \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min_{x, t} \;& t \\ \text{s.t.} \;& f_i(x) \le 0, i = 1, \dots, m \\ & Ax = b \\ & f_i(x) - t \le 0 \end{aligned}$$