Optimization Theory and Algorithms

Instructor: Prof. LIAO, Guocheng （廖国成）

Email: liaogch6@mail.sysu.edu.cn

School of Software Engineering
Sun Yat-sen University

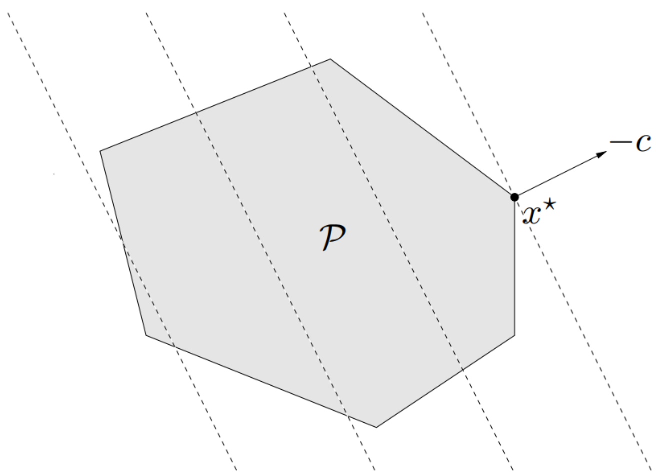
Outline

- Linear programming
- Quadratic programming
- Quadratically constrained quadratic programming
- Second-order cone programming
- Semidefinite programming
- Conic programming

Linear Programming (LP)

$$\begin{array}{rl} \text{min} & c^T x \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

- Affine objective and constraint functions
- minimize an affine function over a polyhedron



- Solution: (i) $-\infty$; (ii) at a vertex

Linear Programming: standard form

$$\begin{aligned} \min\ & c^T x \\ \text{subject to}\ & Ax = b \\ & x \succcurlyeq 0 \end{aligned}$$

- The only inequalities are $x \succcurlyeq 0$
- Converting general form to standard form:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx \preccurlyeq h \\ & Ax = b \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \succcurlyeq 0 \end{aligned}$$

➢ Introduce slack variables s for the inequalities:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx \preccurlyeq h \\ & Ax = b \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx + s = h \\ & Ax = b \\ & s \succcurlyeq 0 \end{aligned}$$

➢ Decompose the variable x as the difference of two non-negative variables

$$x = x^+ - x^-$$

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx + s = h \\ & Ax = b \\ & s \succcurlyeq 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min\ & c^T x^+ - c^T x^- \\ \text{s.t.}\ & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & s \succcurlyeq 0, x^+ \succcurlyeq 0, x^- \succcurlyeq 0 \end{aligned}$$

LP: examples

Diet problem: To find the cheapest combination of foods that satisfies some nutritional requirements.

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & Ax \geqslant b \\
& x \geqslant 0
\end{aligned}
$$

- x_j: units of food j; c_j: per-unit price of food j
- A_{ij}: content of nutrient i in per unit of food j
- b_i: minimum required intake of nutrient i

LP: examples

Transportation: Ship commodities from given sources to destinations at minimum cost

$$
\begin{aligned}
\min_{x} \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} \leq s_i, \; i = 1, \ldots, m \\
& \sum_{i=1}^{m} x_{ij} \geq d_j, \; j = 1, \ldots, n, \quad x \geq 0
\end{aligned}
$$

- x_{ij}: units shipped from i to j
- c_{ij}: per-unit shipping cost from i to j
- s_i: supply at source $i, i = 1, \ldots, m$
- d_j: demand at destination $j, j = 1, \ldots, n$

LP: examples

Piecewise-linear minimization $\min \max_{i=1,\ldots,m} a_i^T x + b_i$

Equivalent LP:

$$\begin{aligned} \text{min} \quad & t \\ \text{s.t.} \quad & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{aligned}$$

Absolute value minimization

$$\begin{aligned} \text{min} \quad & |c^T x + d| \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Equivalent LP:

$$\begin{aligned} \text{min} \quad & t \\ \text{s.t.} \quad & c^T x + d \leq t \\ & -c^T x - d \leq t \\ & Ax = b \end{aligned}$$

LP: examples

L_∞-norm minimization

$$\begin{aligned} \min \quad & \|x\|_\infty \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

$$\|x\|_\infty \triangleq \max_i |x_i|$$

Equivalent LP:

$$\begin{aligned} \min_{t\in\mathbb{R}, x\in\mathbb{R}^n} \quad & t \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \\ & x \preccurlyeq t\boxed{\mathbf{1}} \\ & -t\mathbf{1} \preccurlyeq x \end{aligned}$$

$$\begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

L_1-norm minimization

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

$$\|x\|_1 \triangleq \sum_i |x_i|$$

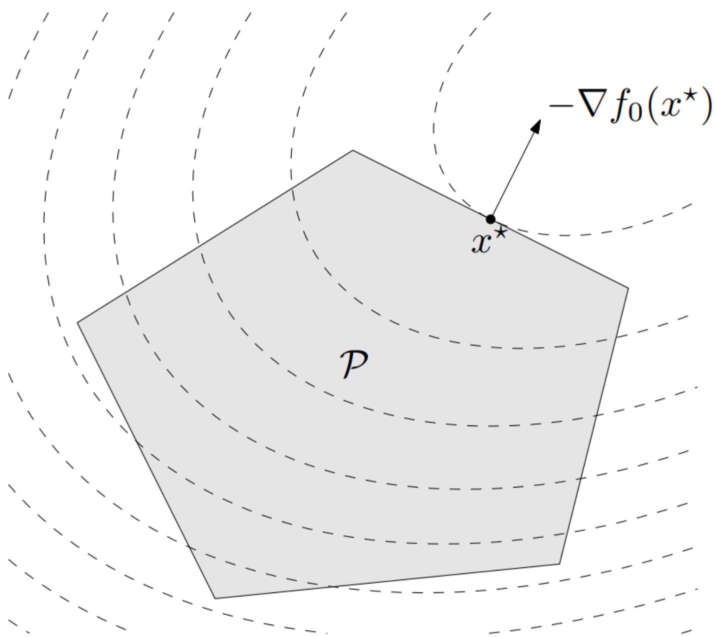
Equivalent LP:

$$\begin{aligned} \min_{t\in\mathbb{R}^n, x\in\mathbb{R}^n} \quad & \mathbf{1}^T t \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \\ & x \preccurlyeq t \\ & -t \preccurlyeq x \end{aligned}$$

Quadratic Programming (QP)

$$
\begin{aligned}
\text{min} \quad & \frac{1}{2} x^T P x + q^T x + r \\
\text{s.t.} \quad & Gx \preccurlyeq h \\
& Ax = b
\end{aligned}
$$

- $P \in \mathbb{S}_+^n$, so the objective is convex function
- Minimize a convex quadratic function over a polyhedron



QP: examples

Least-squares and regression

$$\begin{array}{ll} \min\limits_{x} & ||Ax - b||_2^2 = x^T A^T A x - 2b^T A x + b^T b \\ \text{subject to} & l_i \leq x_i \leq u_i, i = 1, \ldots, n \end{array}$$

Linear programming with random cost

Deterministic

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

Non-deterministic

$$\begin{array}{ll} \min & \mathbf{E}[c^T x] + \gamma \mathbf{var}[c^T x] = \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

- c is random vector with mean $\bar{c}$ and covariance Σ
- $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$

QP: examples

Portfolio optimization

$$\begin{aligned} \min \quad & x^T \Sigma x \\ \text{subject to} \quad & R^T x \geq r_{min} \\ & \mathbf{1}^T x = B \\ & x \succcurlyeq 0 \end{aligned}$$

- Price changes of all invested assets has mean $R \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n\times n}$
- r_{min}: minimum return.
- $\mathbf{1} \in \mathbb{R}^n$: every component is 1;
- B: budget.

Quadratically constrained quadratic programming (QCQP)

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & Ax = b\end{array}$$

- $P \in \mathbb{S}_+^n$, so the objective and constraint functions are convex
- Minimize a convex quadratic function over a intersection of m ellipsoids and an affine set

QCQP: example

Portfolio optimization

$$\begin{array}{ll} \text{minimize} & x^T\Sigma_0 x \\ \text{subject to} & x^T\Sigma_i x \le d_i,\ i = 1, \dots, m \\ & R^T x \ge r_{min} \\ & \mathbf{1}^T x = B \\ & x \succcurlyeq 0 \end{array}$$

- There are a few estimations of the covariance of the price changes, Σ_i, $i = 0, \dots, m$

Second-order cone programming (SOCP)

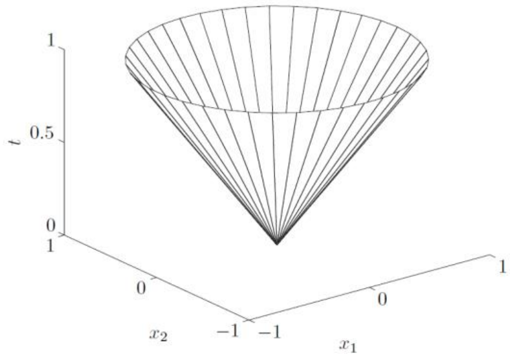
$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \dots, m \\
& Fx = g,
\end{array}
$$

- $A_i \in \mathbb{R}^{n_i \times n}$
- Inequalities are second-order cone constraints

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbb{R}^{n_i+1}$$

- If $A_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP.

$\{(x,t) \mid \|x\|_2 \le t\}$ is *second-order cone, also called ice cream cone.*



SOCP: examples

Robust linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

- a_i is inaccurate, but are known in ellipsoids: $a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$

$\bar{a}_i \in \mathbb{R}^n$, $P_i \in \mathbb{R}^{n \times n}$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

Equivalent SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- $\sup\limits_{||u||_2 \leq 1} \bar{a}_i^T x + (P_i u)^T x = \bar{a}_i^T x + \sup\limits_{||u||_2 \leq 1} u^T \left(P_i^T x\right) = \bar{a}_i^T x + ||P_i^T x||_2$

SOCP: examples

Robust linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m, \end{array}$$

- a_i is Gaussian with mean $\bar{a}_i$, and covariance Σ_i
- $a_i^T x$ is Gaussian with mean $\bar{a}_i^T x$, and variance $x^T \Sigma_i x = \left\| \Sigma_i^{1/2} x \right\|_2$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \le b_i) \ge \eta, \quad i = 1, \ldots, m, \end{array}$$

Equivalent SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \le b_i, \quad i = 1, \ldots, m \end{array}$$

$\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2}\, dt$ is CDF of $\mathcal{N}(0, 1)$

$$\Pr\left(a_i^T x \le b_i\right) = \Pr\left(\frac{a_i^T x - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2} \le \frac{b_i - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2} \right) \ge \eta \Leftrightarrow \frac{b_i - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2} \ge \Phi^{-1}(\eta)$$

Semidefinite programming (SDP)

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & \boxed{x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preccurlyeq 0} \end{aligned}$$

linear matrix inequality (LMI)

$Y \preccurlyeq 0 \Leftrightarrow -Y \succcurlyeq 0$

- $F_i, G \in \mathbb{S}^k$
- Multiple LMI is equivalent to single LMI:

$$x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preccurlyeq 0 \qquad x_1 H_1 + x_2 H_2 + \cdots + x_n H_n + L \preccurlyeq 0$$

$$\Updownarrow$$

$$x_1 \begin{bmatrix} F_1 & 0 \\ 0 & H_1 \end{bmatrix} + x_2 \begin{bmatrix} F_2 & 0 \\ 0 & H_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} F_n & 0 \\ 0 & H_n \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & L \end{bmatrix} \preccurlyeq 0$$

Semidefinite programming (SDP): standard form

$$\begin{aligned} \text{min} \quad & \operatorname{tr}(CX) \\ \text{subject to} \quad & \operatorname{tr}(A_i X) = b_i, \quad \text{for } i = 1, \dots, p \\ & X \succcurlyeq \mathbf{0} \end{aligned}$$

- $\operatorname{tr}(Z)$: sum of matrix Z diagonal elements; $\operatorname{tr}(CX) = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$
- Converting general form to standard form:

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preccurlyeq 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \text{min} \quad & \operatorname{Tr}(CX) \\ \text{subject to} \quad & \operatorname{Tr}(A_i X) = b_i, \quad \text{for } i = 1, \dots, p \\ & X \succcurlyeq \mathbf{0} \end{aligned}$$

- Introduce slack variable $S \succcurlyeq 0$ for the inequalities
- Decompose the variable x as the difference of two non-negative variables

$$x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$$

- Construct block matrix out of x^+, x^-, S as semidefinite matrix.

SDP: example

Eigenvalue minimization

$$\min \quad \lambda_{\max}(F(x))$$

- $F(x) = \sum_{i=1}^{k} x_i F_i,\ F_i \in \mathbb{S}^k$
- $t \geq \lambda_{max}(Z)$ if and only if $tI - Z \succcurlyeq \mathbf{0}$.

Equivalent SDP:

$$\begin{array}{rl} \min & t \\ \text{subject to} & tI - F(x) \succcurlyeq \mathbf{0} \end{array}$$

Matrix norm minimization

$$\min \quad \|F(x)\|_2 = (\lambda_{\max}(F(x)^T F(x)))^{1/2}$$

- l2-norm of a matrix is its maximum singular value: $\|F\|_2 \triangleq \sigma_{\max}(F) = (\lambda_{\max}(F^T F))^{1/2}$
- Schur complement theorem: $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succcurlyeq 0 \Leftrightarrow C \succcurlyeq 0, A - BC^{-1}B^T \succcurlyeq 0$

Equivalent SDP:

$$\begin{array}{rl} \min & t \\ \text{subject to} & t^2 I - F(x)^T F(x) \succcurlyeq \mathbf{0} \end{array} \quad \Longleftrightarrow \quad \begin{array}{rl} \min & t \\ \text{subject to} & \begin{bmatrix} tI & F(x) \\ F(x)^T & tI \end{bmatrix} \succcurlyeq \mathbf{0} \end{array}$$

Connection

LP $\subseteq$ QP $\subseteq$ SOCP $\subseteq$ SDP

LP

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

QP

$$\begin{aligned} \text{min} \quad & \frac{1}{2} x^T P x + q^T x + r \\ \text{s.t.} \quad & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

SOCP

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & \|F_i x + e_i\|_2 \leq g_i^T x + d_i, \ i = 1, \dots, m \\ & Ax = b \end{aligned}$$

SDP

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preccurlyeq 0 \end{aligned}$$

LP and QP

LP $\subseteq$ QP $\subseteq$ SOCP $\subseteq$ SDP

$$\xleftarrow{P = 0}$$

LP

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

QP

$$\begin{aligned} \text{min} \quad & \frac{1}{2} x^T P x + q^T x \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

LP $\subseteq$ QP

QP and SOCP

LP $\subseteq$ QP $\subseteq$ SOCP $\subseteq$ SDP

QP

$$\begin{aligned} \min \quad & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

$\Longleftrightarrow$

SOCP

$$\begin{aligned} \min \quad & q^T x + t \\ \text{subject to} \quad & Gx \preccurlyeq h \\ & Ax = b \\ & \boxed{\frac{1}{2}x^T P x \le t} \end{aligned}$$

$\Updownarrow$

$$\left\| \left(\frac{1}{\sqrt{2}} P^{1/2} x, \frac{1}{2} - \frac{t}{2} \right) \right\|_2 \le \frac{1}{2} + \frac{t}{2}$$

QP $\subseteq$ SOCP

SOCP and SDP

LP $\subseteq$ QP $\subseteq$ SOCP $\subseteq$ SDP $\subseteq$?

SOCP

$$\|x\|_2 \le t$$

$\Longleftrightarrow$

SDP

$$\begin{bmatrix} t & x^T \\ x & tI \end{bmatrix} \succcurlyeq \mathbf{0}$$

SOCP $\subseteq$ SDP

Conic programming

LP $\subseteq$ QP $\subseteq$ SOCP $\subseteq$ SDP $\subseteq$ Conic programming

General conic programming

$$
\begin{aligned}
\text{min} \quad & c^T x \\
\text{subject to} \quad & Fx + g \preccurlyeq_K 0 \\
& Ax = b
\end{aligned}
$$

- $\preccurlyeq_K$: generalized inequality

Proper cone

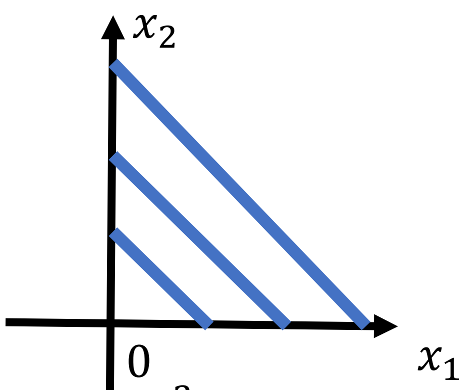
A convex cone $K \subseteq \mathbb{R}^n$ is a proper cone if

- K is closed, i.e., contains its boundary
- K is solid, i.e., has nonempty interior
- K is pointed, i.e., contains no line, or $x \in K$, and $-x \in K \Rightarrow x = 0$

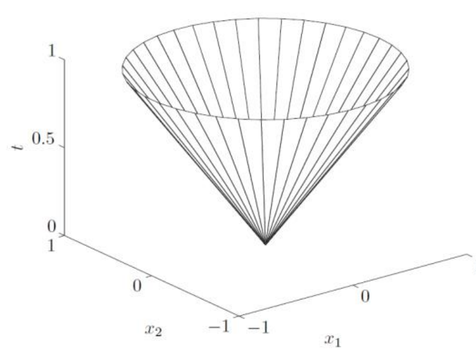
Convex cone:
for any $x_1, x_2 \in K$, $\theta_1, \theta_2 \geq 0$,
$\theta_1 x_1 + \theta_2 x_2 \in K$

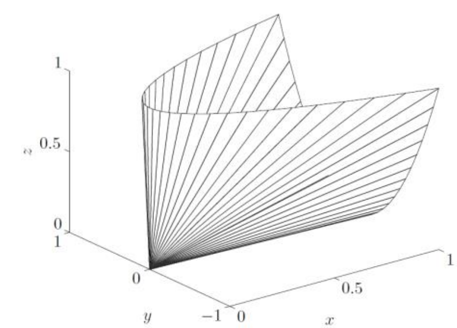
Examples:

- Nonnegative orthant $K = \mathbb{R}^n_+ \triangleq \{x | x_i \geq 0\}$
- Second-order cone $K = Q^{n+1} \triangleq \{(x, t) |\ \|x\|_2 \leq t\}$
- Positive semidefinite cone $K = \mathbf{S}^n_+ \triangleq \{X \in \mathbf{S}^n | z^T X z \geq 0 \text{ for all } z \in \mathbb{R}^n\}$.



$\mathbb{R}^2_+$


Q^3


$\mathbf{S}^2_+$

Generalized inequality

Generalized inequality is defined in a proper cone K to denote a partial ordering

$$x \preccurlyeq_K y \iff y - x \in K \qquad\qquad x \prec_K y \iff y - x \in \mathbf{int}\, K$$

Examples:

- Nonnegative orthant $K = \mathbb{R}_+^n \triangleq \{x | x_i \geq 0\}$: component wise inequality

$$x \preccurlyeq_{\mathbb{R}_+^n} y \iff y - x \in \mathbb{R}_+^n, \text{ or } y_i - x_i \geq 0$$

- Second-order cone $K = Q^{n+1} \triangleq \{(x, t) |\ \|x\|_2 \leq t\}$

$$y \preccurlyeq_{Q^{n+1}} z \iff z - y \in Q^{n+1}, \text{ or } \|z_{1:n} - y_{1:n}\|_2 \leq z_{n+1} - y_{n+1}$$

- Positive semidefinite cone $K = \mathbf{S}_+^n \triangleq \{X \in \mathbf{S}^n | z^T X z \geq 0 \text{ for all } z \in \mathbb{R}^n\}$.

$$X \preccurlyeq_{\mathbf{S}_+^n} Y \iff Y - X \in \mathbf{S}_+^n, \text{ or } Y - X \succcurlyeq 0 \text{ is positive semidefinite matrix}$$

Properties of generalized inequality

Generalized inequality also preserves $\leq$ in $\mathbb{R}$

$\leq$ in $\mathbb{R}$

- Additivity: $x \preccurlyeq_K y$ and $u \preccurlyeq_K v \implies x + u \preccurlyeq_K y + v$
  - $x \leq y$ and $u \leq v \implies x + u \leq y + v$
- Transitivity: $x \preccurlyeq_K y$ and $y \preccurlyeq_K z \implies x \preccurlyeq_K z$
  - $x \leq y$ and $y \leq z \implies x \leq z$
- Anti-symmetric: $x \preccurlyeq_K y$ and $y \preccurlyeq_K x \implies x = y$
  - $x \leq y$ and $y \leq x \implies x = y$
- Homogeneity: $x \preccurlyeq_K y$ and $a \geq 0 \implies ax \preccurlyeq_K ay$
  - $x \leq y$ and $a \geq 0 \implies ax \leq ay$

Conic programming

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & Fx + g \preccurlyeq_K 0 \\ & Ax = b \end{aligned}$$

- $K = \mathbb{R}^n_+ \Longrightarrow$ linear programming (LP)

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & Fx + g \preccurlyeq 0 \\ & Ax = b \end{aligned}$$

- $K = Q^{n+1} \Longrightarrow$ second-order cone programming (SOCP)

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & \|F_i x + e_i\|_2 \leq g_i^T x + d_i, \ i = 1, \dots, m \\ & Ax = b \end{aligned}$$

- $K = \mathbf{S}^n_+ \Longrightarrow$ semidefinite programming (SDP)

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preccurlyeq 0 \\ & Ax = b \end{aligned}$$

Conic programming: standard form

$$\begin{array}{rl} \text{min} & c^T x \\ \text{subject to} & Ax = b \\ & x \succcurlyeq_K 0 \end{array}$$

- LP: $K = \mathbb{R}_+^n$

$$\begin{array}{rl} \text{min} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \text{ (or, } x \in \mathbb{R}_+^n) \end{array}$$

- SOCP: $K = Q^{n+1}$

$$\begin{array}{rl} \text{min} & c^T x \\ \text{subject to} & Ax = b \\ & x \succcurlyeq_{Q^{n+1}} 0 \text{ (or, } x \in Q^{n+1}) \end{array}$$

- SDP: $K = \mathbf{S}_+^n$

$$\begin{array}{rl} \text{min} & \mathrm{Tr}(CX) \\ \text{subject to} & \mathrm{Tr}(A_i X) = b_i, \quad \text{for } i = 1, \dots, p \\ & X \succcurlyeq_{\mathbf{S}_+^n} \mathbf{0} \text{ (or, } X \in \mathbf{S}_+^n) \end{array}$$