Optimization Theory and Algorithms

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Outline

- Equality constrained minimization
- Reduction to an unconstrained minimization
- Newton’s method with equality constrained

Equality constrained minimization problem

$$\begin{aligned} &\text{min} \;\; f(x) \\ &\text{s.t.} \;\; Ax = b \end{aligned}$$

- f is convex and twice continuously differentiable
- Assume optimal point $x^\star$ exists. Let $p^\star = f(x^\star)$ be the optimal value.

Optimality condition (KKT conditions): $x^\star$ is optimal iff there exists a $\nu^\star$ such that

$$\nabla f(x^\star) + A^T \nu^\star = 0, \qquad Ax^\star = b$$

Solution methods:

- Elimination method
- Dual method
- Newton's method

reduce to an unconstrained minimization

Elimination equality constraints

$$\{x|\ Ax = b\} = \{Fz + x'|z \in \mathbb{R}^{n-p}\}$$

- Rank(A)=p
- $F \in \mathbb{R}^{n\times(n-p)}$ is any matrix whose range is the nullspace of A.
- x' is any particular solution of $Ax = b$.

equality constrained minimization

$$\min_x f(x)$$
$$\text{s.t.}\quad Ax = b$$

$$\Downarrow$$

unconstrained minimization

$$\min_z f(Fz + x')$$

Dual method

Primal problem

(equality constrained)

min $f(x)$
s.t. $Ax = b$

Dual function:
$$g(\nu) = -b^T\nu + \min_x (f(x) + \nu^T Ax)$$

Dual problem

(unconstrained)

max $g(\nu)$

Newton's method with equality constraints

Newton's method with no constraints

$$\min_{\Delta x} f(x+\Delta x) \approx \boxed{f(x) + \nabla f(x)^T \Delta x + \frac{1}{2}\Delta x^T \nabla^2 f(x) \Delta x}$$

set the gradient to zero

$$\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

Newton's method with equality constraints

$$\min_{\Delta x} f(x+\Delta x) \approx f(x) + \nabla f(x)^T \Delta x + \frac{1}{2}\Delta x^T \nabla^2 f(x) \Delta x$$

s.t. $A(x+\Delta x) = b$

KKT conditions

$$\nabla f(x) + \nabla^2 f(x) \Delta x_{nt} + A^T \nu^\star = 0$$

$$A \Delta x_{nt} = 0$$

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Newton's method with equality constraints

Newton's method with no constraints

$$\min \; f(x)$$

Optimality condition:

$$\nabla f(x^\star) = 0$$

$x^\star = x + \Delta x_{nt}$

Taylor approximation

$$\nabla f(x) + \nabla^2 f(x) \Delta x_{nt} = 0$$

$$\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

Newton's method with equality constraints

$$\begin{aligned} &\min \; f(x) \\ &\text{s.t.} \;\; Ax = b \end{aligned}$$

Optimality conditions (KKT conditions):

$$\nabla f(x^\star) + A^T \nu^\star = 0, \qquad Ax^\star = b$$

$x^\star = x + \Delta x_{nt}$

Taylor approximation

$$\nabla f(x) + \nabla^2 f(x) \Delta x_{nt} + A^T \nu^\star = 0, \; A(x + \Delta x_{nt}) = b$$

$$\nabla f(x) + \nabla^2 f(x) \Delta x_{nt} + A^T \nu^\star = 0, \;\; A \Delta x_{nt} = 0$$

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Newton's method with equality constraints

- Given a starting point $x \in \mathbf{dom}\ f$
- **Repeat**

1. Compute the *Newton step* and decrement:

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix} \qquad \lambda = \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x) \right)^{1/2}$$

2. Stopping criterion: if $\lambda^2/2 \leq \varepsilon$, break

3. *Line search*: choose a step size t via *backtracking line search*.

4. Update: $x \leftarrow x + t\Delta x_{nt}$