Optimization Theory and Algorithms

Instructor: Prof. LIAO, Guocheng （廖国成）

Email: liaogch6@mail.sysu.edu.cn

School of Software Engineering
Sun Yat-sen University

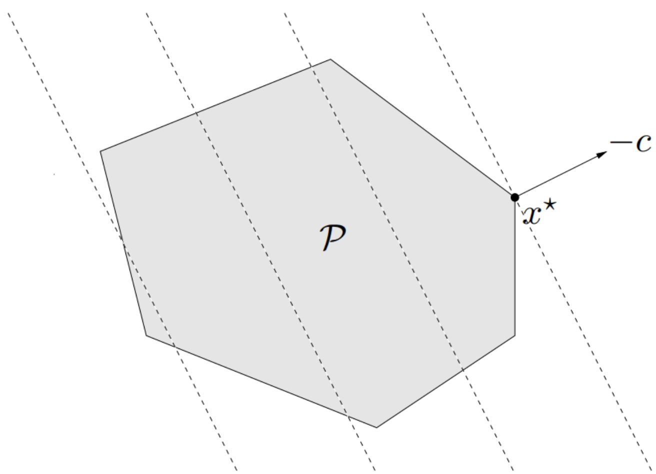
Outline

- Linear programming
- Quadratic programming
- Quadratically constrained quadratic programming
- Second-order cone programming

Linear Programming (LP)

$$
\begin{aligned}
\text{min} \quad & c^T x \\
\text{subject to} \quad & Gx \preccurlyeq h \\
& Ax = b
\end{aligned}
$$

- Affine objective and constraint functions
- minimize an affine function over a polyhedron



- Solution: (i) $-\infty$; (ii) at a vertex

Linear Programming: standard form

$$\begin{aligned} \min\ & c^T x \\ \text{subject to}\ & Ax = b \\ & x \succcurlyeq 0 \end{aligned}$$

- The only inequalities are $x \succcurlyeq 0$
- Converting general form to standard form:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx \preccurlyeq h \\ & Ax = b \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \succcurlyeq 0 \end{aligned}$$

➢ Introducing slack variables s for the inequalities:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx \preccurlyeq h \\ & Ax = b \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx + s = h \\ & Ax = b \\ & s \succcurlyeq 0 \end{aligned}$$

➢ Decompose the variable x as the difference of two non-negative variables

$$x = x^+ - x^-$$

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx + s = h \\ & Ax = b \\ & s \succcurlyeq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min\ & c^T x^+ - c^T x^- \\ \text{s.t.}\ & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & s \succcurlyeq 0, x^+ \succcurlyeq 0, x^- \succcurlyeq 0 \end{aligned}$$

LP: examples

Diet problem: To find the cheapest combination of foods that satisfies some nutritional requirements.

$$\begin{aligned} \min \;\; & c^T x \\ \text{s.t.} \;\; & Ax \geqslant b \\ & x \geqslant 0 \end{aligned}$$

- x_j: units of food j; c_j: per-unit price of food j
- A_{ij}: content of nutrient i in per unit of food j
- b_i: minimum required intake of nutrient i

LP: examples

Transportation: Ship commodities from given sources to destinations at minimum cost

$$
\begin{array}{ll}
\min\limits_{x} & \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} c_{ij} x_{ij} \\
\text{subject to} & \sum\limits_{j=1}^{n} x_{ij} \leq s_i, \; i = 1, \ldots, m \\
& \sum\limits_{i=1}^{m} x_{ij} \geq d_j, \; j = 1, \ldots, n, \quad x \geq 0
\end{array}
$$

- x_{ij}: units shipped from i to j
- c_{ij}: per-unit shipping cost from i to j
- s_i: supply at source i, $i = 1, \ldots, m$
- d_j: demand at destination j, $j = 1, \ldots, n$

LP: examples

Piecewise-linear minimization

$$\min \max_{i=1,\dots,m} a_i^T x + b_i$$

Equivalent LP:

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & a_i^T x + b_i \le t, \quad i = 1, \dots, m \end{aligned}$$

Absolute value minimization

$$\begin{aligned} \min \quad & |c^T x + d| \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Equivalent LP:

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & c^T x + d \le t \\ & -c^T x - d \le t \\ & Ax = b \end{aligned}$$

LP: examples

$\boldsymbol{L_\infty}$**-norm minimization**

$$\begin{array}{rl} \text{min} & \|x\|_\infty \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

$$\|x\|_\infty \triangleq \max_i |x_i|$$

Equivalent LP:

$$\begin{array}{rl} \min\limits_{t\in\mathbb{R}, x\in\mathbb{R}^n} & t \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \\ & x \preccurlyeq t\boxed{\mathbf{1}} \\ & -t\mathbf{1} \preccurlyeq x \end{array}$$

$$\begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

$\boldsymbol{L_1}$**-norm minimization**

$$\begin{array}{rl} \text{min} & \|x\|_1 \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

$$\|x\|_1 \triangleq \sum_i |x_i|$$

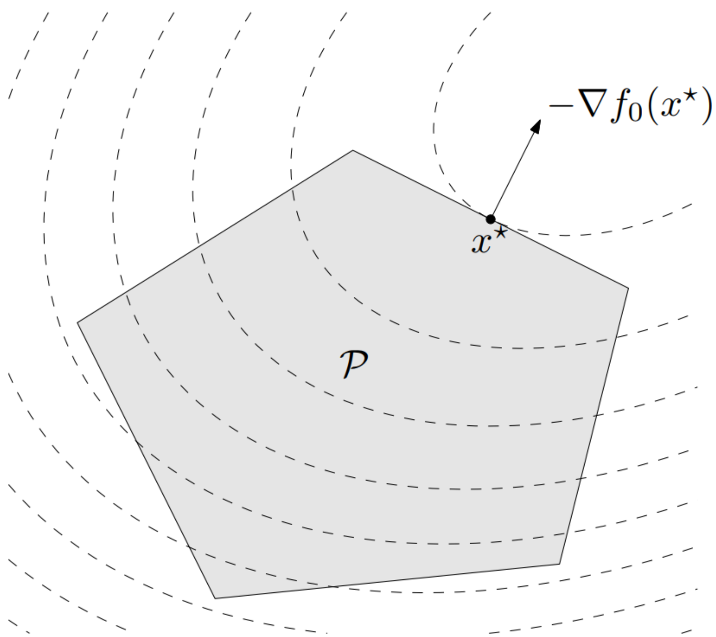
Equivalent LP:

$$\begin{array}{rl} \min\limits_{t\in\mathbb{R}^n, x\in\mathbb{R}^n} & \mathbf{1}^T t \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \\ & x \preccurlyeq t \\ & -t \preccurlyeq x \end{array}$$

Quadratic Programming (QP)

$$\begin{aligned} \min\ & \frac{1}{2}x^T P x + q^T x + r \\ \text{s.t.}\ & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

- $P \in \mathbb{S}_+^n$, so the objective is convex function
- Minimize a convex quadratic function over a polyhedron



QP: examples

Least-squares and regression

$$\begin{array}{ll} \min\limits_{x} & ||Ax - b||_2^2 = x^T A^T A x - 2b^T A x + b^T b \\ \text{subject to} & l_i \leq x_i \leq u_i,\ i = 1, \ldots, n \end{array}$$

Linear programming with random cost

Deterministic

$$\begin{array}{ll} \text{min} & c^T x \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

Non-deterministic

$$\begin{array}{ll} \text{min} & \mathbf{E}[c^T x] + \gamma \mathbf{var}[c^T x] = \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

- c is random vector with mean $\bar{c}$ and covariance Σ
- $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$

QP: examples

Portfolio optimization

$$
\begin{array}{ll}
\text{minimize} & x^T \Sigma x \\
\text{subject to} & R^T x \geq r_{min} \\
& \mathbf{1}^T x = B \\
& x \succcurlyeq 0
\end{array}
$$

- Price changes of all invested assets has mean $R \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n \times n}$
- r_{min}: minimum return.
- $\mathbf{1} \in \mathbb{R}^n$: every component is 1;
- B: budget.

Quadratically constrained quadratic programming (QCQP)

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & Ax = b\end{array}$$

- $P \in \mathbb{S}_+^n$, so the objective and constraint functions are convex
- Minimize a convex quadratic function over a intersection of m ellipsoids and an affine set

QCQP: example

Portfolio optimization

$$
\begin{array}{ll}
\text{minimize} & x^T \Sigma_0 x \\
\text{subject to} & x^T \Sigma_i x \leq d_i,\ i = 1, \dots, m \\
& R^T x \geq r_{min} \\
& \mathbf{1}^T x = B \\
& x \succcurlyeq 0
\end{array}
$$

- There are a few estimation of the covariance of the price changes, $\Sigma_i,\ i = 0, \dots, m$

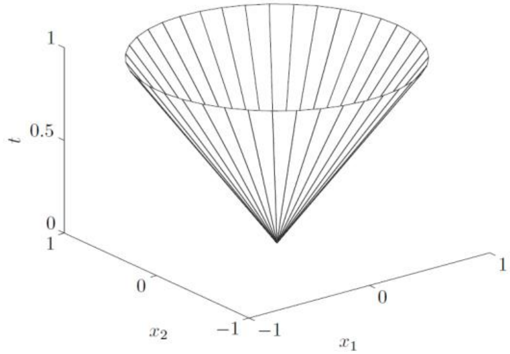
Second-order cone programming (SOCP)

$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\
& Fx = g,
\end{array}
$$

- $A_i \in \mathbb{R}^{n_i \times n}$
- Inequalities are second-order cone constraints

  $(A_i x + b_i, c_i^T x + d_i) \in$ second-order cone in $\mathbb{R}^{n_i+1}$
- If $A_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP.

$\{(x,t) \mid \|x\|_2 \le t\}$ is *second-order cone, also called ice cream cone.*



SOCP: examples

Robust linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

- a_i is inaccurate, but are known in ellipsoids: $a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$

$\bar{a}_i \in \mathbb{R}^n$, $P_i \in \mathbb{R}^{n \times n}$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

Equivalent SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- $\sup_{||u||_2 \leq 1} \bar{a}_i^T x + (P_i u)^T x = \sup_{||u||_2 \leq 1} \bar{a}_i^T x + u^T \left(P_i^T x\right) = \bar{a}_i^T x + ||P_i^T x||_2$

SOCP: examples

Robust linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

- a_i is Gaussian with mean $\bar{a}_i$, and covariance Σ_i
- $a_i^T x$ is Gaussian with mean $\bar{a}_i^T x$, and variance $x^T \Sigma_i x = \left\| \Sigma_i^{1/2} x \right\|_2$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m, \end{array}$$

Equivalent SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

$\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2}\, dt$ is CDF of $\mathcal{N}(0,1)$

$$\Pr\left(a_i^T x \leq b_i\right) = \Pr\left(\frac{a_i^T x - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2} \leq \frac{b_i - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2} \right) \geq \eta \Leftrightarrow \frac{b_i - \bar{a}_i^T x}{\left\| \Sigma_i^{1/2} x \right\|_2} \geq \Phi^{-1}(\eta)$$