SSE5107 Optimization Theory and Algorithms
Homework 1
Due: Oct. 12th, 2021, in class

Problem 1

Explain whether the following sets are convex.

1. A *slab*, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$.
2. A *rectangle*, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, ..., n\}$.
3. A *wedge*, i.e., $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$.
4. The set of points closer to a given point than a given set, i.e.,
$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\},$$
where $S \subseteq \mathbb{R}^n$.
5. The set of points closer to one set than another, i.e.,
$$\{x \mid \mathrm{dist}(x, S) \leq \mathrm{dist}(x, T)\},$$
where $S, T \subseteq \mathbb{R}^n$, and
$$\mathrm{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$
6. The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.
7. The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$, where $a \neq b$ and $0 \leq \theta \leq 1$.

Problem 2

Let $P = \left\{x \in \mathbb{R}^n \mid a_i^T x \leq b_i \ \text{ for } \ i = 1, \ldots, m\right\}$, where $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$ are given. Recall that a ball with center $\bar{x} \in \mathbb{R}^n$ and radius $r > 0$ is defined as the set $B(\bar{x}, r) = \{x \in \mathbb{R}^n \mid \|x - \bar{x}\|_2 \leq r\} = \{\bar{x} + x \in \mathbb{R}^n \mid \|x\|_2 \leq r\}$. We are interested in finding a ball with the largest possible radius, subject to the condition that it is entirely contained within the set P (also known as the largest inscribed ball in P). Give a linear programming formulation of this problem.

Problem 3

Let $S = \left\{x \in \mathbb{R}^n \mid x^T A x + b^T x + c \leq 0\right\}$, where $A \in \mathcal{S}^n, b \in \mathbb{R}^n$, and $c \in \mathbb{R}$ are given.

1. Show that S is convex if $A \succeq \mathbf{0}$. Is the converse true? Explain.
2. Let $H = \left\{x \in \mathbb{R}^n \mid g^T x + h = 0\right\}$, where $g \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $h \in \mathbb{R}$. Show that $S \cap H$ is convex if $A + \lambda g g^T \succeq \mathbf{0}$ for some $\lambda \in \mathbb{R}$.