Optimization Theory and Algorithms

Instructor: Prof. LIAO, Guocheng （廖国成）

Email: liaogch6@mail.sysu.edu.cn

School of Software Engineering
Sun Yat-sen University

Outline

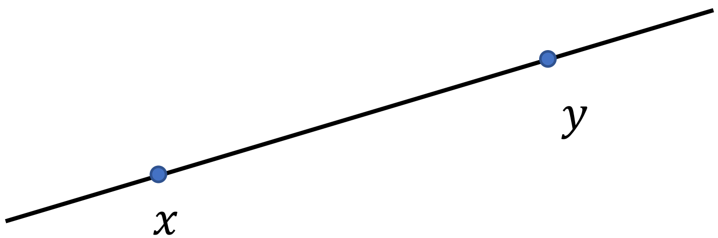
- Affine set
- Convex set
- Convexity-preserving operations
- Separating and supporting hyperplane theorem

Affine set

Definition: a set $S \subseteq \mathbb{R}^n$ is an affine set, if

for any $x, y \in S$, $\boxed{\theta x + (1-\theta) y} \in S$, for all $\theta \in \mathbb{R}$.

A line through x, y, if $x \neq y$.



Affine set contains the line through any two distinct points in the set.

Example: $\{x \mid Ax = b\}$, i.e., solution set of linear equations.

Generalization to more than two points:

Affine combination of $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$:

$$\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k, \text{ where } \sum_{i=1}^{k} \theta_i = 1.$$

Affine set contains the every affine combination of its points in the set.

Affine set: Interpretation

A set $S \subseteq \mathbb{R}^n$ is an affine set $\Longleftrightarrow$ S is the translation of some linear subspace $V \subseteq \mathbb{R}^n$, i.e., S is of the form $\{x\} + V = \{x + v : v \in V\}$ for some $x \in \mathbb{R}^n$.

for any $v_1, v_2 \in V, \alpha, \beta \in \mathbb{R}, \alpha v_1 + \beta v_2 \in V$

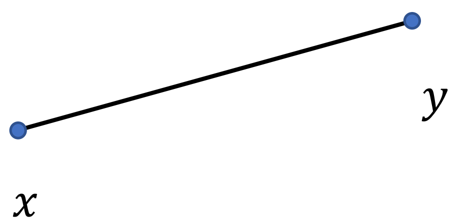
Affine set = subspace + offset

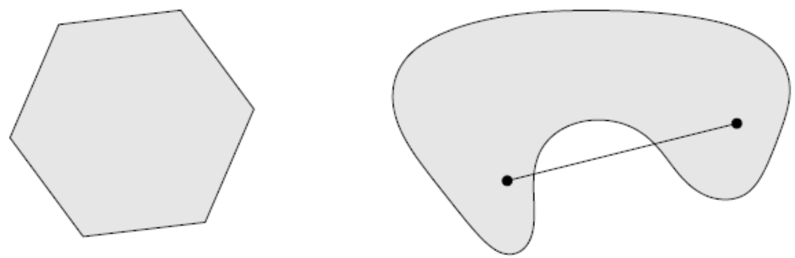
Convex set

Definition: a set $S \subseteq \mathbb{R}^n$ is a convex set, if

for any $x, y \in S$, $\boxed{\theta x + (1-\theta) y} \in S$, for $\theta \in [0,1]$.

Line segment between x, y, if $x \neq y$.





Convex set contains the line segment between any two distinct points in the set.

Generalization to more than two points:

Convex combination of $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$:

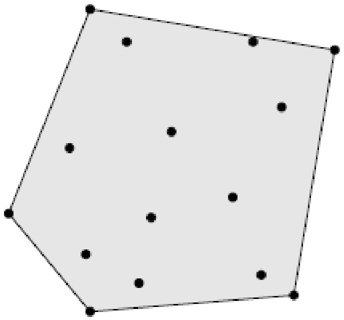
$\sum_{i=1}^{k} \theta_i x_i$, where $\sum_{i=1}^{k} \theta_i = 1$, and $\theta_i \geq 0$ for all i.

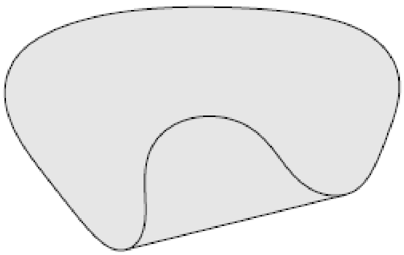
Convex set contains the every convex combination of its points in the set.

Convex hull

The convex hull of a set S is the set of all convex combinations of points in S:

$$conv(S) = \left\{ \sum_{i=1}^{k} \theta_i x_i \,|\, x_i \in S, \sum_{i=1}^{k} \theta_i = 1 \text{ , and } \theta_i \geq 0 \text{ for all } i \right\}$$

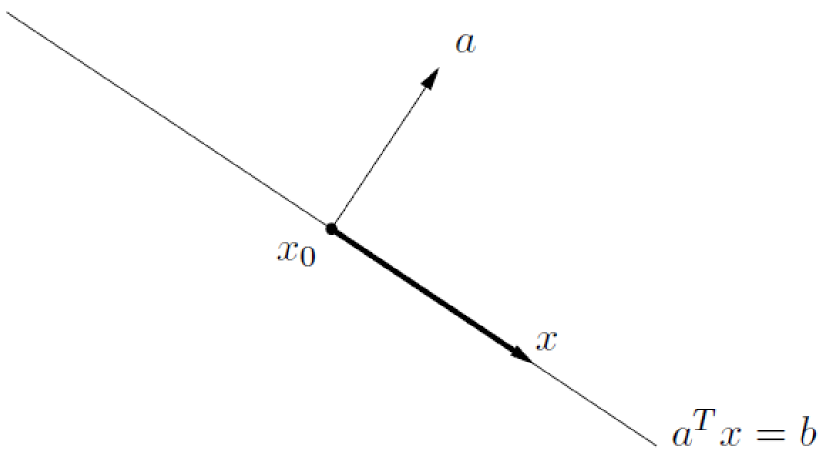


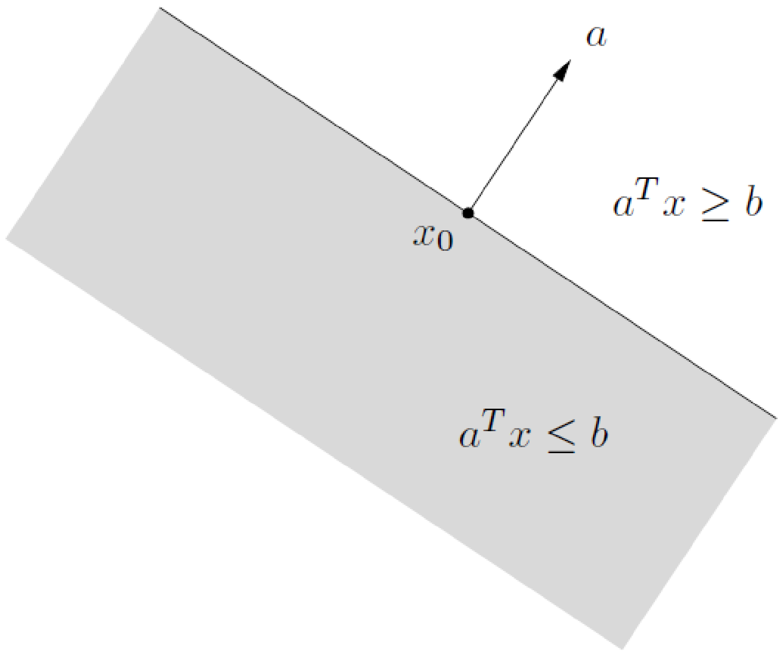


The convex hull of a set S is the smallest convex set that contains S.

Convex set: examples

- Simple examples: empty set $\emptyset$; any single point; line; the whole space $\mathbb{R}^n$.
- Hyperplane $\{x \in \mathbb{R}^n | a^T x = b\}$ ($a \in \mathbb{R}^n, a \neq \mathbf{0}$, $b \in \mathbb{R}$).





- Halfspace $\{x \in \mathbb{R}^n | a^T x \leq b\}$ ($a \in \mathbb{R}^n, a \neq \mathbf{0}$, $b \in \mathbb{R}$).

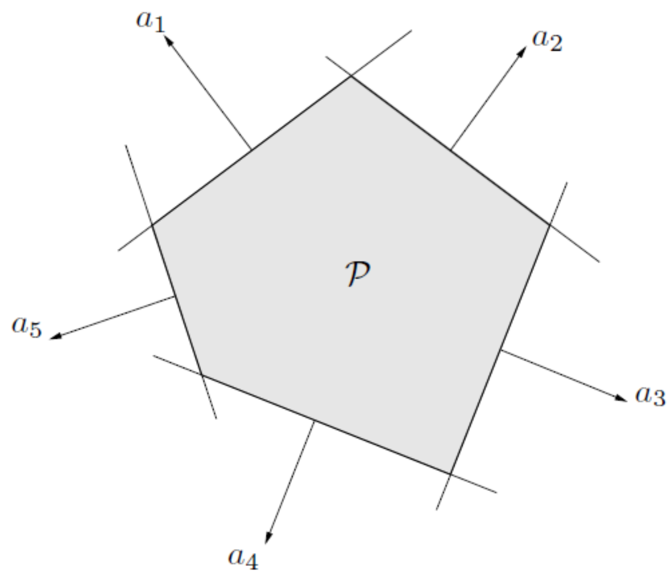
Convex set: examples

- Polyhedron: solution set of finite linear equalities and inequalities

$$\{x|a_i^T x \le b_i, i = 1, \dots, m, c_j^T x = d_i, j = 1, \dots, p\}$$

Matrix form

$$\{x|Ax \preccurlyeq b, Cx = d\}$$



Polyhedron is intersection of finite number of halfspaces and hyperplanes.

Convex set: examples

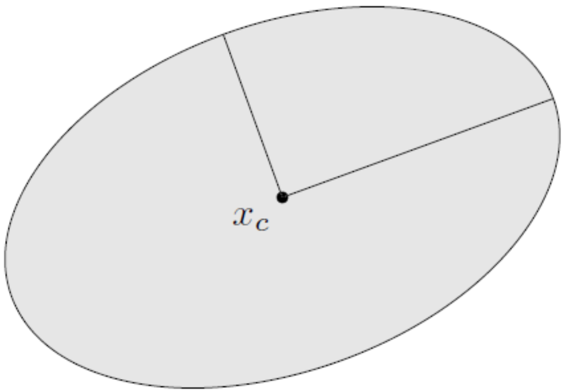
- Euclidean ball

$$B(x_c, r) = \{x|\ ||x - x_c\|_2 \leq r\} = \{x_c + ru|\ ||u\|_2 \leq 1\}$$

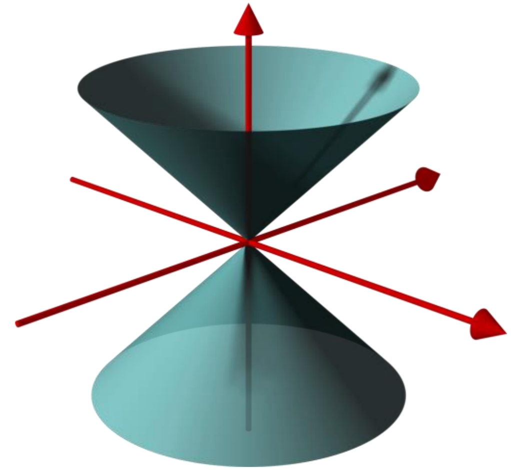
- Ellipsoids

$$E(x_c, Q) = \{x|(x - x_c)^T Q(x - x_c) \leq 1\}$$

x_c is the center of the ellipsoid; $Q \succ 0$, i.e., positive definite matrix.



Cone

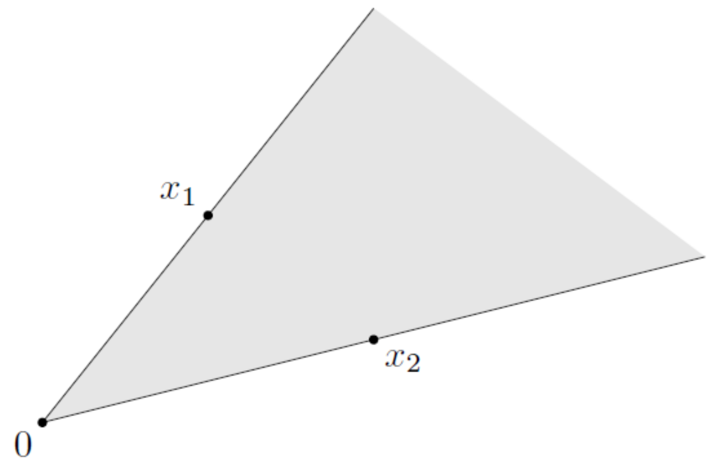


A set $S \subseteq \mathbb{R}^n$ is a cone, if

for any $x \in S$, $\theta x \in S$, for $\theta \geq 0$.

A set $S \subseteq \mathbb{R}^n$ is a convex cone, if

for any $x_1, x_2 \in S$, $\theta_1 x_1 + \theta_2 x_2 \in S$, for $\theta_1, \theta_2 \geq 0$.



Generalization to more than two points:

Conic combination of $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$:

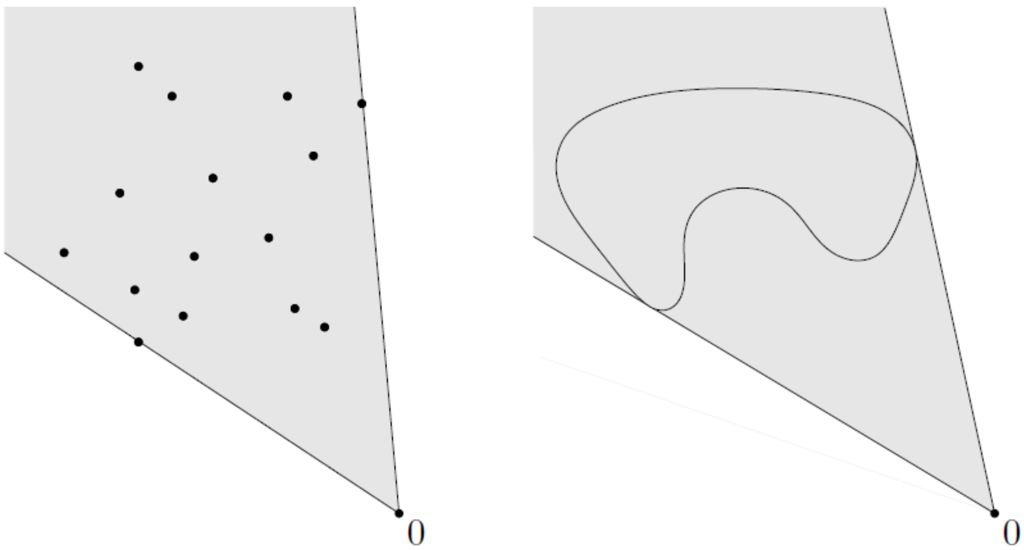
$\sum_{i=1}^{k} \theta_i x_i$, where $\theta_i \geq 0$ for all i.

Convex cone contains the every conic combination of its points in the set.

Conic hull

The conic hull of a set S is the set of all conic combinations of points in S:

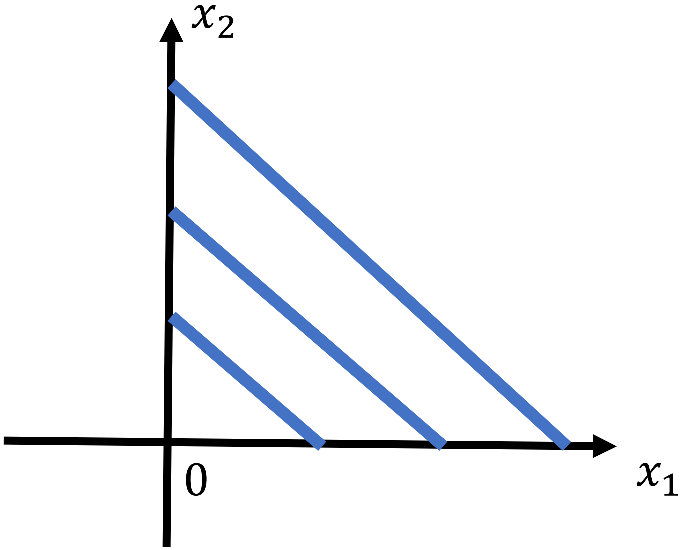
$$conv(S) = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in S, \text{ and } \theta_i \geq 0 \text{ for all } i \right\}$$



The conic hull of a set S is the smallest convex cone that contains S.

Convex cone: example

- Non-negative orthant: $\mathbb{R}_+^n \triangleq \{(x_1, x_2, \dots, x_n) | x_i \geq 0, i = 1, \dots, n\}$



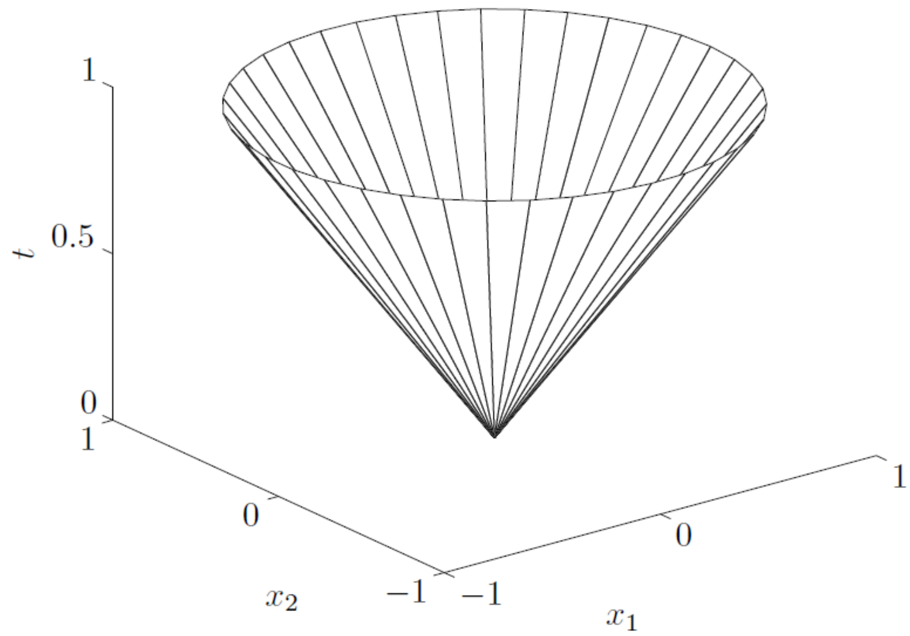
$\mathbb{R}_+^2$

Convex cone: example

- Norm cone: $\{(x,t) |\ ||x|| \leq t\} \subseteq \mathbb{R}^{n+1}$, for a norm $||\cdot||$.

  For a l2-norm (Euclidean norm) $\|\cdot\|_2$:

  $\{(x,t) |\ \|x\|_2 \leq t\}$ is *second-order cone, also called ice cream cone.*



Boundary of second-order cone in $\mathbb{R}^3$, $\{(x_1, x_2, t) |\ (x_1^2 + x_2^2)^{1/2} \leq t\}$.

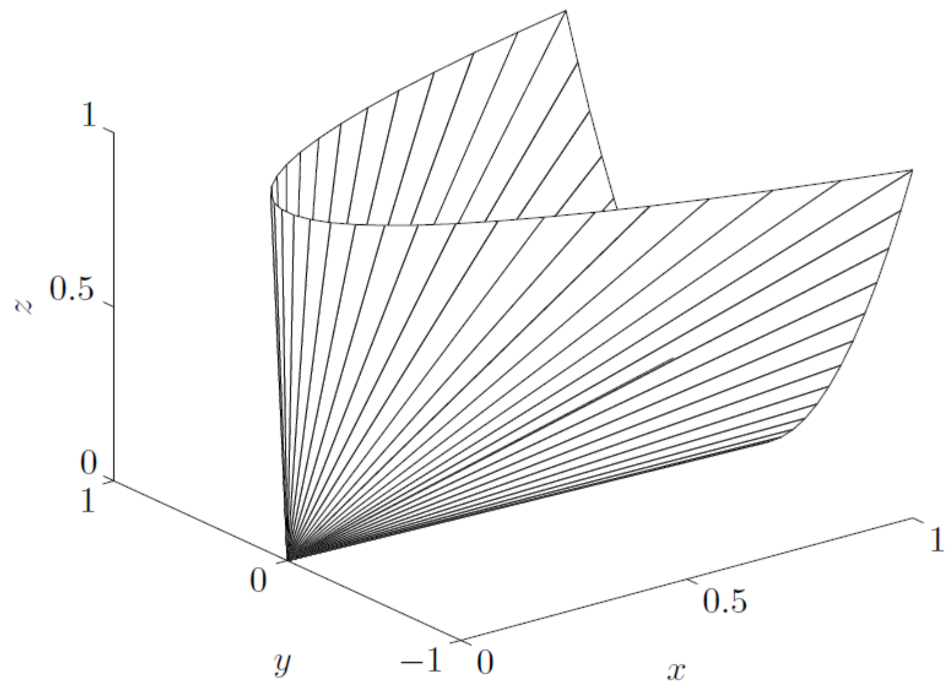
Convex cone: positive semidefinite cone

- Positive semidefinite matrix $\mathbf{S}_+^n \triangleq \{X \in \mathbf{S}^n | X \succcurlyeq \mathbf{0}\}$.

$$z^T X z \geq 0 \text{ for all } z \in \mathbb{R}^n$$

- Example: positive semidefinite cone in $\mathbf{S}_+^2$

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^n \Leftrightarrow x \geq 0, z \geq 0, xz \geq y^2.$$

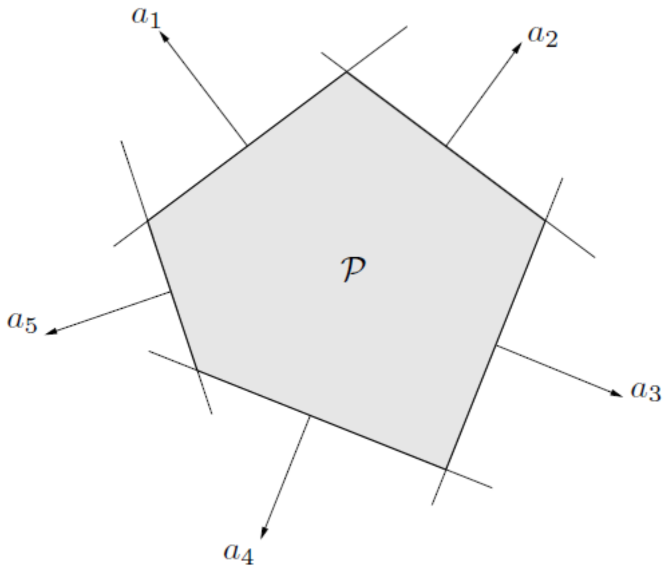


Boundary in $\mathbb{R}^3$

Operations that preserve convexity

- Intersection: the intersection of convex sets is convex.

$$\{x | a_i^T x \leq b_i, i = 1, \dots, m, c_j^T x = d_i, j = 1, \dots, p\}$$



Polyhedron is intersection of finite number of halfspaces and hyperplanes.

Operations that preserve convexity

- Affine mapping: $f(x) = Ax + b$ where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, i.e., $f\colon \mathbb{R}^n \to \mathbb{R}^m$

$$S \subseteq \mathbb{R}^n \text{ is convex } \Longrightarrow f(S) = \{f(x) |\ x \in S\} \text{ is convex}$$

$$T \subseteq \mathbb{R}^n \text{ is convex } \Longrightarrow f^{-1}(T) = \{x |\ f(x) \in T\} \text{ is convex}$$

- Examples:

➢ Scaling ($\{\alpha x | x \in S\}$), translation ($\{x + x_0 | x \in S\}$), projection ($\{x_1 | [x_1, x_2]^T \in S\}$)

➢ Convexity of ellipsoid: $E(x_c, Q) = \{x | (x - x_c)^T Q (x - x_c) \leq 1\}$

Eclidean ball $B(0, r) = \{x | x^T x \leq r^2\}$ is convex

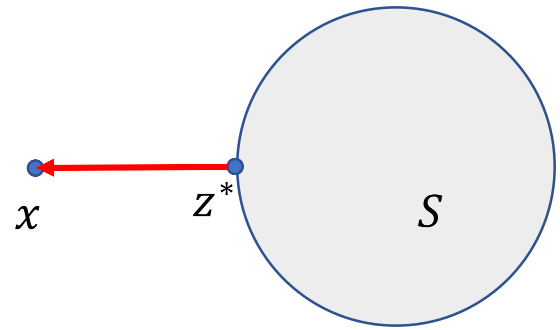
Let $f(x) = rQ^{-1/2} x + x_c$

$$f\big(B(0,r)\big) = \{f(x) | x^T x \leq r^2\} = \{x | (x - x_c)^T Q (x - x_c) \leq 1\} \text{ is convex}$$

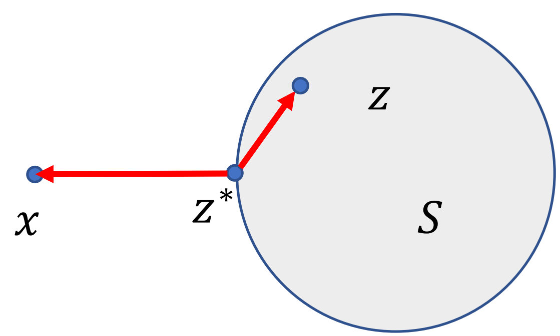
Projection onto closed convex sets

Theorem: Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Then, for every $x \in \mathbb{R}^n$, there exists a unique point $z^* \in S$ that is closest to (in Euclidean norm) x.

Projection of x on S : $\Pi_S(x) = arg \min_{z \in S} \|z - x\|_2$



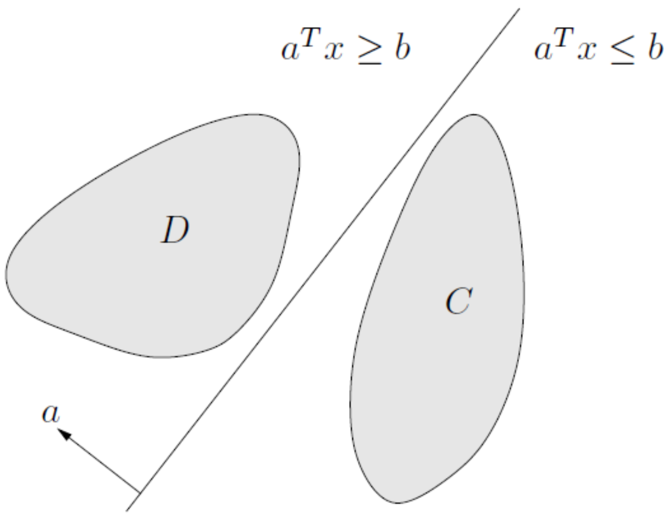
Theorem: Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Given any $x \in \mathbb{R}^n$, we have $z^* = \Pi_S(x)$ iff $z^* \in S$ and $(z - z^*)^T(x - z^*) \leq 0$ for all $z \in S$.



Separating hyperplane theorem

Theorem: If C and D are non-empty, disjoint (i.e., $C \cap D = \emptyset$) convex set, there exists $a \neq 0$ and b such that:

$$a^T x \leq b \text{ for all } x \in C \text{ and } a^T x \geq b \text{ for all } x \in D.$$



Hyperplane $\{x | a^T x = b\}$ separates C and D

$a^T x < b$ for all $x \in C$ and $a^T x > b$ for all $x \in D$ $\longrightarrow$ strictly separates C and D

Theorem (point-set separation): Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Let $x \in \mathbb{R}^n \backslash S$. There exists an $a \in \mathbb{R}^n$ such that $\max_{z \in S} a^T z < a^T x$.

Supporting hyperplane theorem

Let $x_0 \in$ **bd** (S). If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all x in S, then the following hyperplane is a *supporting hyperplane* to S at the point x_0:

boundary

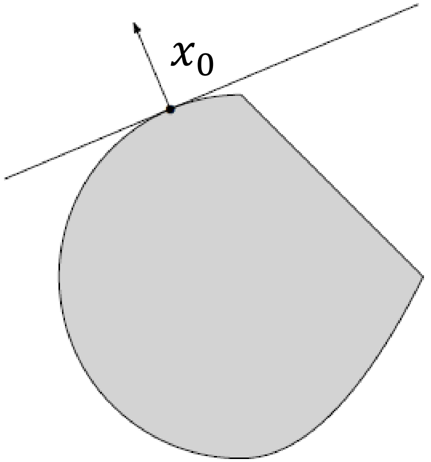
$$\{x | a^T x = a^T x_0\}$$

Theorem: If S is non-empty convex set, there exists a *supporting hyperplane* at every boundary of S.

Interior point: A point $x \in S$ is an interior point of set S, if there exists an $\varepsilon > 0$ such that

$$\{y | \, \|x - y\|_2 \leq \varepsilon\} \subseteq S.$$

A ball centered at x that lies entirely in S.



Interior of set S **int**(S) : the set of all interior points.

Closure of set S: **cl** $(S) \triangleq \mathbb{R}^n \setminus$ **int** $(\mathbb{R}^n \setminus S)$,

i.e., set S + its boundary

Boundary of set S: **bd** $(S) \triangleq$ **cl** $(S) \setminus$ **int** (S)