SSE5107 Optimization Theory and Algorithms
Mid-term quiz

Problem 1 (20 points): True or False

1. A hyperplane is a convex set. True or False
2. A quadratic function is a convex function. True or False
3. A locally optimal point of a convex problem is also globally optimal. True or False
4. A convex problem has a unique optimal value and a unique optimal point. True or False
5. All the eigenvalues of a positive semidefinite matrix are non-negative. True or False

Problem 2 (40 points): Single Choice

1. Make a **single choice** for each given set A. Convex B. Non-convex
   (a) $\{\bar{x} + Px \mid ||x||_2 \leq 1\}$, where $\bar{x} \in \mathcal{R}^n$ and $P \in \mathbb{R}^{n\times n}$.
   (b) $\{\sum_{i=1}^3 \alpha_i x_i \mid \sum_{i=1}^3 \alpha_i = 1, \alpha_i \geq 0, i = 1,2,3\}$, where $x_i \in \mathbb{R}^n$, $i = 1,2,3$.
   (c) A set S such that for any $x \in S$, θx is also in set S, for any $\theta \in [0,1]$.
   (d) $\{x \in \mathbb{R}^n \mid c^T x \leq 0, x^T(W - cc^T)x \leq 0\}$, where $W \in \mathbb{S}_+^n$ and $c \in \mathbb{R}^n$.
   (e) $\{x \in \mathbb{R}^n \mid x^T y \leq 0, \forall y \in C\}$, where C is a given set.
2. Make a **single choice** for each given function A. Convex B. Concave C. Both D. Neither
   (a) $f(x) = ||x||_p$, where $p > 1$.
   (b) $f(X) = \mathrm{Tr}(AXB)$, where $A, B, X \in \mathcal{R}^{n\times n}$.
   (c) $f(X) = \mathrm{Tr}(AX^{-1})$, where **dom** $f = \mathbb{S}_{++}^n$ and $A \in \mathbb{S}_+^n$.
   (d) $g(y) = \sup_{x\in \mathbf{dom}\ f} y^T x - f(x)$, for any given function $f(x)$.
   (e) $f(X) = \log\det(X)$, where **dom** $f = \mathbb{S}_+^n$.

Problem 3 (20 points)

1. Consider the following general linear programming

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^T x \\ s.t. \quad & Ax = b \\ & Gx \preceq h, \end{aligned}$$

   where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $G \in \mathbb{R}^{k\times n}$, and $h \in \mathbb{R}^k$. Convert the above problem into a *standard-form* linear programming.

2. Consider the following general semidefinite programming:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ s.t. \quad & Ax = b \\ & \sum_{i=1}^{n} F_i x_i + G \preceq 0, \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $G \in \mathbb{S}^k$ and $F_i \in \mathbb{S}^k$, for $i = 1, 2, ..., n$. Convert the above problem into a *standard-form* semidefinite programming.

Problem 4 (20 points)

Consider a convex problem of the general form:

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & x \in \mathcal{X}. \end{aligned}$$

Here $\mathcal{X} \subset \mathbb{R}^n$ is a convex set, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable function on $\mathcal{X}$.

1. Give a necessary and sufficient condition for $x^* \in \mathcal{X}$ to be an optimal solution of the above problem.
2. Prove the sufficiency of the condition in question 1.
3. Present more specific optimality conditions (no need to prove) for the following case:

$$\mathcal{X} = \{x \mid x_i \in [l_i, u_i], i = 1, ..., n\}, \quad l_i \leq u_i, \forall i.$$