

Optimization Theory and Algorithms

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Outline

- Equality constrained minimization
- Reduction to an unconstrained minimization
- Newton's method with equality constrained

Equality constrained minimization problem

$$\begin{array}{ll}\min & f(x) \\ \text{s.t.} & Ax = b\end{array}$$

- f is convex and twice continuously differentiable
- Assume optimal point x^* exists. Let $p^* = f(x^*)$ be the optimal value.

Optimality condition (KKT conditions): x^* is optimal iff there exists a v^* such that

$$\nabla f(x^*) + A^T v^* = 0, \quad Ax^* = b$$

Solution methods:

- Elimination method
 - Dual method
 - Newton's method
- } reduce to **an unconstrained** minimization

Elimination equality constraints

$$\{x \mid Ax = b\} = \{Fz + x' \mid z \in \mathbb{R}^{n-p}\}$$

- $\text{Rank}(A)=p$
- $F \in \mathbb{R}^{n \times (n-p)}$ is any matrix whose range is the nullspace of A .
- x' is any particular solution of $Ax = b$.

equality constrained minimization

$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & Ax = b \end{array}$$



unconstrained minimization

$$\min_z f(Fz + x')$$

Dual method

Primal problem
(equality constrained)

$$\begin{array}{ll}\min & f(x) \\ \text{s.t.} & Ax = b\end{array}$$



Dual function:

$$g(v) = -b^T v + \min_x (f(x) + v^T Ax)$$

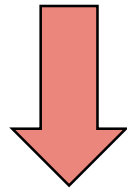
Dual problem
(unconstrained)

$$\max g(v)$$

Newton's method with equality constraints

Newton's method with **no constraints**

$$\min_{\Delta x} f(x + \Delta x) \approx f(x) + \nabla f(x)^T \Delta x + \frac{1}{2} \Delta x^T \nabla^2 f(x) \Delta x$$

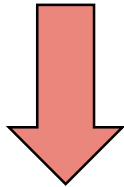


set the gradient to zero

$$\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

Newton's method with **equality constraints**

$$\begin{aligned} \min_{\Delta x} f(x + \Delta x) &\approx f(x) + \nabla f(x)^T \Delta x + \frac{1}{2} \Delta x^T \nabla^2 f(x) \Delta x \\ \text{s.t. } A(x + \Delta x) &= b \end{aligned}$$



KKT conditions

$$\begin{aligned} \nabla f(x) + \nabla^2 f(x) \Delta x_{nt} + A^T \nu^* &= 0 \\ A \Delta x_{nt} &= 0 \end{aligned}$$

A large blue arrow pointing to the right, indicating the final system of equations.
$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

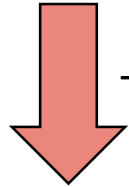
Newton's method with equality constraints

Newton's method with **no constraints**

$$\min f(x)$$

Optimality condition:

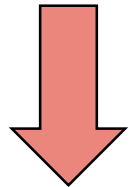
$$\nabla f(x^*) = 0$$



$$x^* = x + \Delta x_{nt}$$

Taylor approximation

$$\nabla f(x) + \nabla^2 f(x) \Delta x_{nt} = 0$$



$$\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

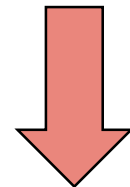
Newton's method with **equality constraints**

$$\min f(x)$$

$$\text{s.t. } Ax = b$$

Optimality conditions (KKT conditions):

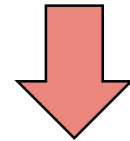
$$\nabla f(x^*) + A^T v^* = 0, \quad Ax^* = b$$



$$x^* = x + \Delta x_{nt}$$

Taylor approximation

$$\nabla f(x) + \nabla^2 f(x) \Delta x_{nt} + A^T v^* = 0, \quad A(x + \Delta x_{nt}) = b$$



$$\nabla f(x) + \nabla^2 f(x) \Delta x_{nt} + A^T v^* = 0, \quad A \Delta x_{nt} = 0$$



$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ v \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Newton's method with equality constraints

- Given a starting point $x \in \mathbf{dom} f$
 - **Repeat**
1. Compute the *Newton step* and decrement:

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ v \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix} \quad \lambda = (\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x))^{1/2}$$

2. **Stopping criterion:** if $\lambda^2/2 \leq \varepsilon$, break
3. *Line search:* choose a step size t via *backtracking line search*.
4. Update: $x \leftarrow x + t\Delta x_{nt}$