

Optimization Theory and Algorithms

Instructor: Prof. LIAO, Guocheng (廖国成)

Email: liaogch6@mail.sysu.edu.cn

**School of Software Engineering
Sun Yat-sen University**

Outline

- Affine set
- Convex set
- Convexity-preserving operations
- Separating and supporting hyperplane theorem

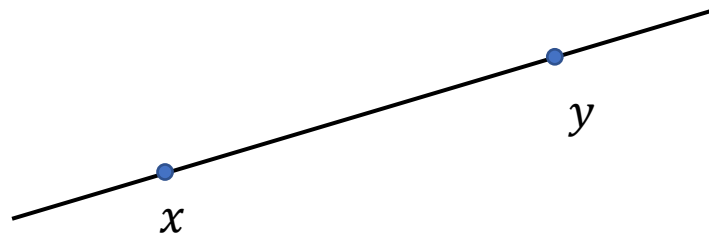
Affine set

Definition: a set $S \subseteq \mathbb{R}^n$ is an affine set, if

for any $x, y \in S$, $\theta x + (1 - \theta)y \in S$, for all $\theta \in \mathbb{R}$.



A line through x, y , if $x \neq y$.



Affine set contains the **line through any two distinct points** in the set.

Example: $\{x | Ax = b\}$, i.e., solution set of linear equations.

Generalization to more than two points:

Affine combination of $x_1, x_2, \dots, x_k \in \mathbb{R}^n$:

$\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$, where $\sum_{i=1}^k \theta_i = 1$.

Affine set contains the **every affine combination of its points** in the set.



Affine set: Interpretation

A set $S \subseteq \mathbb{R}^n$ is
an affine set



S is the translation of some **linear subspace** $V \subseteq \mathbb{R}^n$, i.e., S is of
the form $\{x\} + V = \{x + v : v \in V\}$ for some $x \in \mathbb{R}^n$.

for any $v_1, v_2 \in V, \alpha, \beta \in \mathbb{R}, \alpha v_1 + \beta v_2 \in V$

Affine set = subspace + offset

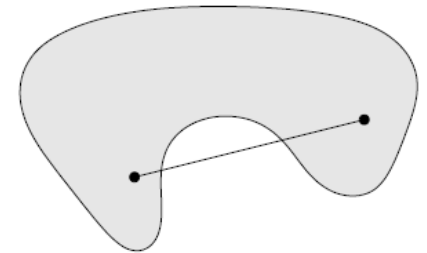
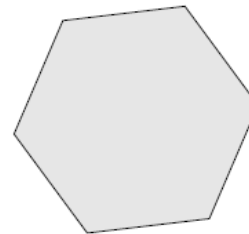
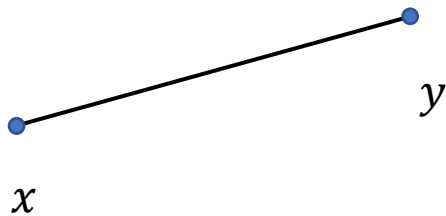
Convex set

Definition: a set $S \subseteq \mathbb{R}^n$ is a convex set, if

for any $x, y \in S$, $\theta x + (1 - \theta)y \in S$, for $\theta \in [0, 1]$.



Line segment between x, y , if $x \neq y$.



Convex set contains the **line segment between any two distinct points** in the set.

Generalization to more than two points:

Convex combination of $x_1, x_2, \dots, x_k \in \mathbb{R}^n$:

$\sum_{i=1}^k \theta_i x_i$, where $\sum_{i=1}^k \theta_i = 1$, and $\theta_i \geq 0$ for all i .

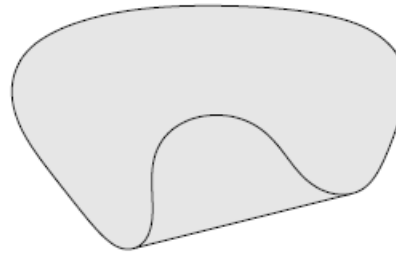
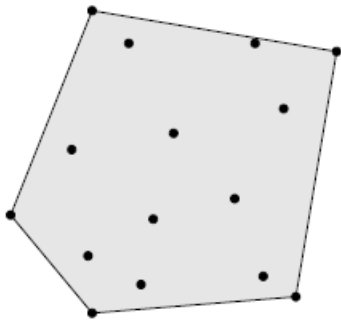
Convex set contains the **every convex combination of its points** in the set.



Convex hull

The convex hull of a set S is the set of all convex combinations of points in S :

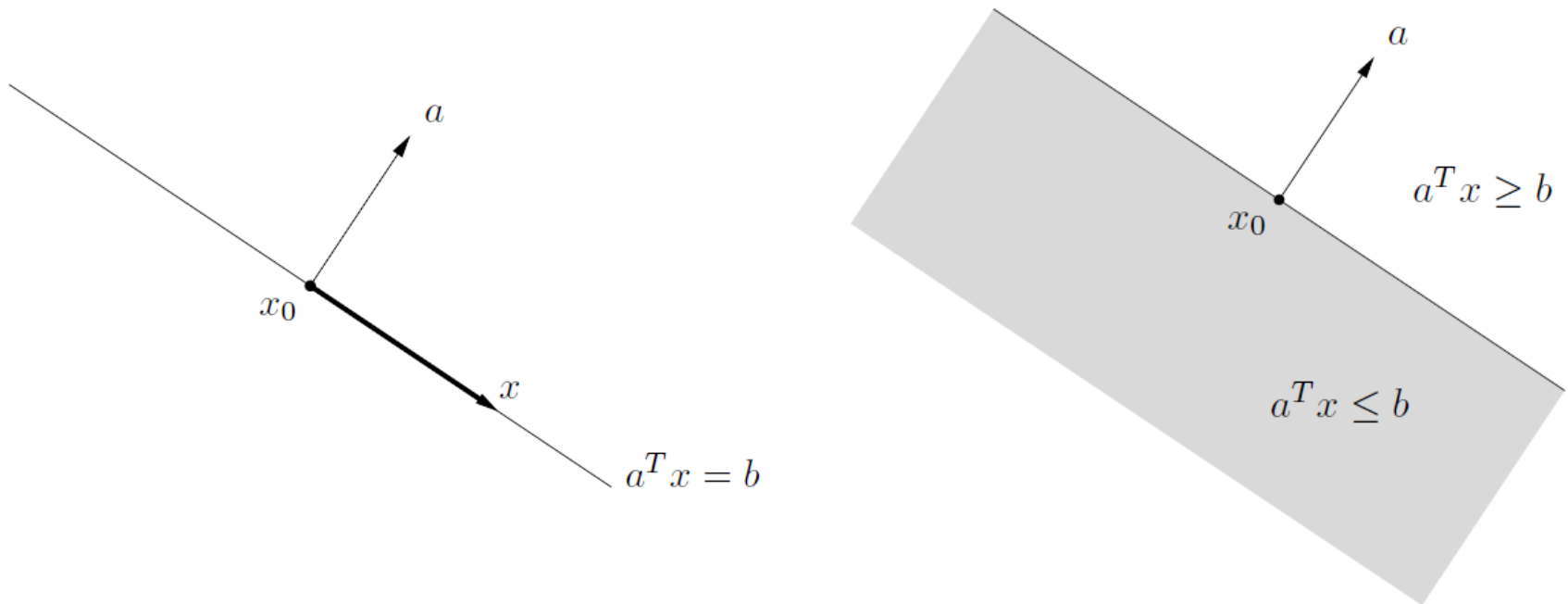
$$\text{conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in S, \sum_{i=1}^k \theta_i = 1, \text{ and } \theta_i \geq 0 \text{ for all } i \right\}$$



The convex hull of a set S is the **smallest** convex set that contains S .

Convex set: examples

- Simple examples: empty set \emptyset ; any single point; line; the whole space \mathbb{R}^n .
- Hyperplane $\{x \in \mathbb{R}^n | a^T x = b\}$ ($a \in \mathbb{R}^n, a \neq \mathbf{0}, b \in \mathbb{R}$).

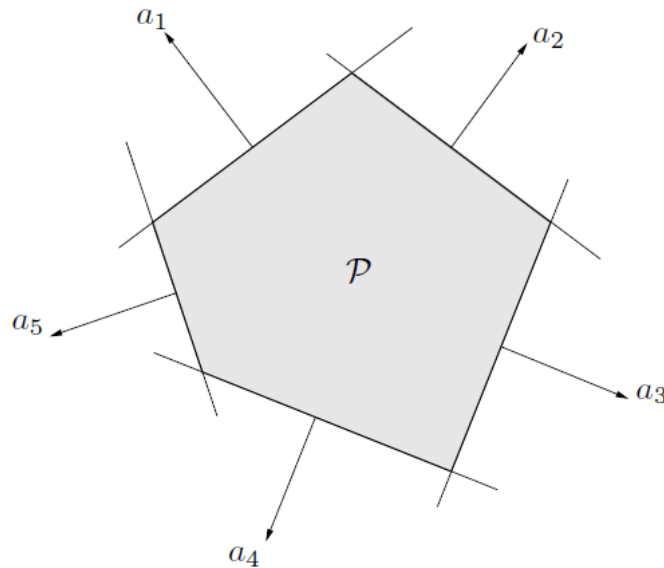


- Halfspace $\{x \in \mathbb{R}^n | a^T x \leq b\}$ ($a \in \mathbb{R}^n, a \neq \mathbf{0}, b \in \mathbb{R}$).

Convex set: examples

- Polyhedron: solution set of finite linear equalities and inequalities

$$\{x | a_i^T x \leq b_i, i = 1, \dots, m, c_j^T x = d_j, j = 1, \dots, p\}$$



Matrix form

$$\{x | Ax \preceq b, Cx = d\}$$

Polyhedron is intersection of finite number of halfspaces and hyperplanes.

Convex set: examples

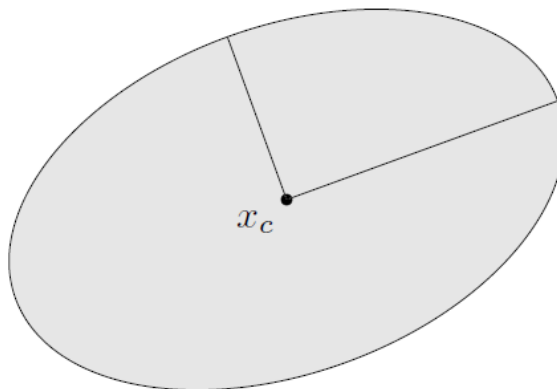
- Euclidean ball

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

- Ellipsoids

$$E(x_c, Q) = \{x \mid (x - x_c)^T Q (x - x_c) \leq 1\}$$

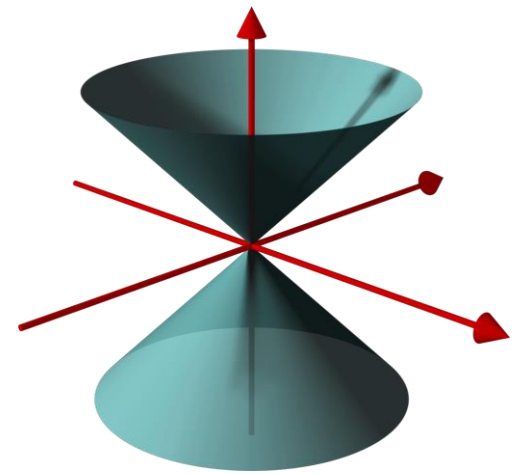
x_c is the center of the ellipsoid; $Q \succ 0$, i.e., positive definite matrix.



Cone

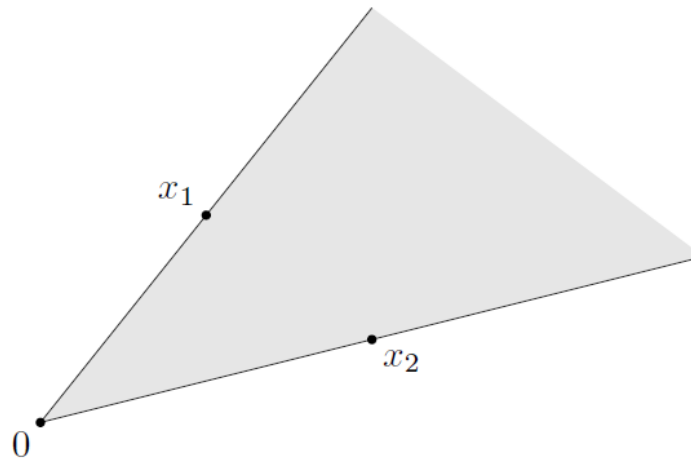
A set $S \subseteq \mathbb{R}^n$ is a **cone**, if

for any $x \in S$, $\theta x \in S$, for $\theta \geq 0$.



A set $S \subseteq \mathbb{R}^n$ is a **convex cone**, if

for any $x_1, x_2 \in S$, $\theta_1 x_1 + \theta_2 x_2 \in S$, for $\theta_1, \theta_2 \geq 0$.



Generalization to more than two points:

Conic combination of $x_1, x_2, \dots, x_k \in \mathbb{R}^n$:

$$\sum_{i=1}^k \theta_i x_i, \text{ where } \theta_i \geq 0 \text{ for all } i.$$

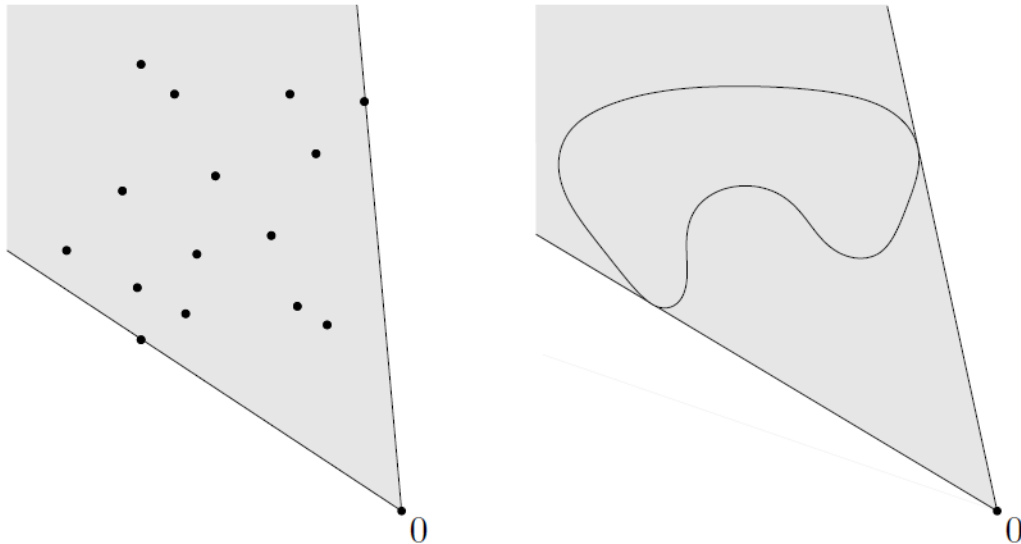
Convex cone contains the **every conic**
of its points in the set.



Conic hull

The conic hull of a set S is the set of all conic combinations of points in S :

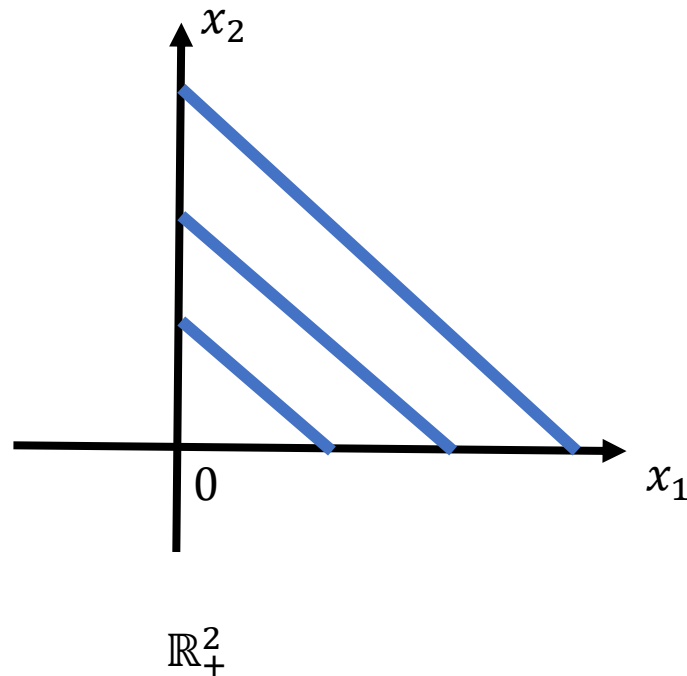
$$\text{conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in S, \text{ and } \theta_i \geq 0 \text{ for all } i \right\}$$



The convex hull of a set S is the **smallest** convex cone that contains S .

Convex cone: example

- Non-negative orthant: $\mathbb{R}_+^n \triangleq \{(x_1, x_2, \dots, x_n) | x_i \geq 0, i = 1, \dots, n\}$

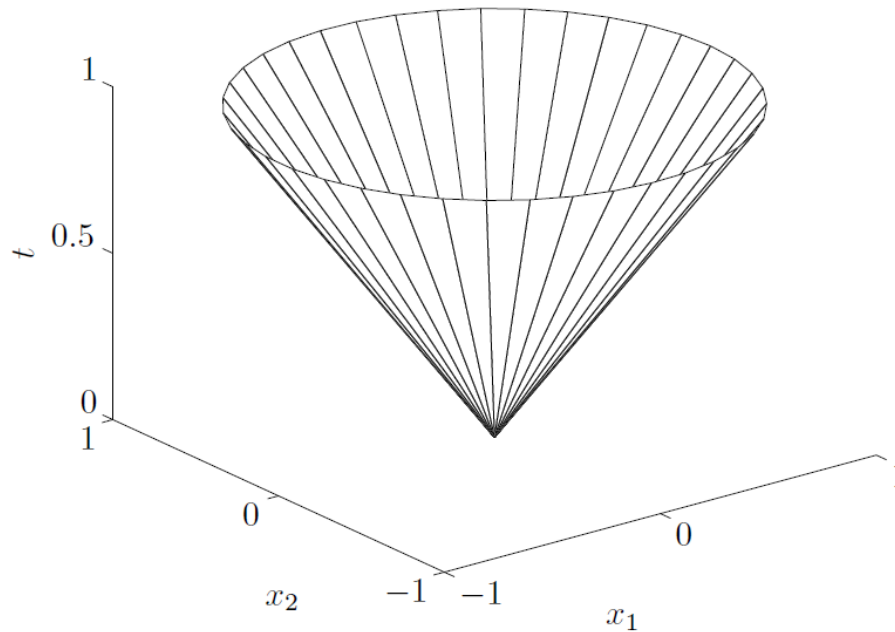


Convex cone: example

- Norm cone: $\{(x, t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$, for a norm $\|\cdot\|$.

For a l2-norm (Euclidean norm) $\|\cdot\|_2$:

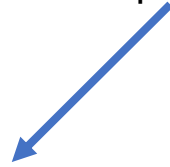
$\{(x, t) \mid \|x\|_2 \leq t\}$ is *second-order cone*, also called *ice cream cone*.



Boundary of second-order cone in \mathbb{R}^3 , $\{(x_1, x_2, t) \mid (x_1^2 + x_2^2)^{1/2} \leq t\}$.

Convex cone: positive semidefinite cone

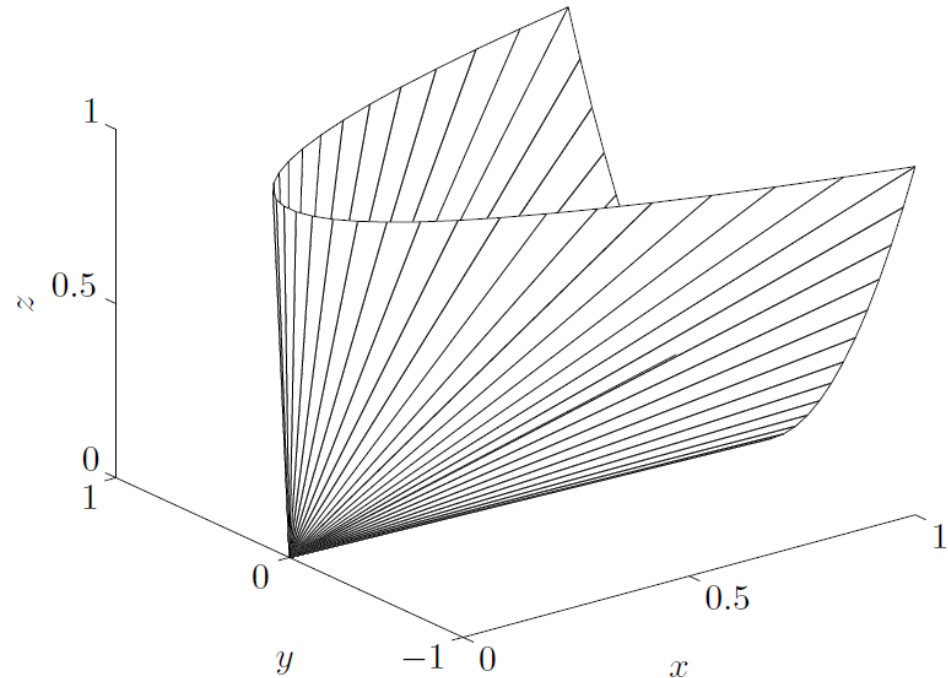
- Positive semidefinite matrix $\mathbf{S}_+^n \triangleq \{X \in \mathbf{S}^n | X \succcurlyeq \mathbf{0}\}$.



$$z^T X z \geq 0 \text{ for all } z \in \mathbb{R}^n$$

- Example: positive semidefinite cone in \mathbf{S}_+^2

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2 \Leftrightarrow x \geq 0, z \geq 0, xz \geq y^2.$$

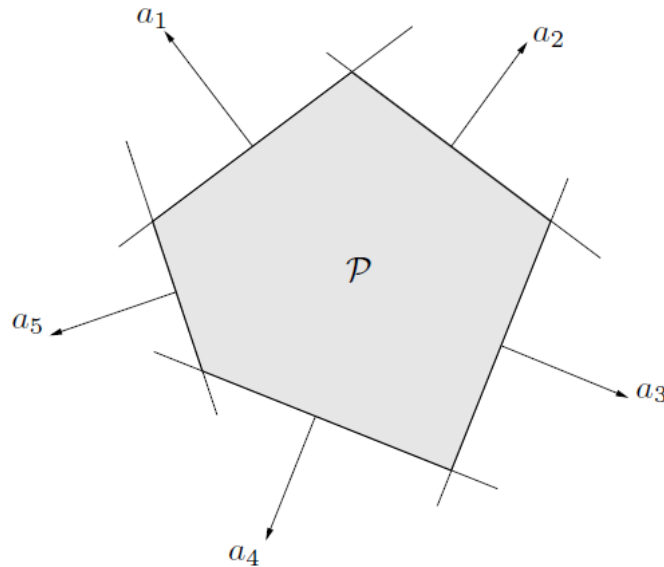


Boundary in \mathbb{R}^3

Operations that preserve convexity

- Intersection: the intersection of convex sets is convex.

$$\{x | a_i^T x \leq b_i, i = 1, \dots, m, c_j^T x = d_j, j = 1, \dots, p\}$$



Polyhedron is intersection of finite number of halfspaces and hyperplanes.

Operations that preserve convexity

- Affine mapping: $f(x) = Ax + b$ where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, i.e., $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$S \subseteq \mathbb{R}^n \text{ is convex} \Rightarrow f(S) = \{f(x) \mid x \in S\} \text{ is convex}$$

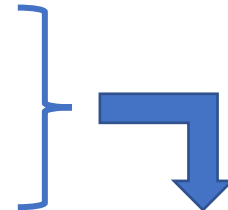
$$T \subseteq \mathbb{R}^m \text{ is convex} \Rightarrow f^{-1}(T) = \{x \mid f(x) \in T\} \text{ is convex}$$

- Examples:

- Scaling ($\{\alpha x \mid x \in S\}$), translation ($\{x + x_0 \mid x \in S\}$), projection ($\{x_1 \mid [x_1, x_2]^T \in S\}$)
- Convexity of ellipsoid: $E(x_c, Q) = \{x \mid (x - x_c)^T Q (x - x_c) \leq 1\}$

$$\text{Euclidean ball } B(0, r) = \{x \mid x^T x \leq r^2\} \text{ is convex}$$

$$\text{Let } f(x) = rQ^{-1/2}x + x_c$$

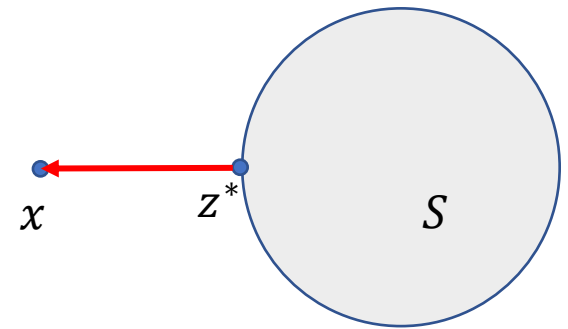


$$f(B(0, r)) = \{f(x) \mid x^T x \leq r^2\} = \{x \mid (x - x_c)^T Q (x - x_c) \leq 1\} \text{ is convex}$$

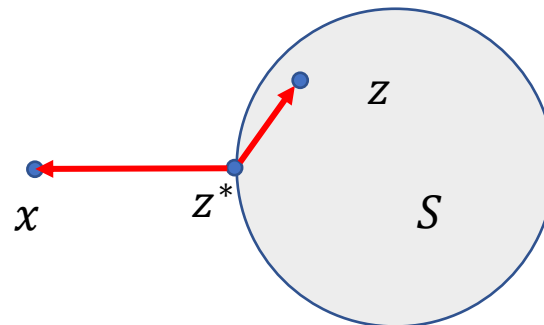
Projection onto closed convex sets

Theorem: Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Then, for every $x \in \mathbb{R}^n$, there exists a **unique** point $z^* \in S$ that is closest to (in Euclidean norm) x .

Projection of x on S : $\Pi_S(x) = \arg \min_{z \in S} \|z - x\|_2$



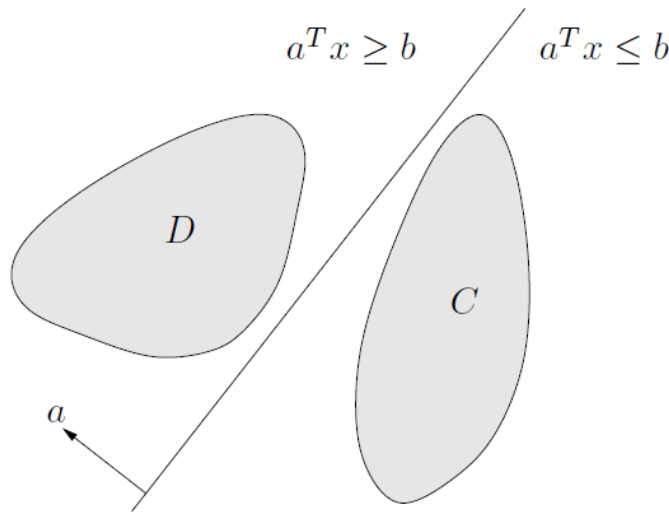
Theorem: Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Given any $x \in \mathbb{R}^n$, we have $z^* = \Pi_S(x)$ iff $z^* \in S$ and $(z - z^*)^T (x - z^*) \leq 0$ for all $z \in S$.



Separating hyperplane theorem

Theorem: If C and D are non-empty, disjoint (i.e., $C \cap D = \emptyset$) convex set, there exists $a \neq 0$ and b such that:

$$a^T x \leq b \text{ for all } x \in C \text{ and } a^T x \geq b \text{ for all } x \in D.$$



Hyperplane $\{x | a^T x = b\}$
separates C and D

$a^T x < b$ for all $x \in C$ and $a^T x > b$ for all $x \in D$ \longrightarrow strictly separates C and D

Theorem (point-set separation): Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Let $x \in \mathbb{R}^n \setminus S$. There exists an $a \in \mathbb{R}^n$ such that $\max_{z \in S} a^T z < a^T x$.

Supporting hyperplane theorem

Let $x_0 \in \text{bd}(S)$. If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all x in S , then the following hyperplane is a *supporting hyperplane* to S at the point x_0 :

boundary

$$\{x | a^T x = a^T x_0\}$$

Theorem: If S is non-empty convex set, there exists a *supporting hyperplane* at **every** boundary of S .

Interior point: A point $x \in S$ is an **interior point** of set S , if there exists an $\varepsilon > 0$ such that

$$\{y | \|x - y\|_2 \leq \varepsilon\} \subseteq S.$$

A ball centered at x that lies **entirely** in S .

Interior of set S $\text{int}(S)$: the set of all interior points.

Closure of set S : $\text{cl}(S) \triangleq \mathbb{R}^n \setminus \text{int}(\mathbb{R}^n \setminus S)$,
i.e., set S + its boundary

Boundary of set S : $\text{bd}(S) \triangleq \text{cl}(S) \setminus \text{int}(S)$

