# Optimization Theory and Algorithms

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# Outline

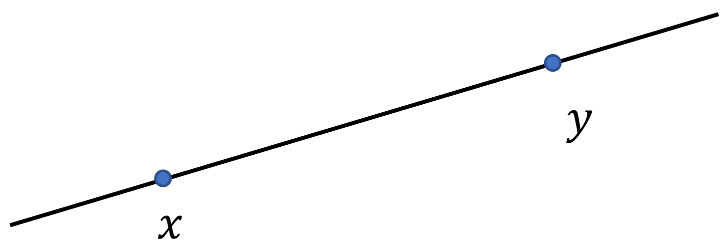
- Affine set
- Convex set
- Convexity-preserving operations
- Separating and supporting hyperplane theorem

# Affine set

Definition: a set $S \subseteq \mathbb{R}^n$ is an affine set, if

for any $x, y \in S$, $\boxed{\theta x + (1-\theta) y} \in S$, for all $\theta \in \mathbb{R}$.

A line through $x, y$, if $x \neq y$.




Affine set contains the line through any two distinct points in the set.

Example: $\{x \mid Ax = b\}$, i.e., solution set of linear equations.

Generalization to more than two points:

Affine combination of $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$:

$$\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k, \text{ where } \sum_{i=1}^{k} \theta_i = 1.$$

Affine set contains the every affine combination of its points in the set.

# Affine set: Interpretation

A set $S \subseteq \mathbb{R}^n$ is an affine set $\Leftrightarrow$ $S$ is the translation of some linear subspace $V \subseteq \mathbb{R}^n$, i.e., $S$ is of the form $\{x\} + V = \{x + v : v \in V\}$ for some $x \in \mathbb{R}^n$.

for any $v_1, v_2 \in V, \alpha, \beta \in \mathbb{R}, \alpha v_1 + \beta v_2 \in V$

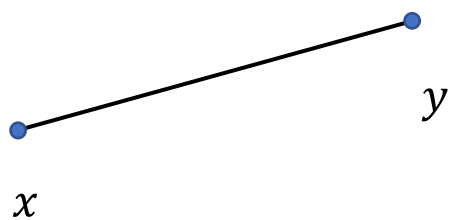
Affine set = subspace + offset

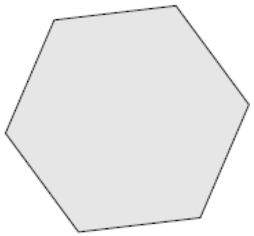
# Convex set

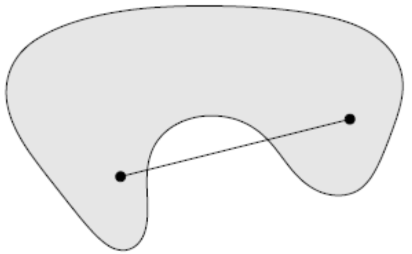
Definition: a set $S \subseteq \mathbb{R}^n$ is a convex set, if

for any $x, y \in S$, $\boxed{\theta x + (1-\theta) y} \in S$, for $\theta \in [0,1]$.

Line segment between $x, y$, if $x \neq y$.





Convex set contains the line segment between any two distinct points in the set.

Generalization to more than two points:

Convex combination of $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$:

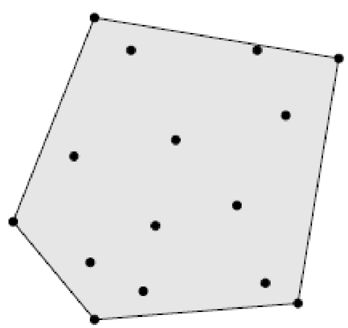
$\sum_{i=1}^{k} \theta_i x_i$, where $\sum_{i=1}^{k} \theta_i = 1$, and $\theta_i \geq 0$ for all $i$.

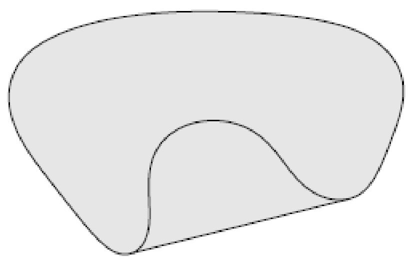
Convex set contains the every convex combination of its points in the set.

# Convex hull

The convex hull of a set $S$ is the set of all convex combinations of points in $S$:

$$conv(S) = \left\{ \sum_{i=1}^{k} \theta_i x_i \,|\, x_i \in S, \sum_{i=1}^{k} \theta_i = 1, \text{ and } \theta_i \geq 0 \text{ for all } i \right\}$$

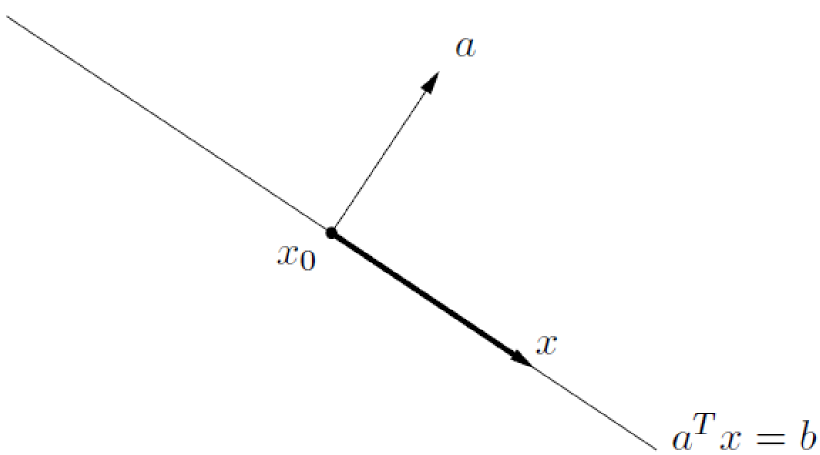


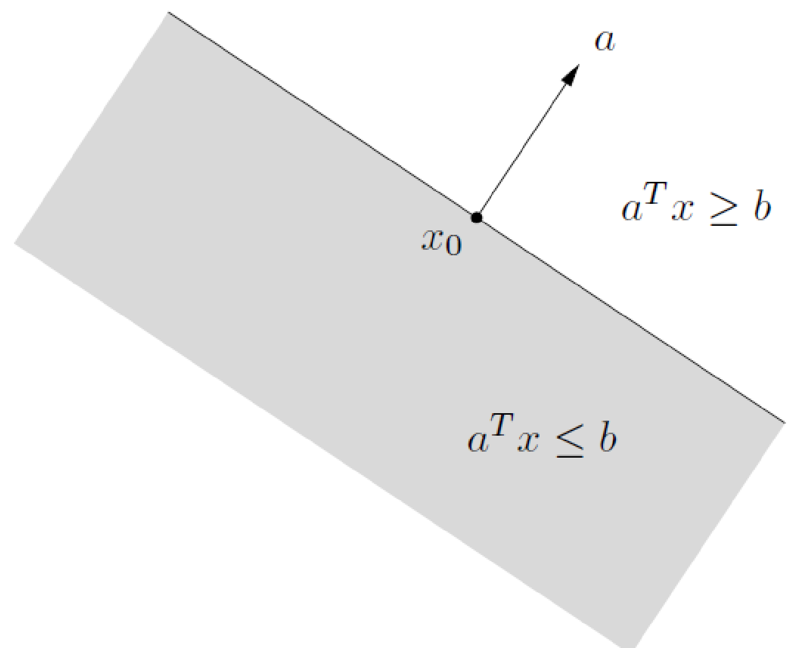


The convex hull of a set $S$ is the smallest convex set that contains $S$.

# Convex set: examples

- Simple examples: empty set $\emptyset$; any single point; line; the whole space $\mathbb{R}^n$.
- Hyperplane $\{x \in \mathbb{R}^n | a^T x = b\}$ ($a \in \mathbb{R}^n, a \neq \mathbf{0}, b \in \mathbb{R}$).




- Halfspace $\{x \in \mathbb{R}^n | a^T x \leq b\}$ ($a \in \mathbb{R}^n, a \neq \mathbf{0}, b \in \mathbb{R}$).

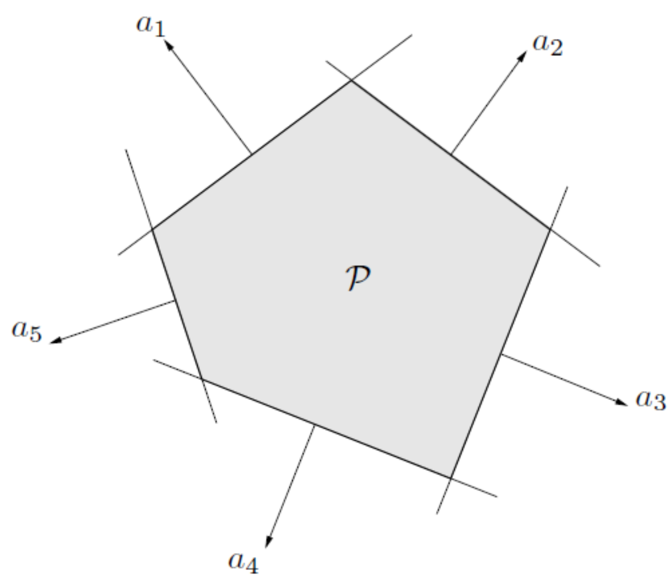
# Convex set: examples

- Polyhedron: solution set of finite linear equalities and inequalities

$$\{x | a_i^T x \leq b_i, i = 1, \dots, m, c_j^T x = d_i, j = 1, \dots, p\}$$

Matrix form

$$\{x | Ax \preccurlyeq b, Cx = d\}$$




Polyhedron is intersection of finite number of halfspaces and hyperplanes.

# Convex set: examples

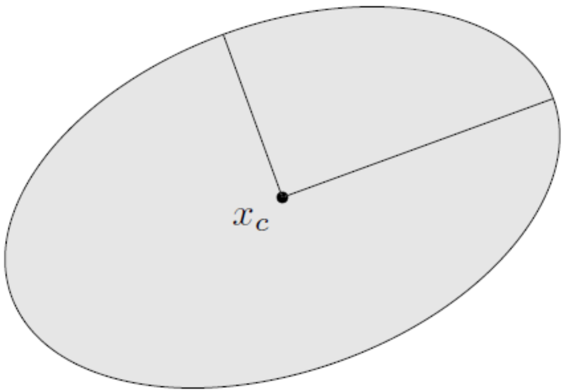
- Euclidean ball

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

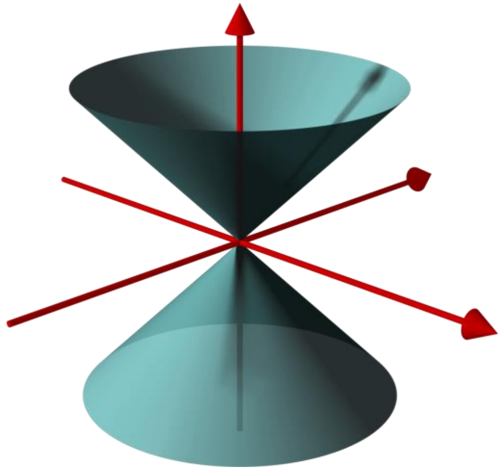
- Ellipsoids

$$E(x_c, Q) = \{x \mid (x - x_c)^T Q (x - x_c) \le 1\}$$

$x_c$ is the center of the ellipsoid; $Q \succ 0$, i.e., positive definite matrix.



# Cone

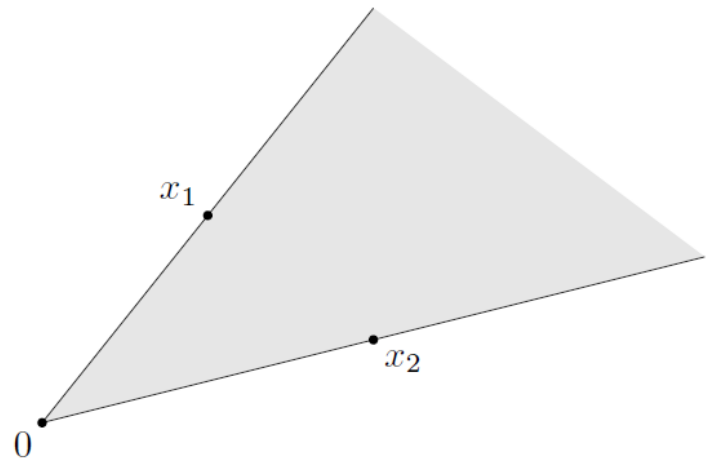


A set $S \subseteq \mathbb{R}^n$ is a cone, if

for any $x \in S$, $\theta x \in S$, for $\theta \geq 0$.

A set $S \subseteq \mathbb{R}^n$ is a convex cone, if

for any $x_1, x_2 \in S$, $\theta_1 x_1 + \theta_2 x_2 \in S$, for $\theta_1, \theta_2 \geq 0$.




Generalization to more than two points:

Conic combination of $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$:

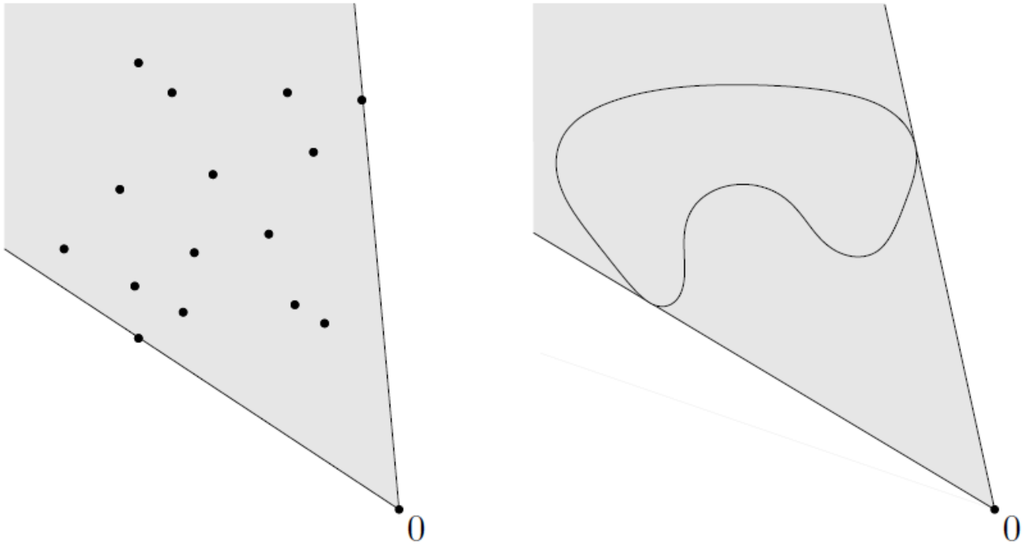
$\sum_{i=1}^{k} \theta_i x_i$, where $\theta_i \geq 0$ for all $i$.

Convex cone contains the every conic of its points in the set.

# Conic hull

The conic hull of a set $S$ is the set of all conic combinations of points in $S$:

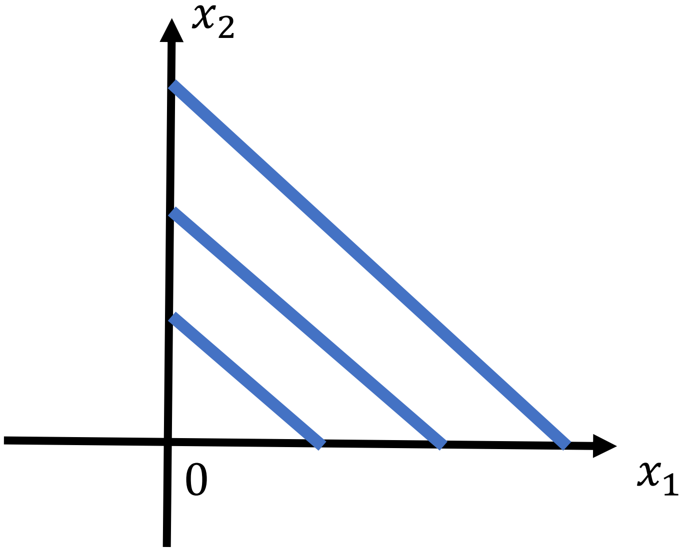
$$conv(S) = \left\{ \sum_{i=1}^{k} \theta_i x_i \,|\, x_i \in S, \text{ and } \theta_i \geq 0 \text{ for all } i \right\}$$




The conic hull of a set $S$ is the smallest convex cone that contains $S$.

# Convex cone: example

- Non-negative orthant: $\mathbb{R}_+^n \triangleq \{(x_1, x_2, \dots, x_n) | x_i \geq 0, i = 1, \dots, n\}$



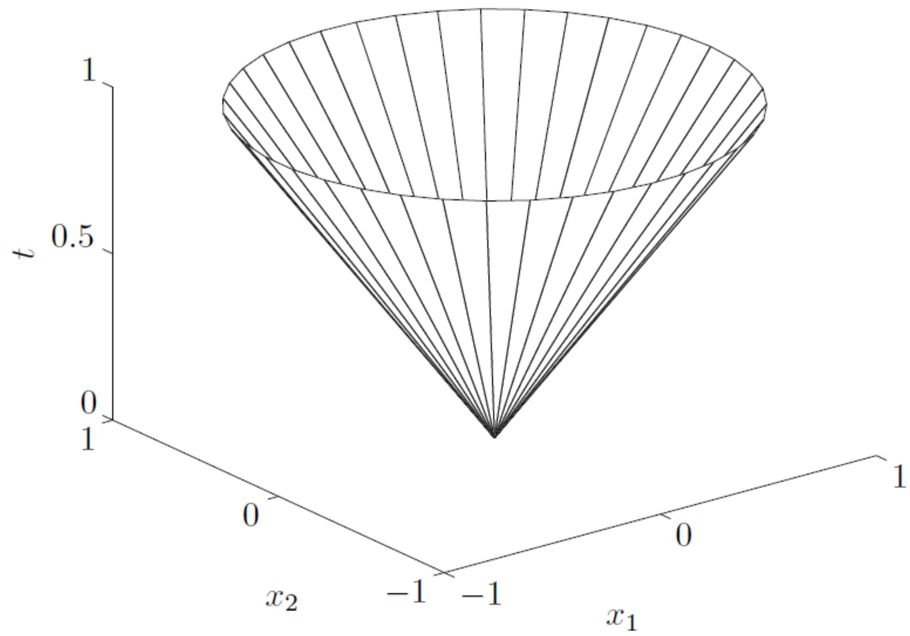

$\mathbb{R}_+^2$

# Convex cone: example

- Norm cone: $\{(x,t) \mid \|x\| \le t\} \subseteq \mathbb{R}^{n+1}$, for a norm $\|\cdot\|$.

  For a l2-norm (Euclidean norm) $\|\cdot\|_2$:

  $\{(x,t) \mid \|x\|_2 \le t\}$ is *second-order cone, also called ice cream cone.*




Boundary of second-order cone in $\mathbb{R}^3$, $\{(x_1, x_2, t) \mid (x_1^2 + x_2^2)^{1/2} \le t\}$.

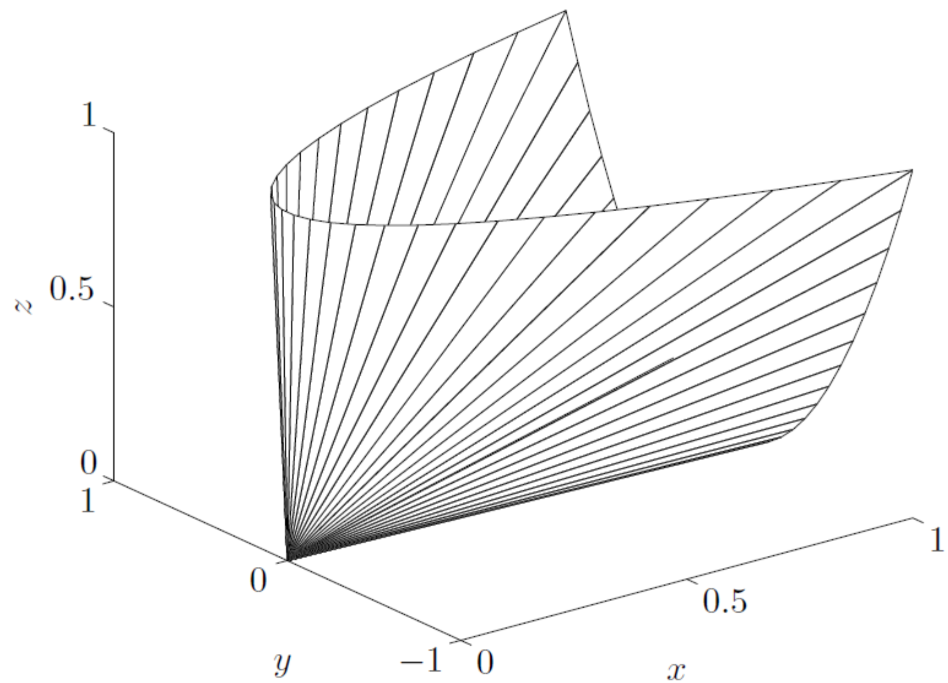
# Convex cone: positive semidefinite cone

- Positive semidefinite matrix $\mathbf{S}_+^n \triangleq \{X \in \mathbf{S}^n | X \succcurlyeq \mathbf{0}\}$.

$$z^T X z \geq 0 \text{ for all } z \in \mathbb{R}^n$$

- Example: positive semidefinite cone in $\mathbf{S}_+^2$

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^n \Leftrightarrow x \geq 0, z \geq 0, xz \geq y^2.$$

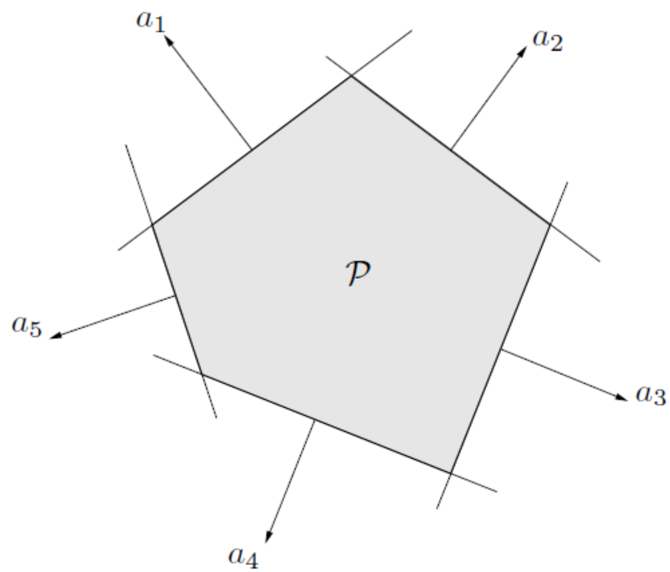



Boundary in $\mathbb{R}^3$

# Operations that preserve convexity

- Intersection: the intersection of convex sets is convex.

$$\{x | a_i^T x \le b_i, i = 1, \dots, m, c_j^T x = d_i, j = 1, \dots, p\}$$




Polyhedron is intersection of finite number of halfspaces and hyperplanes.

# Operations that preserve convexity

- Affine mapping: $f(x) = Ax + b$ where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, i.e., $f\colon \mathbb{R}^n \to \mathbb{R}^m$

$S \subseteq \mathbb{R}^n$ is convex $\Longrightarrow f(S) = \{f(x) \mid x \in S\}$ is convex

$T \subseteq \mathbb{R}^n$ is convex $\Longrightarrow f^{-1}(T) = \{x \mid f(x) \in T\}$ is convex

- Examples:

➢ Scaling ($\{\alpha x \mid x \in S\}$), translation ($\{x + x_0 \mid x \in S\}$), projection ($\{x_1 \mid [x_1, x_2]^T \in S\}$)

➢ Convexity of ellipsoid: $E(x_c, Q) = \{x \mid (x - x_c)^T Q (x - x_c) \leq 1\}$

Eclidean ball $B(0, r) = \{x \mid x^T x \leq r^2\}$ is convex

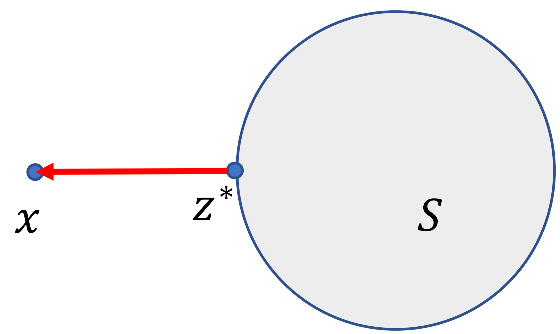
Let $f(x) = rQ^{-1/2}x + x_c$

$f\big(B(0,r)\big) = \{f(x) \mid x^T x \leq r^2\} = \{x \mid (x - x_c)^T Q (x - x_c) \leq 1\}$ is convex

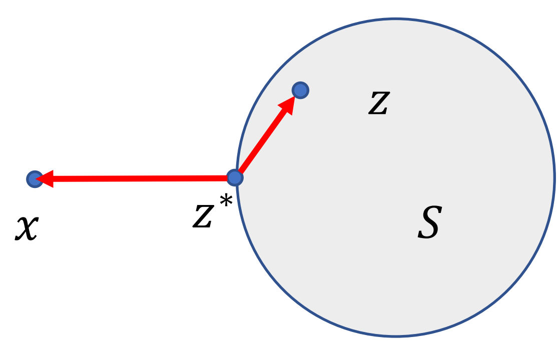
# Projection onto closed convex sets

Theorem: Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Then, for every $x \in \mathbb{R}^n$, there exists a unique point $z^* \in S$ that is closest to (in Euclidean norm) $x$.

Projection of $x$ on $S : \Pi_S(x) = arg \min_{z \in S} \|z - x\|_2$



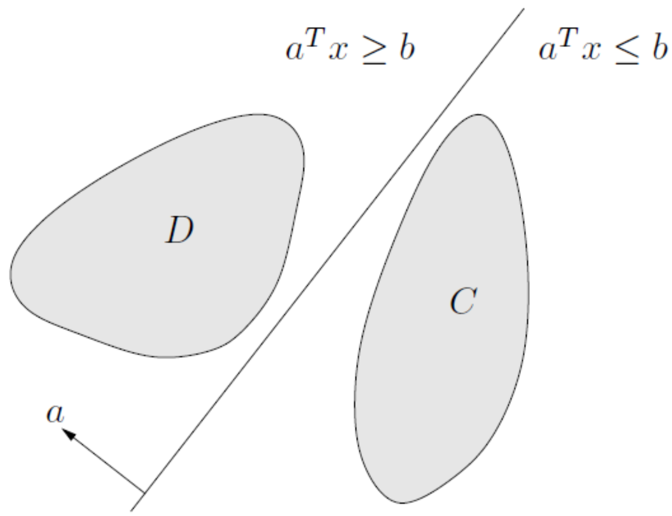

Theorem: Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Given any $x \in \mathbb{R}^n$, we have $z^* = \Pi_S(x)$ iff $z^* \in S$ and $(z - z^*)^T(x - z^*) \leq 0$ for all $z \in S$.



# Separating hyperplane theorem

Theorem: If $C$ and $D$ are non-empty, disjoint (i.e., $C \cap D = \emptyset$) convex set, there exists $a \neq 0$ and $b$ such that:

$$a^T x \leq b \text{ for all } x \in C \text{ and } a^T x \geq b \text{ for all } x \in D.$$




Hyperplane $\{x | a^T x = b\}$ separates $C$ and $D$

$a^T x < b$ for all $x \in C$ and $a^T x > b$ for all $x \in D$ $\longrightarrow$ strictly separates $C$ and $D$

Theorem (point-set separation): Let $S \subseteq \mathbb{R}^n$ be non-empty, closed and convex. Let $x \in \mathbb{R}^n \backslash S$. There exists an $a \in \mathbb{R}^n$ such that $\max_{z \in S} a^T z < a^T x$.

# Supporting hyperplane theorem

Let $x_0 \in$ **bd** $(S)$. If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x$ in $S$, then the following hyperplane is a *supporting hyperplane* to $S$ at the point $x_0$:

boundary

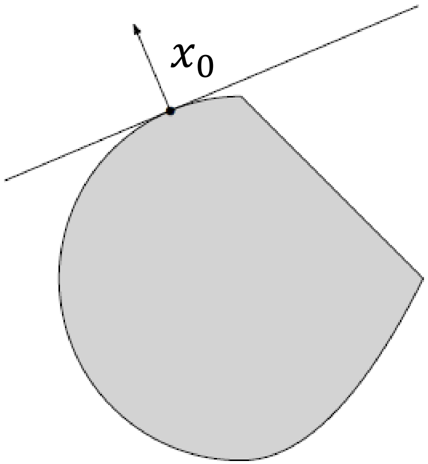
$$\{x | a^T x = a^T x_0\}$$

Theorem: If $S$ is non-empty convex set, there exists a *supporting hyperplane* at every boundary of $S$.

Interior point: A point $x \in S$ is an interior point of set $S$, if there exists an $\varepsilon > 0$ such that

$$\{y |\ \|x - y\|_2 \leq \varepsilon\} \subseteq S.$$

A ball centered at $x$ that lies entirely in $S$.




Interior of set $S$ **int**$(S)$ : the set of all interior points.

Closure of set $S$: **cl** $(S) \triangleq \mathbb{R}^n \setminus$ $\textbf{int}\ (\mathbb{R}^n \setminus S)$,

i.e., set $S$ + its boundary

Boundary of set $S$: **bd** $(S) \triangleq$ **cl** $(S) \setminus \textbf{int}\ (S)$