Optimization Theory and Algorithms

Instructor: Prof. LIAO, Guocheng （廖国成）

Email: liaogch6@mail.sysu.edu.cn

School of Software Engineering
Sun Yat-sen University

Outline

- Inequality constrained minimization
- Logarithmic barrier function and central path
- Barrier method

Equality constrained minimization problem

$$\begin{aligned}
\text{min} \ \ & f_0(x) \\
\text{s.t.} \ \ & f_i(x) \leq 0, i = 1, \dots, m \\
& Ax = b
\end{aligned}$$

- f is convex and twice continuously differentiable
- Assume optimal point $x^\star$ exists. Let $p^\star = f(x^\star)$ be the optimal value.
- Assume Slater's condition holds, i.e., strong duality holds.

Optimality condition (KKT conditions): $x^\star$ is optimal iff there exists a $\lambda^\star$ and $\nu^\star$ such that

- $\lambda_i^\star f_i(x^\star) = 0, i = 1, \dots, m$
- $\nabla f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star \nabla f_i(x^\star) + A^T \nu^\star = 0$
- $\lambda^\star \geq 0$
- $f_i(x^\star) \leq 0, i = 1, \dots, m, Ax^\star = b$

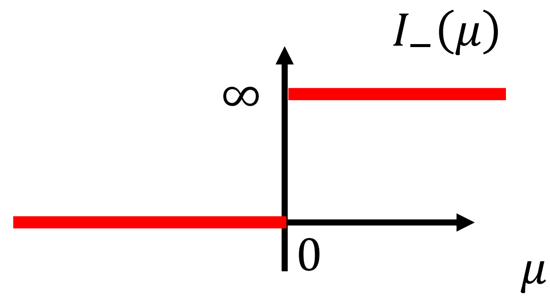
Logarithmic barrier

not differentiable

$$\begin{aligned} \min\ & f_0(x) \\ \text{s.t.}\ & f_i(x) \le 0, i = 1, \dots, m \\ & Ax = b \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min\ & f_0(x) + \textstyle\sum_{i=1}^{m} I_-(f_i(x)) \\ \text{s.t.}\ & Ax = b \end{aligned}$$

I_- is the indicator function for the nonpositive reals

$$I_-(\mu) = \begin{cases} 0, & \text{if } \mu \le 0 \\ \infty, & \text{if } \mu > 0 \end{cases}$$

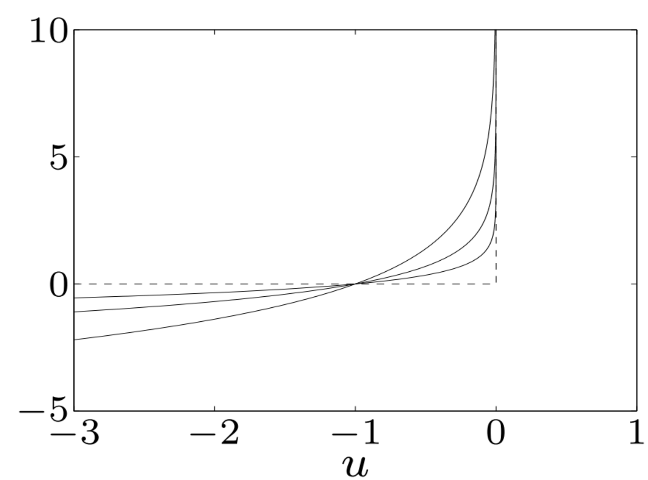


approximation

$$\begin{aligned} \min\ & f_0(x) + \textstyle\sum_{i=1}^{m} I_-(f_i(x)) \\ \text{s.t.}\ & Ax = b \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min\ & f_0(x) - (1/t) \textstyle\sum_{i=1}^{m} \log(-f_i(x)) \\ \text{s.t.}\ & Ax = b \end{aligned}$$

$$\widehat{I_-}(\mu) = -(1/t) \sum_{i=1}^{m} \log(-\mu)$$

- Convex
- Differentiable
- As t increases, the approximation is more accurate



Central path

$$\begin{aligned} \text{min} \;\; & f_0(x) - (1/t)\textstyle\sum_{i=1}^m \log(-f_i(x)) \\ \text{s.t.} \;\; & Ax = b \end{aligned}$$

- $\phi(x) = -\sum_{i=1}^m \log(-f_i(x))$: logarithmic barrier function

Multiply the objective with t

$$\begin{aligned} \text{min} \;\; & tf_0(x) + \phi(x) \\ \text{s.t.} \;\; & Ax = b \end{aligned}$$

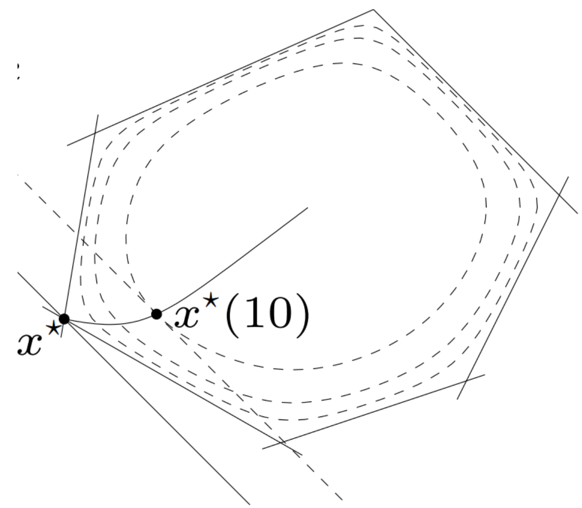
- For $t > 0$, $x^\star(t)$ is the solution of the above problem
- Central path: $x^\star(t)$, $t > 0$:

$$Ax^\star(t) = b$$

$$f_i(x^\star(t)) < 0$$

$$\boxed{t\nabla f_0(x^\star(t)) + \nabla\phi(x^\star(t)) + A^T\nu'} = 0$$

$$t\nabla f_0(x^\star(t)) + \sum_{i=1}^m \frac{1}{-f_i\big(x^\star(t)\big)} \nabla f_i(x^\star(t)) + A^T\nu' = 0$$



Approximation gap

$$\begin{aligned}\min\ & tf_0(x) + \phi(x)\\ \text{s.t.}\ & Ax = b\end{aligned}$$

$$\begin{aligned}p^\star = \min\ & f_0(x)\\ \text{s.t.}\ & f_i(x) \le 0, i = 1, \dots, m\\ & Ax = b\end{aligned}$$

Lagrangian

- $L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \nu^T(Ax - b)$

- $Ax^\star(t) = b$
- $f_i(x^\star(t)) < 0$
- $t\nabla f_0(x^\star(t)) + \sum_{i=1}^m \frac{1}{-f_i(x^\star(t))} \nabla f_i(x^\star(t)) + A^T\nu' = 0$

Lower bound of the optimal value $p^\star$: $f_0(x^\star(t)) \le p^\star + m/t$

convergence as $t \to \infty$

- $\nabla f_0(x^\star(t)) + \sum_{i=1}^m \frac{1}{-tf_i(x^\star(t))} \nabla f_i(x^\star(t)) + A^T\nu'/t = 0$
- Define $\lambda_i^\star(t) = -1/tf_i\big(x^\star(t)\big)$, and $\nu_i^\star(t) = \nu'/t$
- $x^\star(t)$ minimizes Lagrangian $L(x, \lambda^\star(t), \nu^\star(t)) = f_0(x) + \sum_{i=1}^m \lambda_i^\star(t) f_i(x^\star) + \nu_i^\star(t)^T(Ax - b)$
- $g(\lambda^\star(t), \nu^\star(t)) = f_0(x^\star(t)) + \sum_{i=1}^m \lambda_i^\star(t) f_i(x^\star) + \nu_i^\star(t)^T(Ax - b) = f_0\big(x^\star(t)\big) - m/t$

$$f_0\big(x^\star(t)\big) - m/t = g(\lambda^\star(t), \nu^\star(t)) \le p^\star$$

Interpretation via modified KKT conditions

$x^\star(t), \lambda^\star(t), \nu^\star(t)$ satisfy

- Approximate complementary slackness: $\lambda_i^\star(t) f_i\big(x^\star(t)\big) = -1/t$, $i = 1, \ldots, m$
- Lagrangian optimality: $\nabla f_0(x^\star(t)) + \sum_{i=1}^m \lambda_i^\star(t) \nabla f_i(x^\star(t)) + A^T \nu^\star(t) = 0$
- Dual feasibility: $\lambda^\star(t) \geq 0$
- Primal feasibility: $f_i(x^\star(t)) \leq 0, i = 1, \ldots, m, Ax^\star(t) = b$

Barrier method

- Given strictly feasible x, $t > 0$,$u > 1$, tolerance $\epsilon > 0$
- **Repeat**

1. *Centering step.*
   Starting at x, compute $x^\star(t)$ by solving the following problem (Newton's method)

$$\begin{aligned} &\text{min} \;\; tf_0(x) + \phi(x) \\ &\text{s.t.} \;\; Ax = b \end{aligned}$$

2. Update: $x \leftarrow x^\star(t)$

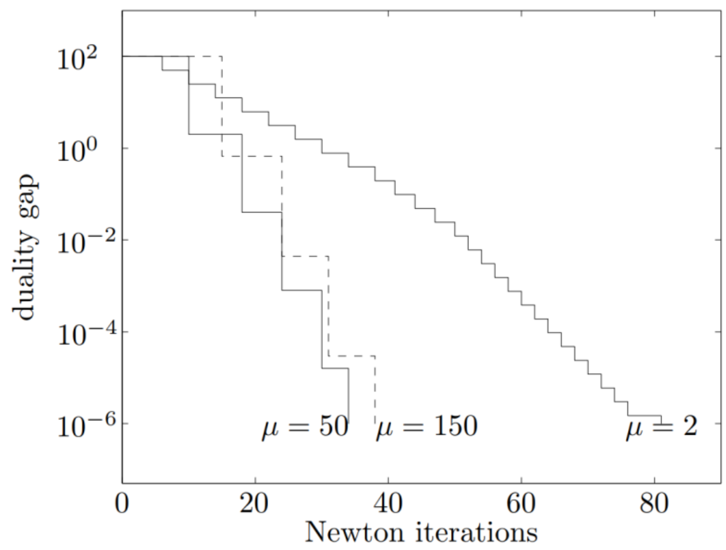
3. Stopping criterion: if $m/t \leq \varepsilon$, break

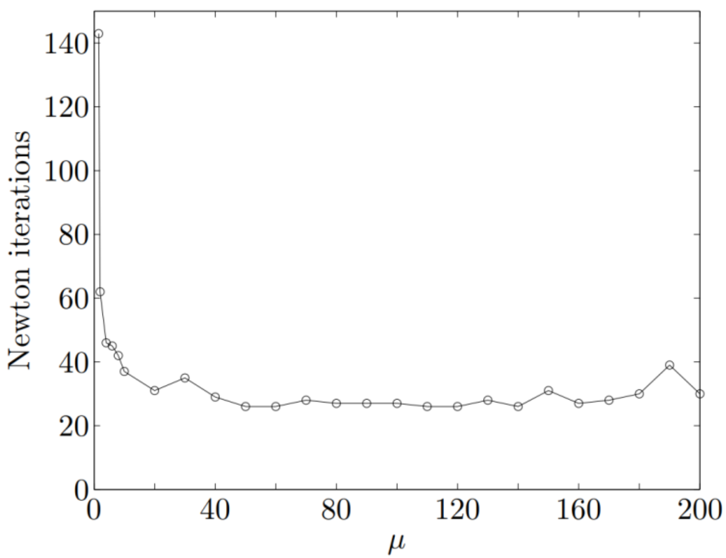
4. *Increase* t. $t \leftarrow ut$

- Centering usually use Newton's method, starting at current x.
- Choice of u: large u means fewer outer iterations, more inner Newton iterations.

Example

Inequality form LP ($m = 100$ inequalities, $n = 50$ variables)





- Staircase shape:
  horizontal portion is the number of Newton steps of that inner iterations; vertical portion is u
- Total number of Newton iterations is nor very sensitive for $u \geq 10$.

Newton step for modified KKT equations

$$\begin{aligned} &\text{min} \;\; t f_0(x) + \phi(x) \\ &\text{s.t.} \;\; Ax = b \end{aligned}$$

Compute Newton step by solving linear equations:

$$\begin{bmatrix} t\nabla^2 f(x) + \nabla^2 \phi(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -t\nabla f_0(x) - \nabla \phi(x) \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

Modified KKT equations

- $\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T \nu = 0$
- $Ax = b$
- $\lambda_i f_i(x) = -1/t$, $i = 1, \ldots, m$

Newton step for modified KKT equations

Newton's method with equality constraints

$$\min \ f(x)$$
$$\text{s.t.} \ \ Ax = b$$

Optimality conditions (KKT conditions):

$$\nabla f(x^\star) + A^T \nu^\star = 0, \qquad Ax^\star = b$$

$x^\star = x + \Delta x_{nt}$

Taylor approximation

$$\nabla f(x) + \nabla^2 f(x)\Delta x_{nt} + A^T \nu^\star = 0,$$
$$A(x + \Delta x_{nt}) = b$$

$$\nabla f(x) + \nabla^2 f(x)\Delta x_{nt} + A^T \nu^\star = 0,$$
$$A\Delta x_{nt} = 0$$

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Newton's method with inequality constraints

$$\min \ f_0(x)$$
$$\text{s.t.} \ \ f_i(x) \le 0, i = 1, \dots, m$$
$$Ax = b$$

Appro. optimality conditions (modified KKT equations)

- $\nabla f_0(x^\star) + \sum_{i=1}^{m} \frac{1}{-t f_i(x^\star)} \nabla f_i(x^\star) + A^T \nu^\star = 0$
- $Ax^\star = b$

$x^\star = x + \Delta x_{nt}$

Taylor approximation

- $(\nabla^2 f(x) + \nabla^2 \phi(x))\Delta x_{nt} + A^T \nu^\star = - \nabla f_0(x) - 1/t \nabla \phi(x),$
- $A\Delta x_{nt} = 0$

$$\begin{bmatrix} \nabla^2 f(x) + (1/t)\nabla^2 \phi(x) & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f_0(x) - (1/t)\nabla \phi(x) \\ 0 \end{bmatrix}$$

Interpretation of Newton's method

Non-linear equations: $F(x^\star) = 0$

$x^\star = x + \Delta x_{nt}$

Taylor approximation

Linear equations of Δx_{nt}:

$$F(x^\star) \approx F(x) + DF(x)\Delta x_{nt} = 0$$

$$\Delta x_{nt} = -(DF(x))^{-1}F(x)$$