# Predictive Model: Generative Classifiers

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# Revision

Given $Y$ = output $j \in \{1, ...K\}$, $X$ = inputs x.
A **discriminative** Learning Model:

$$\mathbb{P}(Y = j|X = x) = \text{some estimate}$$

- kNN, wkNN: nonparametric
  $h_D(\text{x}) = \text{argmax}_{j \in \{1,...,K\}} \mathbb{P}(Y = j|X = \text{X})$
  - numeric input
  - distance, scaling, curse of dimensionality
- LR, multinomial LR, ANN: parametric
  $h_D(\text{x}) = I(\mathbb{P}(Y = 1|X = \text{x}) > 0.5)$ (cut-off=0.5)
  - may not convergence
  - complex maths theory, e.g. ANOVA

# Outline

# Discriminative & Generative Learning Models

- *Discriminative learning*: It assumes the conditional probability $\mathbb{P}(Y|X)$ is known and tries to estimate from it.
- *Generative learning*: It assumes the data is generated by the overall probability $\mathbb{P}(X, Y)$ and tries to estimate the output based on the prior probability of the output and the likelihood function: $P(X, Y) = P(X|Y)P(Y)$.

  Examples of generative supervised learning models:
  - Naive Bayes
  - Discriminative Analysis, e.g. LDA, QDA

# Discriminative & Generative Models (cont)

The probabilistic framework that underlie the discriminative models is the **Maximum Likelihood Estimation (MLE)** which assumes $\theta$ are the constant parameters of a statistical model

$$\hat{\theta} = \underset{\theta}{\text{argmax}}\, L((\mathrm{x}_1, y_1), \cdots, (\mathrm{x}_n, y_n); \theta).$$

The probabilistic framework that underlie the generative models is the **Maximum a Posteriori (MAP)** which assumes $\theta$ are random variables being updated based on observed data:

$$\begin{aligned}\hat{\theta} &= \text{argmax}_\theta\, P(\theta|D) \\ &= \underset{\theta}{\text{argmax}}\, P((\mathrm{x}_y, y_1), \cdots, (\mathrm{x}_n, y_n) \mid \theta) P(\theta).\end{aligned}$$

# Generative Models

According to the Bayes Theorem

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)},$$

we can 'symbolically' (not mathematically because $X$ is usually deterministic) link the discriminative model and the generative model by

$$\mathbb{P}(Y = y|\mathrm{X} = \mathrm{x}) = \frac{\mathbb{P}(\mathrm{X} = \mathrm{x}|Y = y)\mathbb{P}(Y = y)}{\mathbb{P}(\mathrm{X} = \mathrm{x})}. \quad (1)$$

Note that '$\mathbb{P}$' is regarded as "probability density" function $f$ when x and $y$ are continuous.

# Generative Models (cont)

When the response variable $Y$ is categorical and has $K$ distinct values 1, ..., $K$, then (1) becomes

$$\mathbb{P}(Y = j|\mathrm{X} = \mathrm{x}) = \frac{\mathbb{P}(\mathrm{X} = \mathrm{x}|Y = j)\mathbb{P}(Y = j)}{\sum_{k=1}^{K} \mathbb{P}(\mathrm{X} = \mathrm{x}|Y = k)\mathbb{P}(Y = k)}, \tag{2}$$

where $j \in \{1, \cdots, K\}$;

- $\mathbb{P}(Y = j)$ is called the **prior probability**, i.e. the probability that a randomly chosen observation comes from the $j$th class of response variable $Y$;

# Generative Models (cont)

- $\mathbb{P}(Y = k|\mathrm{X} = \mathrm{x})$, called the **posterior probability**, is the probability that the new observation belongs to the $k$th class, given the predictor value for that observation;
- $\mathbb{P}(\mathrm{X} = \mathrm{x}) = \sum_{k=1}^{K} \mathbb{P}(\mathrm{X} = \mathrm{x}|Y = k)\mathbb{P}(Y = k)$ is regarded as a constant w.r.t the response class $j$;
- $\mathbb{P}(\mathrm{X} = \mathrm{x}|Y = j)$ is called the **likelihood function**, a density function of X for an observation that comes from the $j$th class. It is relatively large if there is a high probability that an observation in the $j$th class has $\mathrm{X} \approx \mathrm{x}$.

# Generative Models (cont)

The generative classifier derived from (2) is

$$
\begin{aligned}
h_D(\mathrm{x}) &= \underset{j \in \{1,\ldots,K\}}{\operatorname{argmax}} \mathbb{P}(Y = j | \mathrm{X} = \mathrm{x}) \\
&= \underset{j \in \{1,\ldots,K\}}{\operatorname{argmax}} \frac{\mathbb{P}(\mathrm{X} = \mathrm{x} | Y = j)\mathbb{P}(Y = j)}{\mathbb{P}(\mathrm{X} = \mathrm{x})} \\
&= \underset{j \in \{1,\ldots,K\}}{\operatorname{argmax}} \mathbb{P}(\mathrm{X} = \mathrm{x} | Y = j)\mathbb{P}(Y = j) \\
&= \underset{j \in \{1,\ldots,K\}}{\operatorname{argmax}} [\ln \mathbb{P}(\mathrm{X} = \mathrm{x} | Y = j) + \ln \mathbb{P}(Y = j)]
\end{aligned}
\tag{3}
$$

```
nbD  = naive_bayes(Y ~ ., D, options)
Yhat = predict(nbD,  newX, type="class")
Prob = predict(nbD,  newX, type="prob")  # Get log-posterior prob
Yhat = predict(ldaD, newX)$class
Prob = predict(ldaD, newX)$posterior     # log?
```

# Outline

# Naïve Bayes Classifiers

A simple generative classifier (2) with **strong (naive) independence assumptions** on the likelihood function:

$$\begin{aligned}
&\mathbb{P}(\mathbf{X} = \mathbf{x} | Y = j) \\
&= \mathbb{P}(X_1 = x_1, X_2 = x_2, ..., X_p = x_p | Y = j) \\
&= \mathbb{P}(X_1 = x_1 | Y = j) \times \cdots \times \mathbb{P}(X_p = x_p | Y = j) \\
&= \prod_{i=1}^{p} \mathbb{P}(X_i = x_i | Y = j).
\end{aligned}$$

leads to the **naïve Bayes classifier (NB)** (`https://en.wikipedia.org/wiki/Naive_Bayes_classifier`).

# Naïve Bayes Classifiers (cont)

According to (3), the **naïve Bayes classifier** can have two forms: one form is used in this course:

$$h_D(\mathrm{x}) = \underset{j\in\{1,\ldots,K\}}{\operatorname{argmax}} \mathbb{P}(Y=j)\prod_{i=1}^{p}\mathbb{P}(X_i=x_i|Y=j) \quad (4)$$

and one form is used in computer implementation:

$$h_D(\mathrm{x}) = \underset{j\in\{1,\ldots,K\}}{\operatorname{argmax}} \ln\mathbb{P}(Y=j)+\left[\sum_{i=1}^{p}\ln\mathbb{P}(X_i=x_i|Y=j)\right]. \quad (5)$$

# Naïve Bayes Classifiers (cont)

The estimate of the prior distribution $\mathbb{P}(Y = j)$ using MLE is as follows:

$$\widehat{\mathbb{P}(Y = j)} = \frac{\#\{i \ : \ y_i = j\}}{n}. \tag{6}$$

However, it is possible to choose

$$\mathbb{P}(Y = j) = \frac{1}{K}$$

**if** we know the outcome should be uniformly distributed.

# Naïve Bayes Classifiers (cont)

The features $X_i$ can be categorical or numeric:

- One $X_i$ is categorical — Categorical NB
- One $X_i$ is numeric — Gaussian NB, Non-parametric NB (using kernel density approximation rather than normal distribution)
- All $X_i$ are binary — Bernoulli NB
- All $X_i$ are integral — Multinomial NB & Complement NB(?)
- One $X_i$ is non-negative integral — Poisson (distribution) NB

# Categorical NB

The feature $X_i$

- is categorical;
- takes on $M_i$ possible values: $\{c_1^{(i)}, \cdots, c_{M_i}^{(i)}\} =: \mathscr{C}_i$

The conditional distribution is assumed to be

$$\mathbb{P}(X_i = c | Y = j) = \frac{n_{X_i=c \ \& \ Y=j}}{n_{Y=j}}, \quad c \in \mathscr{C}_i. \tag{7}$$

# Categorical NB (cont)

**Example** 1: (Final Exam May 2019, Q3(d))

Table Q3(d) shows a data set containing 7 observations with 3 categorical predictors, $X_1$, $X_2$ and $X_3$.

| Observation | $X_1$ | $X_2$ | $X_3$ | $Y$ |
|---|---|---|---|---|
| 1 | C | No | 0 | Positive |
| 2 | A | Yes | 1 | Positive |
| 3 | B | Yes | 0 | Negative |
| 4 | B | Yes | 0 | Negative |
| 5 | A | No | 1 | Positive |
| 6 | C | No | 1 | Negative |
| 7 | B | Yes | 1 | Positive |

Table Q3(d)

Without Laplace smoothing, predict the response, $Y$ for an observation with $X_1 = \text{B}$, $X_2 = \text{Yes}$ and $X_3 = 1$ using Naïve Bayes approach. (5 marks)

# Categorical NB (cont)

**Solution**: Let '+' denote 'Positive' and '-' denote 'Negative'.

| prior, $\mathbb{P}(Y)$ | $\mathbb{P}(X_1\|Y)$ | $\mathbb{P}(X_2\|Y)$ | $\mathbb{P}(X_3\|Y)$ | prop, $\Pi$ | $\hat{Y}$ |
|---|---|---|---|---|---|
| $\mathbb{P}(+) = \frac{4}{7}$ | $\mathbb{P}(\text{B}\|+) = \frac{1}{4}$ | $\mathbb{P}(\text{Yes}\|+) = \frac{1}{2}$ | $\mathbb{P}(1\|+) = \frac{3}{4}$ | 0.0536 | |
| $\mathbb{P}(-) = \frac{3}{7}$ | $\mathbb{P}(\text{B}\|-) = \frac{2}{3}$ | $\mathbb{P}(\text{Yes}\|-) = \frac{2}{3}$ | $\mathbb{P}(1\|-) = \frac{1}{3}$ | 0.0635 | ✓, − |

Since $\mathbb{P}(Y = -|X) > \mathbb{P}(Y = +|X)$, $Y$ has higher probability to be "Negative".

# Categorical NB (cont)

Consider the following case given in
`https://machinelearningmastery.com/naive-bayes-tutorial-for-machine-learning/`

| Weather | Car | Y |
|---|---|---|
| sunny | working | go-out |
| rainy | broken | go-out |
| sunny | working | go-out |
| sunny | working | go-out |
| sunny | working | go-out |
| rainy | broken | stay-home |
| rainy | broken | stay-home |
| sunny | working | stay-home |
| sunny | broken | stay-home |
| rainy | broken | stay-home |

Construct the categorical Naïve Bayes model for the above data.

# Categorical NB (cont)

**Solution**: Let $X_1$=Weather, $X_2$=Car. The categorical Naïve Bayes model:

$$\mathbb{P}(Y = j|\mathrm{X} = \mathrm{x})$$
$$\propto \mathbb{P}(Y = j) \times \mathbb{P}(X_1 = x_1|Y = j) \times \mathbb{P}(X_2 = x_2|Y = j)$$

$$\text{where Prior, } \mathbb{P}(Y) = \begin{cases} 0.5, & Y = out \\ 0.5, & Y = stay \end{cases}$$

$$\mathbb{P}(X_1|Y = out) = \begin{cases} \frac{4}{5}, & X_1 = sunny \\ \frac{1}{5}, & X_1 = rainy \end{cases},$$

$$\mathbb{P}(X_1|Y = stay) = \begin{cases} \frac{2}{5}, & X_1 = sunny \\ \frac{3}{5}, & X_1 = rainy \end{cases}$$

# Categorical NB (cont)

**Solution** (cont):

$$\mathbb{P}(X_2|Y = out) = \begin{cases} \frac{4}{5}, & X_2 = working \\ \frac{1}{5}, & X_2 = broken \end{cases},$$

$$\mathbb{P}(X_2|Y = stay) = \begin{cases} \frac{1}{5}, & X_2 = working \\ \frac{4}{5}, & X_2 = broken \end{cases}$$

# Categorical NB (cont)

Implementation in Python: `CategoricalNB` The probability of category $c$ in feature $X_i$ given class $j$ is estimated with "Laplace smoothing":

$$\mathbb{P}(X_i = c|Y = j;\ \alpha) = \frac{n_{X_i=c\ \&\ Y=j} + \alpha}{n_{Y=j} + \alpha d_i} \tag{8}$$

where $\alpha$ is a **smoothing parameter** and $d_i$ is the number of available categories of feature $X_i$ defined above. It has the following form:

```
from sklearn.naive_bayes import CategoricalNB
CategoricalNB(alpha=1.0, fit_prior=True,
  class_prior=None)
```

# Gaussian Naïve Bayes

For continuous inputs $X_i$ in (4), it is assume that $X_i$ is Gaussian:

$$\mathbb{P}(X_i = x_i | Y = j) = \frac{1}{\sigma_j \sqrt{2\pi}} \exp\left(-\frac{(x_i - \mu_j)^2}{2\sigma_j^2}\right). \quad (9)$$

It is defined by a mean and standard deviation specific to $X_i$ and $j$. The estimations of the mean and standard deviation are

$$\mu_j = \mathbb{E}[X_i | Y = j], \quad \sigma_j^2 = \mathbb{E}[(X_i - \mu_j)^2 | Y = j]. \quad (10)$$

# Gaussian Naïve Bayes (cont)

The maximum likelihood estimator for (10) are

$$\widehat{\mu}_j = \frac{1}{\sum_{i=1}^{n} I(y_i = j)} \sum_{k=1}^{n} X_{ki} I(Y = j),$$

$$s_j = \frac{1}{\left( \sum_{i=1}^{n} I(y_i = j) \right) - 1} \sum_{k=1}^{n} (X_{ki} - \widehat{\mu}_j)^2 I(y_i = j). \tag{11}$$

# Gaussian Naïve Bayes (cont)

**Example** 2:

The table below shows the data collected for predicting whether a customer will default on the credit card or not:

| customer | balance | student | Default |
|---|---|---|---|
| 1 | 500 | No | N |
| 2 | 1980 | Yes | Y |
| 3 | 60 | No | N |
| 4 | 2810 | Yes | Y |
| 5 | 1400 | No | N |
| 6 | 300 | No | N |
| 7 | 2000 | Yes | Y |
| 8 | 940 | No | N |
| 9 | 1630 | No | Y |
| 10 | 2170 | Yes | Y |

# Gaussian Naïve Bayes (cont)

(a) Compute the probability density of customer with balance 2080, given Default=Y.

(b) Compute the probability of customer who is a student, given Default=Y.

(c) Calculate the "probability density" of default for a student customer with balance 2080 by using the Naïve Bayes assumption.

# Gaussian Naïve Bayes (cont)

Note: This question just asks for specific answer without the full model, so we don't need to write the full model.

(a) **Solution**:

$$\mathbb{P}(\texttt{balance} = 2080 \mid \texttt{Default} = Y)$$
$$= \frac{1}{s_Y\sqrt{2\pi}} \exp\left(-\frac{(2080-\mu_Y)^2}{2s_Y^2}\right) = 0.0009162$$

where $\mu_Y = \frac{1980+2810+2000+1630+2170}{5} = 2118$;
$s_Y = 433.7857$

# Gaussian Naïve Bayes (cont)

(b) **Solution**: $\mathbb{P}(\texttt{student} = \textit{Yes} \mid \texttt{Default} = Y) = \frac{4}{5}$

(c) **Solution**:

$\mathbb{P}(\texttt{student} = \textit{Yes} \mid \texttt{Default} = N) = \frac{0}{5} = 0$

$$\begin{aligned}
&\mathbb{P}(\texttt{Default} = Y \mid \texttt{balance} = 2080,\ \texttt{student} = \textit{Yes}) \\
=&\frac{\mathbb{P}(\texttt{balance} = 2080,\ \texttt{student} = \textit{Yes} \mid \texttt{Default} = Y)\mathbb{P}(\texttt{Default} = \texttt{Y})}{\mathbb{P}(\texttt{balance} = 2080,\ \texttt{student} = \textit{Yes}) =: \mathbb{P}(\ldots)} \\
=&\frac{\mathbb{P}(\texttt{balance} = 2080,\ \texttt{student} = \textit{Yes} \mid \texttt{Default} = Y)\mathbb{P}(\texttt{Default} = \texttt{Y})}{\mathbb{P}(\ldots \mid \texttt{Default} = Y)\mathbb{P}(\texttt{Default} = \texttt{Y}) + \mathbb{P}(\ldots \mid \texttt{Default} = N)\mathbb{P}(\texttt{Default} = \texttt{N})} \\
=&\frac{0.0009162 \times \frac{4}{5} \times \frac{5}{10}}{0.0009162 \times \frac{4}{5} \times \frac{5}{10} + \mathbb{P}(\texttt{balance} = 2080 | \texttt{Default} = N) \times 0 \times \frac{5}{10}} = 1
\end{aligned}$$

# Gaussian Naïve Bayes (cont)

Remark on **Example** 2: Let $x_1$=balance, $x_2$=student, $y$=Default. The Naive Bayes Model is

$$h_D(x_1, x_2) = \underset{j}{\text{argmax}}\, \mathbb{P}(x_1|y = j)\mathbb{P}(x_2|y = j)\mathbb{P}(y = j).$$

where the prior $\mathbb{P}(y) = \begin{cases} 0.5 & y = N \\ 0.5 & y = Y \end{cases}$

$$\mathbb{P}(x_2|y = N) = \begin{cases} 1 & x_2 = No \\ 0 & x_2 = Yes \end{cases}$$

$$\mathbb{P}(x_2|y = Y) = \begin{cases} 1/5 & x_2 = No \\ 4/5 & x_2 = Yes \end{cases}$$

# Gaussian Naïve Bayes (cont)

Remark on **Example** 2 (cont):

$$\mathbb{P}(x_1|y = N) = \frac{1}{\sqrt{2\pi}(533.6666)} \exp\left\{-\frac{(x_1 - 640)^2}{2(533.6666)^2}\right\}$$

$$\mathbb{P}(x_1|y = Y) = \frac{1}{\sqrt{2\pi}(433.7857)} \exp\left\{-\frac{(x_1 - 2118)^2}{2(433.7857)^2}\right\}$$

# Gaussian Naïve Bayes (cont)

**Example** 3: (Final Exam Jan 2019, Q3(c))
A more efficient marketing strategy can be achieved by targeting the customers who have higher probability to complete a purchase. Hence, you have been asked to predict whether a customer will buy the product based on their demographic data such as age, race, gender and income. Table Q3(c) shows the data collected from previous records.

# Gaussian Naïve Bayes (cont)

**Example** 3: (cont)

| Cust. | Age | Race | Gender | Income | Result |
|---|---|---|---|---|---|
| 1 | 52 | Indian | Male | RM 11,500 | Not Buy |
| 2 | 22 | Chinese | Female | RM 6,500 | Buy |
| 3 | 30 | Chinese | Male | RM 8,000 | Buy |
| 4 | 26 | Malay | Male | RM 8,500 | Buy |
| 5 | 27 | Indian | Female | RM 6,500 | Buy |
| 6 | 32 | Chinese | Female | RM 9,500 | Not Buy |
| 7 | 33 | Indian | Male | RM 4,000 | Not Buy |
| 8 | 50 | Malay | Female | RM 10,000 | Buy |
| 9 | 31 | Chinese | Female | RM 5,500 | Buy |
| 10 | 27 | Malay | Male | RM 9,200 | Not Buy |

Table Q3(c)

# Gaussian Naïve Bayes (cont)

**Example** 3: (cont)

(i) State an assumption used in Naïve Bayes approach. (1 mark)

(ii) Using Naïve Bayes approach without Laplace smoothing, predict whether a Malay female customer, aged 29, with income RM7,800, will buy the product. (9 marks)

# Gaussian Naïve Bayes (cont)

*Gaussian Naïve Bayes (Classifier)* is available in Python as `GaussianNB` of the form:

```
from sklearn.naive_bayes import GaussianNB
GaussianNB(priors=None, var_smoothing=1e-09)
```

R only supports the mixed naïve Bayes models with categorical and Gaussian and lacks support for multinomial NB and complement NB. Popular R functions are: `naivebayes::naive_bayes`, `e1071::naiveBayes` (slow), `bnlearn::naive.bayes` (which can only handle categorical data), `klaR::NaiveBayes` (slow).

```
naive_bayes(formula, data, prior = NULL, laplace = 0,
  usekernel = FALSE, usepoisson = FALSE,
  subset, na.action = stats::na.pass, ...)
```

# Gaussian Naïve Bayes (cont)

**Example** 4: The Iris dataset (`https://archive.ics.uci.edu/ml/datasets/Iris`) consists four attributes: sepal length, sepal width, petal length and petal width, all in cm; and a label of 3 classes (Iris Setosa, Iris Versicolour, Iris Virginica).

```
1  # https://www.python-course.eu/naive_bayes_classifier_scikit.php
2  from sklearn import datasets, metrics, naive_bayes
3  dataset = datasets.load_iris()
4  model = naive_bayes.GaussianNB()
5  model.fit(dataset.data, dataset.target)
6  print(model)
7  expected  = dataset.target
8  predicted = model.predict(dataset.data)
9  # summarise the fit of the model
0  print(metrics.confusion_matrix(predicted, expected))
1  print(metrics.classification_report(predicted, expected))
```

# Laplace Smoothing

An issue faced by a Naïve Bayes classifier with "discrete" data is the numerator in (7) being zero, i.e. $n_{X=c,Y=j} = 0$. In this case, the posterior probability will become zero regardless of the value of other density functions and the Naïve Bayes classifier will fail.

# Laplace Smoothing (cont)

**Example** 5: By using the data from **Example** 2, perform the following tasks.

(a) Compute the probability density of customer with balance 2080, given Default=N.

(b) Compute the probability of customer who is a student, given Default=N.

(c) Calculate the "probability density" of non-default for a student customer with balance 2080 by using the Naïve Bayes assumption.

# Laplace Smoothing (cont)

(a) Solution: $\mathbb{P}(\texttt{balance} = 2080 \mid \texttt{Default} = N) = \frac{1}{s_N\sqrt{2\pi}} \exp(-\frac{(2080-\mu_N)^2}{2s_N^2}) = 1.9616 \times 10^{-5}$

where $\mu_N = \frac{500+60+1400+300+940}{5} = 640; \quad s_N = 533.6666$

(b) Solution:

$\mathbb{P}(\texttt{student} = \textit{Yes} \mid \texttt{Default} = N) = \frac{0}{5} = 0$

(c) Solution:

$$\mathbb{P}(\texttt{Default} = N \mid \texttt{balance} = 2080,\ \texttt{student} = \textit{Yes})$$

$$= \frac{1.9616 \times 10^{-5} \times 0 \times \frac{5}{10}}{\mathbb{P}(\texttt{balance} = 2080,\ \texttt{student} = \textit{Yes})} = 0.$$

# Laplace Smoothing (cont)

This situation is normally happened to categorical variable. To avoid the stated problem, *Laplace smoothing* or `https://en.wikipedia.org/wiki/Additive_smoothing` is applied to the Naïve Bayes classifier. Laplace smoothing modified the density function of categorical variable by adding $\alpha$ (by default, $\alpha = 1$) to each variable per class:

$$\mathbb{P}(X = x_i | Y = k) = \frac{n_{X=x_i;Y=k} + \alpha}{n_{Y=k} + d\alpha} \tag{12}$$

where $d$ is the number of classes in the categorical variable $X$.

# Laplace Smoothing (cont)

**Example** 6: Redo **Example** 5 by applying Laplace smoothing.
**Solution**: The "continuous" variable is the same:

$$\mathbb{P}(\texttt{balance} = 2080 \mid \texttt{Default} = N) = 1.9616 \times 10^{-5}$$

The categorical variable needs the Laplace smoothing ($\alpha = 1$, $d = 2$ for `student`=Yes or No)

$$\mathbb{P}(\texttt{student} = \textit{Yes} \mid \texttt{Default} = N) = \frac{0+1}{5+2}$$

# Laplace Smoothing (cont)

**Example** 6: (cont)
Therefore,

$$\begin{aligned}
&\mathbb{P}(\texttt{Default} = N \mid \texttt{balance} = 2080,\ \texttt{student} = Yes) \\
=&\frac{\mathbb{P}(2080, Yes \mid N)\mathbb{P}(N)}{\mathbb{P}(2080, Yes \mid N)\mathbb{P}(N) + \mathbb{P}(2080, Yes \mid Y)\mathbb{P}(Y)} \\
=&\frac{1.9616 \times 10^{-5} \times \frac{1}{7} \times \frac{5}{10}}{1.9616 \times 10^{-5} \times \frac{1}{7} \times \frac{5}{10} + 0.000916154 \times \frac{5}{7} \times \frac{5}{10}} \\
=&\frac{1.401151e-06}{1.401151e-06 + 0.0003271979} = 0.004264014
\end{aligned}$$

# Multinomial NB

The Naïve Bayes algorithm for multinomially distributed data is called a *multinomial Naïve Bayes classifier*: Application: **text classification**

$$
\begin{aligned}
& h_D(\texttt{document}) \\
&= \underset{j=1,\cdots,K}{\text{argmax}}\, \mathbb{P}(\texttt{document}|Y=j)\mathbb{P}(Y=j) \\
&= \underset{j=1,\cdots,K}{\text{argmax}}\, \mathbb{P}(wc_1, wc_2, \cdots, wc_p|Y=j)\mathbb{P}(Y=j)
\end{aligned}
$$

where $wc_i$ is the number of times the word $X_i$, $i = 1, \cdots, p$, occurred in the document, $p$ is the size of the vocabulary.

# Multinomial NB (cont)

Possible entries of "classes" for `document` are "scientific", "economic", "management", etc. A naïve estimate for $\mathbb{P}(Y = j)$ is

$$\mathbb{P}(Y = j) \approx \frac{\text{number of documents of class } j}{\text{number of documents, } n};$$

$$\mathbb{P}(X_i = wc_i | Y = j) \approx \frac{\begin{array}{c}\text{total number of the occurrences of} \\ \text{the word } X_i \text{ in documents of class } j\end{array}}{\text{total number of words } X_1, \cdots, X_p \text{ in documents of class } j} =: \theta_{ji}.$$

Note: $\sum_{i=1}^{p} \theta_{ji} = 1$.

# Multinomial NB (cont)

A more robust estimate of the parameters $\theta_j := (\theta_{j1}, \ldots, \theta_{jp})$ is given by a smoothed version of MLE:

$$\mathbb{P}(X_i = wc_i | Y = j) \approx \frac{N_{ji} + \alpha}{N_j + \alpha d_i}$$

where $N_{ji} = \sum_{y_i = j} wc_i$ is the number of times feature $i$ appears in a sample of class $j$ in the training set $D$, and $N_j = \sum_{i=1}^{n} N_{ji}$ is the total count of all features for class $j$.
For the smoothing priors $\alpha \geq 0$,
$\alpha < 1$ is called *Lidstone smoothing*,
$\alpha = 1$ is called *Laplace smoothing*.

# Multinomial NB (cont)

The conditional probability is

$$\mathbb{P}(\mathsf{X} = \mathsf{x}|Y = j) = \frac{(\sum_{i=1}^{p} wc_i)!}{wc_1! \times \cdots \times wc_p!} \prod_{i=1}^{p} \theta_{ji}^{wc_i}. \quad (13)$$

According to `https://scikit-learn.org/stable/modules/naive_bayes.html`, the Multinomial Naïve Bayes classifiers is implemented as `MultinomialNB`.

```
from sklearn.naive_bayes import MultinomialNB
MultinomialNB(alpha=1.0, fit_prior=True,
   class_prior=None)
```

R's multinomial Naïve Bayes is supported by `fastNaiveBayes::fnb.multinomial(x,y,laplace=1)`.

# Complement Multinomial NB

A *complement Naïve Bayes* (CNB) algorithm is an adaptation of the standard MNB algorithm that is particularly suited for imbalanced data sets.
The procedure for calculating the weights is as follows:

$$\widehat{\theta}_{ji} = \frac{\alpha_i + \sum_{k:y_j \neq j} d_{ij}}{\alpha + \sum_{k:y_j \neq j} \sum_s d_{sj}} \Rightarrow w'_{ji} = \ln \widehat{\theta}_{ji} \Rightarrow w_{ji} = \frac{w'_{ji}}{\sum_k |w'_{jk}|}$$

where the summations are over all documents $k$ not in class $j$, $d_{ij}$ is either the count or tf-idf value (term frequency-inverse document frequency, see `https://en.wikipedia.org/wiki/Tf-idf`) of term $i$ in document $j$.

# Complement Multinomial NB (cont)

In Python's Sklearn, CNB is implemented as ComplementNB and has the form:

```
from sklearn.naive_bayes import *
ComplementNB(alpha=1.0, fit_prior=True,
  class_prior=None, norm=False)
```

# Bernoulli NB

Bernoulli Naïve Bayes is used when the data is distributed according to multivariate Bernoulli distributions i.e., $x_i$ is a binary value.
The conditional probability is

$$\mathbb{P}(\mathrm{X} = \mathrm{x}|Y = j) = \prod_{i=1}^{p} \theta_{ji}^{x_i}(1 - \theta_{ji}^{x_i})^{1-x_i} \tag{14}$$

It is implemented in Python as `BernoulliNB`:

```
from sklearn.naive_bayes import *
BernoulliNB(alpha=1.0, binarize=0.0,
  fit_prior=True, class_prior=None)
```

# Examples

```
d.f = read.csv(text='
ham,1,"Hi sir, just want to ask you if the formula xxx is OK?"
spam,1,"Maxis great deal is here"
ham,2,"If I solve the problem the following way ... is it OK?"
ham,3,"You solution is correct.  Great job"
spam,2,"Discount 20% from Maxis when dinning at ..."
ham,4,"The maximum value for ... is the coefficient for the model..."
spam,3,"Win a phone when subscribing to Maxis new plan ..."
',col.names=c("Y","id", "content"))
library(tm)   # VCorpus, VectorSource, DocumentTermMatrix, inspect
corpus = VCorpus(VectorSource(d.f$content))
# The DocumentTermMatrix can be slow for large data
# and the stemming is too primitive and brutal!
dtm = DocumentTermMatrix(corpus, control = list(
  tolower = TRUE, removeNumbers = TRUE,
  removePunctuation = TRUE, stemming = TRUE
))   # Statistical model
print(as.matrix(dtm))
library(fastNaiveBayes)
mnnb = fnb.multinomial(x=freqTrain, y=d.f$Y, laplace=1)
#mnnb = fastNaiveBayes(x=freqTrain, y=d.f$Y)  # Use Bernoulli NB? Why
mnnb$present
```

# Examples (cont)

Real-world application is complex. There are a lot of Python examples on the Internet. For example,

```
# https://jakevdp.github.io/PythonDataScienceHandbook/05.05-naive-bay
from sklearn.datasets import fetch_20newsgroups
# Downloads https://ndownloader.figshare.com/files/5975967 and
# put in under ~/scikit_learn_data/
data = fetch_20newsgroups()
data.target_names

categories = ['talk.religion.misc', 'soc.religion.christian',
              'sci.space', 'comp.graphics']
train = fetch_20newsgroups(subset='train', categories=categories)
test  = fetch_20newsgroups(subset='test',  categories=categories)

from sklearn.feature_extraction.text import TfidfVectorizer,CountVect
from sklearn.naive_bayes import MultinomialNB
from sklearn.pipeline import make_pipeline
```

# Examples (cont)

```
1
2  # https://machinelearningmastery.com/prepare-text-data-machine-learni
3  # https://scikit-learn.org/stable/modules/feature_extraction.html
4  vectoriser = CountVectorizer()
5  vectoriser.fit(train.data)
6  print(vectoriser.vocabulary_)
7  feature_table = vectoriser.transform(train.data)
8  model = make_pipeline(TfidfVectorizer(), MultinomialNB())
9  model.fit(train.data, train.target)
0  labels = model.predict(test.data)
1
2  from sklearn.metrics import confusion_matrix
3  import seaborn as sns, matplotlib.pylab as plt
4  mat = confusion_matrix(test.target, labels)
5  sns.heatmap(mat.T, square=True, annot=True, fmt='d', cbar=False,
6              xticklabels=train.target_names,
7              yticklabels=train.target_names)
8  plt.xlabel('true label')
9  plt.ylabel('predicted label')
```

# Outline

# Bayesian Network

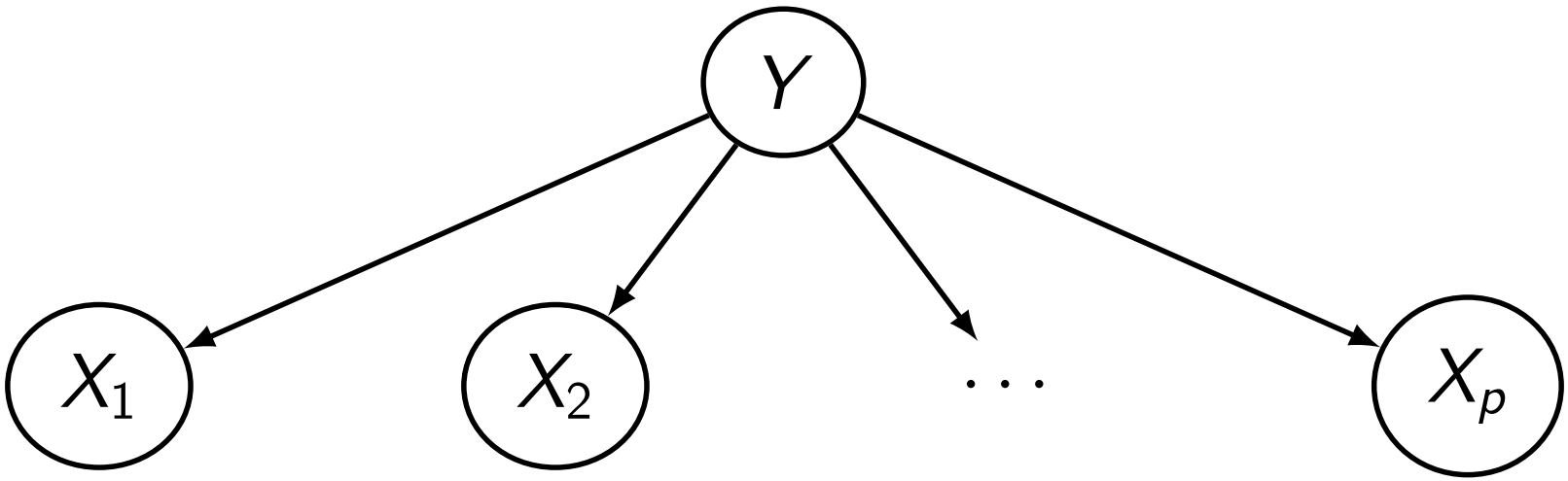
Things not coming out in Final Assessment:

*Computations related to Bayesian Network, Ensemble Methods, SVM*

A *Bayesian network* (or *belief network*) $B = \langle N, A, \Theta \rangle$ is a directed acyclic graph (DAG) $\langle N, A \rangle$ with a conditional probability distribution for each node, collectively represented by $\Theta$. Each node $n \in N$ represents a random variable $X_i$ (which represents a feature) and each arc $a \in A$ represents a probabilistic dependency between features.

# Bayesian Network (cont)

The Naïve Bayes classifier is a special class of Bayesian networks with the following DAG:



# Bayesian Network (cont)

The statistical meaning of this Bayesian network can be described by the conditional independence set $Q = \{(X_i, pa(X_i), nde(X_i)|i = 1, \cdots, p\}$ where $pa(x)$ is the set of all parents of node $x$, $de(x)$ is the set of all descendents of node $x$, and $nde(x) = N \setminus \{x\} \setminus pa(x) \setminus de(x)$ is the set of all non-descendants of node $x$. The joint probability distribution of $X_1, \cdots, X_p$ can be factored based on the probability decomposition of the Bayesian network:

$$\mathbb{P}(X_1, \cdots, X_p) = \prod_{i=1}^{p} \mathbb{P}(X_i|pa(X_i)). \tag{15}$$

# Bayesian Network (cont)

This means that the "Probability Distribution Table" (on the left of (15)) can be factored into a product of much smaller Conditional Probability Tables (on the right of (15)).
Bayesian networks explicitly express conditional independencies in probability distributions and allows computation of probabilities distributions using the chain rule:

$$\mathbb{P}(A|B,C) = \mathbb{P}(A|C), \quad \mathbb{P}(A,B|C) = \mathbb{P}(A|C)\mathbb{P}(B|C)$$

where $A$ and $B$ are **conditionally independent** given $C$.

# Outline

# Discriminant Analysis (DA) Theory (cont)

Naive Bayes is a "parametric" & generative model.
Discriminant analysis is another "parametric" & generative model.
The difference: the "assumptions" on the likelihood functions.
Discriminant analysis assumes that $\mathbb{P}(\mathrm{X} = \mathrm{x}|Y = k)$ is approximated by multivariate Gaussian distribution:

$$f_j(\mathrm{x}) = \frac{1}{(2\pi)^{p/2}\sqrt{|\mathrm{C}_j|}} \exp\left\{-\frac{1}{2}(\mathrm{x} - \boldsymbol{\mu}_j)^T \mathrm{C}_j^{-1}(\mathrm{x} - \boldsymbol{\mu}_j)\right\} \tag{16}$$

# DA Theory (cont)

DA models are generative predictive models (3) which assumes that the predictors $\mathrm{X} = (X_1, X_2, \cdots, X_p)$ to be numeric and are drawn from a *multivariate Gaussian* (or a *multivariate normal*) distribution, $Normal(\boldsymbol{\mu}, \mathrm{C})$. Therefore, the "$\mathbb{P}$" in (3) should be regarded as "probability density" because the predictors are numeric.

# DA Theory (cont)

Two important DA models are the *linear discriminant analysis (LDA)* models and *quadratic discriminant analysis (QDA)* models.
In the QDA models, $\mathbb{P}(\mathrm{X} = \mathrm{x}|Y = j)$ is assumed to have the following form:

$$\mathbb{P}(\mathrm{X} = \mathrm{x}|Y = j) = \frac{1}{(2\pi)^{p/2}\sqrt{|\mathrm{C}_j|}} \exp\left\{-\frac{1}{2}(\mathrm{x} - \boldsymbol{\mu}_j)^T \mathrm{C}_j^{-1}(\mathrm{x} - \boldsymbol{\mu}_j)\right\} \quad (17)$$

where $\boldsymbol{\mu}$ and $\mathrm{C}_j$ are the class-specific mean vector and the class-specific covariance matrix respectively.

# QDA Theory

By substituting (17) into (3), we have

$$
\begin{aligned}
h_D(\mathrm{x}) = & \underset{j\in\{1,\ldots,K\}}{\operatorname{argmax}} \left[ \ln \mathbb{P}(Y=j) + \ln \frac{1}{(2\pi)^{p/2}\sqrt{|\mathrm{C}_j|}} \times \right. \\
& \left. \exp\left\{ -\frac{1}{2}(\mathrm{x}-\boldsymbol{\mu}_j)^T \mathrm{C}_j^{-1}(\mathrm{x}-\boldsymbol{\mu}_j) \right\} \right] \\
= & \underset{j\in\{1,\ldots,K\}}{\operatorname{argmax}} \left[ \ln \mathbb{P}(Y=j) - \frac{1}{2}\ln|\mathrm{C}_j| - \right. \\
& \left. \frac{1}{2}(\mathrm{x}-\boldsymbol{\mu}_j)^T \mathrm{C}_j^{-1}(\mathrm{x}-\boldsymbol{\mu}_j) \right]
\end{aligned}
\tag{18}
$$

# QDA Theory (cont)

From here, we can obtain the "discriminant functions" for QDA:

$$\delta_j(\mathrm{x}) = \ln \mathbb{P}(Y = j) - \frac{1}{2} \ln |\mathrm{C}_j| - \frac{1}{2}(\mathrm{x} - \boldsymbol{\mu}_j)^T \mathrm{C}_j^{-1}(\mathrm{x} - \boldsymbol{\mu}_j) \tag{19}$$

where $j = 1, \cdots, K$.
Note that the "boundary" between classes is a quadratic surface according to (19).

# QDA Theory (cont)

**Parameter Estimation**

The prior distribution

$$\widehat{\mathbb{P}(Y = j)} = \frac{\#\{i \ : \ y_i = j\}}{n}$$

The class-specific covariance matrix

$$\widehat{\mathsf{C}}_j = \frac{1}{n_j - 1} \sum_{i=1}^{n} (\mathsf{x}_i - \widehat{\boldsymbol{\mu}}_j)(\mathsf{x}_i - \widehat{\boldsymbol{\mu}}_j)^T I(y_i = j). \qquad (20)$$

# LDA Theory

In the LDA models, $\mathbb{P}(\mathrm{X} = \mathrm{x}|Y = j)$ is assumed to be (16) with a common covariance matrix C:

$$\mathrm{C}_1 = \mathrm{C}_2 = \cdots = \mathrm{C}_K =: \mathrm{C}.$$

In such a case, the generative model (18) becomes (let $\pi_j = \mathbb{P}(Y = j)$)

$$\begin{aligned} h_D(\mathrm{x}) &= \underset{j\in\{1,\ldots,K\}}{\operatorname{argmax}} \left[\ln \pi_j - \frac{1}{2}\ln|\mathrm{C}| - \frac{1}{2}(\mathrm{x}-\boldsymbol{\mu}_j)^T\mathrm{C}^{-1}(\mathrm{x}-\boldsymbol{\mu}_j)\right] \\ &= \underset{j\in\{1,\ldots,K\}}{\operatorname{argmax}} \left[\ln \pi_j - \frac{1}{2}\left(\mathrm{x}^T\mathrm{C}^{-1}\mathrm{x} - \boldsymbol{\mu}_j^T\mathrm{C}^{-1}\mathrm{x} - \mathrm{x}^T\mathrm{C}^{-1}\boldsymbol{\mu}_j + \boldsymbol{\mu}_j^T\mathrm{C}^{-1}\boldsymbol{\mu}_j\right)\right] \\ &= \underset{j\in\{1,\ldots,K\}}{\operatorname{argmax}} \left[\ln \pi_j - \frac{1}{2}\left(-2\boldsymbol{\mu}_j^T\mathrm{C}^{-1}\mathrm{x} + \boldsymbol{\mu}_j^T\mathrm{C}^{-1}\boldsymbol{\mu}_j\right)\right] \\ &= \underset{j\in\{1,\ldots,K\}}{\operatorname{argmax}} \left[\ln \pi_j + \boldsymbol{\mu}_j^T\mathrm{C}^{-1}\mathrm{x} - \frac{1}{2}\boldsymbol{\mu}_j^T\mathrm{C}^{-1}\boldsymbol{\mu}_j\right]. \end{aligned} \tag{21}$$

# LDA Theory (cont)

From here, we obtain the *discriminant function* for LDA:

$$\delta_j(\mathrm{X}) = \ln \mathbb{P}(Y = j) - \frac{1}{2}\boldsymbol{\mu}_j^T \mathrm{C}^{-1}\boldsymbol{\mu}_j + \boldsymbol{\mu}_j^T \mathrm{C}^{-1}\mathrm{x}. \quad (22)$$

Note that the "boundary" between classes is a linear space according to (22). The name of LDA is from the fact that the discriminant functions, $\delta_j(\mathrm{x})$ are linear functions of x.

# LDA Theory (cont)

The estimation of the parameters in the discriminant function (22) using the **standard moment estimators** is as follows.

- The prior probability $\pi_j$ is approximated by the fraction of training samples of class $j$:

$$\widehat{\pi}_j = \frac{\sum_{i=1}^{n} I(y_i = j)}{n}. \tag{23}$$

- The centre of each class $\boldsymbol{\mu}_j$ has an estimation:

$$\widehat{\boldsymbol{\mu}}_j = \sum_{i=1}^{n} \mathrm{x}_i I(y_i = j) \Big/ \sum_{i=1}^{n} I(y_i = j). \tag{24}$$

# LDA Theory (cont)

- The common covariance matrix C is an unbiased estimate of its covariance matrix of the vectors of deviations $(\mathrm{x}_1 - \widehat{\boldsymbol{\mu}}_{y_1})$, $(\mathrm{x}_2 - \widehat{\boldsymbol{\mu}}_{y_2})$, $\cdots$, $(\mathrm{x}_n - \widehat{\boldsymbol{\mu}}_{y_n})$:

$$\widehat{\mathrm{C}} = \frac{1}{n-K} \sum_{j=1}^{K} \sum_{i=1}^{n} (\mathrm{x}_i - \widehat{\boldsymbol{\mu}}_j)(\mathrm{x}_i - \widehat{\boldsymbol{\mu}}_j)^T I(y_i = j) \quad (25)$$

# LDA Theory (cont)

LDA method has the following advantages:

- When the classes are well-separated, it will be more stable as compared to other models.
- If $n$ is small and the distribution of the predictors X is approximately normal in each of the classes, LDA is more stable than logistic regression.
- When we have more than two response classes, LDA is more popular.

# LDA or QDA?

In real problems, population parameters are usually unknown and estimated from training data as the sample means and sample covariance matrices. While QDA accommodates more flexible decision boundaries compared to LDA, the number of parameters needed to be estimated also increases faster than that of LDA.

For LDA, $(p+1)$ parameters are needed to construct the discriminant function in (22). For a problem with $K$ classes, we theoretically need $K-1$ discriminant functions by arbitrarily choosing one class to be the base class (subtracting the base class likelihood from all other classes). Hence, the total number of estimated parameters for LDA is $(K-1)(p+1)$.

# LDA or QDA? (cont)

On the other hand, for each QDA discriminant function in (19), mean vector has $p$ parameters, covariance matrix has $p(p+1)/2$ parameters, and the class prior has 1 parameter. So we need to estimate $(K-1) \times (\frac{p(p+3)}{2} + 1)$ parameters. Therefore, the number of parameters estimated in LDA increases linearly with $p$ while that of QDA increases quadratically with $p$!

We would expect QDA to have worse performance than LDA when the problem dimension is large.

# LDA Examples

**Example** 7: (Final Exam May 2019, Q4)

(a) State the advantages of using linear discriminant analysis in classification as compared to logistic regression. (3 marks)

**Solution**:

- LDA is more stable when the classes are well-separated.
- LDA is more stable when $n$ is small and the distribution of the predictors $X$ is approximately normal in each of the classes.
- LDA is better when there are more than two response classes.

# LDA Examples (cont)

**Example** 7: (cont)

(b) (i) State two assumptions made in linear discriminant analysis. (2 marks)

(ii) Bayes' theorem states that posterior probability to estimate probability of a new observation belongs to the $j$th class can be written as

$$\mathbb{P}(Y = j|X = x) = \frac{\pi_j \mathbb{P}(x|Y = j)}{\sum_{i=1}^{K} \pi_i \mathbb{P}(x|Y = i)}.$$

where $\pi_j = \mathbb{P}(Y = j)$. Classification is done by assigning an observation to the class which posterior probability, $\mathbb{P}(Y = j|X = x)$ is the largest. (continue next)

# LDA Examples (cont)

**Example** 7: (cont)

(b) (ii) (cont) Linear discriminant analysis involves assigning the observation to the class for which discriminant function, $\delta_j(X)$ is the largest. For linear discriminant analysis with one predictor, the discriminant function is

$$\delta_j(x) = x \cdot \frac{\mu_j}{\sigma^2} - \frac{\mu_j^2}{2\sigma^2} + \ln(\pi_j).$$

With assumptions stated in Q4(b)(i), show how discriminant function, $\delta_j(x)$ can be equivalent to posterior probability, $\mathbb{P}(Y = j|X = x)$ in linear discriminant analysis with one predictor.

(10 marks)

# LDA Examples (cont)

**Example** 7: (cont)

(c) A teacher is preparing an extra class for the students who are predicted to fail in their final exam. The teacher would like to predict the performance of the current students in final exam (fail/pass). You are to build a model for the teacher by using 500 previous students' record. Below shows some information and analysis of the previous record:

(I) The coursework consisted of Assignment, Quiz and Test.

(II) Average mark for students who passed in Assignment was 73.9; whereas average mark for students who failed in Assignment was 51.4.

# LDA Examples (cont)

**Example** 7: (cont)

(c)

(III) Average mark for students who passed in Quiz was 68.2; whereas average mark for students who failed in Quiz was 42.3.

(IV) Average mark for students who passed in Test was 63.7; whereas average mark for students who failed in Test was 35.6.

(V) There were 380 students passed the final exam.

(VI) The inverse of the group covariance matrix for the collected data is

$$C^{-1} = \begin{bmatrix} 0.0022 & 0.0132 & 0.0095 \\ 0.0132 & 0.0074 & 0.0108 \\ 0.0095 & 0.0108 & 0.0180 \end{bmatrix}$$

where $x_1$ = Assignment; $x_2$ = Quiz; $x_3$ = Test.

# LDA Examples (cont)

**Example** 7: (cont)

(c) Using linear discriminant analysis, predict the final exam performance (pass/fail) of a current student who scored 55.7 marks in Assignment, 49.8 marks in Quiz and 52.6 marks in Test. (10 marks)

**Solution**: Let pass $= 1$ and fail $= 0$.

Prior probability (given in item (V) and using (6)):

$$\widehat{\pi}_1 = \frac{380}{500} = 0.76; \quad \widehat{\pi}_0 = 1 - 0.76 = 0.24$$

# LDA Examples (cont)

**Example** 7: (cont)

(c) **Solution** (cont):

Mean vector (given in items (II), (III) and (IV) and calculate using (24)):

$$\widehat{\mu}_1 = [73.9,\ 68.2,\ 63.7]; \quad \widehat{\mu}_0 = [51.4,\ 42.3,\ 35.6]$$

Discriminant function, $\delta_j(X)$:

$$\widehat{\delta}_j(X) = \ln \pi_j + \widehat{\mu}_j C^{-1} \left[ \mathrm{x}^T - \frac{1}{2}\widehat{\mu}_j^T \right],$$

$$\widehat{\delta}_1(X) = \ln(0.76) - \frac{435.8066}{2} + [1.6680,\ 2.1681,\ 2.5852]\mathrm{x}^T$$

$$\widehat{\delta}_0(X) = \ln(0.24) - \frac{166.5589}{2} + [1.0096,\ 1.3760,\ 1.5859]\mathrm{x}^T$$

# LDA Examples (cont)

**Example** 7: (cont)

- **Solution** (cont): For a new observation, $x^* = [55.7,\ 49.8,\ 52.6]$,

$$\widehat{\delta}_1(x^*) = 118.6828$$
$$\widehat{\delta}_0(x^*) = 123.4746$$

Since $\delta_1(x^*) < \delta_0(x^*)$, the new observation should be assigned to class 0, the student will "more likely to" fail in final exam.

# LDA Examples (cont)

**Example** 8:

Table below shows the data collected:

| Customer | Balance | Default |
|---|---|---|
| 1 | 500 | N |
| 2 | 1980 | Y |
| 3 | 60 | N |
| 4 | 2810 | Y |
| 5 | 1400 | N |
| 6 | 300 | N |
| 7 | 2000 | Y |
| 8 | 940 | N |
| 9 | 1630 | Y |
| 10 | 2170 | Y |

# LDA Examples (cont)

**Example** 8: (cont)
Use the data and the predictive model LDA to predict if a customer with balance 1500 will default in his credit card?

Are you able to obtain these?

$$\delta_1(1500) = 3.2565, \quad \delta_2(1500) = 2.5003.$$

# LDA Implementations

LDA and QDA are implemented:

- in R's `MASS` package as `lda` and `qda` respectively.
- in Python's `sklearn.discriminant_analysis` as `LinearDiscriminantAnalysis` and `QuadraticDiscriminantAnalysis`.

# LDA Implementations (cont)

**Example** 9:
Redo **Example** 8 using R. Are you able to find the discriminant functions?
The "boundary" according to **Example** 8 is

$$-10.17773 + 0.008956171x = -1.559164 + 0.002706303x$$
$$\Rightarrow x = 1379$$

# LDA Implementations (cont)

## **Example** 9: (cont)

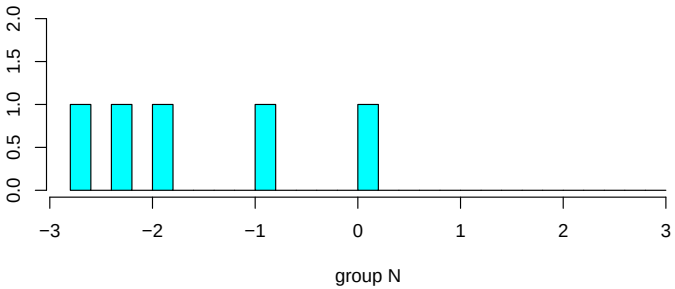
```
1  library(MASS)  # for lda
2
3  d.f = data.frame(
4      balance = c(500 , 1980, 60  , 2810, 1400, 300 , 2000, 940 , 1630,
5      default = c("N", "Y", "N", "Y", "N", "N", "Y", "N", "Y", "Y")
6  )
7  m = lda(default ~ ., data = d.f)
8  print(m); plot(m)
9
0  deltaN = log(m$prior[1]) - 0.5*m$scaling*m$means[1]**2
1  deltaY = log(m$prior[2]) - 0.5*m$scaling*m$means[2]**2
2  cat("deltaN(x) =", deltaN, "+", m$scaling*m$means[1], "x\n")
3  cat("deltaY(x) =", deltaY, "+", m$scaling*m$means[2], "x\n")
4  cat("boundary=",-(deltaN - deltaY)/(m$scaling*(m$means[1]-m$means[2])
```

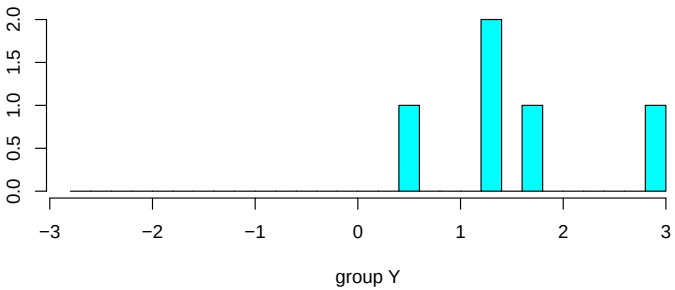
The "discriminant functions" and "boundary" according to MASS::lda are (?) $\delta_N(x) = -421.8348 + 1.316068x, \quad \delta_Y(x) = -4613.02 + 4.355361x \Rightarrow x = 1379$

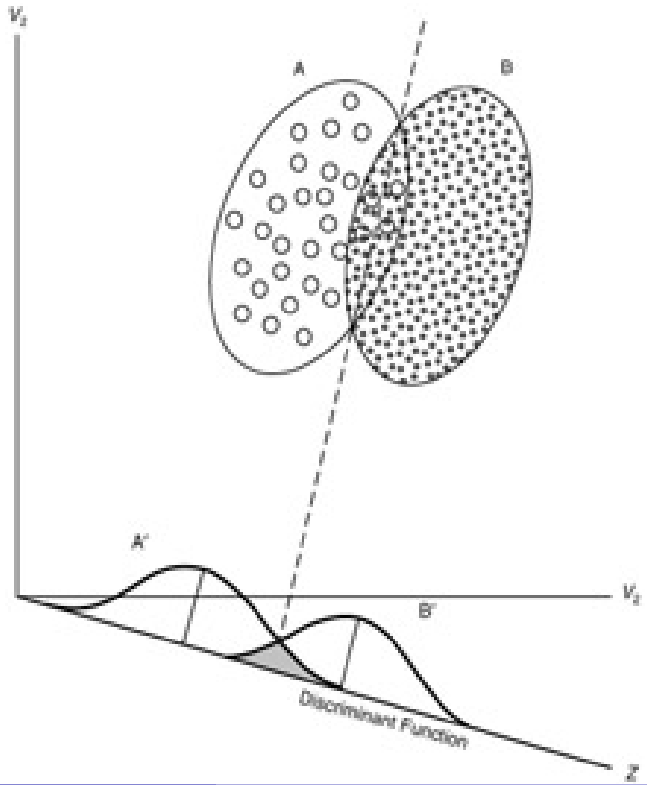
# LDA Implementations (cont)

**Example** 9: (cont)

The graph on the left below shows the set of histograms of the defaults on the Fisher linear discriminant. It can be regarded as the projection of the high-dimensional data as histogram.



# LDA Implementations (cont)

**Example** 10: On the "flame" data.

```
1  d.f = read.table("flame.txt", header=FALSE)
2  d.f$V3 = factor(d.f$V3)
3  m = MASS::lda(V3 ~ ., d.f)
4
5  ### m$scaling is not enough to obtain ``discrimant functions''
6  #d1 = log(m$prior[1])-0.5*sum(m$scaling*(m$means[,1]**2))
7  #d2 = log(m$prior[2])-0.5*sum(m$scaling*(m$means[,2]**2))
8  #coeff = m$scaling*(m$means[,1]-m$means[,2])
9  ## Equation: d1 - d2 + coeff1 x1 + coeff2 x2 = 0
0  ## => x2 = (d2-d1)/coeff2 + coeff1/coeff2
1  #a = (d2-d1)/coeff[2]; b = -coeff[1]/coeff[2]
2  #plot(d.f$V1, d.f$V2, pch=as.integer(d.f[,3]),
3  #       xlab="x1",ylab="x2",main="Flame Data")
4  #abline(a,b)
```

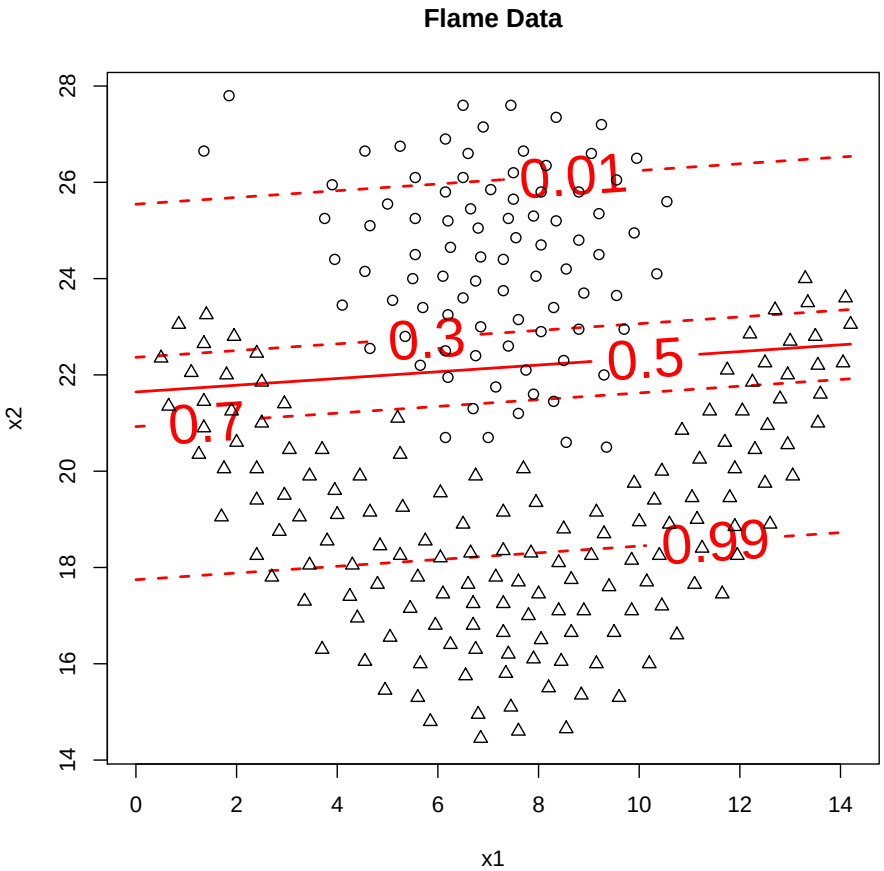
# LDA Implementations (cont)

## **Example** 10: (cont)

```
1  g.x1 = seq(0,max(d.f[,1]),by=0.1)
2  g.x2 = seq(min(d.f[,2]),max(d.f[,2]),by=0.1)
3  d.grid = expand.grid(V1=g.x1, V2=g.x2)
4  prob = predict(m, newdata=d.grid)
5  prob = matrix(prob$posterior[,2], length(g.x1), length(g.x2))
6  contour(g.x1, g.x2, prob, levels=c(0.01,0.3,0.5,0.7,0.99),
7    col="red", labcex=2.5, xlab="x1",ylab="x2",
8    main="Flame Data",lty=c(2,2,1,2,2),lw=2)
9  points(d.f$V1, d.f$V2, pch=as.integer(d.f[,3]),
0    xlab="x1",ylab="x2",main="Flame Data")
```

# LDA Implementations (cont)

## **Example** 10: (cont)



# LDA Examples (cont)

Examples using LDA (and its variants):

- Bankruptcy prediction: Edward Altman's 1968 model (https://en.wikipedia.org/wiki/Altman_Z-score) predicts the probability of company bankruptcy using trained LDA coefficients.
- Facial recognition: While features learned from Principal Component Analysis (PCA) are called Eigenfaces, those learned from LDA are called Fisherfaces (P.N. Belhumeur; J.P. Hespanha; D.J. Kriegman 1997, https://ieeexplore.ieee.org/document/598228)