UECM1304 Tutorial 4: Relations

3 hours

Set Relations, Representations & Properties

1. Let $A = \{a \in \mathbb{R} | -2 \leq a \leq 3\}$ and $B = \{b \in \mathbb{R} | 1 \leq b \leq 5\}$. Sketch the given set in the Cartesian plane $\mathbb{R}^2$ for (i) $A \times B$; (b) $B \times A$.

2. Define a relation R on $\mathbb{R}$ as follows:

   xRy if and only if x, y satisfy the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

   (a) Which of the following ordered pairs belong to R?

   i. (2, 0) .... $\boxed{}$
   ii. (0, 2) .... $\boxed{}$
   iii. (0, 3) .... $\boxed{}$
   iv. (0, 0) .... $\boxed{}$
   v. $(1, \frac{3\sqrt{3}}{2})$ .... $\boxed{}$

   (b) Find $R(\{1, 7\})$ and $R(\{3, 4, 5\})$.

3. Find the domain, range, matrix representation of the relation R.

   (a) $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $R = \{(a, 1), (a, 2), (b, 1), (c, 2), (d, 1)\}$.
   (b) $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 6, 9\}$; aRb if and only if $b = a^2$.
   (c) $A = \{1, 2, 3, 4, 8\}$, $B = \{1, 4, 6, 9\}$; aRb if and only if a divides b.
   (d) $A = \{1, 2, 3, 4, 5\} = B$; aRb if and only if $a \leq b$.

4. Suppose R and S are reflexive relations on a set A. Prove or disprove each of the following:

   (a) $R \cup S$ is reflexive.
   (b) $R \cap S$ is reflexive.
   (c) $S \circ R := \{(a, c) : \exists b((a, b) \in R \wedge (b, c) \in S\}$ is reflexive.

5. Give an example of a relation on a set that is

   (a) symmetric and anti-symmetric.
   (b) neither symmetric nor anti-symmetric.

Closure of Binary Relations

6. Let $A = \{a, b, c, d, e\}$ and R and S be the relations on A described by

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_S = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

   Use Warshall's algorithm to compute the transitive closure of the relation $R \cup S$.

7. Let R be a relation on the set $A = \{a_1, a_2, a_3, a_4, a_5\}$ with a matrix representation:

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

   (a) Write down the listing tuples (or Roster notation) representation of R.

   (b) Compute $M_{cl_{trn}(R)}$ as in Warshall's algorithm and then sketch the digraph representation of $cl_{trn}(R)$.

   (c) Is R transitive? Explain your answer.

Equivalence Relations

16. If R and S are two relations on $\mathbb{R}$ such that for $x, y \in \mathbb{R}$, xRy iff $x < y$ and xSy iff $x > y$. Find (i) $R \cap S$ (ii) $R \cup S$ (iii) $S^{-1} := \{(y, x) : (x, y) \in S\}$.

17. Let $A = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ and define a relation R on A as follows:

$$\forall (a, b) \in A, \forall (c, d) \in A, (a, b)R(c, d) \leftrightarrow ab = cd.$$

   (a) Verify that R is an equivalence relation on A.

   (b) Determine the equivalence class $[(2, 3)]$ by listing all its elements.

18. Let R be the relation on $A = \{2, 4, 6, 8\}$ defined by $xRy \leftrightarrow \gcd(x, y) = 2$.

   (a) Write R as a set of ordered pairs.

   (b) Determine whether R is an equivalence relation.

19. Given $M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$. Compute A/R.

20. Let $S = \{1, 2, 3, 4\}$ and $A = S \times S$. Define the equivalence relation R on A as follows: $(a, b)R(c, d)$ if and only if $a + b = c + d$. Compute A/R.

21. Define a binary relation R on $\mathbb{R}$ as follows:

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} | y = |x|\}.$$

   Determine whether R is reflexive, symmetric and transitive.

22. Let R be an equivalence relation on $\mathbb{Z}$ such that for $x, y \in \mathbb{Z}$, xRy iff $7|x - y$. Which of the following equivalence classes

$$[3], [-7], [12], [0], [-2], [17]$$

   are equal?

Partial Order Relations

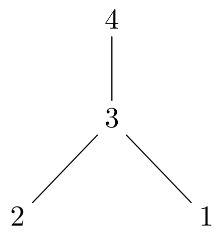
23. Determine whether the relation R is a partial order on $\mathbb{Z}$.

(a) aRb if and only if $a = 3b$.

(b) aRb if and only if $a^2|b$.

(c) aRb if and only if $a = b^k$ for some positive integers k.

24. Describe the ordered pairs in the relation $\preceq$ determined by the Hasse diagram on the set $A = \{1, 2, 3, 4\}$.



25. Consider the poset $(A, |)$ with $|$ the divisibility relation. Draw the Hasse diagram of the poset and determine which posets are linearly ordered.

(a) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

(b) $A = \{3, 6, 12, 72\}$

(c) $A = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60\}$.

26. Let $\preceq$ be a relation over the set $A = \{1, 2, 3, 4, 5\}$ such that

$$1 \preceq 1,\ 1 \preceq 2,\ 1 \preceq 3,\ 1 \preceq 4,\ 1 \preceq 5,\ 2 \preceq 2,\ 2 \preceq 5,\ 3 \preceq 3,\ 3 \preceq 5,\ 4 \preceq 4,\ 4 \preceq 5,\ 5 \preceq 5.$$

Show that $(A, \preceq)$ is a poset.