Tut 8: PCA Dimensional Reduction

Feb 2026

When variances $\text{Var}(x_{\cdot j})$ for features/columns $x_{\cdot j}$ differ a lot, we need to perform scaling:

`pca$scale:` $\sqrt{\frac{\sum_i (x_{ij} - \overline{x}_{\cdot j})^2}{n-1}}$

However, you do not need to scale the data unless it is stated in the question.

Original data: X; Data shifted to centre: $\widetilde{X}$

`pca$center:` $\overline{x}_{\cdot j}$
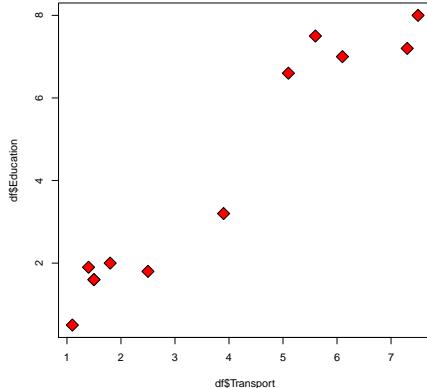
`pca$sdev:` $\sqrt{\lambda_i}$

`pca$rotation:` $[\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots]$

`pca$x:` $[\widetilde{X}\boldsymbol{e}_1, \widetilde{X}\boldsymbol{e}_2, \cdots]$

---

1. You are given 12 communities that were rated according to transportation and education — the higher the score the better. For example, a better transportation system will score higher. Higher education facilities will score higher as well. The table below shows the score for 12 communities in the two criteria:

Obs	Transportation	Education
1	1.1	0.5
2	3.9	3.2
3	1.5	1.6
4	5.6	7.5
5	2.5	1.8
6	7.3	7.2
7	1.4	1.9
8	6.1	7.0
9	1.5	1.6
10	5.1	6.6
11	1.8	2.0
12	7.5	8.0

(a) Use a computer software (e.g. R or Excel) to plot the above scatterplot which is based on the above table.

> *Solution.* A simple R script:
>
> ```
> d.f = data.frame(
>         Transport = c(1.1,3.9,1.5,5.6,2.5,7.3,1.4,6.1,1.5,5.1,1.8,7.5),
>         Education = c(0.5,3.2,1.6,7.5,1.8,7.2,1.9,7.0,1.6,6.6,2.0,8.0)
> )
> plot(d.f$Transport,d.f$Education,type='p',pch=23,bg="red",cex=2)
> ```
> □

(b) Generate two principal components for the data.

> *Solution.* Calculating using R script:
>
> ```
> Transport = c(1.1,3.9,1.5,5.6,2.5,7.3,1.4,6.1,1.5,5.1,1.8,7.5)
> Education = c(0.5,3.2,1.6,7.5,1.8,7.2,1.9,7.0,1.6,6.6,2.0,8.0)
> X = data.frame(Transport, Education)
> PC = prcomp(X)
> ```

```
print(PC)
```

```
Standard deviations:
[1] 3.7504618 0.4861164

Rotation:
                PC1        PC2
Transport 0.6429319 -0.7659234
Education 0.7659234  0.6429319
```

Manual calculation:

i. Shift $\boldsymbol{X}$ to centre, i.e. find $\mu_1 = 3.775$, $\mu_2 = 4.075$ and generate table $\boldsymbol{X}^*$ below.

x_1	-2.675	0.125	-2.275	1.825	-1.275	3.525	-2.375
			2.325	-2.275	1.325	-1.975	3.725
x_2	-3.575	-0.875	-2.475	3.425	-2.275	3.125	-2.175
			2.925	-2.475	2.525	-2.075	3.925

ii. Calculate the covariance matrix for $\boldsymbol{X}^*$, i.e.

$$C = \frac{1}{12-1}(\boldsymbol{X}^*)^T\boldsymbol{X}^* = \begin{bmatrix} 5.952955 & 6.810227 \\ 6.810227 & 8.349318 \end{bmatrix}$$

iii. Find the eigenvalues and eigenvectors of C which characterises the "variance" of the data $\boldsymbol{X}^*$, i.e.

$$|C - \lambda I| = (5.952955 - \lambda)(8.349318 - \lambda) - 6.810227^2 = \lambda^2 - 14.302273\lambda + 3.323923 = 0$$

Using calculator, we obtain

$$\lambda_1 = 14.065963, \ \lambda_2 = 0.236310$$

iv. We then find the eigenvectors corresponding to the eigenvalues λ_1 and λ_2 respectively:

$$\boldsymbol{e}_1 = \frac{1}{\sqrt{6.810227^2 + (8.113008)^2}} \begin{bmatrix} 6.810227 \\ -(5.952955 - 14.065963) \end{bmatrix} = \begin{bmatrix} 0.6429319 \\ 0.765923 \end{bmatrix}$$

$$\boldsymbol{e}_2 = \frac{1}{\sqrt{6.810227^2 + (-5.716645)^2}} \begin{bmatrix} 6.810227 \\ -(5.952955 - 0.236310) \end{bmatrix} = \begin{bmatrix} 0.765923 \\ -0.6429319 \end{bmatrix}$$
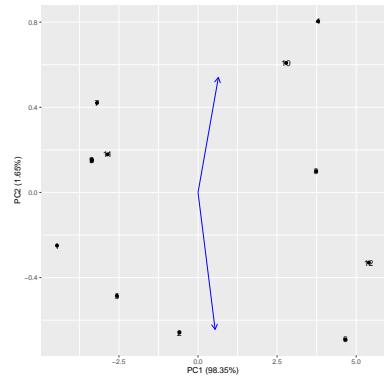
Observe that when $\boldsymbol{e}_1 = [a, b]$ and $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$, $\boldsymbol{e}_2 = [b, -a]$ is an answer.

Note: $\boldsymbol{e}_1$ **and** $\boldsymbol{e}_2$ **correspond to** `PC1` **and** `PC2` **in the** `Rotation` **of** `prcomp`.

v. Calculate the "principal components":
$PC_1 = \sum_{i=1}^{2} e_{i1}(X_i - \mathbb{E}(X_i)) = 0.6429319x_1^* + 0.7659234x_2^*$
$PC_2 = \sum_{i=1}^{2} e_{i2}(X_i - \mathbb{E}(X_i)) = 0.7659234x_1^* - 0.6429319x_2^*$

PC_1	PC_2
-4.4580188	-0.24963649
-0.5898165	-0.65830582
-3.3583304	0.15121924
3.7966382	0.80423156
-2.5622138	-0.48611775
4.6598454	-0.69071771
-3.1928465	0.42069114
3.7351425	0.09980394
-3.3583304	0.15121924
2.7858412	0.60855455
-2.8590814	0.17861498
5.4011705	-0.32955688

By rotating the "principal components" and shift it to the centre (μ_1, μ_2), we can "recover" the original data. $\square$
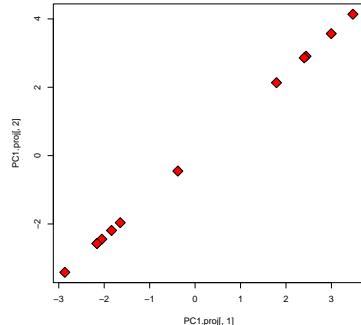
(c) Choose one suitable principal component to represent the data.

Solution. It must be the first principal component, i.e. PC_1. $\square$

(d) Plot your data with the principal component you chose in (c).

Solution. Projecting the centred data $\boldsymbol{X}^*$ to the space span by PC1:

$x_1^{\#}$	$x_2^{\#}$
-2.8662	-3.4145
-0.3792	-0.4518
-2.1592	-2.5722
2.4410	2.9079
-1.6473	-1.9625
2.9960	3.5691
-2.0528	-2.4455
2.4014	2.8608
-2.1592	-2.5722
1.7911	2.1337
-1.8382	-2.1898
3.4726	4.1369

$\square$

(e) With the eigenvalues computed in (b), calculate the proportion of variance explained by each component and the cumulative proportion.

Solution. `print(summary(PC))`

```
1                         PC1      PC2
2  Standard deviation     3.7505  0.48612
3  Proportion of Variance 0.9835  0.01652
4  Cumulative Proportion  0.9835  1.00000
```

Manual calculation:

	Eigenvalue	PVE	Cumulative PVE
PC1	14.0660	$\frac{14.0660}{14.3023} = 0.9835$	0.9835
PC2	0.2363	$\frac{0.2363x}{14.3023} = 0.0165$	1
$\lambda_1 + \lambda_2$	14.3023		

$\square$

(f) With a targeted explained variation of 95%, how many principal components should be considered? State the total variation explained.

Solution. One principal component, PC1. Total variance explained is 98.35%. $\square$

2. (Final Exam May 2020 Sem, Q4(a)) Given the following data with 8 observations in Table 4.1:

Table 4.1: Data with 2 features.

Obs	x	y
A	5.51	5.35
B	20.82	24.03
C	-0.77	-0.57
D	19.30	19.39
E	14.24	12.77
F	9.74	9.68
G	11.59	12.06
H	-6.08	-5.22

Find the first principle component and project the data $(5.51, 5.35)$ to the space span by the first principal component. (4 marks)

Solution. First, we need to find the mean: $\overline{x} = 9.29375, \quad \overline{y} = 9.68625$ .... [0.5 mark]

and shift the data to centre at the mean:

$$X = \begin{array}{c|c|c}
\text{Obs} & \text{x} & \text{y} \\
\hline
\text{A} & -3.78375 & -4.33625 \\
\text{B} & 11.52625 & 14.34375 \\
\text{C} & -10.06375 & -10.25625 \\
\text{D} & 10.00625 & 9.70375 \\
\text{E} & 4.94625 & 3.08375 \\
\text{F} & 0.44625 & -0.00625 \\
\text{G} & 2.29625 & 2.37375 \\
\text{H} & -15.37375 & -14.90625 \\
\end{array}$$

.... [0.5 mark]

Form the covariant matrix and

$$\frac{1}{8-1}X^T X = \begin{bmatrix} 614.8648 & 631.9173 \\ 631.9173 & 661.2402 \end{bmatrix} = \begin{bmatrix} 87.83783 & 90.27390 \\ 90.27390 & 94.46288 \end{bmatrix} \qquad \text{[0.5 mark]}$$

By solving the eigenvalue problem

$$\begin{vmatrix} 87.83783 - \lambda & 90.27390 \\ 90.27390 & 94.46288 - \lambda \end{vmatrix} = \lambda^2 - 182.3007\lambda + 148.0374 = 0 \qquad \text{[1 mark]}$$

leads to the eigenvalues 181.4850, 0.8157

The first principle component corresponds v to the linear algebra problem of the eigenvalue 181.4850

$$\begin{bmatrix} 87.83783 - 181.4850 & 90.27390 \\ 90.27390 & 94.46288 - 181.4850 \end{bmatrix} v = 0$$

i.e.

$$v = \frac{1}{\sqrt{90.27390^2 + 93.64717^2}} \begin{bmatrix} 90.27390 \\ 93.64717 \end{bmatrix} = \begin{bmatrix} 0.69402 \\ 0.71995 \end{bmatrix} \qquad \text{[0.5 mark]}$$

The projection of (5.51, 5.35) to the first principle component space is

$$(-3.78375, -4.33625) \cdot (0.69402, 0.71995) \begin{bmatrix} 0.69402 \\ 0.71995 \end{bmatrix} + \begin{bmatrix} 9.29375 \\ 9.68625 \end{bmatrix} = \begin{bmatrix} 5.3046 \\ 5.5481 \end{bmatrix} \qquad \text{[1 mark]}$$

□

3. (Final Exam Jan 2021 Sem, Q3(a)) Given the following data with 11 observations in Table 3.1:

Table 3.1: Data with two features.

Obs	x	y
1	-5.79	4.91
2	-3.73	4.87
3	-3.25	3.98
4	-2.61	4.09
5	-2.76	4.90
6	2.81	-5.34
7	2.92	-6.15
8	1.97	-4.51
9	5.17	-5.29
10	2.66	-7.10
11	3.47	-4.70

Find the proportions of variance and the principle components. (5 marks)

and shift the data to centre at the mean:

$$X=\begin{array}{c|cc}
\text{Obs} & \text{x} & \text{y} \\
\hline
1 & -5.868182 & 5.85 \\
2 & -3.808182 & 5.81 \\
3 & -3.328182 & 4.92 \\
4 & -2.688182 & 5.03 \\
5 & -2.838182 & 5.84 \\
6 & 2.731818 & -4.40 \\
7 & 2.841818 & -5.21 \\
8 & 1.891818 & -3.57 \\
9 & 5.091818 & -4.35 \\
10 & 2.581818 & -6.16 \\
11 & 3.391818 & -3.76 \\
\end{array}$$

....[0.5 mark]

Form the covariant matrix and

$$\frac{1}{11-1}X^T X = \begin{bmatrix} 138.5108 & -187.3119 \\ -187.3119 & 281.8462 \end{bmatrix} = \begin{bmatrix} 13.85108 & -18.73119 \\ -18.73119 & 28.18462 \end{bmatrix}. \qquad [1 \text{ mark}]$$

By solving the eigenvalue problem

$$\begin{vmatrix} 13.85108 - \lambda & -18.73119 \\ -18.73119 & 28.18462 - \lambda \end{vmatrix} = \lambda^2 - 42.0357\lambda + 39.52995 = 0$$

leads to the eigenvalues $41.073275, 0.962425$ ....[1 mark]

The proportions of variance are

$$\frac{41.073275}{41.073275 + 0.962425} = 0.977105, \quad \frac{0.962425}{41.073275 + 0.962425} = 0.022895 \qquad [0.5 \text{ mark}]$$

The first principle component corresponds $\boldsymbol{v}$ to the linear algebra problem of the eigenvalue 41.073275

$$\begin{bmatrix} 13.85108 - 41.073275 & -18.73119 \\ -18.73119 & 28.18462 - 41.073275 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$$

i.e.

$$\boldsymbol{v} = \frac{1}{\sqrt{(-18.73119)^2 + (27.222195)^2}} \begin{bmatrix} -18.73119 \\ 27.222195 \end{bmatrix} = \begin{bmatrix} -0.566856 \\ 0.823817 \end{bmatrix} \qquad [1 \text{ mark}]$$

The second principle component is orthogonal to the first principle component:

$$\begin{bmatrix} 0.823817 \\ 0.566856 \end{bmatrix} \qquad [0.5 \text{ mark}]$$

$\square$

4. (Final Exam Jan 2023 Sem, Q3(a)) Given the two-dimensional data in Table 3.1.

x_1	x_2
6.0	9.5
2.5	7.5
6.4	10.4
2.1	8.7
5.6	8.7
7.3	8.1

Table 3.1: Two-dimensional data.

Suppose the covariance matrix of the data is

$$\begin{bmatrix} 4.6537 & 0.9623 \\ 0.9623 & 1.0497 \end{bmatrix},$$

find the eigenvalues and normalised eigenvectors of the covariance matrix of the two-dimensional data and write down the principal components of the data in Table 3.1. (8 marks)

Solution. By solving the quadratic equation

$$\begin{vmatrix} 4.6537 - \lambda & 0.9623 \\ 0.9623 & 1.0497 - \lambda \end{vmatrix} = \lambda^2 - 5.7034\lambda + 3.958968 = 0 \qquad \text{[3 marks]}$$

we obtain the eigenvalues of the covariance matrix C:

$$\lambda = 4.8945, \; 0.8089 \qquad \text{[1 mark]}$$

The eigenvectors are obtained by solving linear algebra problems and using the spectral theorem: The normal eigenvector corresponding to $\lambda = 4.8945$ is

$$\begin{bmatrix} 4.6537 - 4.8945 & 0.9623 \\ 0.9623 & 1.0497 - 4.8945 \end{bmatrix} \mathbf{x}_1 = \mathbf{0} \Rightarrow \mathbf{x}_1 = \frac{1}{\sqrt{(0.9623^2 + 0.2408^2)}} \begin{bmatrix} 0.9623 \\ 0.2408 \end{bmatrix}$$

$$= \begin{bmatrix} 0.970089 \\ 0.242749 \end{bmatrix}$$

[2 marks]

By orthogonality, the normal eigenvector corresponding to $\lambda = 0.8089$ is

$$\mathbf{x}_2 = \begin{bmatrix} 0.242749 \\ -0.970089 \end{bmatrix} \qquad \text{[1 mark]}$$

The principal components are

$$PC1 = 0.970089(x_1 - \overline{x_1}) + 0.242749(x_2 - \overline{x_2})$$
$$PC2 = 0.242749(x_1 - \overline{x_1}) - 0.970089(x_2 - \overline{x_2})$$

[1 mark]

Average: 6.32 / 8 marks in Jan 2023; 10% below 4 marks.

□

5. (Final Exam Jan 2023 Sem, Q3(b)) Given the five-dimensional data in Table 3.2.

Obs.	x_1	x_2	x_3	x_4	x_5
A	5.2	7.8	4.9	3.6	3.3
B	7.1	6.4	3.6	4.6	3.9
C	1.3	6.6	2.5	7.3	0.8
D	8.0	7.4	3.3	-0.8	0.9
E	2.7	9.5	2.4	6.6	1.0
F	2.9	10.8	-2.2	3.8	-0.3

Table 3.2: Five-dimensional data.

Suppose the output of the principal component analysis by R is as follows.

```
Centres (1, ..., p=5):
[1]  4.5333 8.0833 2.4167 4.1833 1.6000

Standard deviations (1, .., p=5):
[1]  3.9593 2.9483 1.1729 0.9856 0.4294

Rotation (n x k) = (5 x 5):
         PC1      PC2      PC3       PC4       PC5
[1,]  -0.6499  -0.1170  -0.4415  -0.19537  -0.5752
```

```
[2,]   0.2283  -0.3966  -0.3836   0.78791  -0.1505
[3,]  -0.3866   0.5944   0.3845   0.56568  -0.1714
[4,]   0.5678   0.5815  -0.3444  -0.12603  -0.4527
[5,]  -0.2315   0.3709  -0.6257   0.07177   0.6420
```

Find the **proportions of variance explainced, PVEs**, of the principal component analysis. Then, calculate the PC1 for the point A in Table 3.2. (4 marks)

Solution. The PVEs are

$$PVE_i = \frac{(3.9593^2, 2.9483^2, 1.1729^2, 0.9856^2, 0.4294^2)}{3.9593^2 + 2.9483^2 + 1.1729^2 + 0.9856^2 + 0.4294^2}$$

$$= \frac{(15.6761, 8.6925, 1.3757, 0.9714, 0.1844)}{26.90002} \qquad \text{[2 marks]}$$

$$= (0.5828, 0.3231, 0.0511, 0.0361, 0.0069)$$

The PC1 for point A is

$$PC1(A) = -0.6499 * (5.2 - 4.5333) + 0.2283 * (7.8 - 8.0833)$$
$$- 0.3866 * (4.9 - 2.4167) + 0.5678 * (3.6 - 4.1833) \qquad \text{[2 marks]}$$
$$- 0.2315 * (3.3 - 1.6000) = -2.182757$$

Average: 1.10 / 4 marks in Jan 2023; 66% below 2 marks.
Reason for low marks: Not much pay attention in practical class to relate theory to the output of the R command `prcomp`. Check out page 1 of this tutorial. □

6. (Final Exam May 2023 Sem, Q3(a)) Given the two-dimensional data in Table 3.1.

x_1	x_2
6.0	11.6
5.2	8.7
4.2	12.2
8.6	7.3
4.5	7.1
5.2	9.6

Table 3.1: Two-dimensional data.

Suppose the covariance matrix of the data is

$$\begin{bmatrix} 2.5297 & -1.3223 \\ -1.3223 & 4.5817 \end{bmatrix},$$

find the eigenvalues and normalised eigenvectors of the covariance matrix of the two-dimensional data in Table 3.1. (7 marks)

Solution. By solving the quadratic equation

$$\begin{vmatrix} 2.5297 - \lambda & -1.3223 \\ -1.3223 & 4.5817 - \lambda \end{vmatrix} = \lambda^2 - 7.1114\lambda + 9.841849 = 0 \qquad \text{[3 marks]}$$

we obtain the eigenvalues of the covariance matrix C:

$$\lambda = 5.2294, \ 1.8820 \qquad \text{[1 mark]}$$

The eigenvectors are obtained by solving linear algebra problems and using the spectral

theorem: The normal eigenvector corresponding to $\lambda = 5.2294$ is

$$\begin{bmatrix} 2.5297 - 5.2294 & -1.3223 \\ -1.3223 & 4.5817 - 5.2294 \end{bmatrix} \mathbf{x}_1 = \mathbf{0}$$

$$\Rightarrow \mathbf{x}_1 = \frac{1}{\sqrt{((-1.3223)^2 + 2.6997^2}} \begin{bmatrix} -1.3223 \\ 2.6997 \end{bmatrix} = \begin{bmatrix} -0.4398669 \\ 0.8980630 \end{bmatrix}$$

[2 marks]

By orthogonality, the normalised eigenvector corresponding to $\lambda = 0.8089$ is

$$\mathbf{x}_2 = \begin{bmatrix} 0.8980630 \\ 0.4398669 \end{bmatrix}.$$

[1 mark]

□

7. (Final Exam Jan 2024 Sem, Q4(b)) Given the two-dimensional data in Table 4.3.

Table 4.3: Two-dimensional data

x_1	x_2
4.4	4.8
1.2	11.1
5.0	9.7
7.8	9.6
5.9	6.9
5.8	9.2
3.6	7.6

Suppose the covariance matrix of the data in Table 4.2 is

$$\begin{bmatrix} 4.3014 & -0.7186 \\ -0.7186 & 4.4848 \end{bmatrix},$$

write down the principal components of a data $\mathbf{x}$ related to the data in Table 4.3 by first finding all normalised eigenvectors of the PCA. (10 marks)

Solution. From the quadratic equation

$$\begin{vmatrix} 4.3014 - \lambda & -0.7186 \\ -0.7186 & 4.4848 - \lambda \end{vmatrix} = \lambda^2 - 8.7862\lambda + 18.7745 = 0$$

[3 marks]

we obtain the eigenvalues of the covariance matrix:

$$\lambda = 5.1175, \ 3.6687$$

[1 mark]

The normalised eigenvector corresponding to $\lambda = 5.1175$ is obtained from

$$\begin{bmatrix} 4.3014 - 5.1175 & -0.7186 \\ -0.7186 & 4.4848 - 5.1175 \end{bmatrix} \mathbf{x}_1 = \mathbf{0}$$

$$\Rightarrow \mathbf{x}_1 = \frac{1}{\sqrt{((-0.7186)^2 + (-0.8161)^2}} \begin{bmatrix} -0.7186 \\ -(-0.8161) \end{bmatrix} = \begin{bmatrix} -0.6608518 \\ 0.7505165 \end{bmatrix}$$

[2 marks]

By orthogonality, the normalised eigenvector corresponding to $\lambda = 3.6687$ is

$$\mathbf{x}_2 = \begin{bmatrix} 0.7505165 \\ 0.6608518 \end{bmatrix}$$

[1 mark]

To write down the principal components, we first find the column average

$$(4.814286, \ 8.414286)$$

[1 mark]

8

The first principal component is

$$PC_1(\mathbf{x}) = -0.6608518(x_1 - 4.814286) + 0.7505165(x_2 - 8.414286) \qquad \text{[1 mark]}$$

The second principal component is

$$PC_2(\mathbf{x}) = 0.7505165(x_1 - 4.814286) + +0.6608518(x_2 - 8.414286) \qquad \text{[1 mark]}$$

Average: 5.63 / 10 marks in Jan 2024; 33.33% below 5 marks. (Spend too much time completing everything in Q1 to Q3 and not enough time for Q4(a))
Suggestion: Do the simple questions with more marks first. □

8. (Final Exam May 2023 Sem, Q3(b)) Given the six-dimensional data in Table 3.2.

Obs.	x_1	x_2	x_3	x_4	x_5	x_6
A	1.8	6.0	9.0	5.4	3.6	4.6
B	1.8	3.3	8.5	3.3	5.1	6.3
C	2.1	2.2	7.6	3.1	1.2	5.1
D	-1.5	4.7	8.5	2.8	3.7	3.7
E	1.2	3.0	8.3	5.7	-0.2	5.8
F	2.9	6.6	10.8	4.7	0.9	9.0
G	0.0	5.6	10.0	5.7	2.4	6.5
H	1.6	1.2	8.3	0.8	1.7	6.1

Table 3.2: Six-dimensional data.

Suppose the output of the principal component analysis by R is as follows.

```
Standard deviations (1, .., p=6):
[1]  2.6054  2.1215  1.5351  1.0690  0.6889  0.1514

Rotation (n x k) = (6 x 6):
          PC1       PC2       PC3       PC4       PC5       PC6
[1,]  -0.10535  -0.45161   0.35985  -0.6705   0.4286   0.14886
[2,]  -0.66557   0.35272   0.09264   0.1836   0.5315  -0.32834
[3,]  -0.35074  -0.01992   0.20368   0.3332  -0.0519   0.84934
[4,]  -0.54773   0.07718  -0.49883  -0.4993  -0.4379   0.06436
[5,]   0.09398   0.66379   0.59385  -0.3290  -0.2986   0.02278
[6,]  -0.33772  -0.47400   0.46792   0.2195  -0.5001  -0.37947
```

Find the proportions of variance explained, PVEs, of the principal component analysis. Then, calculate the PC1 for the point B in Table 3.2. (5 marks)

Solution. The PVEs are

$$\begin{aligned}
PVE_i &= \frac{(2.6054^2, 2.1215^2, 1.5351^2, 1.0690^2, 0.6889^2, 0.1514^2)}{2.6054^2 + 2.1215^2 + 1.5351^2 + 1.0690^2 + 0.6889^2 + 0.1514^2} \\
&= \frac{(6.7881 \, 4.5008 \, 2.3565 \, 1.1428 \, 0.4746 \, 0.0229)}{15.28566959} \qquad \text{[2 marks]} \\
&= (0.4441, 0.2944, 0.1542, 0.0748, 0.0310, 0.0015)
\end{aligned}$$

To find the PC1 of B, we first find the centre of the data:

$$(1.2375, 4.0750, 8.8750, 3.9375, 2.3000, 5.8875) \qquad \text{[1 mark]}$$

9

and then calculate the PC1 for point B to be

$$PC1(B) = -0.10535(1.8 - 1.2375) - 0.66557(3.3 - 4.0750)$$
$$- 0.35074(8.5 - 8.8750) - 0.54773(3.3 - 3.9375)$$
$$+ 0.09398(5.1 - 2.3000) - 0.33772(6.3 - 5.8875)$$
$$= 1.061097$$

[2 marks]

☐

9. (Final Exam May 2024 Sem, Q3(a)) Given the two-dimensional data in Table 5.1.

Table 3.1: Two-dimensional data

x_1	x_2
1	3.0
2	5.0
3	7.0
1	3.9
2	4.3
3	4.7

Find the sample covariance matrix for the data in Table 3.1 and then write down the principal components of a data **x** based on the data in Table 3.1 by first finding all normalised eigenvectors of the PCA. (12 marks)

Solution. To find the sample covariance, we first find the column averages

$$\overline{x_1} = 2, \quad \overline{x_2} = 4.65$$

[1 mark]

The sample covariance matrix is

$$C = \frac{1}{6-1} \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 1 \\ -1.65 & 0.35 & 2.35 & -0.75 & -0.35 & 0.05 \end{bmatrix} \begin{bmatrix} -1 & -1.65 \\ 0 & 0.35 \\ 1 & 2.35 \\ -1 & -0.75 \\ 0 & -0.35 \\ 1 & 0.05 \end{bmatrix}$$

[2 marks]

$$= \frac{1}{5} \begin{bmatrix} 4 & 4.8 \\ 4.8 & 9.055 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.96 \\ 0.96 & 1.811 \end{bmatrix}$$

Note: Many students make computation error with the covariance matrix, why?
We now find the eigenvalues for C:

$$|C - \lambda I| = \begin{vmatrix} 0.8 - \lambda & 0.96 \\ 0.96 & 1.811 - \lambda \end{vmatrix} = \lambda^2 - 2.611\lambda + 0.5272 = 0$$

[3 marks]

we obtain the eigenvalues of the covariance matrix:

$$\lambda = 2.390456, \ 0.220544$$

[1 mark]

The normalised eigenvector corresponding to $\lambda = 2.390456$ is obtained from

$$\begin{bmatrix} 0.8 - 2.390456 & 0.96 \\ 0.96 & 4.4848 - 2.390456 \end{bmatrix} \mathbf{x}_1 = \mathbf{0}$$

[2 marks]

$$\Rightarrow \mathbf{x}_1 = \frac{1}{\sqrt{(0.96)^2 + (-1.590456)^2}} \begin{bmatrix} 0.96 \\ -(-1.590456) \end{bmatrix} = \begin{bmatrix} 0.5167605 \\ 0.8561300 \end{bmatrix}$$

By orthogonality, the normalised eigenvector corresponding to $\lambda = 0.220544$ is

$$\mathbf{x}_2 = \begin{bmatrix} 0.8561300 \\ -0.5167605 \end{bmatrix} \qquad \text{[1 mark]}$$

The principal components are

$$PC_1(\mathbf{x}) = 0.5167605(x_1 - 2) + 0.8561300(x_2 - 4.65) \qquad \text{[1 mark]}$$
$$PC_2(\mathbf{x}) = 0.8561300(x_1 - 2) - 0.5167605(x_2 - 4.65) \qquad \text{[1 mark]}$$

Average: 8.76 / 12 marks in May 2024; 14.8% below 6 marks. $\qquad\square$

10. (Final Exam Feb 2025 Sem, Q3(b)) Given the two-dimensional data in Table 3.1.

Table 3.1: Two-dimensional data `d.f.`

Obs.	x	y
A	2.3	3.15
B	2.1	3.05
C	1.0	2.50
D	0.9	0.77
E	0.6	0.68

(a) If the output of `cov(d.f)` is

$$\begin{array}{cc} & \begin{array}{cc} x & \quad y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 0.587 & 0.81100 \\ 0.811 & 1.48145 \end{pmatrix}, \end{array}$$

write down the principal components of a data $\mathbf{x}$ based on the data in Table 3.1 by first finding all normalised eigenvectors of the PCA. State the principal components for the data C in Table 3.1. (10 marks)

Solution. We find the eigenvalues for $C = \mathtt{cov(d.f)}$:

$$|C - \lambda I| = \begin{vmatrix} 0.587 - \lambda & 0.811 \\ 0.811 & 1.48145 - \lambda \end{vmatrix} = \lambda^2 - 2.06845\lambda + 0.2118901 = 0 \quad \text{[3 marks]}$$

we obtain the eigenvalues of the covariance matrix:

$$\lambda = 1.960363, \ 0.1080872 \qquad \text{[1 mark]}$$

The normal eigenvector corresponding to $\lambda = 1.960363$ is obtained from

$$\begin{bmatrix} 0.587 - 1.960363 & 0.811 \\ 0.811 & 1.48145 - 1.960363 \end{bmatrix} \mathbf{x}_1 = \mathbf{0}$$

$$\Rightarrow \mathbf{x}_1 = \frac{1}{\sqrt{((0.811)^2 + (-1.373363)^2}} \begin{bmatrix} 0.811 \\ -(-1.373363) \end{bmatrix} = \begin{bmatrix} 0.50848 \\ 0.86107 \end{bmatrix} \qquad \text{[2 marks]}$$

By orthogonality, the normal eigenvector corresponding to $\lambda = 0.1080872$ is

$$\mathbf{x}_2 = \begin{bmatrix} 0.86107 \\ -0.50848 \end{bmatrix} \qquad \text{[1 mark]}$$

The column means of the data in Table 3.1 are

$$1.38, \quad 2.03$$

and the principal components of the data in Table 3.1 are

$$PC_1(\mathbf{x}) = 0.50848(x - 1.38) + 0.86107(y - 2.03)$$
$$PC_2(\mathbf{x}) = 0.86107(x - 1.38) - 0.50848(y - 2.03)$$

[1 mark]

The principal components for the data C in Table 3.1 are

$$PC_1(C(1, 2.5)) = 0.50848(1 - 1.38) + 0.86107(2.5 - 2.03) = 0.21148$$
$$PC_2(C(1, 2.5)) = 0.86107(1 - 1.38) - 0.50848(2.5 - 2.03) = -0.56619$$

[2 marks]

$\square$