UECM1304 Tutorial 2: Logic and Arguments of Quantified Statements

2 hours

Predicates and Quantifiers

1. Let P be a proposition and $Q(x, y)$ be a predicate. Are the following strings well-formed formula?

   (a) $\sim$ ....

   (b) $(\sim (P)) \wedge (P)$ ....

   (c) $\forall x((P \to Q(x, y)) \vee (P))$ ....

   (d) $\sim \forall x Q(x, x^2) \equiv \exists x \sim Q(x, x^2)$ ....

   (e) $x^2 + y^2 - 3z^2$ ....

   (f) $x^2 + y^2 - 3z^2 = 0$ ....

2. Let a be a constant, f_i be functions and P_i be predicates. Determine the bound and free variables of the following formula.

   (a) $(\forall x_2 P_1(x_1, x_2)) \to P_1(x_2, a)$. ....

   (b) $P_1(x_3) \to \sim \forall x_1 \forall x_2 P_2(x_1, x_2)$. ....

   (c) $\forall x_2(P_1(f_1(x_1, x_2), x_1) \to \forall x_1 P_2(x_3, f_2(x_1, x_2)))$ ....
   ....

Equivalence Forms of the Universal and Existential statements

3. Show that the statements $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logically equivalent in general.

4. Determine whether the statements $\forall x P(x) \wedge \exists x Q(x)$ and $\forall x \exists y(P(x) \wedge Q(y))$ are logically equivalent.

5. Determine whether $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ have the same truth value. Explain.

6. Determine the truth value of the following statements assuming we are interpreting these formulae over the real number domain.

   (a) $\forall x(x > \frac{1}{x})$. ....

   (b) $\exists x(x \in \mathbb{Z} \to \frac{x-3}{x} \notin \mathbb{Z})$. ....

   (c) $\exists m \exists n(m \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge m > 0 \wedge n > 0 \wedge mn \geq m + n)$. ....

   (d) $\forall x \forall y(\sqrt{x + y} = \sqrt{x} + \sqrt{y})$. ....

7. Determine the truth value of each of these statements if they are modelled over the set of integers.

   (a) $\forall n \exists m (n^2 < m)$ ....

   (b) $\exists n \forall m (nm = m)$ ....

   (c) $\exists n \exists m (n^2 + m^2 = 6)$ ....

8. Let $P(x, y)$ be a predicate and the domain of discourse be a nonempty set. Given that $\forall x \exists y P(x, y)$ is true, which of the following are not necessarily true?

   (a) $\exists x \exists y P(x, y)$ ....

   (b) $\forall x \forall y P(x, y)$ ....

   (c) $\exists x \forall y P(x, y)$ ....

9. What are the truth values of $\exists y \forall x (y \geq x)$ and $\forall x \exists y (y \geq x)$ if they are interpreted over the model with the domain of nonnegative integers?

10. Suppose the model M of the predicate $P(x, y)$ has a universe of discourse $D = \{1, 2, 3\}$ and corresponding Boolean function (an equivalent way to describe a relation) P^M. Write the propositions $\exists y \sim P(1, y)$ and $\forall x P(x, 2)$ using disjunctions and conjunctions.

11. Let odd(x) be the predicate "x is an odd positive integer." Determine the truth value of each of the following statements for the model M with domain of positive integers. If the statement is false, provide an explanation or suggest a counterexample.

    (a) $\forall x \forall y (\text{odd}(x + y))$. ....

    (b) $\exists x \forall y (\text{odd}(x + y))$. ....

    (c) $\forall x \exists y (\text{odd}(x + y))$. ....

    (d) $\exists x \exists y (xy + 1 = 0)$. ....

12. Consider the following models

    $M_1 = (\mathbb{N}, \{=\}, \{+^{\mathbb{N}}, \cdot^{\mathbb{N}}\}, 0^{\mathbb{N}})$ where

$$(=^{\mathbb{N}}) = \{(n, n) | n \in \mathbb{N}\},$$
$$+^{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m, n) \mapsto m + n,$$
$$\cdot^{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m, n) \mapsto m \cdot n.$$

    $M_2 = (\mathbb{Q}, \{=\}, \{+^{\mathbb{Q}}, \cdot^{\mathbb{Q}}\}, 0^{\mathbb{Q}})$ where

$$(=^{\mathbb{Q}}) = \{(n, n) | n \in \mathbb{Q}\},$$
$$+^{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (m, n) \mapsto m + n,$$
$$\cdot^{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (m, n) \mapsto m \cdot n.$$

    Determine the truth value of

    (a) $\forall x_1 \forall x_2 \exists x_3 (x_1 + x_2 = x_3)$ and

    (b) $\forall x_1 \forall x_2 ((\exists x_3 (x_1 \cdot x_3 = x_2) \wedge \exists x_4 (x_2 \cdot x_4 = x_1) \to x_1 = x_2)$.

13. Determine whether each of the following statements is true or false over the model of integer sets.

(a) $\forall x \exists y \exists z (x = 7y + 5z)$ ....

(b) $\forall x \exists y \exists z (x = 31y + 41z)$ ....

(c) $\forall x \exists y \exists z (x = 4y + 6z)$ ....

Translating between Formal and Informal Language

14. Consider the predicates

$LT(x, y) : x < y$	$EQ(x, y) : x = y$	$EV(x)$: x is even
$I(x)$: x is an integer	$P(x) : x > 0$	$R(x)$: $x \in \mathbb{R}$.

(a) Write the statement using ONLY these predicates and any needed quantifiers.

i. Every integer is even. ....

ii. Some real numbers are negative integers. ....
....

iii. If $x < y$, then x is not equal to y. ......

iv. There is no largest number. ....

v. If $y > x$ and $0 > z$, then $x \cdot z > y \cdot z$. ....
....

(b) Write the statement $\exists x \sim P(x)$ in English without using variables and symbols. ...
....

15. Let $M(s)$ denote "s is a math major", $C(s)$ denote "s is a computer science student" and $E(s)$ denote "s is an engineering student". Rewrite the following statements by using quantifiers, variables and predicates $M(s)$, $C(s)$ and $E(s)$.

(a) There is an engineering student who is a math major. ..

(b) Every computer science student is an engineering student.

(c) No computer science students are engineering students. .

(d) Some computer science students are engineering students and some are not.
....

16. Let $H(x)$ denote the predicate "x is a human" and $T(x, y)$ denote the predicate "x trusts y". Rewrite the following formula into English sentence without quantifiers and variables.

(a) $\forall x \exists y (H(x) \rightarrow (H(y) \wedge T(x, y)))$ ....

(b) $\exists x \forall y (H(x) \wedge (H(y) \rightarrow \sim T(x, y)))$. ....

(c) $\forall x \forall y (H(x) \rightarrow (H(y) \rightarrow \sim T(x, y))$. ....

Argument with Quantified Statements and Validity

17. Consider the following statement:

$$\exists x (x \in \mathbb{R} \wedge x^2 = 2).$$

Which of the following are equivalent ways of expressing this statement?

(a) The square of each real number is 2. ....

(b) Some real numbers have square 2. .... ☐

(c) If x is a real number, then $x^2 = 2$. .... ☐

18. Let $E(n)$ be the predicate "n is even" and consider the following statement:

$$\forall n(n \in \mathbb{Z} \to (E(n^2) \to E(n))).$$

Which of the following are equivalent ways of expressing this statement?

(a) All integers have even squares and are even. .... ☐

(b) Given any integer whose square is even, that integer is itself even. .... ☐

(c) For all integers, there are some whose square is even. .... ☐

(d) Any integer with even square is even. .... ☐

(e) All even integers have even squares. .... ☐

19. Give a negation for each statement below:

(a) For all integers x, if x is odd, then $x^2 - 1$ is even.

(b) There exists an integer x with $x \geq 2$ such that $x^2 - 4x + 7$ is prime.

(c) For all real numbers x and y, if $x = y$, then $x^2 = y^2$.

(d) There is no easy question in the exam.

(e) If the square of real number x is greater than or equal to 1 then $x > 0$.

20. For the following arguments, state which are valid and which are invalid. Justify your answers.

(a) All healthy people eat an apple a day. John is not a healthy person. Therefore John does not eat an apple a day.

(b) Every student who studies discrete mathematics is good at logic. John studies discrete mathematics. Therefore John is good at logic.

(c) No heavy object is cheap. XYZ is not a heavy object. Therefore XYZ is cheap.

21. Use ONLY the rules of inference to show that

$$\forall x(P(x) \to (Q(x) \wedge R(x))),\ \forall x(P(x) \wedge S(x)) \ \vdash\ \exists x(R(x) \wedge S(x))$$

22. Show that $\sim (\forall x(P(x) \to Q(x))) \Rightarrow \exists x(P(x) \wedge \sim Q(x))$.

23. Use rules of inference to show that

$$\exists x P(x) \to \forall x(P(x) \vee Q(x) \to R(x)),\ \exists x(P(x) \wedge Q(x)) \vdash \exists y R(y)$$

24. Show that $\forall x(Q(x) \to R(x)) \wedge (\exists x(Q(x) \wedge I(x)) \vdash \exists x(R(x) \wedge I(x))$.

25. Prove that the following argument is valid: "Every undergraduate is either an arts student or a science student. Some undergraduates are top students. James is not a science student, but he is a top student. Therefore if James is an undergraduate, he is an arts student."

26. Show that the following argument is valid.

$$\begin{array}{ll} & \exists x(F(x) \wedge S(x)) \to (\forall y(M(y) \to W(y))) \\ & \exists y(M(y) \wedge \sim W(y)) \\ \hline \therefore & \forall x(F(x) \to \sim S(x)) \end{array}$$

27. What is wrong with the following proof?

1	$\forall x \exists y (x > y)$	premise
2	$\exists y (c > y)$	universal instantiation, c arbitrary
3	$(c > s)$	existential instantiation, s specific
4	$\forall x (x > s)$	universal generalisation
5	$\exists y \forall x (x > y)$	existential generalisation

28. Derive the following rule using laws of equivalence:

$$\sim (\forall x (x \in D \rightarrow (\forall y (y \in E \rightarrow P(x, y))) \equiv \exists x \exists y (x \in D \wedge (y \in E \wedge \sim P(x, y))$$

29. Show that $\forall x[(C(x) \wedge \exists y(T(y) \wedge L(x, y))) \rightarrow \exists y(D(y) \wedge B(x, y))] \equiv \forall x \forall y \exists z[(C(x) \wedge T(y) \wedge L(x, y)) \rightarrow (D(z) \wedge B(x, z))]$

30. Write a negation for the following statement:

For all real numbers $y > 0$, there exists a real number $z > 0$ such that if $a - z < x < a + z$ then $L - y < f(x) < L + y$.