UECM1304/UECM1303 Test 1 Marking Guide

Name: Student ID: Mark: /20

COURSE CODE & COURSE TITLE: UECM1304/3 DISCRETE MATHEMATICS WITH APPLICATIONS
FACULTY: LKC FES, UTAR COURSE: AM, AS, SE
SESSION: MAY 2019 LECTURER: KOAY HANG LEEN, LIEW HOW HUI

Instruction: Answer all questions in the space provided. **If you do not write your answer in the space provided, you will get ZERO mark**. An answer without working steps may also receive ZERO mark.

1. CO1: Recognise statements and quantified statements. C1

(a) Given the atomic statements p, q and r. **State** the truth table of the following statement

$$\sim (p \vee (q \wedge \sim r)) \rightarrow (\sim p \wedge \sim q \wedge \sim r).$$

Recognise whether the statement is a tautology, contingency or contradiction (4.5 marks)

Ans. The truth table is stated below.

p	q	r	$\sim (p \vee (q \wedge \sim r)) \rightarrow (\sim p \wedge \sim q \wedge \sim r)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

.... [0.5 × 8=4 marks]

Since there is a truth assignment in which the statement is true and there is a truth assignment in which the statement is false, the statement is a **contingency**. .. [0.5 mark]

(b) Given the domain of discourse is $\mathbb{R}$. Translate the following quantified statement

$$\forall x \exists y (x > 0 \rightarrow (y > 0 \wedge x = y^2))$$

to English sentence. Marks will be deducted if your English sentence is more than 18 words. (1 mark)

Ans. Every positive number is the square of some positive number. .... [1 mark]

(c) Given that p, q, r are atomic statements, for the statement

$$(p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r),$$

identify the logical equivalent statement with no more than 8 logical connectives from the set $\{\sim, \wedge, \vee\}$. If you use the logical connectives $\to$ and $\leftrightarrow$, marks will be deducted.

(2 marks)

Ans. $(p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge (r \vee \sim r)) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

[2,3 distributive law; 0.4 mark]

$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge \sim q) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

[2, negation and identity; 0.4 mark]

$\equiv ((p \vee \sim p) \wedge q \wedge \sim r) \vee (p \wedge \sim q) \vee (\sim p \wedge q \wedge r)$ [1,4 distributive law; 0.3 mark]

$\equiv (q \wedge \sim r) \vee (p \wedge \sim q) \vee (\sim p \wedge q \wedge r)$ [1, negation and identity; 0.4 mark]

$\equiv (q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q) \vee (\sim p \wedge q \wedge r)$ [1, absorption law; 0.2 mark]

$\equiv (q \wedge \sim r) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$ [2,4, distributive, negation, identity; 0.3 mark]

(d) Use the **laws of logical equivalence** to transform the following quantified statement

$$(\exists x \forall y (p(x,y))) \vee \sim \exists y (q(y) \to \forall z r(z))$$

to prenex normal form. (2 marks)

Ans. $(\exists x \forall y (p(x,y))) \vee \sim \exists y (q(y) \to \forall z r(z))$

$\equiv \quad (\exists x \forall y (p(x,y))) \vee \forall y \sim (q(y) \wedge \sim \forall z r(z))$ Generalised De Morgan [0.3 mark]

$\equiv \quad (\exists x \forall y (p(x,y))) \vee \forall y (q(y) \wedge \sim \forall z r(z))$ Implication & De Morgan [0.4 mark]

$\equiv \quad (\exists x \forall y (p(x,y))) \vee \forall y (q(y) \wedge \exists z \sim r(z))$ Generalised De Morgan [0.3 mark]

$\equiv \quad (\exists x \forall y (p(x,y))) \vee \forall y \exists z (q(y) \wedge \sim r(z))$ Free variable law [0.4 mark]

$\equiv \quad \exists x \forall y (p(x,y) \vee \forall y_2 \exists z (q(y_2) \wedge \sim r(z)))$ Independent quantifier law [0.3 mark]

$\equiv \quad \exists x \forall y \forall y_2 \exists z (p(x,y) \vee (q(y_2) \wedge \sim r(z)))$ Independent quantifier law [0.3 mark]

(e) The definition of "pointwise convergence" of a sequence of functions $\{f_n\}$ to a function f on an interval $A \subset \mathbb{R}$ can be defined by the following quantified statement

$$\forall x \forall \epsilon \exists N \forall n \Big[(x \in A) \to \Big[(\epsilon > 0) \to \Big((N \in \mathbb{N}) \wedge ((n \geq N) \to |f_n(x) - f(x)| < \epsilon) \Big) \Big] \Big].$$

Write the negation of this statement in prenex normal form, i.e. apply $\sim$ to the quantified statement and then write it into the logically equivalent prenex normal form. (0.5 mark)

Ans. The negation of the quantified statement is

$$\exists x \exists \epsilon \forall N \exists n \Big[(x \in A) \wedge \Big[(\epsilon > 0) \wedge \Big((N \in \mathbb{N}) \to ((n \geq N) \wedge |f_n(x) - f(x)| \geq \epsilon) \Big) \Big] \Big].$$ [0.5 mark]

2. CO2. Determine the validity of an argument. C2

(a) Given the following argument:

$$\begin{array}{rr} & p \vee q \\ & p \wedge q \to r \\ & q \wedge \sim r \\ \hline \therefore & \sim p \end{array}$$

Either use the comparison table to **defend** that the argument is valid or **give a counter example** to show that the argument is invalid. (4 marks)

Ans. The comparison table is stated below: .... [3.5 marks]

p	q	r	$p \vee q$	$p \wedge q \to r$	$q \wedge \sim r$	$\sim p$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	T	T	F	F
T	F	F	T	T	F	F
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

Scanning through rows 1 to 4 of the comparison table, there is no truth assignment for which the premises are true but the conclusion is false. Therefore the argument is valid.
.... [0.5 mark]

(b) Let p, q, r and s be atomic statements. Use the **laws of logical equivalence and implication** to *explain* the validity of the following argument:

$$\begin{array}{rr} & \sim p \vee q \\ & \sim q \vee s \\ \hline \therefore & \sim p \vee r \vee s \end{array}$$

(2 marks)

Ans.

ϕ_1	$\sim p \vee q$	premise
ϕ_2	$\sim q \vee s$	premise
ψ_1	$p \to q$	ϕ_1, Implication law .... [0.4 mark]
ψ_2	$q \to s$	ϕ_2, Implication law .... [0.4 mark]
ψ_3	$p \to s$	ψ_1, ψ_2, Transitivity .... [0.4 mark]
ψ_4	$\sim p \vee s$	ψ_3, Implication law .... [0.4 mark]
$\therefore$	$\sim p \vee r \vee s$	ψ_4, Generalisation, Commutative law [0.4 mark]

(c) Use **only** the **rules of inference** and fitch style proof to **infer** the argument

$$p \to \sim (q \vee r),\ q \vdash \sim p.$$

[**Warning:** If you use any other rules, you will receive ZERO for this question.] (2 marks)

Ans.

1	$p \to \sim (q \vee r)$	premise	
2	q	premise	
3	$\quad p$	assumption	[0.4 mark]
4	$\quad \sim (q \vee r)$	1,3 $\to$E	[0.4 mark]
5	$\quad q \vee r$	2 $\vee$I	[0.4 mark]
6	$\quad \perp$	4,5 $\neg$E	[0.4 mark]
7	$\sim p$	3–6 $\neg$I	[0.4 mark]

(d) Let $P(x)$ and $Q(x)$ be predicates. Use **either** the *laws of logical equivalence and implication* **or** *rules of inference* to *explain* the validity of the following argument:

$$\forall x(Q(x) \to P(x)),\ \exists x Q(x) /\therefore \exists x(Q(x) \wedge P(x)).$$ (2 marks)

Ans. Using the laws of logical equivalence and implication, the inference goes as follows:

ϕ_1	$\forall x(Q(x) \to P(x))$	premise	
ϕ_2	$\exists x Q(x)$	premise	
ψ_1	$Q(s)$	ϕ_2 existential initialisation	[0.4 mark]
ψ_2	$Q(s) \to P(s)$	ϕ_1 universal initialisation	[0.4 mark]
ψ_3	$P(s)$	ψ_1, ψ_2 MP	[0.4 mark]
ψ_4	$Q(s) \wedge P(s)$	ψ_1, ψ_3 conjuction	[0.4 mark]
$\therefore$	$\exists x(Q(x) \wedge P(x))$	ψ_4 existential generalisation	[0.4 mark]

Laws of Logical Equivalence and Implication

Let p, q and r be atomic statements, T be a tautology and F be a contradiction. Suppose the variable x has no free occurrences in ξ and is substitutable for x in ξ. Then

1.	Double negative law:	$\sim(\sim p) \equiv p.$
2.	Idempotent laws:	$p \wedge p \equiv p;$ $\quad p \vee p \equiv p.$
3.	Universal bound laws:	$p \vee T \equiv T;$ $\quad p \wedge F \equiv F.$
4.	Identity laws:	$p \wedge T \equiv p;$ $\quad p \vee F \equiv p.$
5.	Negation laws:	$p \vee \sim p \equiv T;$ $\quad p \wedge \sim p \equiv F.$
6.	Commutative laws:	$p \wedge q \equiv q \wedge p;$ $\quad p \vee q \equiv q \vee p.$
7.	Absorption laws:	$p \vee (p \wedge q) \equiv p;$ $\quad p \wedge (p \vee q) \equiv p.$
8.	Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r);$ $\quad (p \vee q) \vee r \equiv p \vee (q \vee r).$
9.	Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r);$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$
10.	De Morgan's laws:	$\sim(p \wedge q) \equiv \sim p \vee \sim q;$ $\quad \sim(p \vee q) \equiv \sim p \wedge \sim q.$
11.	Implication law:	$p \to q \equiv \sim p \vee q$
12.	Biconditional law:	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p).$
13.	Modus Ponens (MP in short):	$p \to q,\ p \models q$
14.	Modus Tollens (MT in short):	$p \to q,\ \sim q \models \sim p$
15.	Generalisation:	$p \models p \vee q;$ $\quad q \models p \vee q$
16.	Specialisation:	$p \wedge q \models p;$ $\quad p \wedge q \models q$
17.	Conjunction:	$p,\ q \models p \wedge q$
18.	Elimination:	$p \vee q,\ \sim q \models p;$ $\quad p \vee q,\ \sim p \models q$
19.	Transitivity:	$p \to q,\ q \to r \models p \to r$
20.	Contradiction Rule:	$\sim p \to F \models p$
21.	Quantified de Morgan laws:	$\sim \forall x \phi \equiv \exists x \sim \phi;$ $\sim \exists x \phi \equiv \forall x \sim \phi;$
22.	Quantified conjunctive law:	$\forall x(\phi \wedge \psi) \equiv (\forall x\ \phi) \wedge (\forall x\ \psi);$
23.	Quantified disjunctive law:	$\exists x(\phi \vee \psi) \equiv (\exists x\ \phi) \vee (\exists x\ \psi);$
24.	Quantifiers swapping laws:	$\forall x \forall y \phi \equiv \forall y \forall x \phi;$ $\quad \exists x \exists y \phi \equiv \exists y \exists x \phi;$
25.	Independent quantifier law:	$\xi \equiv \forall x \xi \equiv \exists x \xi;$
26.	Variable renaming laws:	$\forall x \phi \equiv \forall y \phi[y/x];$ $\quad \exists x \phi \equiv \exists y \phi[y/x];$
27.	Free variable laws:	$\forall x(\xi \wedge \psi) \equiv \xi \wedge (\forall x \psi);$ $\quad \exists x(\xi \wedge \psi) \equiv \xi \wedge (\exists x \psi);$ $\forall x(\xi \vee \psi) \equiv \xi \vee (\forall x \psi);$ $\quad \exists x(\xi \vee \psi) \equiv \xi \vee (\exists x \psi);$
28.	Universal instantiation:	$\forall x \phi \Rightarrow \phi[a/x];$
29.	Universal generalisation:	$\phi[a/x] \Rightarrow \forall x \phi;$
30.	Existential instantiation:	$\exists x \phi \Rightarrow \phi[s/x];$
31.	Existential generalisation:	$\phi[s/x] \Rightarrow \exists x \phi.$

Rules of Inference

Let ϕ, ψ, ξ be any well-formed formulae. Then

1.	$\wedge$-introduction:	$\phi,\ \psi \vdash \phi \wedge \psi$
2.	$\wedge$-elimination:	$\phi \wedge \psi \vdash \phi$ or $\phi \wedge \psi \vdash \psi$
3.	$\rightarrow$-introduction:	$\boxed{\phi,\ \cdots,\ \psi} \vdash (\phi \rightarrow \psi)$
4.	$\rightarrow$-elimination:	$\phi \rightarrow \psi,\ \phi \vdash \psi$
5.	$\vee$-introduction:	$\phi \vdash \phi \vee \psi$ or $\psi \vdash \phi \vee \psi$
6.	$\vee$-elimination:	$\phi \vee \psi,\ \boxed{\phi,\ \cdots,\ \xi},\ \boxed{\psi,\ \cdots,\ \xi} \vdash \xi$
7.	$\neg$-introduction or $\sim$-introduction:	$\boxed{\sim \phi,\ \cdots,\ \bot} \vdash \phi$ or $\boxed{\phi,\ \cdots,\ \bot} \vdash \sim \phi$
8.	$\neg$-elimination or $\sim$-elimination:	$\phi,\ \sim \phi \vdash \bot$
9.	$\forall$-introduction:	$\phi(a) \vdash \forall x \phi(x)$
10.	$\forall$-elimination:	$\forall x \phi(x) \vdash \phi(t)$
11.	$\exists$-introduction:	$\phi(t) \vdash \exists x \phi(x)$
12.	$\exists$-elimination:	$\exists x \phi(x), \boxed{\phi(s) \cdots \xi} \vdash \xi$

The term t is free with respect to x in ϕ and $[t/x]$ means "t replaces x".