PART A: Answer **ALL** questions.

Q1. (a) Let p, q, r be atomic statements. **State** the truth table for the following compound statement

$$\sim(p \to ((p \vee q) \wedge r)).$$

Use the truth table to **recognise** whether the compound statement is a tautology, contingency or contradiction. (10 marks)

Ans. The truth table is stated below. .... [8 marks]

p	q	r	$(p \vee q) \wedge r$	$p \to ((p \vee q) \wedge r)$	$\sim(p \to ((p \vee q) \wedge r))$
T	T	T	T	T	F
T	T	F	F	F	T
T	F	T	T	T	F
T	F	F	F	F	T
F	T	T	T	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

It is sometimes true, sometimes false, depending on the truth assignment, by definition, the compound statement is a *contingency*. .... [2 marks]

(b) Show that the statement $(p \to q \vee r)$ and the statement $(p \wedge q \to r)$ are not logically equivalent. (4 marks)

Ans. One can either construct a truth table or just give a counterexample below to show that they are not equivalent:

p	q	r	$p \to q \vee r$	$p \wedge q \to r$
T	T	T	T	T
T	T	F	T	F

.... [2 marks]

When $v(p)$ =T, $v(q)$ =T and $v(r)$ =F, the two statements has different truth values and they are not logically equivalent. [2 marks]

(c) Simplify the following statement

$$((p \vee q) \to (p \wedge q)) \vee (\sim p \wedge q).$$

to a logically equivalent statement with no more than TWO(2) logical connectives from the set $\{\sim, \wedge, \vee\}$ by stating the law used in each step of the simplification. (5 marks)

Ans. The steps are shown below:

$((p \vee q) \to (p \wedge q)) \vee (\sim p \wedge q)$
$\equiv (\sim(p \vee q) \vee (p \wedge q)) \vee (\sim p \wedge q)$ [Implication law, 1 mark]
$\equiv (\sim p \wedge \sim q) \vee (p \wedge q) \vee (\sim p \wedge q)$ [de Morgan law, 1 mark]
$\equiv (\sim p \wedge \sim q) \vee (p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge q)$ [Idempotent law, 1 mark]
$\equiv \sim p \wedge (q \vee \sim q) \vee ((p \vee \sim p) \wedge q)$ [Distributive law, 1 mark]
$\equiv \sim p \vee q$ [Negation and identity, 1 mark]

(d) Let $F(u,x,y)$, $G(y,v)$ and $H(x)$ be predicates. **List** down the steps and the logical equivalent rules to transform the following quantified statement

$$\sim\left[\forall x\exists y F(u,x,y)\rightarrow\exists x\big(\sim\forall y G(y,v)\rightarrow H(x)\big)\right]$$

to prenex normal form. (6 marks)

Ans. The steps and rules are listed below:

$\sim\left[\forall x\exists y F(u,x,y)\rightarrow\exists x\big(\sim\forall y G(y,v)\rightarrow H(x)\big)\right]$

$\equiv\ \sim\left[\sim\forall x\exists y F(u,x,y)\vee\exists x\big(\sim\forall y G(y,v)\rightarrow H(x)\big)\right]$ [Implication law, 0.5 mark]

$\equiv\forall x\exists y F(u,x,y)\bigwedge\sim\exists x\big(\sim\forall y G(y,v)\vee H(x)\big)$

[de Morgan law, double negative, 1 mark]

$\equiv\forall x\exists y F(u,x,y)\bigwedge\left[\forall x\sim\big(\sim\forall y G(y,v)\rightarrow H(x)\big)\right]$

[Generalised de Morgan law, 0.5 mark]

$\equiv\forall x\exists y F(u,x,y)\bigwedge\left[\forall x\sim\big(\forall y G(y,v)\vee H(x)\big)\right]$

[Implication law, double negative, 1 mark]

$\equiv\forall x\exists y F(u,x,y)\bigwedge\left[\forall x\big(\forall y\sim G(y,v)\wedge\sim H(x)\big)\right]$

[Generalised de Morgan law, 0.5 mark]

$\equiv\forall x\exists y F(u,x,y)\bigwedge\left[\forall x\forall y\big(\sim G(y,v)\wedge\sim H(x)\big)\right]$ [Free variable law, 0.5 mark]

$\equiv\forall x\left[\exists y F(u,x,y)\bigwedge\forall y\big(\sim G(y,v)\wedge\sim H(x)\big)\right]$ [Quantified conjunctive law, 0.5 mark]

$\equiv\forall x\left[\exists y F(u,x,y)\bigwedge\forall z\big(\sim G(z,v)\wedge\sim H(x)\big)\right]$ [Quantifier renaming law, 0.5 mark]

$\equiv\forall x\exists y\forall z\left[F(u,x,y)\wedge\sim G(z,v)\wedge\sim H(x)\right]$ [Free variable law, 1 mark]

Q2. (a) Let p, q, r be atomic statements. Use a truth table or a comparison table to show that

$$(p\rightarrow r)\wedge(q\rightarrow r)\equiv(p\vee q)\rightarrow r.$$ (9 marks)

Ans. The comparison table is given below.

p	q	r	$(p\rightarrow r)\wedge(q\rightarrow r)$	$(p\vee q)\rightarrow r$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	T	T

.... [8 marks]

Since the last two columns are the same for all different assignments, therefore, the two statements $(p\rightarrow r)\wedge(q\rightarrow r)$ and $(p\vee q)\rightarrow r$ are logically equivalent. ...... [1 mark]

(b) Simplify the following statement to a logically equivalent statement with no more than TWO(2) logical connectives from the set $\{\wedge,\ \vee\}$ by stating the law used in each step of the simplification:

$$(\sim p\wedge q)\vee(\sim p\wedge r)\vee(p\wedge\sim q\wedge r)\vee(q\wedge r).$$ (7 marks)

Ans. The simplification is shown below:

$(\sim p \wedge q) \vee (\sim p \wedge r) \vee (p \wedge \sim q \wedge r) \vee (q \wedge r)$.
$\equiv (\sim p \wedge q) \vee \left[\sim p \vee (p \wedge \sim q) \vee q\right] \wedge r$. [Distributive law on last 3 terms, 2 marks]
$\equiv (\sim p \wedge q) \vee \left[(\sim p \vee p) \wedge (\sim p \vee \sim q) \vee q\right] \wedge r$. [Distributive law on $\sim p \vee$, 1 mark]
$\equiv (\sim p \wedge q) \vee \left[(\sim p \vee \sim q) \vee q\right] \wedge r$. [Negation, Identity, 1 mark]
$\equiv (\sim p \wedge q) \vee \left[\sim p \vee \mathrm{T}\right] \wedge r$. [Associativity, Negation, 1 mark]
$\equiv (\sim p \wedge q) \vee \mathrm{T} \wedge r$. [Universal bound, 1 mark]
$\equiv (\sim p \wedge q) \vee r$. [Identity, 1 mark]

(c) Given the following quantified statement:

$$\forall x \forall y \left[((x>0) \wedge (y>0)) \rightarrow \left(\sqrt{x+y} = \sqrt{x} + \sqrt{y}\right)\right]. \qquad (*)$$

(i) Translate the quantified statement into an informal English sentence. (2 marks)
Ans. The square root of the sum of two numbers is equal to the sum of the square roots of the two numbers

(ii) Determine whether the quantified statement is true or false in the domain of real numbers. You need to defend your answer. (2 marks)
Ans. The quantified statement is *false*. .... [1 mark]
To defend, we write a counterexample: Let $x = y = 1$, $\sqrt{x+y} = \sqrt{2} \neq \sqrt{1} + \sqrt{1} = 2$. .... [1 mark]

(iii) Write down the negation of the quantified statement $(*)$ in prenex normal form. (5 marks)
Ans. By applying the generalised de Morgan law, the negation of $(*)$ is logically equivalent to

$$\exists x \exists y \sim \left[((x>0) \wedge (y>0)) \rightarrow \left(\sqrt{x+y} = \sqrt{x} + \sqrt{y}\right)\right].$$

In prenex normal form, it can be written as

$$\exists x \exists y \left[(x>0) \wedge (y>0) \wedge \left(\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}\right)\right]. \qquad \text{[5 marks]}$$

PART B: Answer **ALL** questions.

Q3. (a) Use **truth table** to explain whether the following argument is valid or invalid:

$$\begin{array}{rr} & (p \vee q) \to (p \wedge q) \\ & \sim (p \vee q) \\ \hline \therefore & \sim (p \wedge q) \end{array}$$

(9 marks)

Ans. The truth table is

p	q	$(p \vee q) \to (p \wedge q)$	$\sim (p \vee q)$	$\sim (p \wedge q)$
T	T	T	F	F
T	F	F	F	T
F	T	F	F	T
F	F	T	T	T

. . . . [$4 \times 2 = 8$ marks]

We observe that when the premises are true (row 4), the conclusion is true, therefore, the argument is **valid**. . . . . [1 mark]

(b) Infer the argument

$$p \vee q,\ p \to r,\ \sim s \to \sim q \vdash r \vee s$$

syntatically by stating the **rules of inference** in each step. (6 marks)

Ans.

1	$p \vee q$	premise
2	$p \to r$	premise
3	$\sim s \to \sim q$	premise
4	p	assumption
5	r	2,4 $\to$E
6	$r \vee s$	5 $\vee$I
7	q	assumption
8	$\sim s$	assumption
9	$\sim q$	3,8 $\to$E
10	$\perp$	7,9 $\neg$E
11	s	8–10 $\neg$I
12	$r \vee s$	11 $\vee$I
13	$r \vee s$	1,4–6,6–12 $\vee$E

The p-assumption . . . . [2 marks]
The q-assumption . . . . [3 marks]
Line 12 . . . . [1 mark]

(c) Show that the following argument

$$\begin{array}{rr} & \forall x(F(x) \to \sim G(x)) \\ & \exists x(H(x) \wedge G(x)) \\ \hline \therefore & \exists x(H(x) \wedge \sim F(x)) \end{array}$$

is valid using the rules of logical equivalence and implication. (5 marks)

Ans. The semantic deduction is shown below

ϕ_1	$\forall x(F(x) \to \sim G(x))$	premise
ϕ_2	$\exists x(H(x) \wedge G(x))$	premise
ψ_1	$H(s) \wedge G(s)$	ϕ_2, existential instantiation .... [1 mark]
ψ_2	$F(s) \to \sim G(s)$	ϕ_1, universal instantiation
ψ_3	$G(s)$	ψ_1, specialisation .... [1 mark]
ψ_4	$\sim F(s)$	ψ_2, ψ_3, MT .... [1 mark]
ψ_5	$H(s)$	ψ_1, specialisation .... [1 mark]
ψ_6	$H(s) \wedge \sim F(s)$	ψ_3, ψ_4 conjunction
$\therefore$	$\exists x(H(x) \wedge \sim F(x)$	ψ_6, existential generalisation .... [1 mark]

(d) Let $R(x,y)$ be a predicate with two variables. Infer the argument involving quantified statements

$$\forall x \forall y(R(x,y) \to \sim R(y,x)) \vdash \forall x(\sim R(x,x))$$

syntatically by stating the **rules of inference** in each step. (5 marks)

Ans. Let t be an arbitrary term independent of variables x and y.

1	$\forall x \forall y(R(x,y) \to \sim R(y,x))$	premise
2	$\forall y(R(t,y) \to \sim R(y,t))$	1 $\forall$-elimination .... [1 mark]
3	$R(t,t) \to \sim R(t,t)$	2 $\forall$-elimination
4	$\quad R(t,t)$	assumption .... [1 mark]
5	$\quad \sim R(t,t)$	3,4 $\to$I .... [1 mark]
6	$\quad \perp$	4,5 $\neg$E
7	$\sim R(t,t)$	4–6 $\neg$I .... [1 mark]
8	$\forall x(\sim R(x,x))$	7 $\forall$-introduction ....[1 mark]

Q4. (a) Prove by mathematical induction that $17^n - 6^n$ is divisible by 11 for every positive integer n. (8 marks)

Ans. **Base step**: When $n = 1$,

$$17^1 - 6^1 = 11 = 11 \times 1 \Rightarrow 11 \mid (17^1 - 6^1).$$

Inductive step: Suppose that the predicate $P(k)$ is valid when $n = k$, i.e.

$$11 \mid (17^k - 6^k) \Rightarrow 17^k - 6^k = 11m$$

for some integer m. When $n = k+1$,

$$\begin{aligned} 17^{k+1} - 6^{k+1} &= 17^k \times 17 - 6^k \times 6 \\ &= 17^k \times 11 + 17^k \times 6 - 6^k \times 6 \quad = 17^k \times 11 + 6 \times 11m = 11(17^k + 6m) \end{aligned}$$

which implies $11 \mid (17^{k+1} - 6^{k+1})$.
By the principle of mathematical induction, $17^n - 6^n$ is divisible by 11 for every positive integer n.

(b) Use a proof by contraposition to show that if n is an integer and $n^2 + 5$ is odd, then n is even. (5 marks)

Ans. Let n be an integer. Suppose n is odd, then there is an integer k such that $n = 2k+1$ and

$$n^2 + 5 = (2k+1)^2 + 5 = 4k^2 + 4k + 1 + 5 = 2(2k^2 + 2k + 3)$$

which shows that $n^2 + 5$ is even.

(c) Use the Euclidean algorithm to prove or disprove that $\gcd(198, 54)$ is prime. (4 marks)

Ans. $\gcd(198,54) = \gcd(54,36) = \gcd(36,18) = 18$
18 is not a prime. The statement "$\gcd(198,54)$ is prime" is disproved.

(d) Prove or disprove the following congruence relations.

(i) $-122 \equiv 5 \pmod 7$ (3 marks)
Ans. $5 - (-122) \bmod 7 = 127 \bmod 7 = 1$. Therefore, $7 \nmid (5 - (-122))$, so $-122 \not\equiv 5 \pmod 7$ and it is disproved.

(ii) $3^{2019} \equiv 27 \pmod{40}$ (5 marks)
Ans. The computation below shows that $3^{2019} \equiv 27 \pmod{40}$ is true (Python `3**2019 % 40` also confirms this).

x^2	$q/2$	$q \bmod 2$	m2
$3^2 \equiv_{40} 9$	2019/2 = 1009	1	3
$9^2 \equiv_{40} 1$	1009/2 = 504	1	$3 \times 9 \equiv_{40} = 27$
$1^2 \equiv_{40} 1$	504/2 = 252	0	27
...			

Q5. (a) Let $R = \{(x,y) \in \mathbb{N}^* \times \mathbb{N}^* \mid xy = 1\}$, where $\mathbb{N}^*$ is the set of positive integers. Determine whether R is reflexive, symmetric, or transitive. Hence, determine whether R is an equivalence relation. Justify your answers. (7 marks)

Ans. $R = \{(1,1)\}$.
Since $(2,2) \notin R$, R is not reflexive.
Since there is no symmetric pair in R, R is symmetric.
R is transitive because there is only one loop.

R is not an equivalence relation because it is not reflexive.

(b) Let R be the relation on $A = \{1,2,5,6,7,11\}$ defined by

$$xRy \text{ if } x \equiv y \pmod 5.$$

Write out the equivalence classes of R and verify that they partition A. (5 marks)

Ans.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \\ 7 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 5 & 6 & 7 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

The equivalence classes of R is $\{1,6,11\}$, $\{2,7\}$, $\{5\}$.
They partition A because their pair intersections are empty and the union is A.

(c) Let R be a relation defined on the set A whose matrix is

$$M_R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Use M_R to explain why R is not transitive. Then use the Warshall's algorithm to find the transitive closure of R. (6 marks)

Ans. From M_R we see $(3,4),(4,3) \in R$ but no $(3,3) \in R$. So R is not transitive.

Step 1: $M_R^{(1)} = M_R$.

$$\text{Step 2: } M_R^{(2)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & \boxed{1} & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

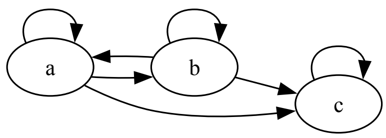
$$\text{Step 3: } M_R^{(3)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{array} = M_R^{(4)} \text{ in step 4.}$$

$cl_{trn}(R) = \{(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)\}$.

(d) (i) Define what it means for a relation R on a set A to be a partial order. (3 marks)
Ans. A relation R is said to be partial order if R is reflexive, anti-symmetric and transitive.

(ii) Let $R = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,c)\}$ a relation on $A = \{a,b,c\}$. Draw the directed graph of R and use it to explain why R is not a partial order. (4 marks)

Ans. The directed graph of R is



It is not symmetric because we have $(a,b) \in R$ and $(b,a) \in R$ in which R violates antisymmetry.

Laws of Logical Equivalence and Implication

Let p, q and r be atomic statements, T be a tautology and F be a contradiction. Suppose the variable x has no free occurrences in ξ and is substitutable for x in ξ. Then

1.	Double negative law:	$\sim(\sim p) \equiv p$.	
2.	Idempotent laws:	$p \wedge p \equiv p$;	$p \vee p \equiv p$.
3.	Universal bound laws:	$p \vee T \equiv T$;	$p \wedge F \equiv F$.
4.	Identity laws:	$p \wedge T \equiv p$;	$p \vee F \equiv p$.
5.	Negation laws:	$p \vee \sim p \equiv T$;	$p \wedge \sim p \equiv F$.
6.	Commutative laws:	$p \wedge q \equiv q \wedge p$;	$p \vee q \equiv q \vee p$.
7.	Absorption laws:	$p \vee (p \wedge q) \equiv p$;	$p \wedge (p \vee q) \equiv p$.
8.	Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$;	$(p \vee q) \vee r \equiv p \vee (q \vee r)$.
9.	Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$; $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.	
10.	De Morgan's laws:	$\sim(p \wedge q) \equiv \sim p \vee \sim q$;	$\sim(p \vee q) \equiv \sim p \wedge \sim q$.
11.	Implication law:	$p \to q \equiv \sim p \vee q$	
12.	Biconditional law:	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.	
13.	Modus Ponens (MP in short):	$p \to q,\ p \models q$	
14.	Modus Tollens (MT in short):	$p \to q,\ \sim q \models \sim p$	
15.	Generalisation:	$p \models p \vee q$; $\ q \models p \vee q$	
16.	Specialisation:	$p \wedge q \models p$; $\ p \wedge q \models q$	
17.	Conjunction:	$p,\ q \models p \wedge q$	
18.	Elimination:	$p \vee q,\ \sim q \models p$; $\ p \vee q,\ \sim p \models q$	
19.	Transitivity:	$p \to q,\ q \to r \models p \to r$	
20.	Contradiction Rule:	$\sim p \to F \models p$	
21.	Quantified de Morgan laws:	$\sim \forall x\phi \equiv \exists x \sim \phi$; $\sim \exists x\phi \equiv \forall x \sim \phi$;	
22.	Quantified conjunctive law:	$\forall x(\phi \wedge \psi) \equiv (\forall x\ \phi) \wedge (\forall x\ \psi)$;	
23.	Quantified disjunctive law:	$\exists x(\phi \vee \psi) \equiv (\exists x\ \phi) \vee (\exists x\ \psi)$;	
24.	Quantifiers swapping laws:	$\forall x \forall y \phi \equiv \forall y \forall x \phi$;	$\exists x \exists y \phi \equiv \exists y \exists x \phi$;
25.	Independent quantifier law:	$\xi \equiv \forall x \xi \equiv \exists x \xi$;	
26.	Variable renaming laws:	$\forall x\phi \equiv \forall y \phi[y/x]$;	$\exists x\phi \equiv \exists y \phi[y/x]$;
27.	Free variable laws:	$\forall x(\xi \wedge \psi) \equiv \xi \wedge (\forall x \psi)$; $\forall x(\xi \vee \psi) \equiv \xi \vee (\forall x \psi)$;	$\exists x(\xi \wedge \psi) \equiv \xi \wedge (\exists x \psi)$; $\exists x(\xi \vee \psi) \equiv \xi \vee (\exists x \psi)$;
28.	Universal instantiation:	$\forall x \phi \Rightarrow \phi[a/x]$;	
29.	Universal generalisation:	$\phi[a/x] \Rightarrow \forall x \phi$;	
30.	Existential instantiation:	$\exists x \phi \Rightarrow \phi[s/x]$;	
31.	Existential generalisation:	$\phi[s/x] \Rightarrow \exists x \phi$.	

Rules of Inference

Let ϕ, ψ, ξ be any well-formed formulae. Then

1. $\wedge$-introduction: $\phi,\ \psi \vdash \phi \wedge \psi$
2. $\wedge$-elimination: $\phi \wedge \psi \vdash \phi$ or $\phi \wedge \psi \vdash \psi$
3. $\rightarrow$-introduction: $\boxed{\phi,\ \cdots,\ \psi} \vdash (\phi \rightarrow \psi)$
4. $\rightarrow$-elimination: $\phi \rightarrow \psi,\ \phi \vdash \psi$
5. $\vee$-introduction: $\phi \vdash \phi \vee \psi$ or $\psi \vdash \phi \vee \psi$
6. $\vee$-elimination: $\phi \vee \psi,\ \boxed{\phi,\ \cdots,\ \xi},\ \boxed{\psi,\ \cdots,\ \xi} \vdash \xi$
7. $\neg$-introduction or $\sim$-introduction: $\boxed{\sim\phi,\ \cdots,\ \perp} \vdash \phi$ or $\boxed{\phi,\ \cdots,\ \perp} \vdash \sim\phi$
8. $\neg$-elimination or $\sim$-elimination: $\phi,\ \sim\phi \vdash \perp$
9. $\forall$-introduction: $\phi(a) \vdash \forall x\phi(x)$
10. $\forall$-elimination: $\forall x\phi(x) \vdash \phi(t)$
11. $\exists$-introduction: $\phi(t) \vdash \exists x\phi(x)$
12. $\exists$-elimination: $\exists x\phi(x), \boxed{\phi(s)\cdots\xi} \vdash \xi$

The term t is free with respect to x in ϕ and $[t/x]$ means "t replaces x".