UECM1303 Tutorial 2: Valid and Invalid Arguments

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Predicates & Quantified Statements

1. Let P be a proposition and $Q(x, y)$ be a predicate. Are the following strings well-formed formula?

   (a) $\sim$ ....

   (b) $(\sim (P)) \wedge (P)$ ....

   (c) $\forall x((P \to Q(x, y)) \vee (P))$ ....

   (d) $\sim \forall x Q(x, x^2) \equiv \exists x \sim Q(x, x^2)$ ....

   (e) $x^2 + y^2 - 3z^2$ ....

   (f) $x^2 + y^2 - 3z^2 = 0$ ....

2. Let a be a constant, f_i be functions and P_i be predicates. Determine the bound and free variables of the following formula.

   (a) $(\forall x_2 P_1(x_1, x_2)) \to P_1(x_2, a)$. ....

   (b) $P_1(x_3) \to \sim \forall x_1 \forall x_2 P_2(x_1, x_2)$. ....

   (c) $\forall x_2(P_1(f_1(x_1, x_2), x_1) \to \forall x_1 P_2(x_3, f_2(x_1, x_2)))$ ....
   ....

3. Consider the predicates

   $LT(x, y) : x < y$ $EQ(x, y) : x = y$ $EV(x)$: x is even
   $I(x)$: x is an integer $P(x) : x > 0$ $R(x)$: $x \in \mathbb{R}$.

   (a) Write the statement using ONLY these predicates and any needed quantifiers.

     i. Every integer is even. ....

     ii. Some real numbers are negative integers. ....
   ....

     iii. If $x < y$, then x is not equal to y. ......

     iv. There is no largest number. ....

     v. If $y > x$ and $0 > z$, then $x \cdot z > y \cdot z$. ....
   ....

   (b) Write the statement $\exists x \sim P(x)$ in English without using variables and symbols. ...
   ....

4. Let $M(s)$ denote "s is a math major", $C(s)$ denote "s is a computer science student" and $E(s)$ denote "s is an engineering student". Rewrite the following statements by using quantifiers, variables and predicates $M(s)$, $C(s)$ and $E(s)$.

(a) There is an engineering student who is a math major. .. ⬚

(b) Every computer science student is an engineering student. ⬚

(c) No computer science students are engineering students. . ⬚

(d) Some computer science students are engineering students and some are not.
.... ⬚

5. Let $H(x)$ denote the predicate "x is a human" and $T(x, y)$ denote the predicate "x trusts y". Rewrite the following formula into English sentence without quantifiers and variables.

(a) $\forall x \exists y (H(x) \rightarrow (H(y) \wedge T(x, y)))$ .... ⬚

(b) $\exists x \forall y (H(x) \wedge (H(y) \rightarrow \sim T(x, y)))$. .... ⬚

(c) $\forall x \forall y (H(x) \rightarrow (H(y) \rightarrow \sim T(x, y)))$. .... ⬚

6. Consider the following statement:
$$\exists x (x \in \mathbb{R} \wedge x^2 = 2).$$

Which of the following are equivalent ways of expressing this statement?

(a) The square of each real number is 2. .... ⬚

(b) Some real numbers have square 2. .... ⬚

(c) If x is a real number, then $x^2 = 2$. .... ⬚

7. (Difficult, requires logic programming) Express the following summation using predicate:
$$S(n) = 1^2 + 2^2 + \cdots + n^2.$$

Logical Equivalence

8. Determine whether each of the following statements is true or false over the model of integer sets.

(a) $\forall x \exists y \exists z (x = 7y + 5z)$ .... ⬚

(b) $\forall x \exists y \exists z (x = 31y + 41z)$ .... ⬚

(c) $\forall x \exists y \exists z (x = 4y + 6z)$ .... ⬚

9. Determine the truth value of the following statements assuming we are interpreting these formulae over the real number domain.

(a) $\forall x (x > \frac{1}{x})$. .... ⬚

(b) $\exists x (x \in \mathbb{Z} \rightarrow \frac{x-3}{x} \notin \mathbb{Z})$. .... ⬚

(c) $\exists m \exists n (m \in \mathbb{Z} \wedge n \in \mathbb{Z} \wedge m > 0 \wedge n > 0 \wedge mn \geq m + n)$. .... ⬚

(d) $\forall x \forall y (\sqrt{x + y} = \sqrt{x} + \sqrt{y})$. .... ⬚

10. Let $E(n)$ be the predicate "n is even" and consider the following statement:
$$\forall n (n \in \mathbb{Z} \rightarrow (E(n^2) \rightarrow E(n))).$$

Which of the following are equivalent ways of expressing this statement?

(a) All integers have even squares and are even. .... ⬚

(b) Given any integer whose square is even, that integer is itself even. ....

(c) For all integers, there are some whose square is even. ....

(d) Any integer with even square is even. ....

(e) All even integers have even squares. ....

11. Give a negation for each statement below:

(a) For all integers x, if x is odd, then $x^2 - 1$ is even.

(b) There exists an integer x with $x \geq 2$ such that $x^2 - 4x + 7$ is prime.

(c) For all real numbers x and y, if $x = y$, then $x^2 = y^2$.

(d) There is no easy question in the exam.

(e) If the square of real number x is greater than or equal to 1 then $x > 0$.

12. Show that the statements $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x (P(x) \wedge Q(x))$ are not logically equivalent in general.

13. Determine whether the statements $\forall x P(x) \wedge \exists x Q(x)$ and $\forall x \exists y (P(x) \wedge Q(y))$ are logically equivalent.

14. Determine whether $\exists x (P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ have the same truth value. Explain.

15. Determine the truth value of each of these statements if they are modelled over the set of integers.

(a) $\forall n \exists m (n^2 < m)$ ....

(b) $\exists n \forall m (nm = m)$ ....

(c) $\exists n \exists m (n^2 + m^2 = 6)$ ....

16. Let $P(x, y)$ be a predicate and the domain of discourse be a nonempty set. Given that $\forall x \exists y P(x, y)$ is true, which of the following are not necessarily true?

(a) $\exists x \exists y P(x, y)$ ....

(b) $\forall x \forall y P(x, y)$ ....

(c) $\exists x \forall y P(x, y)$ ....

17. What are the truth values of $\exists y \forall x (y \geq x)$ and $\forall x \exists y (y \geq x)$ if they are interpreted over the model with the domain of nonnegative integers?

18. Let odd(x) be the predicate "x is an odd positive integer." Determine the truth value of each of the following statements for the model M with domain of positive integers. If the statement is false, provide an explanation or suggest a counterexample.

(a) $\forall x \forall y (\text{odd}(x + y))$. ....

(b) $\exists x \forall y (\text{odd}(x + y))$. ....

(c) $\forall x \exists y (\text{odd}(x + y))$. ....

(d) $\exists x \exists y (xy + 1 = 0)$. ....

19. Consider the following models

$$M_1 = (\mathbb{N}, \{=\}, \{+^{\mathbb{N}}, \cdot^{\mathbb{N}}\}, 0^{\mathbb{N}}) \text{ where}$$

$$(=^{\mathbb{N}}) = \{(n, n)|n \in \mathbb{N}\},$$
$$+^{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m, n) \mapsto m + n,$$
$$\cdot^{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m, n) \mapsto m \cdot n.$$

$$M_2 = (\mathbb{Q}, \{=\}, \{+^{\mathbb{Q}}, \cdot^{\mathbb{Q}}\}, 0^{\mathbb{Q}}) \text{ where}$$

$$(=^{\mathbb{Q}}) = \{(n, n)|n \in \mathbb{Q}\},$$
$$+^{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (m, n) \mapsto m + n,$$
$$\cdot^{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (m, n) \mapsto m \cdot n.$$

Determine the truth value of

(a) $\forall x_1 \forall x_2 \exists x_3 (x_1 + x_2 = x_3)$ and

(b) $\forall x_1 \forall x_2 ((\exists x_3 (x_1 \cdot x_3 = x_2) \wedge \exists x_4 (x_2 \cdot x_4 = x_1) \to x_1 = x_2)$.

20. Derive the following rule using laws of equivalence:

$$\sim (\forall x (x \in D \to (\forall y (y \in E \to P(x, y)))) \equiv \exists x \exists y (x \in D \wedge (y \in E \wedge \sim P(x, y))$$

21. Show that $\forall x[(C(x) \wedge \exists y(T(y) \wedge L(x, y))) \to \exists y(D(y) \wedge B(x, y))] \equiv \forall x \forall y \exists z[(C(x) \wedge T(y) \wedge L(x, y)) \to (D(z) \wedge B(x, z))]$

22. Write a negation for the following statement:

For all real numbers $y > 0$, there exists a real number $z > 0$ such that if $a - z < x < a + z$ then $L - y < f(x) < L + y$.

Logical Implication

23. Show that $\sim (\forall x(P(x) \to Q(x))) \Rightarrow \exists x(P(x) \wedge \sim Q(x))$.

Note that the two quantified statements are actually logically equivalent. Are you able to show the reverse?

24. For the following arguments, state which are valid and which are invalid. Justify your answers.

(a) All healthy people eat an apple a day. John is not a healthy person. Therefore John does not eat an apple a day.

(b) Every student who studies discrete mathematics is good at logic. John studies discrete mathematics. Therefore John is good at logic.

(c) No heavy object is cheap. XYZ is not a heavy object. Therefore XYZ is cheap.

25. Show that the following argument is valid using the laws of logical equivalence and logical implication.

$$\exists x(F(x) \wedge S(x)) \to (\forall y(M(y) \to W(y)))$$
$$\underline{\exists y(M(y) \wedge \sim W(y))}$$
$$\therefore \quad \forall x(F(x) \to \sim S(x))$$

Rules of Inference

26. What is wrong with the following proof?

1	$\forall x \exists y (x > y)$	premise
2	$\exists y (c > y)$	universal instantiation, c arbitrary
3	$(c > s)$	existential instantiation, s specific
4	$\forall x (x > s)$	universal generalisation
5	$\exists y \forall x (x > y)$	existential generalisation

27. Use ONLY the rules of inference to show that

$$\forall x (P(x) \rightarrow (Q(x) \wedge R(x))), \ \forall x (P(x) \wedge S(x)) \ \vdash \ \exists x (R(x) \wedge S(x))$$

28. Use rules of inference to show that

$$\exists x P(x) \rightarrow \forall x (P(x) \vee Q(x) \rightarrow R(x)), \ \exists x (P(x) \wedge Q(x)) \vdash \exists y R(y)$$

29. Show that $\forall x (Q(x) \rightarrow R(x)) \wedge (\exists x (Q(x) \wedge I(x)) \vdash \exists x (R(x) \wedge I(x))$.

30. Prove that the following argument is valid using the rules of inference: "Every under-graduate is either an arts student or a science student. Some undergraduates are top students. James is not a science student, but he is a top student. Therefore if James is an undergraduate, he is an arts student."

31. Write a Prolog program to apply the predicate in Question 7 to perform calculations for $n = 1$, $n = 10$ and $n = 50$.