UECM1304 Test 2 Marking Guide

Name: Student ID: Mark: /20

Course Code & Course Title: UECM1304 Discrete Mathematics with Applications
Faculty: LKC FES, UTAR Course: AM
Trimester: Sample Lecturer: Liew How Hui

Instruction: Answer all questions in the space provided. **If you do not write your answer in the space provided, you will get ZERO mark**. An answer without working steps may also receive ZERO mark.

1. CO3: Demonstrate various proof-techniques. C3

   (a) Use direct proof to show that there are integers m and n such that $7684m + 15283n = 17$. [**Note**: You need to use extended Euclidean algorithm to find out m and n rather than simply guessing. Guessing the answer is regarding as cheating and only 0.5 marks will be awarded.] (4 marks)

   Ans.

   $15283 = 7684 \times 1 + 7599$
   $7684 = 7599 \times 1 + 85$
   $7599 = 85 \times 89 + 34$
   $85 = 34 \times 2 + 17$
   $34 = 17 \times 2 + 0$ [1.5 marks]
   $17 = 85 - 34 \times 2$
   $17 = 85 - (7599 - 85 \times 89) \times 2 = 85 \times 179 - 7599 \times 2$
   $17 = (7684 - 7599) \times 179 - 7599 \times 2 = 7684 \times 179 - 7599 \times 181$
   $17 = 7684 \times 179 - (15283 - 7684) \times 181 = 7684 \times 360 - 15283 \times 181$ [2 marks]

   Therefore, there is $m = 360$ and $n = -181$ such that $7684m + 15283n = 17$. ...[0.5 mark]

(b) Let A and B be two sets. Use contrapositive proof to show that if $A \times B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$. (2 marks)

Ans. Suppose $A \neq \emptyset$ and $B \neq \emptyset$. .... [0.5 mark]

There exist $x \in A$ and $y \in B$. .... [0.5 mark]

Therefore $(x, y) \in A \times B$. So $A \times B \neq \emptyset$ by definition. .... [1 mark]

(c) Use the Principle of Mathematical Induction to prove the equality

$$2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 - 1.$$

for integers $n \geq 2$. (3 marks)

Ans. Base Step:

When $n = 2$, RHS$=\left[\frac{(2 \times (2+1))}{2}\right]^2 - 1 = 3^2 - 1 = 8 = 2^3 =$ LHS .... [1 mark]

Induction Step:

Assume $2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2 - 1$ for $k \geq 2$. .... [0.3 mark]

$$\begin{aligned}
2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 - 1 + (k+1)^3 && \text{[0.5 mark]} \\
&= (k+1)^2\left[\frac{k^2}{4} + k + 1\right] - 1 = \frac{(k+1)^2}{4}\left[k^2 + 4k + 4\right] - 1 && \text{[0.5 mark]} \\
&= \frac{(k+1)^2}{2^2} \times (k+2)^2 - 1 = \left[\frac{(k+1)((k+1)+1)}{2}\right]^2 - 1 && \text{[0.5 mark]}
\end{aligned}$$

Hence, by the Principle of Mathematical Induction, $2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 - 1$ for $n \geq 2$.

.... [0.2 mark]

(d) Use the Principle of Mathematical Induction to prove that the inequality

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$$

for integers $n \geq 0$. (3 marks)

Ans. Base Step:

When $n = 0$, LHS $= 1 \geq 1 + \frac{0}{2} =$ RHS .... [1 mark]

[**Remark**: The left hand side is another notation of $\sum_{i=1}^{2^n} \frac{1}{i}$. Many students do not know how to prove this because **they refuse to put in any amount of efforts to understand series**]

Induction Step:

Assume $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} \geq 1 + \frac{k}{2}$ for $n = k \geq 0$. .... [0.3 mark]

We now look at the series with 2^{k+1} terms

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \underbrace{\frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}}_{2^{k+1} - 2^k = 2^k \text{ extra terms}}$$

$$\geq 1 + \frac{k}{2} + \underbrace{\frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}}_{2^{k+1} - 2^k = 2^k \text{ extra terms}} \qquad [0.5 \text{ mark}]$$

$$\geq 1 + \frac{k}{2} + \left[\underbrace{\frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}}_{2^k \text{ extra terms}}\right] \quad \left(\because \frac{1}{2^k + j} \geq \frac{1}{2^{k+1}} \text{ for } j = 1, ..., 2^k\right) \qquad [0.5 \text{ mark}]$$

$$= 1 + \frac{k}{2} + \frac{2^k}{2^{k+1}} = 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2} \qquad [0.5 \text{ mark}]$$

Hence, by the Principle of Mathematical Induction, $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$ for integers $n \geq 0$ .... [0.2 mark]

2. CO4. Express relations correctly with their mathematical properties. C2

(a) A set $A = \{1, 2, 3, 4\}$ has a relation

$$R = \{(2, 3),\ (3, 2),\ (4, 4)\}$$

i. Is R irreflexive? .... No

ii. Is R symmetric? .... Yes

iii. Is R an equivalence relation? .... No

Provide justifications for the above questions if your answer is No. ($3 \times 0.3 + 0.6$=1.5 mark)

Ans. i. R is not irreflexive because $(4, 4) \in R$. .... [0.3 mark]

iii. R is not an equivalence relation because it is not reflexive, i.e. $(1, 1) \notin R$. [0.3 mark]

(b) Let A be all the factors of 60. The partial order $\preceq$ for any $p, q \in A$ is defined by $p \preceq q$ if p is divisible by q. Sketch the Hasse diagram of the poset $(A, \preceq)$. (1.5 marks)

Ans. From $60 = 2^2 \times 3 \times 5$, we have

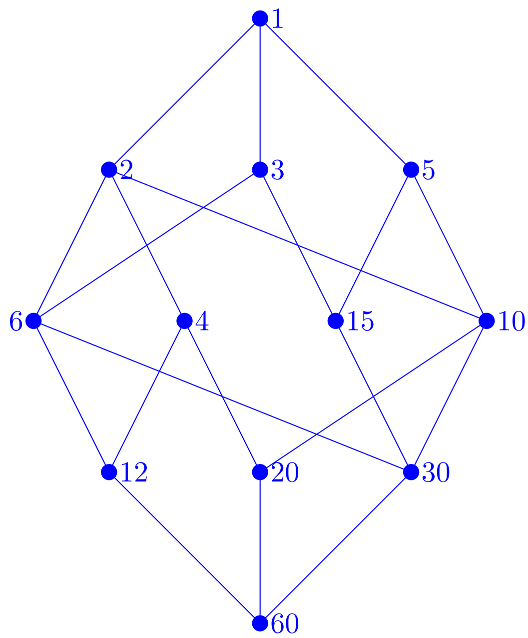
$$A = \{1, 2, 3, 5, 4, 6, 10, 15, 12, 20, 30, 60\}.$$

and the partial order is given by

$$\preceq = \{(p, q)\ :\ p \text{ is divisible by } q\}$$

For example, $(60, 2) \in \preceq$, i.e. 60 is divisible by 2.

The Hasse diagram is



List out all elements of A in proper ordering .... [0.7 mark]

Correct links in the Hasse diagram .... [0.8 mark]

(c) Consider a set $A = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$ with the following relation

$$R = \{(\alpha, \delta),\ (\beta, \beta),\ (\gamma, \epsilon),\ (\delta, \zeta),\ (\epsilon, \gamma),\ (\zeta, \alpha)\}$$

i. Write down the **matrix representation** of R and then (I) determine if the relation R is **reflexive** with justification; (II) determine if the relation R is **anti-symmetric** with justifications. (2 marks)

Ans. $$M_R = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \end{array} \begin{array}{c} \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \zeta \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$ ....[1 mark]

Since $(\beta, \beta) \in R$, R is **not reflexive**. ....[0.5 mark]

Since $(\gamma, \epsilon), (\epsilon, \gamma) \in R$, R is **not anti-symmetric**. ....[0.5 mark]

ii. Apply the **Warshall algorithm** to find the matrix representation of the **transitive closure** of R. (3 marks)

Ans. Step 1: $$M_R = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \end{array} \begin{array}{c} \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \zeta \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$ ....[0.5 mark]

Step 2: $$M_R = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \end{array} \begin{array}{c} \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \zeta \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$ ....[0.5 mark]

Step 3: $$M_R = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \end{array} \begin{array}{c} \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \zeta \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$ ....[0.5 mark]

$$\text{Step 4: } M_R = \begin{array}{c c} & \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \zeta \end{array} \\ \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

.... [0.5 mark]

$$\text{Step 5: } M_R = \begin{array}{c c} & \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \zeta \end{array} \\ \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

.... [0.5 mark]

$$\text{Step 6: } M_R = \begin{array}{c c} & \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \zeta \end{array} \\ \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

.... [0.5 mark]