MELBOURNE
SCHOOL OF
ENGINEERING

MCEN90028
Robotic Systems

Assignment 1:

Forward and Inverse Dynamics of a 5-DOF Jenga Tower Construction Robot

Assignment Group 1

Jiawei Liao	756560
Harfi Dharma Santoso	772503
Omar Sakkal	757372

Department of Mechanical Engineering

March 30, 2020

1 Description of Task

This assignment involves deriving both the forward and inverse kinematics of a 5 DOF robot manipulator designed to stack 54 Jenga blocks into a 9 - storey tower.

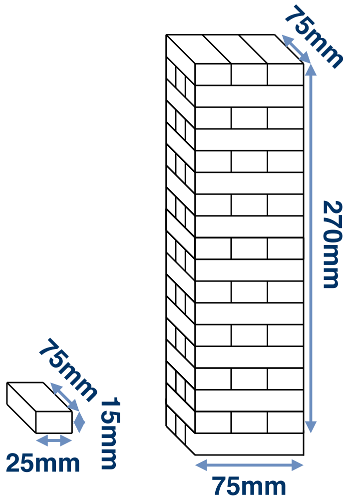

Figure 1: 9 - Storey Jenga Tower to be constructed

2 Forward Kinematics

2.1 Robot Schematics

The kinematic definition of the robot and its "zero position" configuration can be described by the following diagram following the Denavit - Hartenberg (DH) notation.

- Assume the values of d_i's are given in metres and the values of q_i's are given in degrees in this report.
- The end effector frame $\{E\}$ is designed so that Z_5 is parallel to Z_0 when pointing vertically down during its normal operation.

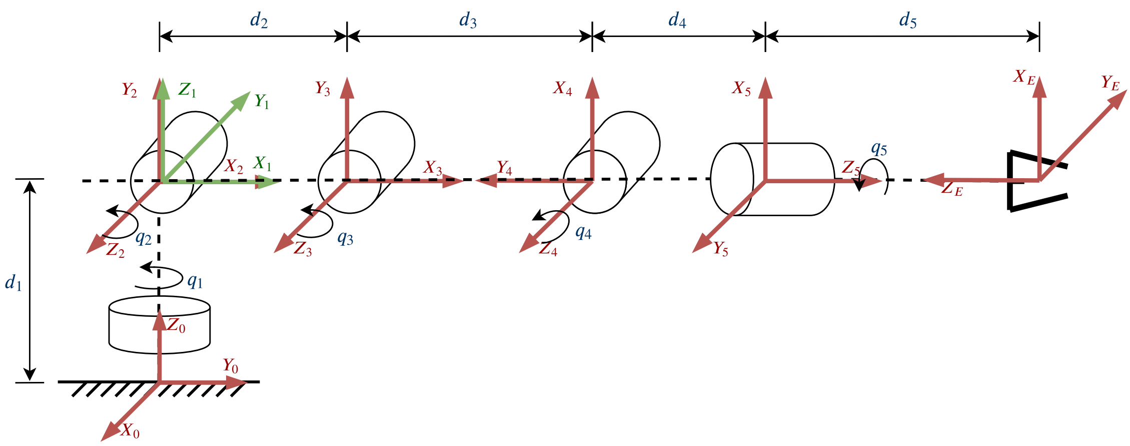

Figure 2: Schematic of the 5 DoF Robot

2.2 Denavit - Hartenberg Table

DH table					
i		$\mathbf{a_{i-1}}$	$\alpha_{\mathbf{i-1}}$	$\mathbf{d_i}$	$\theta_{\mathbf{i}}$
1		0	0°	d_1	$90^\circ + q_1^\circ$
2		0	90°	0	q_2°
3		d_2	0°	0	q_3°
4		d_3	0°	0	$90^\circ + q4^\circ$
5		0	90°	d_4	q_5°
E		0	180°	$-d6$	0°

Table 1: DH table for the 5 DoF robot illustrated in Figure (2)

The DH table describes the forward kinematics relationship between defined frames and allows the overall homogeneous transformation matrix of the end-effector to be obtained in which the pose can be readily ascertained.
Each row contains 4 elements a, α, d and θ which describe the series of actions needed to transform between the $(i-1)$th frame to (i)th frame:

1. Operations needed to align Z_{i-1} to Z_i:
   - Translation of a_{i-1} along X_{i-1} axis
   - Rotation by α_{i-1} about the X_{i-1} axis
2. Operations needed to align X_{i-1} to X_i:
   - Translation of d_i along Z_{i-1} axis
   - Rotation by θ_i about the Z_{i-1} axis

2.3 Transformation Matrices from Frame $\{E\}$ to $\{0\}$

The $4x4$ homogeneous transformation matrix from frame i to frame $i-1$ in a DH table is given as:

$$ {}^{i-1}_{i}T = D_x(a_{i-1})R_x(\alpha_{i-1})D_z(d_i)R_z(\theta_i) \tag{1}$$

where

$$D_x(a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R_x(\alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_{i-1}) & -\sin(\alpha_{i-1}) & 0 \\ 0 & \sin(\alpha_{i-1}) & \cos(\alpha_{i-1}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_z(d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R_z(\theta_i) = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix from end effector frame to zero frame is thus computed through:

$$ {}^{0}_{E}T = {}^{0}_{1}T \cdot {}^{1}_{2}T \cdot {}^{2}_{3}T \cdot {}^{3}_{4}T \cdot {}^{4}_{5}T \cdot {}^{5}_{E}T \tag{2}$$

where ${}^0_1T, {}^1_2T, {}^2_3T, {}^3_4T, {}^4_5T, {}^5_ET$ are respectively

$$
{}^0_1T = \begin{bmatrix} \cos(q_1^\circ + 90^\circ) & -\sin(q_1^\circ + 90^\circ) & 0 & 0 \\ \sin(q_1^\circ + 90^\circ) & \cos(q_1^\circ + 90^\circ) & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1_2T = \begin{bmatrix} \cos(q_2^\circ) & -\sin(q_2^\circ) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin(q_2^\circ) & \cos(q_2^\circ) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^2_3T = \begin{bmatrix} \cos(q_3^\circ) & -\sin(q_3^\circ) & 0 & d_2 \\ \sin(q_3^\circ) & \cos(q_3^\circ) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^3_4T = \begin{bmatrix} \cos(q_4^\circ + 90^\circ) & -\sin(q_4^\circ + 90^\circ) & 0 & d_3 \\ \sin(q_4^\circ + 90^\circ) & \cos(q_4^\circ + 90^\circ) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^4_5T = \begin{bmatrix} \cos(q_5^\circ) & -\sin(q_5^\circ) & 0 & 0 \\ 0 & 0 & -1 & -d_4 \\ \sin(q_5^\circ) & \cos(q_5^\circ) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^5_ET = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

${}^0_ET$ then has elements (where ${}^0_ET(i,j)$ refers to the ith row and jth column):

$$
\begin{aligned}
{}^0_ET(1,1) &= \cos(q_1^\circ)\sin(q_5^\circ) - \cos(q_1^\circ + 90^\circ)(\sin(q_2^\circ + q_3^\circ + q_4^\circ + q_5^\circ)/2 + \sin(q_2^\circ + q_3^\circ + q_4^\circ - q_5^\circ)/2) \\
{}^0_ET(1,2) &= \cos(q_1^\circ + 90^\circ)(\cos(q_2^\circ + q_3^\circ + q_4^\circ + q_5^\circ)/2 - \cos(q_2^\circ + q_3^\circ + q_4^\circ - q_5^\circ)/2) - \cos(q_1^\circ)\cos(q_5^\circ) \\
{}^0_ET(1,3) &= -\cos(q_2^\circ + q_3^\circ + q_4^\circ)\cos(q1^\circ + 90^\circ) \\
{}^0_ET(1,4) &= \cos(q_1^\circ + 90^\circ)(d_3\cos(q_2^\circ + q_3^\circ) + d_2\cos(q_2^\circ) + d_4\cos(q_2^\circ + q_3^\circ + q_4^\circ) + d_5\cos(q_2^\circ + q_3^\circ + q_4^\circ)) \\
{}^0_ET(2,1) &= \sin(q_1^\circ)\sin(q_5^\circ) - \sin(q_1^\circ + 90^\circ)(\sin(q_2^\circ + q_3^\circ + q_4^\circ + q_5^\circ)/2 + \sin(q_2^\circ + q_3^\circ + q_4^\circ - q_5^\circ)/2) \\
{}^0_ET(2,2) &= \sin(q_1^\circ + 90^\circ)(\cos(q_2^\circ + q_3^\circ + q_4^\circ + q_5^\circ)/2 - \cos(q_2^\circ + q_3^\circ + q_4^\circ - q_5^\circ)/2) - \cos(q_1^\circ)\sin(q_5^\circ) \\
{}^0_ET(2,3) &= -\cos(q_2^\circ + q_3^\circ + q_4^\circ)\sin(q1^\circ + 90^\circ) \\
{}^0_ET(2,4) &= \sin(q_1^\circ + 90^\circ)(d_3\cos(q_2^\circ + q_3^\circ) + d_2\cos(q_2^\circ) + d_4\cos(q_2^\circ + q_3^\circ + q_4^\circ) + d_5\cos(q_2^\circ + q_3^\circ + q_4^\circ)) \\
{}^0_ET(3,1) &= \cos(q_2^\circ + q_3^\circ + q_4^\circ + q_5^\circ)/2 + \cos(q_2^\circ + q_3^\circ + q_4^\circ - q_5^\circ)/2 \\
{}^0_ET(3,2) &= \sin(q_2^\circ + q_3^\circ + q_4^\circ + q_5^\circ)/2 - \sin(q_2^\circ + q_3^\circ + q_4^\circ - q_5^\circ)/2 \\
{}^0_ET(3,3) &= -\sin(q_2^\circ + q_3^\circ + q_4^\circ) \\
{}^0_ET(3,4) &= d_1 + d_2\sin(q_2^\circ) + d_3\sin(q_2^\circ + q_3^\circ) + d_4\sin(q_2^\circ + q_3^\circ + q_4^\circ) + +d_5\sin(q_2^\circ + q_3^\circ + q_4^\circ) \\
{}^0_ET(4,:) &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

2.4 Verification of Forward Kinematics Solutions

Two simple configurations (other than the zero position) of the robot manipulator were chosen to allow direct hand calculations of the forward kinematics.
MATLAB solutions via numeric substitution could then be compared to verify the forward kinematics solutions derived in Section (2.3).
Assume generic dimensions and the end effector is always above the ground for all the configurations below.

1. Zero Configuration : $\vec{q} = \begin{bmatrix} 0^\circ & 0^\circ & 0^\circ & 0^\circ & 0^\circ \end{bmatrix}^T$

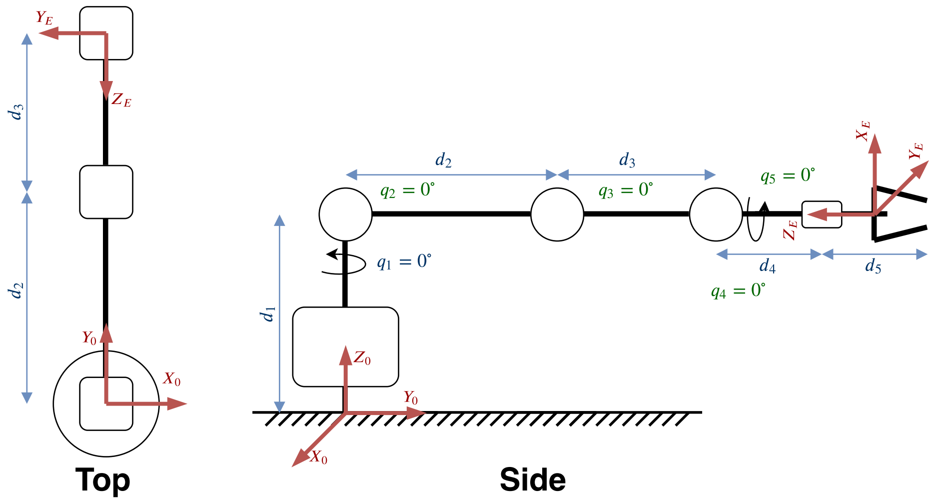

Figure 3: Top/Side view of zero configuration

It is trivial to see that:

$$
{}^{0}\vec{r}_{OE} = \begin{bmatrix} 0 \\ d_2 + d_3 + d4 + d_5 \\ d_1 \end{bmatrix}, \quad {}^{0}_{E}R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}
$$

$$
\implies {}^{0}_{E}T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & d_2 + d_3 + d4 + d_5 \\ 1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

MATLAB results via numerical substitution of expression defined in Section (2.3) is consistent.

$$
\textbf{TE0\_config0} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & d_2 + d_3 + d4 + d_5 \\ 1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

2. Configuration 1 : $\vec{q} = \begin{bmatrix} 30^\circ & 0^\circ & 0^\circ & -90^\circ & 0^\circ \end{bmatrix}^T$

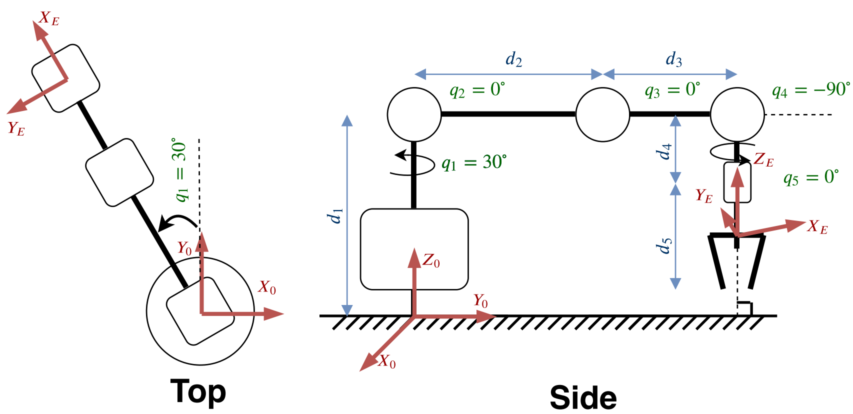

Figure 4: Top/Side view of Configuration 1

It is simple to see that:

$$^0\vec{r}_{OE} = \begin{bmatrix} -(d_2+d_3)\sin(30^\circ) \\ (d_2+d_3)\cos(30^\circ) \\ d_1 \end{bmatrix}, \quad {}^0_ER = \begin{bmatrix} -\sin(30^\circ) & -\cos(30^\circ) & 0 \\ \cos(30^\circ) & -\sin(30^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies {}^0_ET = \begin{bmatrix} -\sin(30^\circ) & -\cos(30^\circ) & 0 & -(d_2+d_3)\sin(30^\circ) \\ \cos(30^\circ) & -\sin(30^\circ) & 0 & (d_2+d_3)\cos(30^\circ) \\ 0 & 0 & 1 & d_1-d_4-d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

MATLAB results via numerical substitution of expression defined in Section (2.3) is consistent.

$$\textbf{TE0\_config1} = \begin{bmatrix} -1/2 & -3^{(1/2)}/2 & 0 & -d_2/2-d_3/2 \\ 3^{(1/2)}/2 & -1/2 & 0 & (3^{(1/2)}*(d_2+d_3))/2 \\ 0 & 0 & 1 & d_1-d_4-d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Configuration 2: $\vec{q} = \begin{bmatrix} 0^\circ & 45^\circ & -90^\circ & -45^\circ & 90^\circ \end{bmatrix}^T$

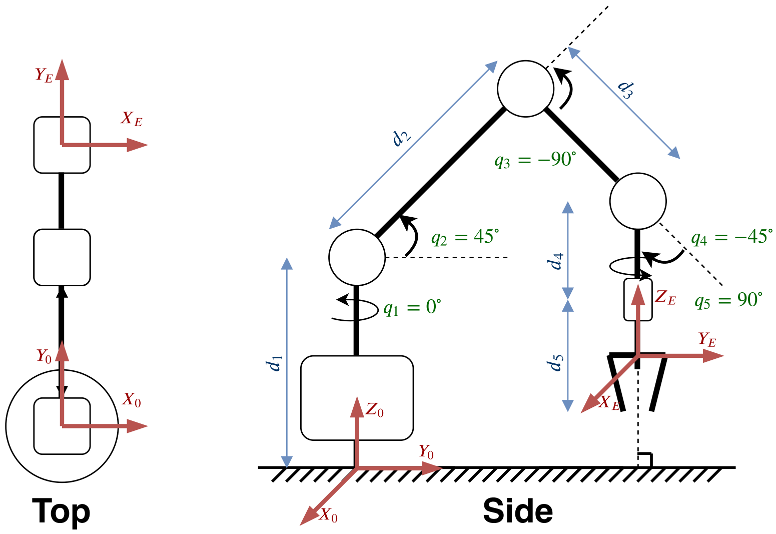

Figure 5: Top/Side view of Configuration 2

This configuration aligns frame $\{E\}$ with frame $\{0\}$ so it is straight forward to see that:

$$^0\vec{r}_{OE} = \begin{bmatrix} 0 \\ d_2\cos 45^\circ + d_3\cos 45^\circ \\ d_1 + d_2\sin 45^\circ - d_3\sin 45^\circ - d_4 - d_5 \end{bmatrix}, \quad {}^0_ER = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies {}^0_ET = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2\cos 45^\circ + d_3\cos 45^\circ \\ 0 & 0 & 1 & d_1 + d_2\sin 45^\circ - d_3\sin 45^\circ - d_4 - d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

MATLAB results via numerical substitution of expression defined in Section (2.3) is consistent.

$$\mathbf{TE0\_config2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & (2^{(1/2)} * d_2)/2 + (2^{(1/2)} * d_3)/2 \\ 0 & 0 & 1 & d_1 - d_4 - d_5 + (2^{(1/2)} * d_2)/2 - (2^{(1/2)} * d_3)/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.5 Evaluating Reachable Workspace via forward dynamics

The MATLAB Function robot_simplews_plot.m can plot the 2-D robot workspace from the top view (attached to Frame $\{0\}$) and the cross-section view (attached to Frame $\{1\}$). The Jenga tower is represented as a pink rectangle in the plots and its center base was placed at $(0, 0.2, 0)$.

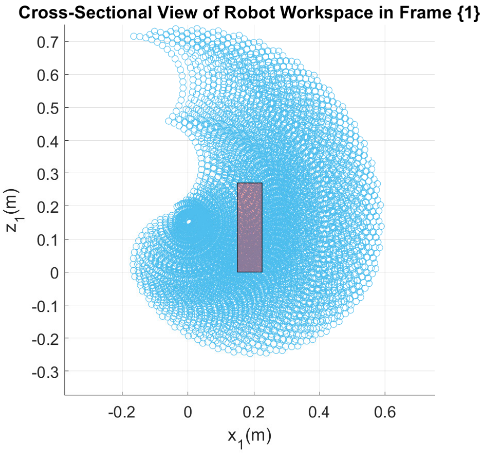

Figure 6: workspace cross-section view

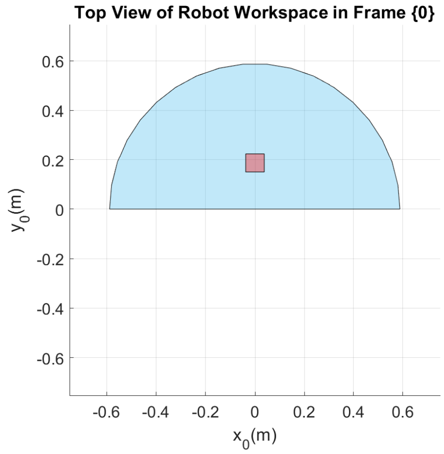

Figure 7: workspace top view

The parameters used for the workspace plot is given below:

Robot Parameter Table												
	Dimensions							**Angle Limits**				
	d_1	d_2	$d-3$	d_4	d_5			q_1	q_2	q_3	q_4	q_5
(m)	0.15	0.15	0.2	0.1	0.05		**(ˆ{\circ})**	-90, 90	0, 120	-150, -15	-150, -15	-90,90

Table 2: Robot Parameters Table

2.6 Justification of DH parameter and joint limits settings

2.6.1 Mechanical Dimensions

- ($\mathbf{d_1 = 0.15m}$): The base height $d_1 = 0.15m$ allows the robot to operate more efficiently in the elbow-down manifold. The internal turning mechanism also needs abundant volume.
- ($\mathbf{d_2 = 0.2m}$): The second arm is to be designed using a parallelogram linkage to control joints 2 and 3 near the origin of Frame $\{1\}$.
- ($\mathbf{d_3 = 0.25m}$): $d_3 + d_2 = 0.35m$ could provide a maximum sweep span of 0.35m. This is adequate for operating the Jenga tower which has dimensions of $(0.075m * 0.075m * 0.13m)$. The lengths must be constrained to reduce the cost of the structure and the motors.

- $(\mathbf{d_4 = 0.1m, d_5 = 0.05m})$: The end-effector length is sufficient for reaching the table in the elbow-down configuration, where the size of a small gripper is included.

2.6.2 Joint Limits

- $(\mathbf{-90^\circ \leq q_1 \leq 90^\circ})$: The robot was designed to operate in the first two quadrants of the X-Y plane using a 180° Servo Motor.
- $(\mathbf{0^\circ \leq q_2 \leq 120^\circ})$: It was difficult to achieve $q_2 < 0$ as this joint sits on the surface of the base, and is unnecessary for the elbow-down configuration.
- $(\mathbf{-135^\circ \leq q_3 \leq -15^\circ})$: Exclude elbow-up manifold completely.
- $(\mathbf{-135^\circ \leq q_4 \leq -15^\circ})$: End effector points down at all times during elbow-down configuration.
- $(\mathbf{-90^\circ \leq q_5 \leq 90^\circ})$: Allow the end-effector to change the orientation of the Jenga block.

3 Inverse Kinematics

The inverse kinematics function $f_{ik}([x; y; z])$ computes the needed joint rotations $\vec{Q}$ needed for the robot to arrive at the desired end-effector position $[x; y; z]$.

$$\vec{Q_k} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = f_{ik}(\begin{bmatrix} x \\ y \\ z \end{bmatrix})$$

For simplification purposes, a constraint is introduced for the end-effector to always point straight down. This implies that the Z_E axis is always parallel to the Z_0 axis.
The end-effector's position was then restricted to quadrants 1 and 2 of the X-Y plane to reduce the solution manifolds and due to the existing motor limitations.

3.1 Derivation of Explicit Inverse Kinematics solutions

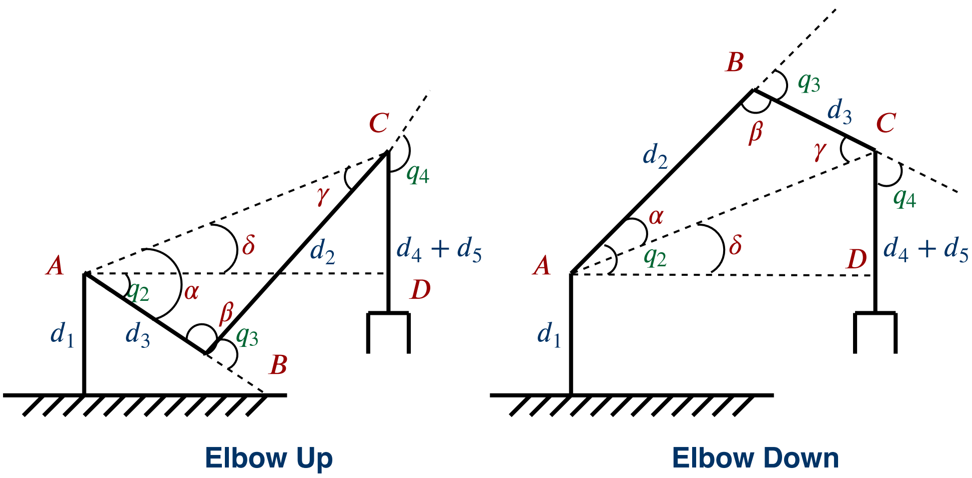

Figure 8: Robot Schematic in Elbow Down/Up Configuration

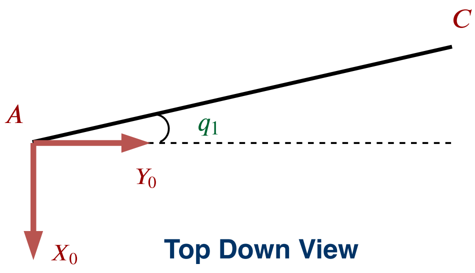

Figure 9: Robot Schematic Top View

- q_1

  q_1 can be directly solved by looking at Figure (9).

$$q_1 = \arctan(-x_c/y_c) = \arctan(-x/y) \tag{3}$$

- q_5 is irrelevant to the robot reaching task so it will be treated as zero.

$$q_5 = 0 \tag{4}$$

- q_3

  Using Figure ((8)) as reference:

$$L_{AC} = \sqrt{(x_c^2 + y_c^2 + (z_c - d_1^2))} = \sqrt{(x^2 + y^2 + (z + d_4 + d_5 - d_1)^2)}$$

  Using cosine rule for β:

$$\begin{aligned} L_{AC}^2 &= L_{AB}^2 + L_{BC}^2 - 2L_{AB}L_{BC}\cos(\beta) \\ \implies \cos(\beta) &= \frac{d_2^2 + d_3^2 - x^2 - y^2 - (z + d_4 + d_5 - d_1)^2}{2d_2d_3} \\ \implies \beta &= \arccos(\frac{d_2^2 + d_3^2 - x^2 - y^2 - (z + d_4 + d_5 - d_1)^2}{2d_2d_3}) \end{aligned}$$

  From Figure (8) we can see that

$$\beta = 180° - q_3$$

  Using trigonometry identities:

$$\cos(q_3) = -\cos(\pi - q_3) = -\cos\beta = \arccos(\frac{-d_2^2 - d_3^2 + x^2 + y^2 + (z + d_4 + d_5 - d_1)^2}{2d_2d_3})$$

  This gives two solutions of q_3 for elbow up/down configuration

$$q_3 = +\arccos(\frac{x^2 + y^2 + (z + d_4 + d_5 - d1)^2 - d_2^2 - d_3^2}{2d_2d_3}), \quad \text{elbow up} \tag{5}$$

$$q_3 = -\arccos(\frac{x^2 + y^2 + (z + d_4 + d_5 - d1)^2 - d_2^2 - d_3^2}{2d_2d_3}), \quad \text{elbow down} \tag{6}$$

- q_2

  Using the sine rule with β and α

$$\frac{d_3}{\sin\alpha} = \frac{L_{AC}}{\sin\beta}$$
$$\implies \alpha = \arcsin\left(\frac{d_3 \sin\beta}{L_{AC}}\right)$$
$$\implies \alpha = \arcsin\left(\frac{d_3\sqrt{1-(\cos\beta)^2}}{\sqrt{(x^2+y^2+(z+d_4+d_5-d_1)^2)}}\right)$$
$$\implies \alpha = \arcsin\left(\frac{\sqrt{(2d_2d_3)^2-(d_2^2+d_3^2-x^2-y^2-(z+d_4+d_5-d_1)^2)^2}}{2d_2\sqrt{(x^2+y^2+(z+d_4+d_5-d_1)^2)}}\right)$$

  Extending point A horizontally to intersect the Z_E axis of the end-effector, as seen in Figure (8), allows us to compute δ

$$\delta = \arctan\left(\frac{L_{CD}}{L_{AD}}\right) = \arctan\left(\frac{z+d_4+d_5-d_1}{\sqrt{(x^2+y^2)}}\right)$$

  This gives two solutions of q_2 for elbow up/down configuration

$$q_2 = -\alpha + \delta, \quad \text{elbow up} \tag{7}$$
$$q_2 = \alpha + \delta, \quad \text{elbow down} \tag{8}$$

  Using the properties of q_3 solutions, we could do a trick to use a single expression for solving q_2

$$q_2 = -\frac{d_3}{\sin q_3} + \arctan\left(\frac{z+d_4+d_5-d_1}{\sqrt{(x^2+y^2)}}\right) \tag{9}$$

- q_4

  The end-effector pointing constraint implies that:

$$q_2 + 90^\circ + 180^\circ + q_3 + 180^\circ + q_4 + 90^\circ + 90^\circ = 540^\circ, \quad \text{inner angle sum of pentagon}$$

$$\implies q_4 = -90^\circ - q_2 - q_3 \tag{10}$$

  The two manifolds of the IK solutions are summarised here:

$$\vec{Q}(\text{elbow\_up}) = \begin{bmatrix} \arctan(-x/y) \\ -\arcsin\left(\frac{\sqrt{(2d_2d_3)^2-(d_2^2+d_3^2-x^2-y^2-(z+d_4+d_5-d_1)^2)^2}}{2d_2\sqrt{(x^2+y^2+(z+d_4+d_5-d_1)^2)}}\right) + \arctan\left(\frac{z+d_4+d_5-d_1}{\sqrt{(x^2+y^2)}}\right) \\ +\arccos\left(\frac{x^2+y^2+(z+d_4+d_5-d1)^2-d_2^2-d_3^2}{2d_2d_3}\right) \\ -90^\circ - q_2 - q_3 \\ 0 \end{bmatrix} \tag{11}$$

$$\vec{Q}(\text{elbow\_down}) = \begin{bmatrix} \arctan(-x/y) \\ +\arcsin\left(\frac{\sqrt{(2d_2d_3)^2-(d_2^2+d_3^2-x^2-y^2-(z+d_4+d_5-d_1)^2)^2}}{2d_2\sqrt{(x^2+y^2+(z+d_4+d_5-d_1)^2)}}\right) + \arctan\left(\frac{z+d_4+d_5-d_1}{\sqrt{(x^2+y^2)}}\right) \\ -\arccos\left(\frac{x^2+y^2+(z+d_4+d_5-d1)^2-d_2^2-d_3^2}{2d_2d_3}\right) \\ -90^\circ - q_2 - q_3 \\ 0 \end{bmatrix} \tag{12}$$

Due to the constraints introduced in Section (2.6) and that any part of the robot should be above the X-Y plane, only the elbow-down manifold solution is relevant during its operation.

3.2 Verification of Inverse Kinematics Solutions

The new home position of the Robot requires $q_4 = -90°$ due to the constraint.
The two configurations in Section (2.4 (2,3)) will be reused to validate the IK solutions derived in Section (11).
The dimensions set in table (2) are used to get the end-effector position in Frame $\{0\}$ through forward kinematics.
The MATLAB Function IK_robot.m takes in the positions to compute the corresponding elbow-down solutions of $\vec{Q}$.
The computed $\vec{Q}$ could then be checked against the configurations for validation.

1. Home Configuration : $\vec{q} = \begin{bmatrix} 0° & 0° & 0° & -90° & 0° \end{bmatrix}^T$

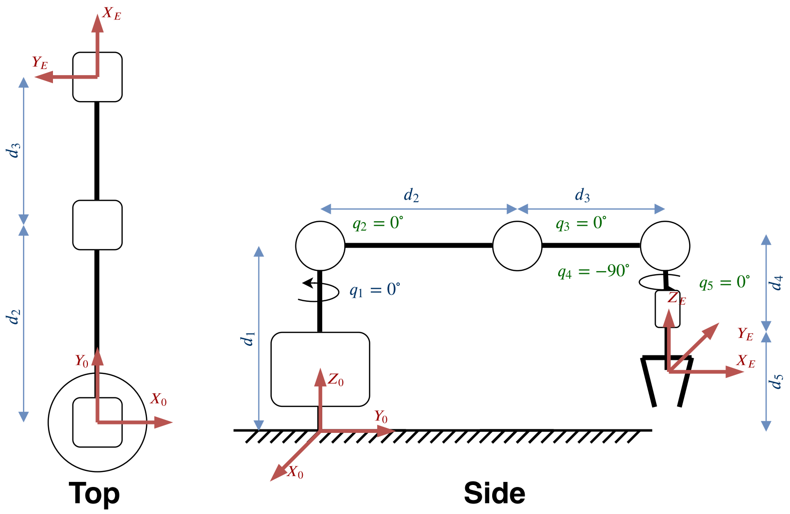

Figure 10: Top/Side view of home configuration

It is trivial to see that:

$$^0\vec{r}_{OE} = \begin{bmatrix} 0 \\ d_2 + d_3 \\ d_1 - d_4 - d_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.45 \\ 0 \end{bmatrix}$$

Ik_robot($^0\vec{r}_{OE}$) gives consistent results of $\vec{q}$.

$$\mathbf{Q20} = \begin{bmatrix} 30.0000° \\ 0.0000° \\ 0° \\ -90.0000° \\ 0° \end{bmatrix}$$

2. Configuration 1 : $\vec{q} = \begin{bmatrix} 30° & 0° & 0° & -90° & 0° \end{bmatrix}^T$

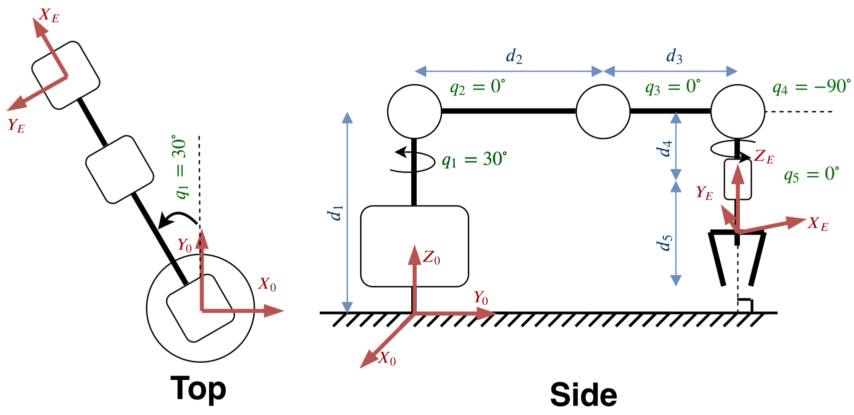

Figure 11: Top/Side view of Configuration 1

Forward Kinematics or hand calculation could lead to:

$$
{}^0\vec{r}_{OE} = \begin{bmatrix} -(d_2 + d_3)\sin 30° \\ (d_2 + d_3)\cos 30° \\ d_1 - d_4 - d_5 \end{bmatrix} = \begin{bmatrix} -0.225 \\ 0.3897 \\ 0 \end{bmatrix}
$$

Ik_robot(${}^0\vec{r}_{OE}$) gives consistent results of $\vec{q}$.

$$
\mathbf{Q20} = \begin{bmatrix} 0° \\ 0.0000° \\ 0° \\ -90.0000° \\ 0° \end{bmatrix}
$$

3. Configuration 2: $\vec{q} = \begin{bmatrix} 0° & 45° & -90° & -45° & 90° \end{bmatrix}^T$

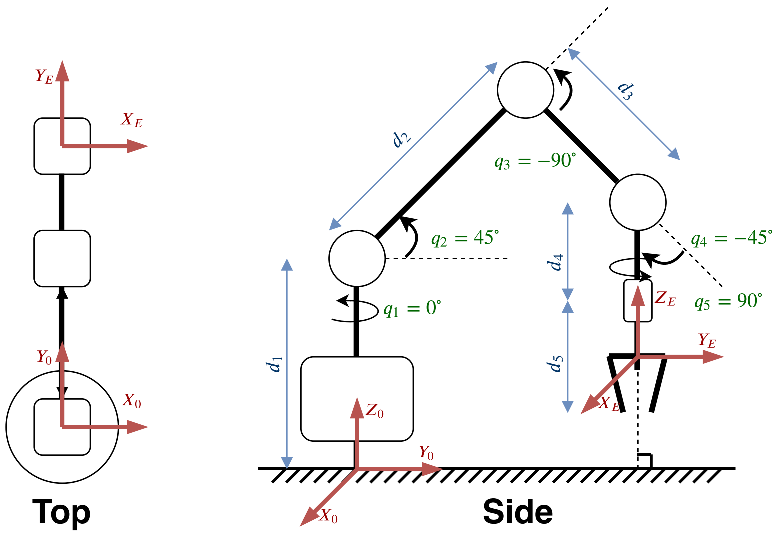

Figure 12: Top/Side view of Configuration 2

Forward Kinematics and easy hand calculation could lead to:

$$^{0}\vec{r}_{OE} = \begin{bmatrix} 0 \\ d_2 \cos 45° + d_3 \cos 45° \\ d_1 + d_2 \sin 45° - d_3 \sin 45° - d_4 - d_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3182 \\ -0.0354 \end{bmatrix}$$

Ik_robot($^{0}\vec{r}_{OE}$) gives consistent results of $\vec{q}$ except for q_5. q_5 is treated as 0 in the IK solution as it has no impact on the end-effector position.

$$\mathbf{Q20} = \begin{bmatrix} 0° \\ 45° \\ -90° \\ -45° \\ 0° \end{bmatrix}$$

3.3 IK solutions mapped to Jenga Tower Vertices

The 8 vertices of the Jenga Tower are placed at the following co-ordinates:

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0.25 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0.325 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0.075 \\ 0.25 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0.075 \\ 0.325 \\ 0 \end{bmatrix},$$
$$\vec{v}_5 = \begin{bmatrix} 0 \\ 0.25 \\ 0.27 \end{bmatrix}, \vec{v}_6 = \begin{bmatrix} 0 \\ 0.325 \\ 0.27 \end{bmatrix}, \vec{v}_7 = \begin{bmatrix} 0.075 \\ 0.25 \\ 0.27 \end{bmatrix}, \vec{v}_8 = \begin{bmatrix} 0.075 \\ 0.325 \\ 0.27 \end{bmatrix}$$

The inverse kinematics solutions for the elbow-up manifold are:

$$qu_1 = \begin{bmatrix} 0 \\ -66.4218 \\ 113.5772 \\ -137.1564 \\ 0 \end{bmatrix}, qu_2 = \begin{bmatrix} 0 \\ -50.2512 \\ 88.2092 \\ -127.9580 \\ 0 \end{bmatrix}, qu_3 = \begin{bmatrix} -16.6992 \\ -64.0868 \\ 110.1055 \\ -136.0187 \\ 0 \end{bmatrix}, qu_4 = \begin{bmatrix} -12.9946 \\ -48.3016 \\ 84.9802 \\ -126.6786 \\ 0 \end{bmatrix},$$
$$qu_5 = \begin{bmatrix} 0 \\ 7.2925 \\ 70.7919 \\ -168.0844 \\ 0 \end{bmatrix}, qu_6 = \begin{bmatrix} 0 \\ 17.1137 \\ 40.5138 \\ -147.6275 \\ 0 \end{bmatrix}, qu_7 = \begin{bmatrix} -16.6992 \\ 8.0666 \\ -67.3407 \\ -165.4073 \\ 0 \end{bmatrix}, qu_8 = \begin{bmatrix} -12.9946 \\ 19.3354 \\ 35.2641 \\ -144.5994 \\ 0 \end{bmatrix}$$

The inverse kinematics solutions for the elbow-down manifold are:

$$qu_1 = \begin{bmatrix} 0 \\ 66.4218 \\ -113.5772 \\ -42.8436 \\ 0 \end{bmatrix}, qu_2 = \begin{bmatrix} 0 \\ 50.2512 \\ -88.2092 \\ -52.0420 \\ 0 \end{bmatrix}, qu_3 = \begin{bmatrix} -16.6992 \\ 64.0868 \\ -110.1055 \\ 43.9813 \\ 0 \end{bmatrix}, qu_4 = \begin{bmatrix} -12.9946 \\ 48.3016 \\ -84.9802 \\ 53.3214 \\ 0 \end{bmatrix},$$
$$qu_5 = \begin{bmatrix} 0 \\ 87.1127 \\ -70.7919 \\ -106.3208 \\ 0 \end{bmatrix}, qu_6 = \begin{bmatrix} 0 \\ 62.3238 \\ -40.5138 \\ -111.8100 \\ 0 \end{bmatrix}, qu_7 = \begin{bmatrix} -16.6992 \\ 83.8738 \\ -67.3407 \\ -106.5331 \\ 0 \end{bmatrix}, qu_8 = \begin{bmatrix} -12.9946 \\ 58.6445 \\ -35.2641 \\ -113.3805 \\ 0 \end{bmatrix}$$

All the solutions for elbow-down manifold fall within the joint limitations defined in table (2). The elbow-up manifold solutions are invalid due to negative values of q_2.
The function robot_3dplot.m can be used to check the solutions by plotting the robot in 3D.

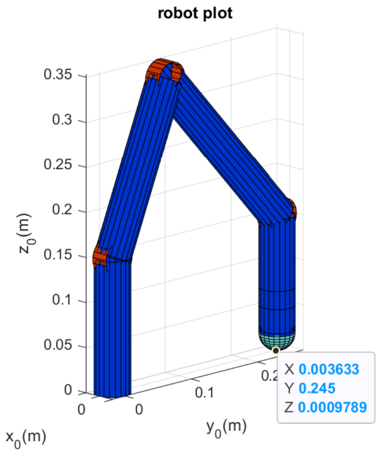

Figure 13: $\vec{v}_1$

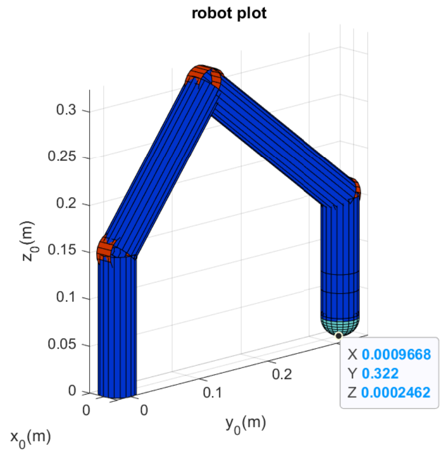

Figure 14: $\vec{v}_2$

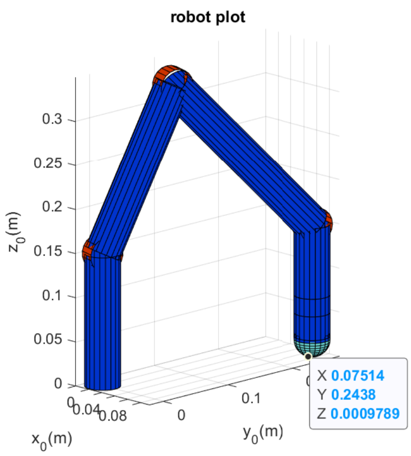

Figure 15: $\vec{v}_3$

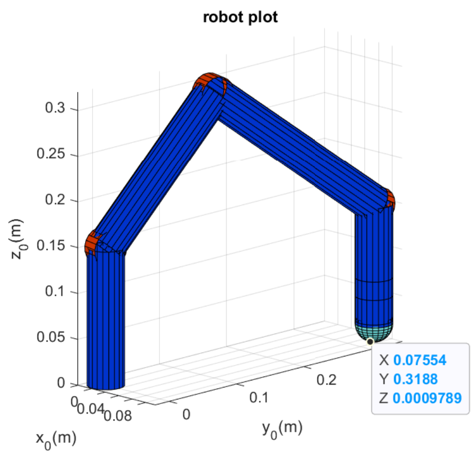

Figure 16: $\vec{v}_4$

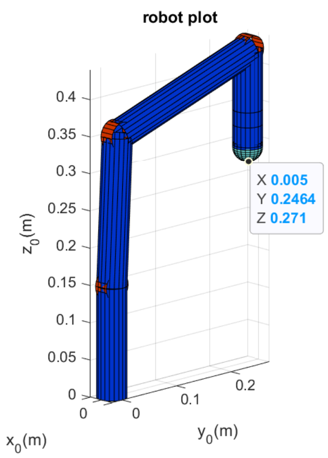

Figure 17: $\vec{v}_5$

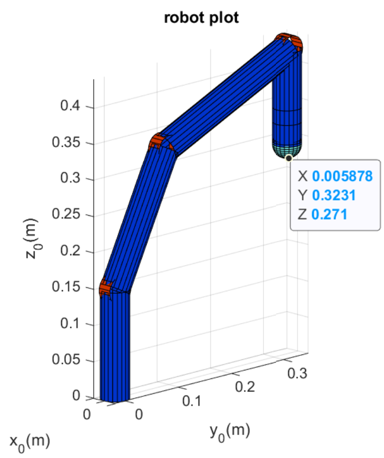

Figure 18: $\vec{v}_6$

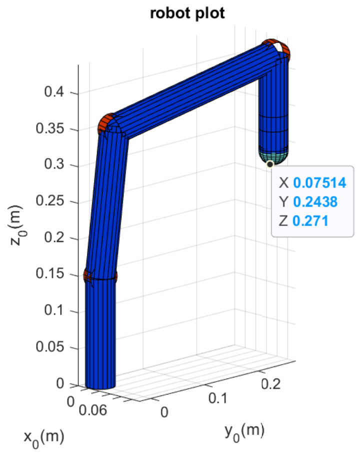

Figure 19: $\vec{v}_7$

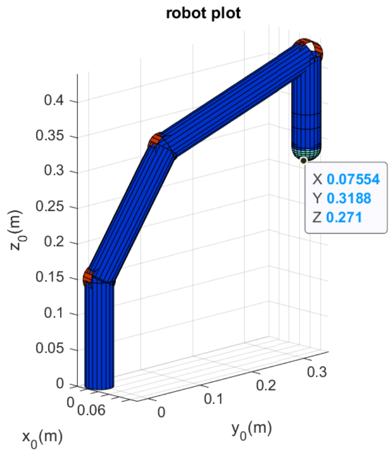

Figure 20: $\vec{v}_8$