MATHEMATICS

What is the dimension of a complex Stiefel manifold

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The Stiefel manifold is defined as follows.

$$V_k(\mathbb{C}^n) = \{X \in \mathbb{C}^{n\times k} | X^* X = I_k\}$$

where $n \geqslant k \geqslant 1$.

According to the pulic knowledge on Wikipedia on "Steifel manfiold", the dimension of manifold is given by

$$\dim V_k(\mathbb{C}^n) = 2nk - k^2 \ .$$

Actually, the above dimension is only correct when $V_k(\mathbb{C}^n)$ is considered a real manifold embedded in a $2nk$-dimensional real vector space $\mathbb{C}^{n\times k}$.

More naturally, the vector space $\mathbb{C}^{n\times k}$ should be considered as complex rather than real, and the dimension of the complex vector space $\mathbb{C}^{n\times k}$ is nk rather than $2nk$.

If the manifold $V_k(\mathbb{C}^n)$ is considered as complex rather than real, the dimension, given by Wikipedia, I contend, is not correct.

I can not find useful information on the complexification of $V_k(\mathbb{C}^n)$.

I also content, for example, when $k = 1$, there is not such a thing as the complex Stiefel manifold, only real Stiefel manifold exists.

Under what condition, a complex Stiefel manifold exists?

What is the original source giving the resulf $\dim V_k(\mathbb{C}^n) = 2nk - k^2$? Is there anybody who can help?

differential-geometry stiefel-manifolds

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Liang Liao
31 3

2 What do you mean by "complexification of $V_k(\mathbb{C}^n)$"? The manifold $V_k(\mathbb{C}^n)$ is a real submanifold of $\mathbb{C}^{n\times k}$. For example, like you mentioned, when $k = 1$ then $V_k(\mathbb{C}^n)$ can be identified with S^{2n-1} which is an odd (real) dimensional sphere do it definitely doesn't have a structure of a complex manifold (which must be of even real dimension). – levap 2 days ago

3 As to the formula for the dimension, if you consider the map F from $C^{n\times k}$to the subspace of self-

adjoint $k \times k$ matrices then $V_k(\mathbb{C}^n) = F^{-1}(I_k)$ and I_k is a regular value of F so $\dim V_k(\mathbb{C}^n) = 2nk - k^2$. – levap 2 days ago

Given a point $X \in V_k(\mathbb{C}^n)$, one can have the tangent space and normal space at X. A random vector $Z \in \mathbb{C}^{n \times k}$ can be projected onto the tangent space and normal space. Let Z_T and Z_N be the non-trivial projections of Z. I would like to have the tangent space and normal space be complex rather than be real. If the range space and normal space can be complex, one should have $tr\,(Z_T)^H Z_N = 0$. I call that the complexification of the Stiefel manifold. Is $tr\,(Z_T)^H Z_N = 0$ possible? – Liang Liao 2 days ago

1 If I understand you correctly, you are asking when the (real) tangent space of $V_k(\mathbb{C}^n)$ is a complex vector subspace of $\mathbb{C}^{n \times k}$. This never happens. Consider the point $X \in V_k(\mathbb{C}^n)$ whose columns are the first k elements of the standard basis of $\mathbb{C}^n$. The tangent space at X is $T_X(V_k(\mathbb{C}^n)) = \{V \in M_{n \times k}(\mathbb{C}) \,|\, V + V^* = 0\}$ and this is never a complex vector subspace (if $V \in T_X(V_k(\mathbb{C}^n))$ then $V^* = -V$ but $(iV)^* = -i(-V) = iV$ so iV is not in the tangent space). – levap 2 days ago

If the tangent space can not be complex. To define the orthogonality between two vectors A and B respectively in the tangent space and normal space, the elegant inner product $trA^H B$ usually used for the complex vector space $\mathbb{C}^{n \times k}$ had to be abandoned. If that is the situation, an inner product for real space has to be adopted. I wonder the inner product for the $2nk$-dimensional real space $\mathbb{C}^{n \times k}$ will look awkward. – Liang Liao 2 days ago

Let $A = A_1 + i \cdot A_2 \in \mathbb{C}^{n \times k}$ where $A_1 = real(A)$ and $A_2 = imag(A)$ and $B = B_1 + i \cdot B_2 \in \mathbb{C}^{n \times k}$. The inner product is something like this ---- $\langle A, B \rangle = tr(A_1^T B_1) + tr(A_2^T B_2)$. – Liang Liao 2 days ago

Let us continue this discussion in chat. – Liang Liao 2 days ago