MATHEMATICS

What is the dimension of a complex Stiefel manifold

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Let $V_k(\mathbb{C}^n) = \{X|X^*X = I_k\}$ be the Stiefel manifold. According to Wikipedia, (https://en.wikipedia.org/wiki/Stiefel_manifold) the dimension is given by

$$\dim V_k(\mathbb{C}^n) = 2nk - k^2$$

I wonder the given dimension is a mistake. I wonder it should had been given by

$$\dim V_k(\mathbb{C}^n) = nk - k^2$$

The number k^2 should be the dimension of the normal space of a point of the manifold.

Can anyone provide the original source of the perhapse wrong dimension $\dim V_k(\mathbb{C}^n) = 2nk - k^2$? A careful check on the correctness is needed.

Please help me if I am not correct. Thanks.

stiefel-manifolds

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edited 17 mins ago

asked 3 hours ago

L Liang Liao
31 2

2 Answers

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Let me offer a perspective different to the quotient perspective. The complex Stiefel manfiold $V_k(\mathbb{C}^n)$ can be embedded in the vector space $\mathbb{C}^{n\times k}$, which is isomorphic to the space $\mathbb{C}^{nk}$. The normal space $N_X V_k(\mathbb{C}^n)$ has a dimension k^2, and is isomorphic to a k^2-dimensional subspace N of $\mathbb{C}^{np}$.

The tangent space $T_X V_k(\mathbb{C}^n)$, orthogonal to the normal space $N_X V_k(\mathbb{C}^n)$, is isomorphic to the orthogonal complement of N in C^{np}.

Since $T_X V_k(\mathbb{C}^n) \cong N^\perp$, the following equations hold.

$$\dim T_X V_k(\mathbb{C}^n) = \dim N^\perp = nk - k^2$$

The following identity always holds

$$\dim V_k(\mathbb{C}^n) \equiv \dim T_X V_k(\mathbb{C}^n) = nk - k^2 \ .$$

Please correct me if I am wrong. I really appreciate any help you can provide.

Note that a dimension is defined by the number of independent spanning complex vectors (sometimes in the form of complex matrices, for example, in the case of an embedded complex Stiefel manifold).

A convenient verification case is that when $k = 1$. Under this condition the complex manifold $V_k(\mathbb{C}^D)$ is a complex sphere in $\mathbb{C}^n$. Thus, $\dim V_k(\mathbb{C}^D)|_{k=1} = nk - k^2 = n - 1$.

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5 revs
Liang Liao

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The dimension $\dim V_k(\mathbb{C}^n) = 2nk - k^2$ is indeed correct, one way to see this is to use the bijection $V_k(\mathbb{C}^n) \cong U(n)/U(n-k)$, where $U(n)$ is the unitary group.

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answered 2 hours ago

Riquelme
571 1 4 10

Please see my following post. Please correct me if I am wrong. Thanks., – Liang Liao 29 mins ago