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Local homeomorphism

In mathematics, more specifically topology, a **local homeomorphism** is a function between topological spaces that, intuitively, preserves local (though not necessarily global) structure. If $f : X \to Y$ is a local homeomorphism, X is said to be an **étale space** over Y. Local homeomorphisms are used in the study of sheaves. Typical examples of local homeomorphisms are covering maps.

A topological space X is **locally homeomorphic** to Y if every point of X has a neighborhood that is homeomorphic to an open subset of Y. For example, a manifold of dimension n is locally homeomorphic to $\mathbb{R}^n$.

If there is a local homeomorphism from X to Y, then X is locally homeomorphic to Y, but the converse is not always true. For example, the two dimensional sphere, being a manifold, is locally homeomorphic to the plane $\mathbb{R}^2$, but there is no local homeomorphism between them (in either direction).

Contents

Formal definition

Let X and Y be topological spaces. A function $f : X \to Y$ is a local homeomorphism[1] if for every point $x \in X$ there exists an open set U containing x, such that the image $f(U)$ is open in Y and the restriction $f\big|_U : U \to X$ is a homeomorphism (where the respective subspace topologies are used on U and on $f(U)$).

Examples and sufficient conditions

By definition, every homeomorphism is also a local homeomorphism.

If U is an open subset of Y equipped with the subspace topology, then the inclusion map $i : U \to Y$ is a local homeomorphism. Openness is essential here: the inclusion map of a non-open subset of Y never yields a local homeomorphism.

Let $f : \mathbb{R} \to S^1$ be the map that wraps the real line around the circle (i.e. $f(t) = e^{it}$ for all $t \in \mathbb{R}$. This is a local homeomorphism but not a homeomorphism.

Let $f : S^1 \to S^1$ be the map that wraps the circle around itself n times (i.e. has winding number n). This is a local homeomorphism for all non-zero n, but a homeomorphism only in the cases where it is bijective, i.e. when $n = 1$ or -1.

Generalizing the previous two examples, every covering map is a local homeomorphism; in particular, the universal cover $p : C \to Y$ of a space Y is a local homeomorphism. In certain situations the converse is true. For example: if X is Hausdorff and Y is locally compact and Hausdorff and $p : X \to Y$ is a proper local homeomorphism, then p is a covering map.

There are local homeomorphisms $f : X \to Y$ where Y is a Hausdorff space and X is not. Consider for instance the quotient space $X = (\mathbb{R} \sqcup \mathbb{R}) / \sim$, where the equivalence relation $\sim$ on the disjoint union of two copies of the reals identifies every negative real of the first copy with the corresponding negative real of the second copy. The two copies of 0 are not identified and they do not have any disjoint neighborhoods, so X is not Hausdorff. One readily checks that the natural map $f : X \to \mathbb{R}$ is a local homeomorphism. The fiber $f^{-1}(\{y\})$ has two elements if $y \geq 0$ and one element if $y < 0$.

Similarly, we can construct a local homeomorphisms $f : X \to Y$ where X is Hausdorff and Y is not: pick the natural map from $X = \mathbb{R} \sqcup \mathbb{R}$ to $Y = (\mathbb{R} \sqcup \mathbb{R}) / \sim$ with the same equivalence relation $\sim$ as above.

If $f : X \to Y$ is a local homeomorphism and U is an open subset of X, then the restriction $f|_U : U \to Y$ is also a local homeomorphism.

If $f : X \to Y$ and $g : Y \to Z$ are local homeomorphisms, then the composition $g \circ f : X \to Z$ is also a local homeomorphism.

If $f : X \to Y$ is continuous, $g : Y \to Z$ is a local homeomorphism, and $g \circ f : X \to Z$ a local homeomorphism, then f is also a local homeomorphism.

It is shown in complex analysis that a complex analytic function $f : U \to \mathbb{C}$ (where U is an open subset of the complex plane $\mathbb{C}$) is a local homeomorphism precisely when the derivative $f'(z)$ is non-zero for all $z \in U$. The function $f(x) = z^n$ on an open disk around 0 is not a local homeomorphism at 0 when $n \geq 2$. In that case 0 is a point of "ramification" (intuitively, n sheets come together there).

Using the inverse function theorem one can show that a continuously differentiable function $f : U \to \mathbb{R}^n$ (where U is an open subset of $\mathbb{R}^n$) is a local homeomorphism if the derivative $D_x f$ is an invertible linear map (invertible square matrix) for every $x \in U$. (The converse is false, as shown by the local homeomorphism $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$). An analogous condition can be formulated for maps between differentiable manifolds.

Suppose $f : X \to Y$ is a continuous open surjection between two Hausdorff second-countable spaces where X is a Baire space and Y is a normal space. Let O be the union of all open subsets U of X such that the restriction $f|_U : U \to Y$ is an injective map, where the assumptions on f imply that the (potentially empty) set O is also the (unique) largest open subset of X such that $f|_O : O \to Y$ is a local homeomorphism. If every fiber of f is a discrete subspace of X then O is a dense subset of X. For example, if $f : \mathbb{R} \to [0, \infty)$ is defined to be the polynomial $f(x) = x^2$ then the maximal open set O from this theorem is $O = \mathbb{R} \setminus \{0\}$, which shows that it is possible for O to be a *proper* subset of f's domain. Because every fiber of every non-constant polynomial is finite (and thus a discrete, and even compact, subspace), this example generalizes to such polynomials whenever the mapping induced by it is an open map; and if it is not an open map then it is nevertheless still straightforward to apply the theorem (possibly multiple times) by choosing domain(s) based on an appropriate consideration of the map's local minimums/maximums.

Properties

Every local homeomorphism is a continuous and open map. A bijective local homeomorphism is therefore a homeomorphism.

Every fiber of a local homeomorphism $f : X \to Y$ is a discrete subspace of its domain X.

A local homeomorphism $f : X \to Y$ transfers "local" topological properties in both directions:

- X is locally connected if and only if $f(X)$ is;
- X is locally path-connected if and only if $f(X)$ is;
- X is locally compact if and only if $f(X)$ is;
- X is first-countable if and only if $f(X)$ is.

As pointed out above, the Hausdorff property is not local in this sense and need not be preserved by local homeomorphisms.

The local homeomorphisms with codomain Y stand in a natural one-to-one correspondence with the sheaves of sets on Y; this correspondence is in fact an equivalence of categories. Furthermore, every continuous map with codomain Y gives rise to a uniquely defined local homeomorphism with codomain Y in a natural way. All of this is explained in detail in the article on sheaves.

Generalizations and analogous concepts

The idea of a local homeomorphism can be formulated in geometric settings different from that of topological spaces. For differentiable manifolds, we obtain the local diffeomorphisms; for schemes, we have the formally étale morphisms and the étale morphisms; and for toposes, we get the étale geometric morphisms.

See also

- Homeomorphism – Isomorphism of topological spaces in mathematics
- Local diffeomorphism

Citations

1. Munkres, James R. (2000). Topology (2nd ed.). Prentice Hall. ISBN 0-13-181629-2.

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