MATHEMATICS

What is the dimension of a complex Stiefel manifold

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Let $V_k(\mathbb{C}^n) = \{X|X^*X = I_k\}$ be the Stiefel manifold. According to Wikipedia, (https://en.wikipedia.org/wiki/Stiefel_manifold) the dimension is given by

$$\dim V_k(\mathbb{C}^n) = 2nk - k^2$$

I wonder the given dimension is a mistake. I wonder it should had been given by

$$\dim V_k(\mathbb{C}^n) = nk - k^2$$

The number k^2 should be the dimension of the normal space of a point of the manifold.

Can anyone provide the original source of the perhapse wrong dimension $\dim V_k(\mathbb{C}^n) = 2nk - k^2$? A careful check on the correctness is needed.

Please help me if I am not correct. Thanks.

stiefel-manifolds

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edited 48 mins ago

asked 3 hours ago

Liang Liao
31 2

2 Answers

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Let me offer a perspective different to the quotient perspective. The complex Stiefel manfiold $V_k(\mathbb{C}^n)$ can be embedded in the vector space $\mathbb{C}^{n\times k}$, which is isomorphic to the space $\mathbb{C}^{nk}$. The normal space $N_X V_k(\mathbb{C}^n)$ has a dimension k^2, and is isomorphic to a k^2-dimensional subspace N of $\mathbb{C}^{np}$.

The tangent space $T_X V_k(\mathbb{C}^n)$, orthogonal to the normal space $N_X V_k(\mathbb{C}^n)$, is isomorphic to the orthogonal complement of N in C^{np}.

Since $T_X V_k(\mathbb{C}^n) \cong N^\perp$, the following equations hold.

$$\dim T_X V_k(\mathbb{C}^n) = \dim N^\perp = nk - k^2$$

The following identity always holds

$$\dim V_k(\mathbb{C}^n) \equiv \dim T_X V_k(\mathbb{C}^n) = nk - k^2 \ .$$

Please correct me if I am wrong. I really appreciate any help you can provide.

Note that a dimension is defined by the number of independent spanning complex vectors (sometimes in the form of complex matrices, for example, in the case of an embedded complex Stiefel manifold).

A convenient example is given when $k = 1$. Under this condition the complex manifold $V_k(\mathbb{C}^D)$ is a complex sphere in $\mathbb{C}^n$. Thus, dim $V_k(\mathbb{C}^D)|_{k=1} = nk - k^2 = n - 1$.

A misleading example is given when $n = 1$ and $k = 1$. Under this condition, the complex Stiefel manifold is reduced to a circle in the complex plane $\mathbb{C}$. One might think a circle in a complex plane is two-dimensional.

Not actually true. A circle in $\mathbb{C}$ is only one-dimensional (i.,e., dim $= 2nk - k^2 = 2 - 1 = 1$) defined over the field of real numbers. A circle in $\mathbb{C}$ is only zero-dimensional defined over complex numbers.

I am talking about the complex Stiefel manifold. Therefore, my question on the dimension that I am talking about is ver complex numbers rather than real numbers.

I contend that Wikipedia is wrong because the dimension of the ambient vector space is defined over real numbers rather than complex numbers. On the other hand, the dimension of normal space k^2 is always defined over complex numbers.

Please correct me if I am wrong.

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7 revs
Liang Liao

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The dimension dim $V_k(\mathbb{C}^n) = 2nk - k^2$ is indeed correct, one way to see this is to use the bijection $V_k(\mathbb{C}^n) \cong U(n)/U(n-k)$, where $U(n)$ is the unitary group.

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answered 2 hours ago
Riquelme
571 1 4 10

Please see my following post. Please correct me if I am wrong. Thanks., – Liang Liao 1 hour ago