MATHEMATICS

Tangent and normal space of the Stiefel manifold

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Let $V_{n,p}$ be the real Stiefel manifold with parameters n and p. Let $X \in V_{n,p}$ and $Y(t)$ curve on $V_{n,p}$ with $Y(0) = X$. Then, $\dot{Y}(0)$ is a tangent vector to $V_{n,p}$ at X. With the standard knowledge on the Stiefel manifold. We know the following equation holds

$$\dot{Y}(0)^T X + X^T \dot{Y}(0) = 0$$

The projections, on the tangent space and the normal space, of a random vector $Z \in \mathbb{R}^{n\times p}$ at X are respectively given by

$$\begin{aligned} T &\doteq \dot{Y}(0) = X\,\mathrm{skew}(X^T Z) + (I - XX^T)Z \\ N &= X\,\mathrm{sym}(X^T Z) \end{aligned}$$

where

$$\begin{aligned} \mathrm{sym}(A) &\doteq (A + A^T)/2 \\ \mathrm{skew}(A) &\doteq (A - A^T)/2 \end{aligned}$$

The above equations are well-understood. However, the above equations need to be modified to extend the above results to the complex manifold.

My numerical code shows there is something wrong to extend the above results to the complex case by simply replacing transposes of real matrices to conjugate transposes of complex matrices.

```
1  function Stiefel_manifold
2      row_num = 11;
3      col_num = 3;
4
5      X = orth(randn(row_num, col_num) + i * randn(row_num, col_num) );
6      Z = randn(row_num, col_num) + i * randn(row_num, col_num);
7
8
9      N = X * mysym(ctranspose(X) * Z);
10     T = X * myskew(ctranspose(X)  * Z) + (eye(row_num) - X * ctranspose(X)) * Z;
11     inner_product = trace(T' * N);
12
13     assert(isequal(round(N + T, 4), round(Z, 4)));
14     assert(norm(X' * T + T' * X, 'fro') < 1e-8)
15     assert(abs(inner_product) < 1e-6);
16
17     return;
18 end
19
20 function result = mysym(A)
21     result = (A + ctranspose(A) ) / 2;
22 end
23
24 function result = myskew(A)
25     result = (A - ctranspose(A)) / 2;
26 end
```

The third assertion (line 15) can not be passed, and it means the tangent vector and norm vector are not orthogonal.

Can anyone provide helps to extend the results to a complex manifold? Thanks.

A reference to the real Stiefel manifold can be found in the paper

Edelman, A., Arias, T.A., Smith, S.T.: The Geometry of Algorithms with Orthogonality Constraints. SIAM Journal on Matrix Analysis and Applications 20(2), 303--353 (1998), Section 2.2

PDF: https://arxiv.org/pdf/physics/9806030.pdf

Big thanks for any help.

PS: Why I am interested in the complex Stiefel manifold? Because I think I can generalize the notion of Stiefel manifold over a novel commutative algebra called t-algebra. I need a complex manifold to generalize the Stiefel manifold over T-Algebra.

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asked Feb 5 at 4:52
Liang Liao
31 3

2 Answers

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Some told me the Steifel manifold can never be complex. Even embedded in a complex space $\mathbb{C}^{nk}$, the Steifel manifold $V_{n,p}(\mathbb{C}) = \{X \in \mathbb{C}^{n\times p} | X^* X = I_p\}$ is still real rather than complex.

I finally accept the above result as accurate. It is interesting to know that the Stiefel manifold can never be complex.

Please correct me if I am wrong.

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answered 2 hours ago
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31 3

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I find this page interesting and might be my answer.
https://mathoverflow.net/questions/84955/exponential-and-logarithm-mapping-on-stiefel-manifold/133084#133084

In the complex case, the inner product is not defined $\langle X, Y\rangle = tr(X^T \cdot Y)$. The inner product might be defined as follows.

$$\langle X, Y\rangle_c \doteq tr(X^T \cdot (I - \frac{1}{2} \cdot UU^T) \cdot Y)$$

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answered Feb 5 at 10:09
Liang Liao
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