

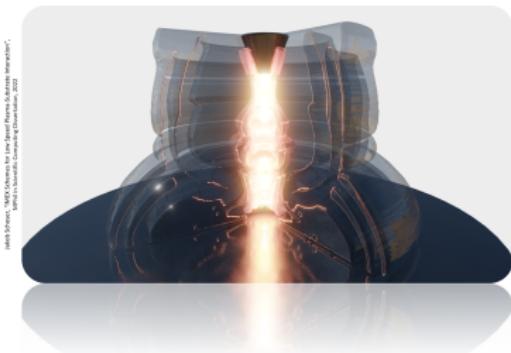
Numerical Methods for Compressible Fluid Dynamics

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Numerical Method for Compressible Fluid Dynamics

- This course focuses on the numerical algorithms for solving compressible (high-speed) materials
- This involves understanding the governing non-linear systems of partial differential equations
- Fortunately we can start simple, with scalar linear problems, and work towards these complex cases
- By the end of the course, we will have studied techniques capable of simulating a wide range of compressible systems (and not just fluid dynamics)



Numerical Method for Compressible Fluid Dynamics

- Seven lectures and six practicals
- Lectures will be recorded for use in revision and practical work
- Apart from today, practicals will be started in the second half of each lecture session
- These will need to be finished outside of lectures
- Each practical will build upon the previous ones, but try to save your code from each practical separately, in case something goes wrong with the modifications!
- Progress in practicals may be used as part of the project decision process
- Start creating a document, or multiple documents, with results from the practicals
- Additional practical sessions are scheduled for after exams

Numerical Method for Compressible Fluid Dynamics

- The lectures:
 - ① Introduction to the equations and basics of formulating numerical methods
 - ② Discretisation of partial differential equations and an initial application to the scalar advection equation
 - ③ Numerical methods for scalar hyperbolic equations and an introduction to non-linear equations
 - ④ Finite volume methods for non-linear scalar hyperbolic equations
 - ⑤ Systems of hyperbolic equations, with a focus on the Euler equations
 - ⑥ Improving the accuracy of the numerical methods for the Euler equations
 - ⑦ Two- and three-dimensional solutions to the Euler equations

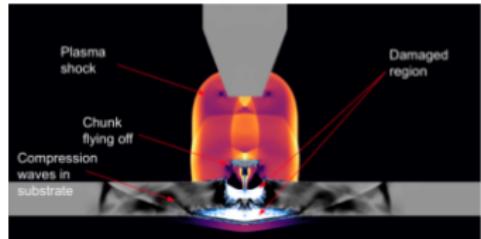
Outline

- 1 Computational continuum modelling
- 2 Equations of flow
- 3 Classifying the equations
- 4 Mathematical properties and discontinuities

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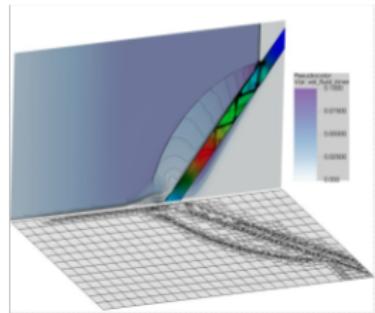
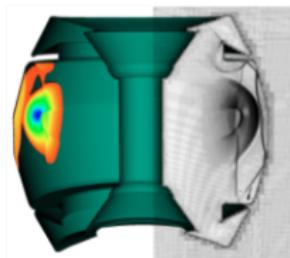
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Why do we need computational continuum modelling?

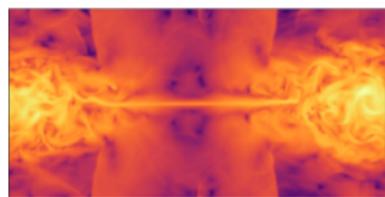


Jakob Schoser

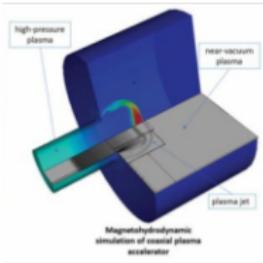
Alexis Farmakalides



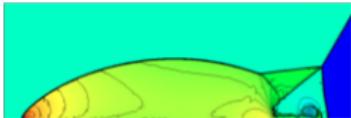
Lara Lu



Xudong Ke Lin



Riccardo Dematte

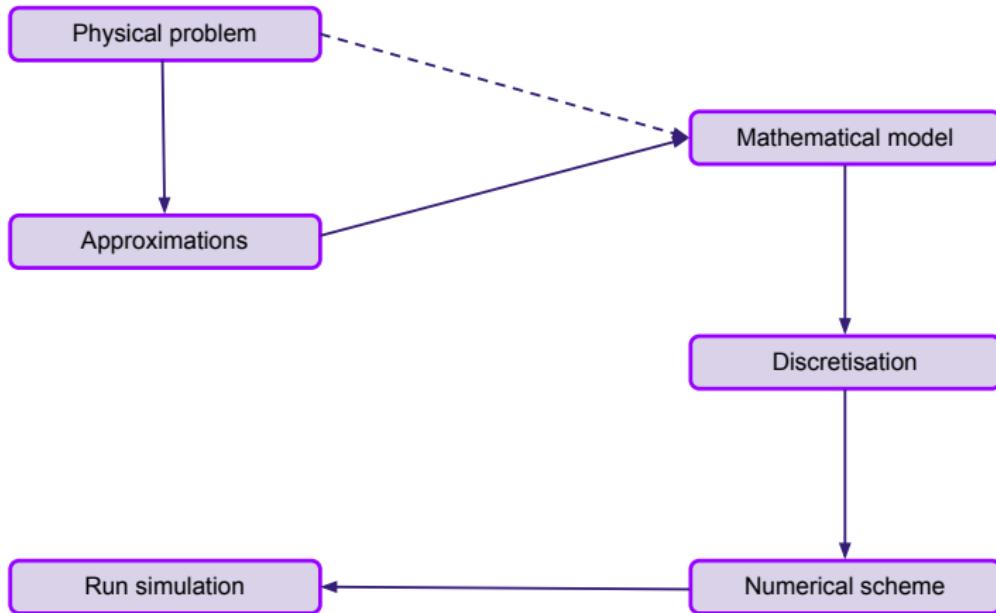


Maria Chrysanthou

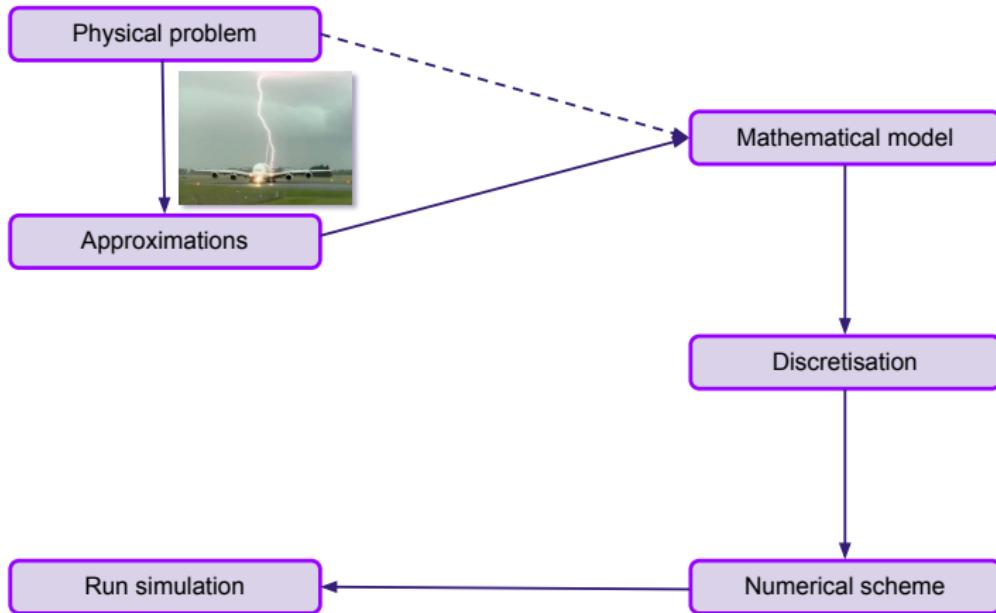
Recommended literature

- There is no set course text, but many good references go into much greater detail than is possible in this course
- For more information on the methods covered in this course, Toro's book "Riemann Solvers and Numerical Methods for Fluid Dynamics" is strongly recommended
- Available electronically from the University Library, and many physical copies exist in the lab
- Laney's "Computational Gasdynamics" offers an accessible introduction to discretisation and numerical methods
- Leveque's "Finite Volume Methods for Hyperbolic Problems" is a more mathematical (and thorough) treatment of many of the same topics

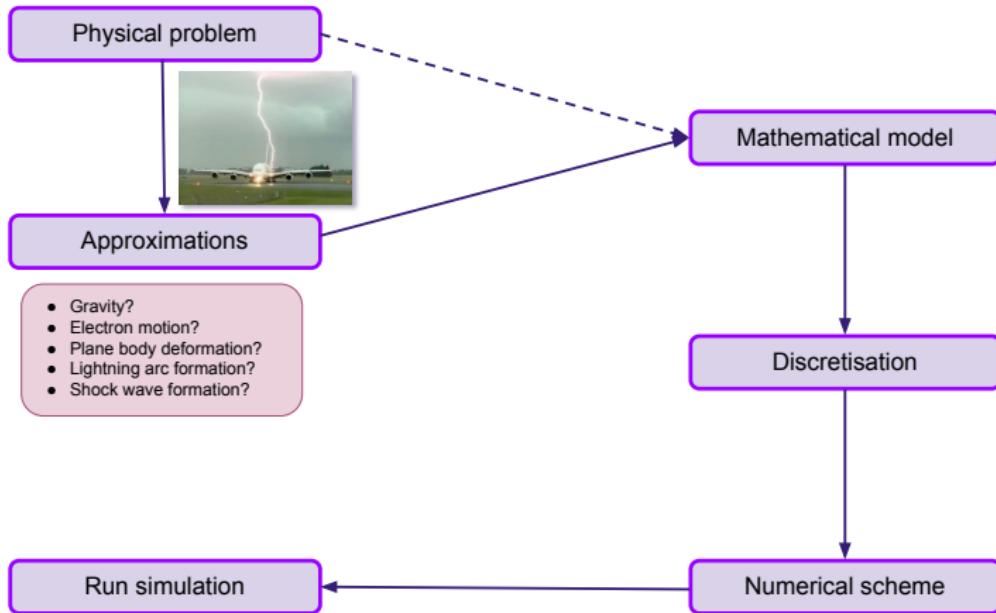
From the real world to a numerical simulation



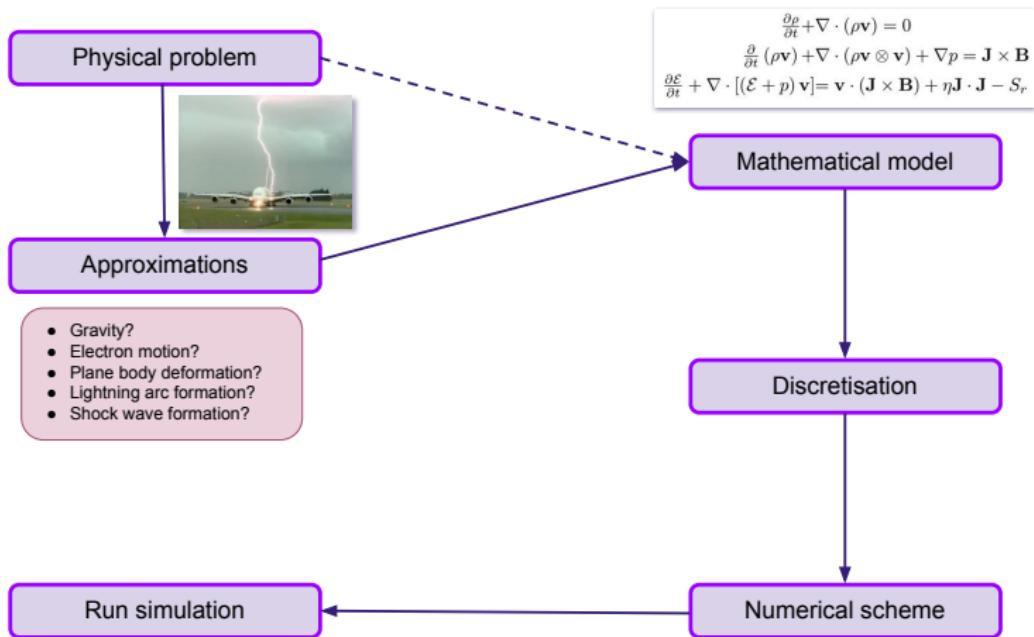
From the real world to a numerical simulation



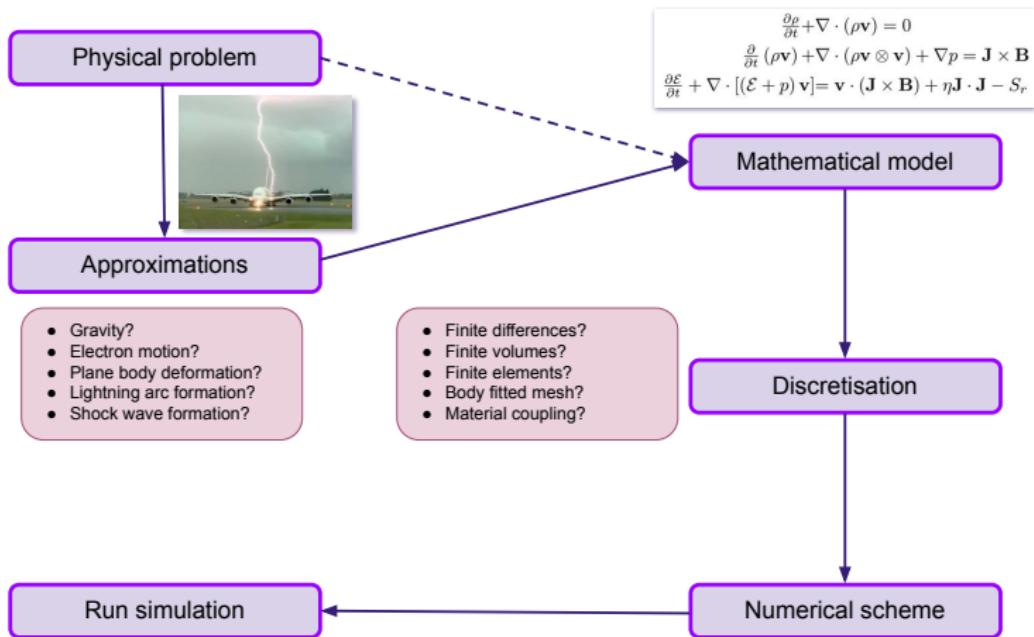
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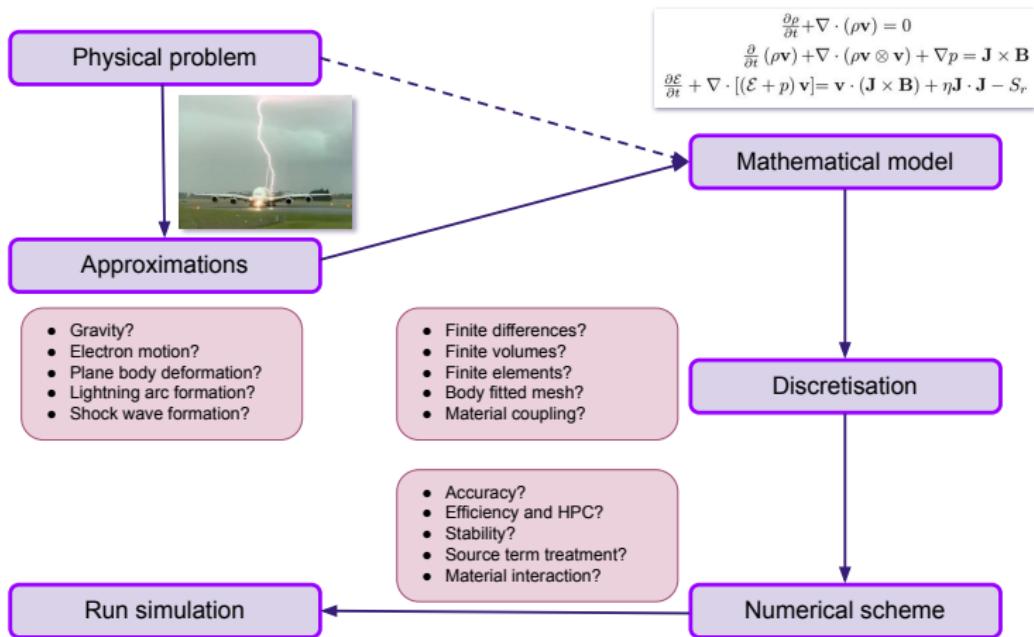
From the real world to a numerical simulation



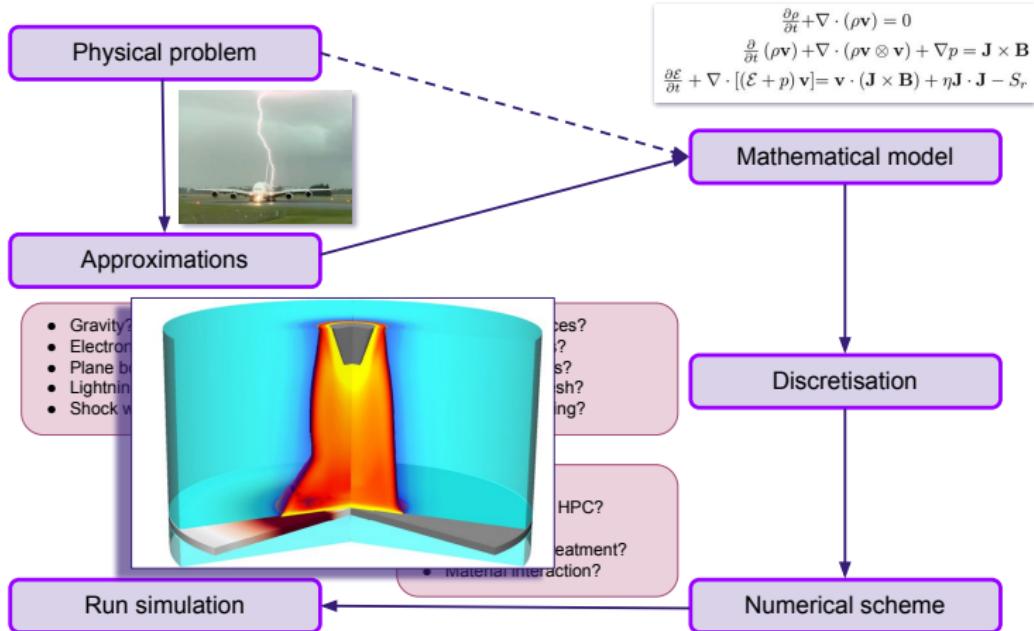
From the real world to a numerical simulation



From the real world to a numerical simulation



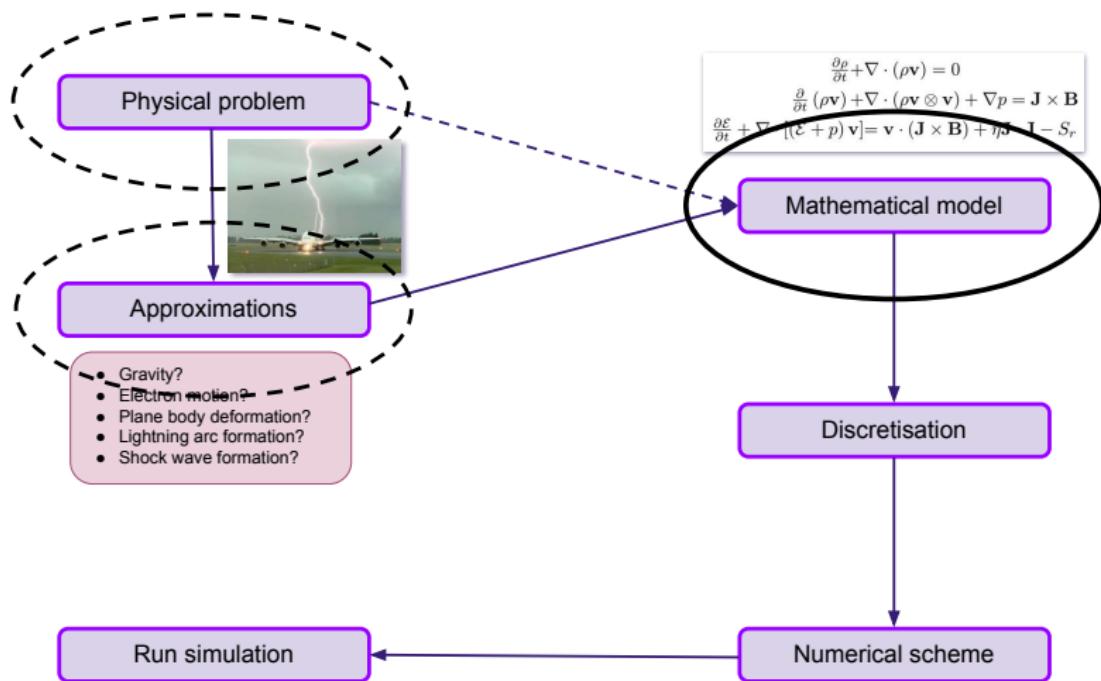
From the real world to a numerical simulation



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Equations of flow



Continuum models

- Many factors are involved when selecting the most appropriate mathematical model for your problem
- Typically these considerations are based on two properties:
 - **Length scale** - do you need to consider individual molecules, atoms, subatomic particles, or are volume-averaged quantities a better option?
 - **Time scale** - how long do you need to simulate behaviour and what physical processes occur over these times
- This course focuses on cases where the underlying mathematical model is a **continuum** described by a **hyperbolic system of conservation equations** (and we shall cover what these terms mean)
- Solving these models is often referred to as **computational fluid dynamics** (CFD), though 'fluid' has rather a broad meaning here
- One of the most important systems of equations for CFD are the **Navier-Stokes** equations

Navier Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \boldsymbol{\tau}) + \rho \mathbf{f} \cdot \mathbf{v}$$

Navier Stokes equations

Density

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$

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Navier Stokes equations

Density Velocity vector

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

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Density Velocity vector Pressure

Navier Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
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Density

Velocity vector

Pressure

Total energy

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graph LR; DE[Density] --> Ddt["∂ρ/∂t"]; VE[Velocity vector] --> DV["∇ · (ρv)"]; P[Pressure] --> Pp["p"]; TE[Total energy] --> DEE["∂E/∂t"];
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Navier Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
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Density
Momentum
Total energy
Velocity vector
Pressure

Navier Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$
$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \boldsymbol{\tau}) + \rho \mathbf{f} \cdot \mathbf{v}$$

Diagram illustrating the components of the Navier-Stokes equations:

- Momentum leads to the first term $\frac{\partial \rho \mathbf{v}}{\partial t}$.
- Density leads to the second term $\frac{\partial \rho}{\partial t}$ and the third term $\nabla \cdot (\rho \mathbf{v})$.
- Velocity vector leads to the fourth term $\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p)$.
- Pressure leads to the fifth term $\nabla \cdot \boldsymbol{\tau}$.
- Total energy leads to the sixth term $\frac{\partial E}{\partial t}$.
- Temperature leads to the seventh term $\nabla \cdot (\kappa \nabla T)$.

Navier Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$
$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \boldsymbol{\tau}) + \rho \mathbf{f} \cdot \mathbf{v}$$

Density
Momentum
Total energy
Velocity vector
Temperature
Pressure
Identity matrix

The diagram illustrates the components of the Navier-Stokes equations. It shows three main equations connected by arrows indicating dependencies. The first equation, involving density, has arrows pointing from 'Density' and 'Momentum'. The second equation, involving momentum, has arrows pointing from 'Velocity vector', 'Identity matrix', and 'Pressure'. The third equation, involving total energy, has arrows pointing from 'Temperature' and 'Pressure'. Labels for each variable are provided in boxes.

Navier Stokes equations

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \tau) + \rho \mathbf{f} \cdot \mathbf{v}$$

Diagram illustrating the components of the Navier-Stokes equations:

- Momentum leads to the first term $\frac{\partial \rho \mathbf{v}}{\partial t}$.
- Total energy leads to the second term $\frac{\partial E}{\partial t}$.
- Density leads to the first term $\frac{\partial \rho \mathbf{v}}{\partial t}$ and the second term $\nabla \cdot (\rho \mathbf{v})$.
- Velocity vector leads to the second term $\nabla \cdot (\rho \mathbf{v})$ and the third term $\nabla \cdot (\mathbf{v} \tau)$.
- Pressure leads to the second term $\nabla \cdot (\mathbf{v} \tau)$ and the fourth term $\rho \mathbf{f} \cdot \mathbf{v}$.
- Identity matrix leads to the third term $\nabla \cdot (\mathbf{v} \tau)$.
- Temperature leads to the second term $\nabla \cdot (\kappa \nabla T)$.
- (deviatoric) Stress tensor leads to the third term $\nabla \cdot (\mathbf{v} \tau)$.

Navier Stokes equations

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \tau) + \rho \mathbf{f} \cdot \mathbf{v}$$

The diagram illustrates the derivation of the Navier-Stokes equations. It starts with two fundamental equations: the continuity equation $\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$ and the total energy equation $\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \tau) + \rho \mathbf{f} \cdot \mathbf{v}$. Various physical quantities are labeled with boxes, and arrows point from these labels to specific terms in the equations. Labels include: Density, Momentum, Total energy, Velocity vector, Pressure, External force vector, Temperature, (deviatoric) Stress tensor, and Identity matrix.

Navier Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$
$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \boldsymbol{\tau}) + \rho \mathbf{f} \cdot \mathbf{v}$$

The diagram illustrates the components of the Navier-Stokes equations. It shows two main equations: the continuity equation and the momentum equation. The continuity equation is $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$. The momentum equation is $\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$. The terms in these equations are connected to various physical quantities. The continuity equation connects Density and Momentum. The momentum equation connects Velocity vector, Identity matrix, Pressure, External force vector, (deviatoric) Stress tensor, Temperature, Thermal conductivity, and Momentum. The total energy term in the momentum equation connects to Total energy.

Navier Stokes equations

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \boldsymbol{\tau}) + \rho \mathbf{f} \cdot \mathbf{v}$$

Navier Stokes equations

Conservation of mass

Conservation of momentum

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \boldsymbol{\tau}) + \rho \mathbf{f} \cdot \mathbf{v}$$

Navier Stokes equations

Conservation of mass

Conservation of momentum

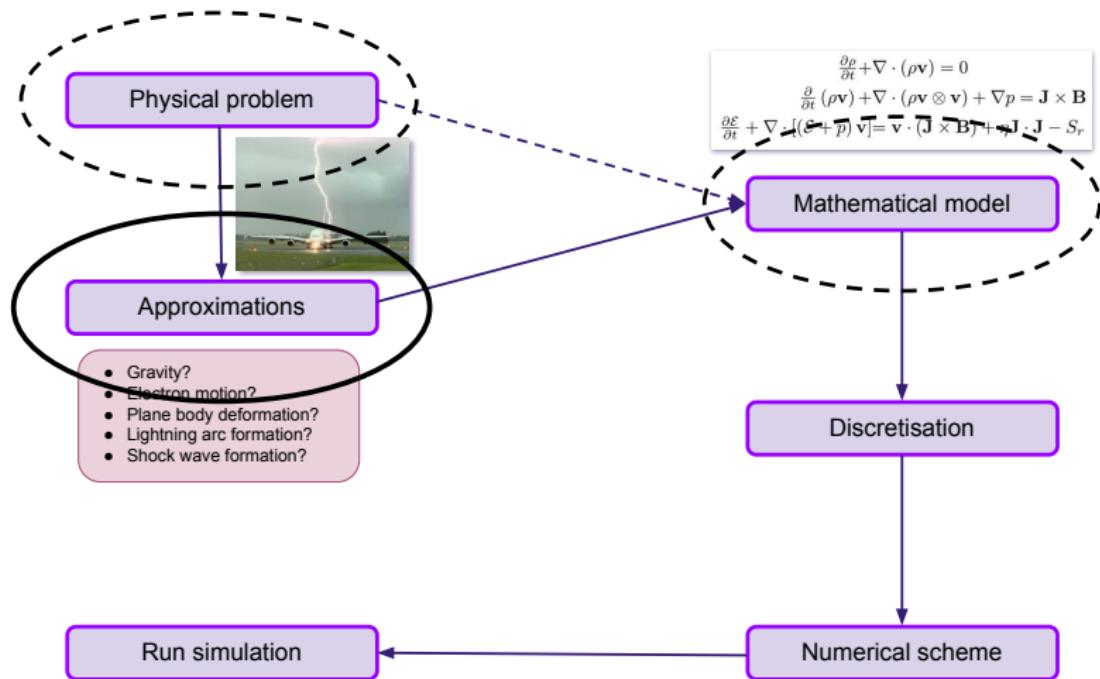
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

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$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\mathbf{v} \boldsymbol{\tau}) + \rho \mathbf{f} \cdot \mathbf{v}$$

Conservation of energy

Should you always solve these equations?



The inviscid limit

- In many cases, viscosity of the material of interest is negligible, as is diffusive heat exchange - this is particularly true when behaviour occurs on **short time scales**
- This is often the case for air, but only over the **appropriate length and time scales**
- Viscosity allows a material to contain internal stresses, for an inviscid material we have $\tau = 0$
- For heat exchange to be negligible, we are assuming $\nabla \cdot (\kappa \nabla T) \ll \nabla \cdot [(E + p) \mathbf{v}]$
- In the absence of other forces ($\mathbf{f} = 0$), this gives us the Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I} p) = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] = 0$$

- We shall revisit these equations later

The incompressible limit

- Another common limit for the Navier Stokes equations is to assume **incompressibility** - density is constant
- This leads to a **decoupling** of the equations, giving

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla p}{\rho} = \nu \nabla^2 \mathbf{v} + \mathbf{g}$$

- Here, ν is kinematic viscosity, and \mathbf{g} gravitational forcing
- The incompressible approximation is associated with slower (**long time scale**) phenomena - gravity can rarely be ignored!
- Additionally, we have a **constraint** on the velocity

$$\nabla \cdot \mathbf{v} = 0$$

- Computational Continuum Modelling 2 will cover these applications

Constant velocity limit

- If a constant velocity is assumed, the first of the Navier Stokes equations becomes

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$$

- The stress tensor is dependent on the gradient of velocity and as a result, the second Navier Stokes equation vanishes
- The energy equation also simplifies

$$\rho c_V \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \nabla \cdot (\kappa \nabla T)$$

- Total energy is a combination of internal energy and kinetic energy

$$E = \rho \varepsilon + \frac{1}{2} \rho v^2, \quad \varepsilon = c_V T$$

- This assumes that specific heat at constant volume is a constant
- You may also have additional energy terms, e.g. chemical energy if something is reacting, or magnetic energy if something is ionised

Constant velocity limit

- The two equations in the constant velocity limit are often considered independently
 - many problems have varying density at constant temperature or vice versa
- If temperature is considered constant,

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$$

is sometimes called the **transport equation**, the **advection equation** or **convection**, where ρ could represent any quantity carried with velocity \mathbf{v}

- If density is considered constant,

$$\rho c_V \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \nabla \cdot (\kappa \nabla T)$$

is a **convection-diffusion** equation

- If $\mathbf{v} = 0$, either through change of frame of reference, or otherwise, then we get the standard **heat equation**

$$\rho c_V \frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T)$$

Constant pressure and steady state equations

- If pressure is constant (and viscosity is negligible), the second Navier Stokes equation can be written

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = 0$$

- This is **Burgers' equation**, often used in traffic flow simulating, and can also include viscosity (a non-linear advection-diffusion equation)
- If the boundaries of a domain are held fixed (e.g. at constant temperature), and there are no sources or sinks, then the solution will eventually tend to all time derivatives vanishing
- The solution at this point is referred to as the **steady state**, and can be obtained directly by solving the equations in the absence of time-derivative
- For example, the heat equation becomes

$$\nabla \cdot (\kappa \nabla T) = q$$

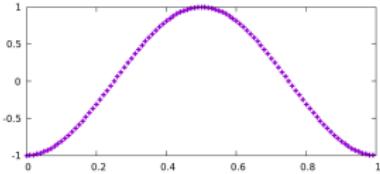
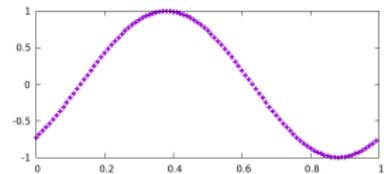
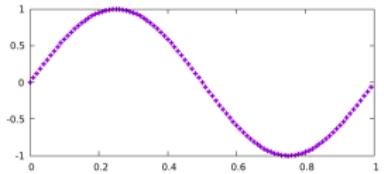
- Note - we have included an external forcing term, the heat flux, in order to describe how the boundaries of the domain are held fixed

In summary

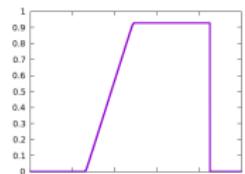
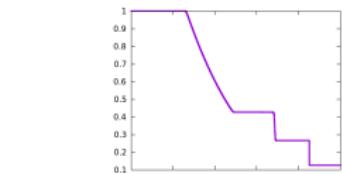
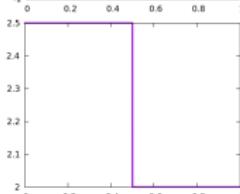
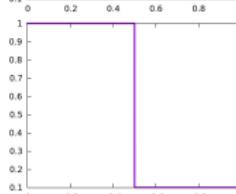
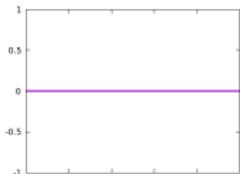
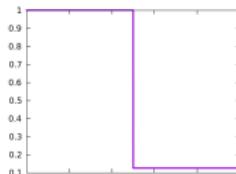
- In many textbooks, the various continuum equations that are solved are often presented separately
- We have shown that from the Navier Stokes equations, we obtained:
 - The advection (or transport) equation
 - The convection-diffusion equation
 - Burger's equation
 - The heat equation
 - The compressible Euler equations
 - The incompressible Euler equations
- With a bit more work, and further assumptions, we could have also found the shallow-water equations and the wave equation
- With a different consideration of the stress tensor, τ , we could recover the equations of solid mechanics

What will we be working towards?

From solutions to scalar linear PDEs



To coupled systems of non-linear PDEs

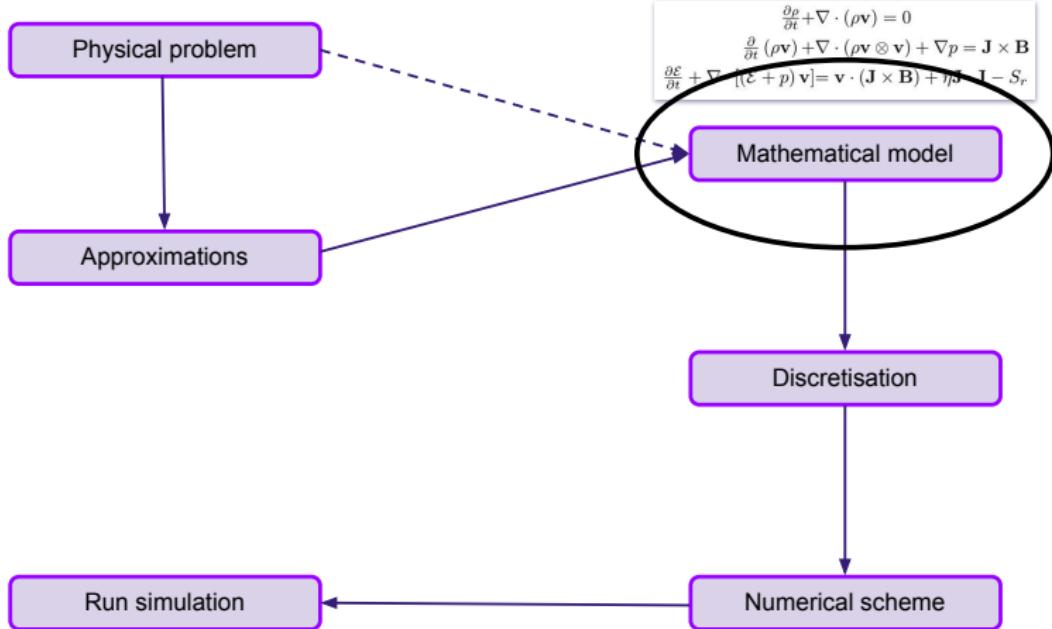


When moving from these simple cases to complex applications, the system of equations will change but the numerical techniques **will remain the same**

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Classifying the equations



Classification of PDEs

- We have shown several different simplifications of the Navier Stokes equation, but since they all have the same original equation, can they be solved by a single numerical technique?

Classification of PDEs

- We have shown several different simplifications of the Navier Stokes equation, but since they all have the same original equation, can they be solved by a single technique?
- **No.** The simplifications and assumptions alter the physics of the system, and alter the **type** of the equations
- This can also be seen as a mathematical change in the underlying system, and these changes mean that different numerical solution techniques will be required
- In order to know how to solve a system of equations, we need to **classify** them
- This classification will tell you both the **physical behaviour** of the system and the **mathematical behaviour** of the equations

Classification of PDEs - physical behaviour

- **Equilibrium problems:**

- If we consider an equilibrium (steady) state to a problem, this is a 'balance' of **all** states at different locations in space
- Such problems are governed by **elliptic equations**

- **Time-varying problems:**

- Evolution in time generally implies a 'speed-of-travel' of physical quantities
- If physical quantities can move without dissipation (diffusion), we have **hyperbolic equations**
- If there is significant diffusion, then we have **parabolic equations**

Classification of PDEs - mathematical behaviour

- At this stage, these definitions are a bit vague
- However, the classifications of elliptic, hyperbolic and parabolic may sound familiar
- Similar terminology used for conic sections

$$Ax^2 + Bxy + Cy^2 + f(x) + g(y) = 0$$

- The sign of the discriminant, $D = B^2 - 4AC$, is used to classify the conic section
 - $D < 0$: Ellipse
 - $D = 0$: Parabola
 - $D > 0$: Hyperbola

Classification of PDEs - mathematical behaviour

$$Ax^2 + Bxy + Cy^2 + f(x) + g(y) = 0$$

- Conic sections and linear, second order, two-dimensional PDEs have a similar form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f(u_x) + g(u_y) + h(u) = 0$$

- Again we can consider $D = B^2 - 4AC$
 - $D < 0$: Elliptic equations
 - $D = 0$: Parabolic equations
 - $D > 0$: Hyperbolic equations
- What happens if your equation is **not** two-dimensional or **not** second order or **not** linear or **not** a single equation?

Classification of PDEs - mathematical behaviour

- We will not go into the full techniques for classifying PDEs, but we shall cover a second case, when you have a system of equations

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

- Recall that this can be written

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \nabla \cdot (\mathbf{u}) = 0$$

where $\partial \mathbf{f} / \partial \mathbf{u}$ is a Jacobian matrix

- The eigenvalues and eigenvectors of this matrix determine the class of the equations:
 - All pure complex** - elliptic equation
 - All real, unique eigenvectors** - hyperbolic equation (equivalent to the Jacobian being diagonalisable)
 - All real, degenerate eigenvectors** - parabolic equation
- Note - this doesn't cover all cases, and non-linear equations may be **mixed-type** (including hyperbolic for some range of variables, parabolic for others)

Hyperbolic PDEs

- For the remainder of this course, we shall focus on hyperbolic PDEs, specifically **non-linear** hyperbolic PDEs
- Two key features differentiate these equations from other types:
 - ① Finite information travel time - a change in one variable in one location will travel out with a finite **wave speed**
 - ② **Discontinuities** can form in the solution, even from smooth initial data
- We shall focus on how this behaviour effects the numerical solution
- We shall see how this leads to specific numerical methods designed to deal with **accurate representation of discontinuities**
- These methods shall be applied to real (though one-dimensional) flow problems

What do we mean by discontinuity?

- A discontinuity is a change in variables with **zero width**; for some quantity u

$$\frac{\partial u}{\partial x} \rightarrow \infty$$

- Discontinuities **between materials** clearly exist everywhere - these do not arise purely from the equations of flow
- This course is concerned with a discontinuity forming **within a single material** - equivalent to a sudden change in the thermodynamic states of a material
- In reality, temperature diffusion prevents true discontinuities from forming (and even material discontinuities have a transition at the atomic scale)
- However, numerical methods require **discrete points** - these transitions can occur at a length scale smaller than the distance between two points

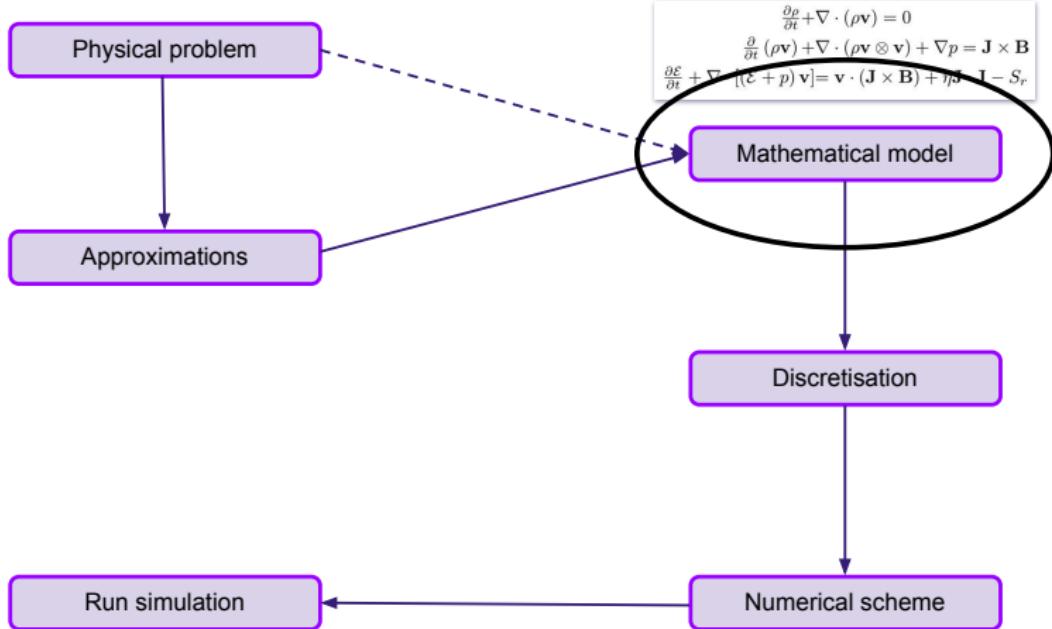
What do we mean by discontinuity?

- If we cannot resolve the scale of a transition, it is more accurate to treat it as a discontinuity, than smear it and hope for the best
- Perhaps the most common example which requires this treatment is a **shock wave**
- Mathematically, the assumptions we make can result in a system which genuinely forms discontinuities
- For example, the compressible Euler equations, which neglect diffusive effects
- As we saw, discontinuities and differential equations aren't compatible, and we need **integral forms** of the equation

Outline

- 1 Computational continuum modelling
- 2 Equations of flow
- 3 Classifying the equations
- 4 Mathematical properties and discontinuities

Mathematical properties



Integral form and conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

- Equations written in the form above are often referred to as being in **conservation form**, and the equation itself is a conservation law
- The reason for terminology becomes apparent when considering the **integral form** of the equations
- Recall this is

$$\int_a^b \mathbf{u}(t_2, x) - \mathbf{u}(t_1, x) dx = - \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(t, b)) - \mathbf{f}(\mathbf{u}(t, a)) dt$$

Integral form and conservation laws

- If we rewrite the integrals

$$\int_a^b \mathbf{u}(t_2, x) dx - \int_a^b \mathbf{u}(t_1, x) dx = - \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(t, b)) dt + \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(t, a)) dt$$

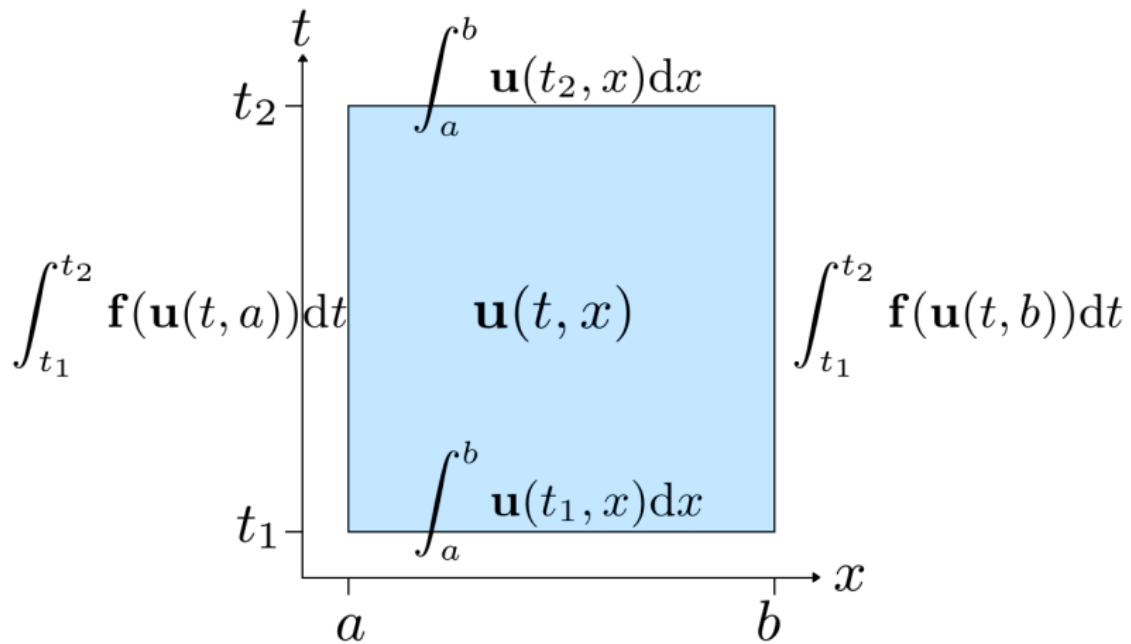
- The integral of $\mathbf{u}(t_1, x)$ over space is the 'total amount' of this quantity in $[a, b]$ at time $t = t_1$
- Similarly, the integral of $\mathbf{u}(t_2, x)$ over space is the 'total amount' of this quantity in $[a, b]$ at time $t = t_2$
- The left hand side of the equation above therefore represents the change in the 'total amount' of \mathbf{u} between $t = t_1$ and $t = t_2$
- The total change in \mathbf{u} is therefore dependent only on two locations in space, a and b

Integral form and conservation laws

$$\int_a^b \mathbf{u}(t_2, x) dx - \int_a^b \mathbf{u}(t_1, x) dx = - \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(t, b)) dt + \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(t, a)) dt$$

- Therefore the integral of $\mathbf{f}(\mathbf{u}(t, a))$ over time measures some form of change in \mathbf{u} at $x = a$
- This can be loosely interpreted as the 'amount' of \mathbf{u} that enters (or leaves) through $x = a$ over a given time
- $\mathbf{f}(\mathbf{u})$ is commonly known as the **flux function**
- \mathbf{u} 'entering' the domain could be literal, through material flow, or through forcing effects on the boundary
- The right hand side of the equation above is then the 'amount' of \mathbf{u} that leaves through the two boundaries, $x = a$ and $x = b$ between $t = t_1$ and $t = t_2$

The four integral quantities



Conservation

$$\int_a^b \mathbf{u}(t_2, x) - \mathbf{u}(t_1, x) dx = - \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(t, b)) - \mathbf{f}(\mathbf{u}(t, a)) dt$$

- This integral form of the equations mathematically states that the change in \mathbf{u} over a time interval in a fixed domain can be calculated by computing the amount of \mathbf{u} that enters or leaves the domain in this interval
- The function \mathbf{f} governs how much \mathbf{u} enters or leaves
- This is **conservation**, no other way to change the value of \mathbf{u} within the domain exists
- As a result, this equation, as well as the partial differential equation form are known as **conservation laws**

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

Jump conditions

- We can continue to use the PDE form of the conservation law by introducing **jump conditions** - knowledge of how the solution changes across discontinuities
- To do this, we start from the PDE equations, and derive a version of the integral form for these equations which takes in to account a discontinuity moving with speed S

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

- Integrating both sides of this equation with respect to x gives

$$\frac{d}{dt} \int_a^b \mathbf{u} dx = - (\mathbf{f}(\mathbf{u}(t, b)) - \mathbf{f}(\mathbf{u}(t, a)))$$

- If there is a single discontinuity $x_S \in (a, b)$, then the above equation still holds for

$$\frac{d}{dt} \left(\int_{x_s(t)}^b \mathbf{u} dx + \int_a^{x_s(t)} \mathbf{u} dx \right) = - (\mathbf{f}(\mathbf{u}(t, b)) - \mathbf{f}(\mathbf{u}(t, a)))$$

Jump conditions

- We note that the discontinuity is moving, therefore its position is a function of time, and in order to simplify the equations above, we use Liebniz's integral rule

$$-\mathbf{u}(t, x_s^+) \frac{\partial x_s}{\partial t} + \int_{x_s(t)}^b \frac{\partial \mathbf{u}}{\partial t} dx + \mathbf{u}(t, x_s^-) \frac{\partial x_s}{\partial t} + \int_a^{x_s(t)} \frac{\partial \mathbf{u}}{\partial t} dx = -(\mathbf{f}(\mathbf{u}(t, b)) - \mathbf{f}(\mathbf{u}(t, a)))$$

- Note that because of the discontinuity, we have two different solutions for \mathbf{u} as we approach from $x > x_s$ (x_s^+) and from $x < x_s$ (x_s^-)
- We now take the limits $a \rightarrow x_s^-$ and $b \rightarrow x_s^+$, and note that $S = \partial x_s / \partial t$ to get

$$-\mathbf{u}(t, x_s^+) S + \mathbf{u}(t, x_s^-) S = -(\mathbf{f}(\mathbf{u}(t, x_s^+)) - \mathbf{f}(\mathbf{u}(t, x_s^-)))$$

- This is the jump condition that any numerical solution to the PDE **must** obey across discontinuities to remain valid, it is commonly written:

$$\mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L) = S (\mathbf{u}_R - \mathbf{u}_L)$$

- These are also known as the **Rankine-Hugoniot conditions**

Does this help us solve these equations?

- We have given an overview of some of the important equations in computational continuum model
- Additionally, we have detailed mathematical properties of these equations, focusing on the hyperbolic equations which will form the rest of this course
- In general, a coupled, non-linear system of equations cannot be solved analytically, we must use **numerical methods**
- We shall now begin to focus on specific equations and see how the mathematical properties described so far influence the numerical methods
- We shall see the importance of finite wave speeds and jump conditions for solving (simple) real-world problems