Systems of hyperbolic equations

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Systems of hyperbolic equations

- For practical applications, we rarely find a problem defined by a single variable, but we have a **coupled system of equations**
- Examples of commonly encountered hyperbolic systems include:
- **The shallow water equations**:

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi\\ \phi u\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\phi u\\ \phi u^2+\frac{1}{2}\phi^2\end{pmatrix}=0$$

- u is a speed and ϕ is related to height
- **The wave equation**:

$$\frac{\partial}{\partial t}\begin{pmatrix}\bar{\rho}\\ \bar{v}\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho_0\bar{v}\\ \frac{a^2}{\rho_0}\bar{\rho}\end{pmatrix}=0 \quad \rightarrow \quad \frac{\partial^2\bar{\rho}}{\partial t^2}=a^2\frac{\partial^2\bar{\rho}}{\partial x^2}$$

- Uses (small) velocity, $\bar{v}$ and perturbations of density, $\bar{\rho}$ where $\rho=\rho_0+\bar{\rho}$, typically solved as a single second order equation

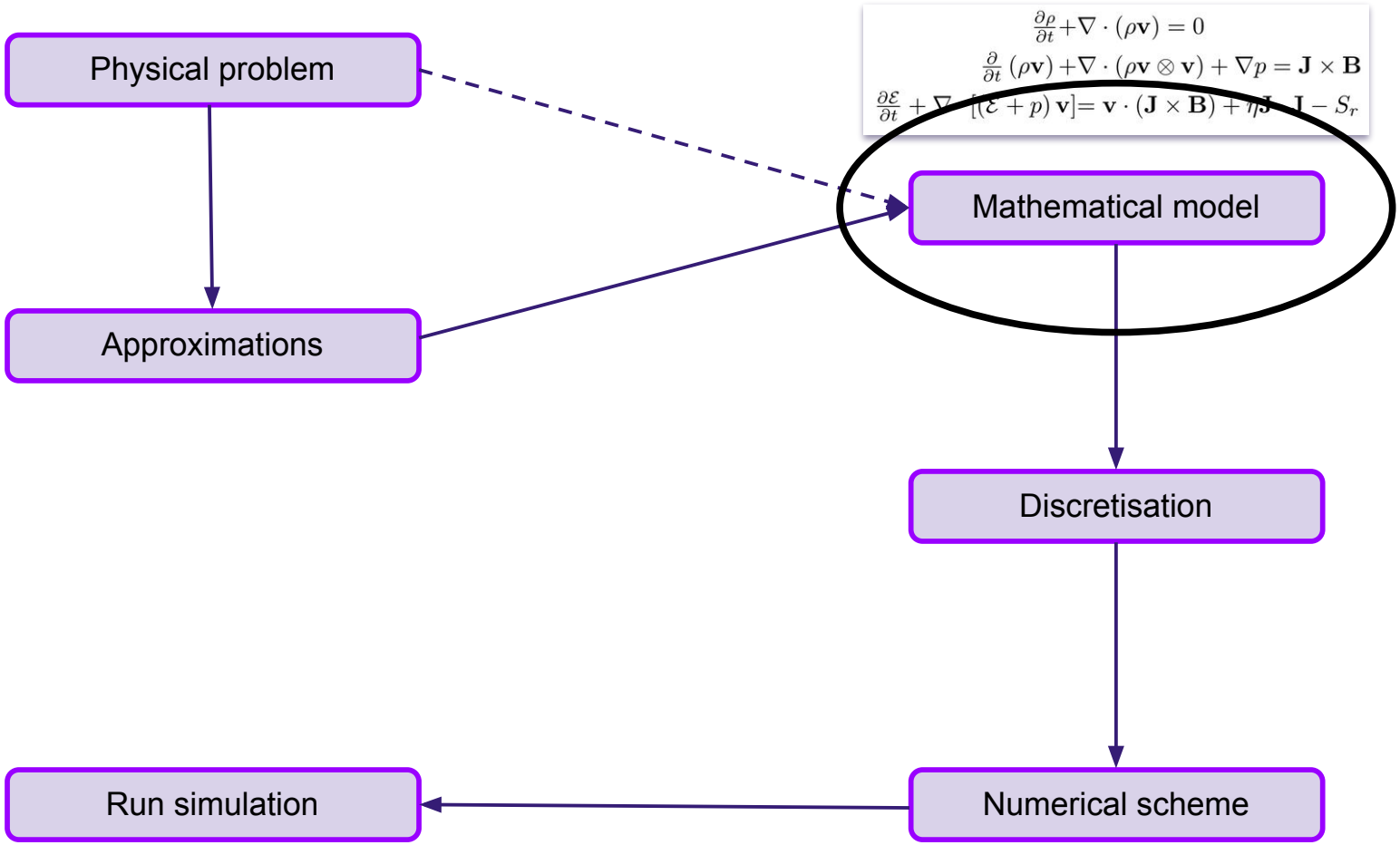
The Euler equations

- For the remainder of this course, we will focus on a chosen system of hyperbolic equations, the Euler equations
- To be slightly more specific, the **inviscid**, **compressible** Euler equations
- There are several reasons for this:
  1. These are a very general system of hyperbolic equations, both the shallow water and wave equations are reductions of this system (vanishing vertical acceleration and small perturbations respectively)
  2. They are commonly found in real-world problems, especially those tackled here in the LSC
  3. The numerical techniques we consider are not specific to these equations, we don't need to consider every set separately
  4. Eventually even I get bored introducing new equations to solve

Outline

Outline

The Euler equations



The Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{I}p) = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\,\mathbf{v}] = 0$$

- We introduced these equations as a reduction of the Navier Stokes equations in the first lecture
- They comprise three conservation laws for density (mass), ρ, momentum $\rho\mathbf{v}$ and total energy, E - note that vector quantities, such as momentum, are a single variable - we cannot consider the components of this vector until we've chosen a coordinate system
- They also contain a fourth variable, pressure, p, currently we have an underdetermined system - we will need to close the system of equations

Deriving the equations

- Before we consider how to close the system of equations, it is worth considering a brief derivation of these equations
- For simplicity (and for reducing the number of lines equations take up) we shall consider the one-dimensional equations
- Recall that a conservative system of PDEs can be written in weak (integral) form

$$\int_a^b \mathbf{u}(t_2, x) - \mathbf{u}(t_1, x)\mathrm{d}x = -\int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(t, b)) - \mathbf{f}(\mathbf{u}(t, a))\mathrm{d}t$$

- The vector quantities here are:

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} \qquad \mathbf{f} = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (E + p)\, v \end{pmatrix}$$

- The integral form gives us an intuitive (though not mathematically rigorous) understanding of where the equations come from, see e.g. Toro for further details

Deriving the equations: conservation of mass

- Given that we are considering a control volume, $x \in [a, b]$, $t \in [t_1, t_2]$, we work with density, $\rho(\mathbf{x})$, (kg/m^3) rather than mass, m (kg)
- This quantity is then independent of the size of the control volume
- The flow of density through the one wall of surface at a given time is the density of the material at that surface multiplied by the velocity
- Therefore, the conservation of mass (density) is expressed as

$$\int_{x_0}^{x_1} \left[\rho\left(x, t_2\right) - \rho\left(x, t_1\right)\right] \mathrm{d}x = -\int_{t_1}^{t_2} \left[\rho\left(x_1, t\right) v\left(x_1, t\right) - \rho\left(x_0, t\right) v\left(x_0, t\right)\right] \mathrm{d}t$$

- This gives us our **flux function** for density, $f_\rho\left(\mathbf{u}\right) = \rho v$

Deriving the equations: conservation of momentum

- Momentum, ρv ($\mathrm{kg\,m^{-2}\,s^{-1}}$), can also be described through the change in this variable via momentum loss/gain through the surface of the control volume
- The momentum lost at a given time, as with mass, is the momentum multiplied by the velocity
- **However**, in this case, momentum of a volume is **also** affected by the pressure, p (Pa or $\mathrm{kg\,m^{-1}\,s^{-2}}$), exerted on the volume, i.e. an additional 'push' of the material in the control volume
- To calculate momentum at time t_2 we need to solve

$$\int_{x_0}^{x_1} \left[\rho\left(x,t_2\right) v\left(x,t_2\right) - \rho\left(x,t_1\right) v\left(x,t_1\right)\right] \mathrm{d}x$$

$$= -\int_{t_1}^{t_2} \left\{\rho\left(x_1,t\right)\left[v_x\left(x_1,t\right)\right]^2 - \rho\left(x_0,t\right)\left[v_x\left(x_0,t\right)\right]^2 + p\left(x_1,t\right) - p\left(x_0,t\right)\right\} \mathrm{d}t$$

- Again, our flux function is clear, $f_{\rho v}(\mathbf{u}) = \rho v^2 + p$

Deriving the equations: conservation of energy

- Total energy, (J or $\mathrm{kg\,m^{-2}\,s^{-2}}$) is conserved, however, total energy comprises kinetic energy, internal energy etc. - these individual quantities are not conserved
- As with mass, it is easier to work with energy per unit volume, E (J $\mathrm{m^{-3}}$ or $\mathrm{kg\,m^{-1}\,s^{-2}}$)
- As before, loss of energy through the surface of the control volume at a given time is simply energy multiplied by velocity
- Additionally, compression of the material in the control volume increases its energy, i.e. pressure change across the surface will lead to energy change within the volume
- To calculate this, we need to solve

$$\int_{x_0}^{x_1} \left[E\left(x,t_2\right) - E\left(x,t_1\right)\right] \mathrm{d}x$$

$$= -\int_{t_1}^{t_2} \left\{\left[E\left(x_1,t\right) + p\left(x_1,t\right)\right] v_x\left(x_1,t\right) - \left[E\left(x_0,t\right) + p\left(x_0,t\right)\right] v_x\left(x_0,t\right)\right\} \mathrm{d}t$$

- This is the expected flux function $f_E(\mathbf{u}) = (E+p)\,v$

The equation of state

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho v\\ \rho v^2+p\\ (E+p)\,v\end{pmatrix}=0$$

- We now consider (briefly) how to close the system
- It is not surprising that we have to give some more information to the system
- The Euler equations apply to a wide range of materials, including air ($\rho \sim \mathcal{O}(1)\,\mathrm{kg/m}^{-3}$) and water ($\rho \sim \mathcal{O}(10^3)\,\mathrm{kg/m}^{-3}$)
- We do not expect two different materials to behave the same; even if we compressed air to around $10^3\,\mathrm{kg/m}^{-3}$, it would still be a gas
- Two possibilities to include material-dependent behaviour - either through a definition of pressure (equation of state) or through additional source terms (viscosity etc.)

The equation of state

- The requirement for an equation of state is one of the key differences between compressible and incompressible systems
- The incompressible system is two equations for two variables ($\mathbf{v}$ and p)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \frac{1}{\rho_0}\nabla p = \mathbf{s}$$

$$\nabla \cdot \mathbf{u} = 0$$

- This is two equations for two unknowns - material properties must enter through $\mathbf{s}$

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho v \\ \rho v^2 + p \\ (E+p)\,v \end{pmatrix} = 0$$

- For the compressible Euler equations, we are often working in a regime where we know viscosity, conductivity etc. are negligible
- Material properties can only enter through the equation of state in this case

The equation of state

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho v\\ \rho v^2+p\\ (E+p)\,v\end{pmatrix}=0$$

- The **equation of state** is a continuum description of the molecular thermodynamics properties of the material of interest
- One common realisation of this is the speed of sound
- Sound is a vibrational wave through a medium, i.e. it compresses (and expands) it, which is, itself, governed by molecular behaviour of the medium
- Equations of state can be written in many ways:

$$p = p(V,T), \quad p = p(\rho,T), \quad p = p(\rho,\varepsilon)$$

- Volume, V, and temperature, T, are useful, intuitive variables
- For practical solutions of the Euler equations, however, **specific internal energy**, ε, is more common

Energy

- The total energy, E, is technically a quantity that cannot be calculated, all we can measure is changes in energy
- As a result, we typically assume only the energy components directly affect by our system of equations, and all others are set to a convenient constant (i.e. zero)
- For the compressible Euler equations, there are two key components, internal energy ($\rho\varepsilon$) and kinetic energy ($\frac{1}{2}\rho v^2$)

$$E = \rho\varepsilon + \frac{1}{2}\rho v^2$$

- **Internal energy** is effectively a sum of the atomistic behaviour of the material in question
- This incorporates the kinetic energy of each of the material's component particles, the potential energy of the chemical and nuclear particle bonds and other electromagnetic and/or solid effects

Specific internal energy

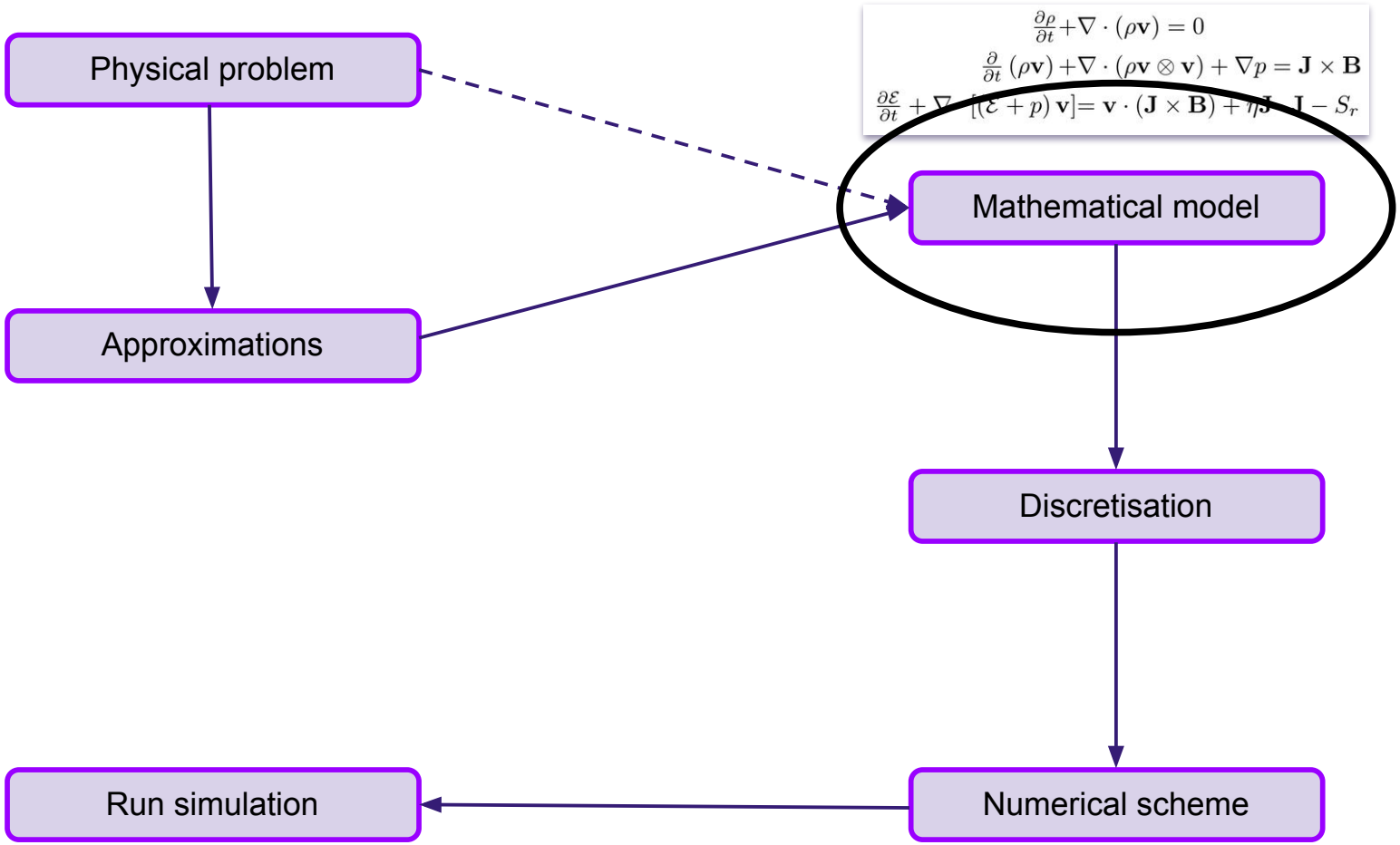
- Fortunately, the laws of thermodynamics mean we don't need to consider anything at an atomistic level
- Typically, we work with **specific internal energy**, ε, (energy per unit mass), again to avoid scaling issues
- Note of caution: in the literature, variable names for energy (total, internal and specific internal) are not consistent; depending on the source, E, e, e_T, ε can be used interchangeably - context, and dimensional analysis, will let you know which variable you are dealing with
- In these lectures, we will focus on a single, simple equation of state - that of an **ideal gas** - more details and derivations will be given in further courses

$$p = (\gamma - 1)\,\rho\varepsilon$$

- $\gamma = c_p/c_v$ is a constant, and this relationship can be derived from the more familiar ideal gas law $pV = nRT$
- For air, and other diatomic molecular gases, $\gamma = 1.4$

Outline

Properties of the Euler equations



Euler equations

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- For the rest of this course, this system of equations is what we shall focus on solving
- As with the other equations we have simulated, we begin with a look at the mathematical properties
- We shall consider a general equation of state for the mathematics, but all practicals will use assume an ideal gas

$$p = (\gamma - 1)\,\rho\varepsilon$$

Euler equations - hyperbolicity

- As with the previous equations we have considered, we want to show that the Euler equations really are hyperbolic
- We need to compute the Jacobian $\partial \mathbf{f} / \partial \mathbf{u}$
- To do this, the flux vector must be written in terms of the conserved variables, ρ, ρv and E
- In general, pressure terms pose a bit of a problem, since, for a general equation of state we have $p = p(\rho, \varepsilon(\rho, \rho v, E))$
- For now, to avoid a lot of unspecified partial derivatives, we will assume the ideal gas equation of state, for which, we can write pressure as

$$p = (\gamma - 1)\,\rho\varepsilon = (\gamma - 1)\left[E - \frac{(\rho v)^2}{2\rho}\right]$$

Euler equations - hyperbolicity

- We can now write our flux vector as

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \rho v \\ \frac{(\rho v)^2}{\rho} + (\gamma - 1)\left[E - \frac{(\rho v)^2}{2\rho}\right] \\ \left\{E + (\gamma - 1)\left[E - \frac{(\rho v)^2}{2\rho}\right]\right\}\frac{\rho v}{\rho} \end{pmatrix} = \begin{pmatrix} \rho v \\ \frac{1}{2}(3-\gamma)\frac{(\rho v)^2}{\rho} + (\gamma-1)E \\ \gamma\frac{\rho v}{\rho}E - \frac{1}{2}(\gamma-1)\frac{(\rho v)^3}{\rho^2} \end{pmatrix}$$

- This leads to the Jacobian matrix (with some simplification)

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)v^2 & (3-\gamma)v & \gamma - 1 \\ -\gamma\frac{v}{\rho}E + (\gamma-1)v^3 & \gamma\frac{E}{\rho} - \frac{3}{2}(\gamma-1)v^2 & \gamma v \end{pmatrix}$$

- Now it is just a simple matter of finding the eigenvalues...

Euler equations - hyperbolicity

- Calculating the eigenvalues of the Jacobian matrix is described as tedious by Leveque - only recommended if you are very bored!
- However, the actual eigenvalues are not too complex

$$\lambda_1 = v - \sqrt{\frac{\gamma p}{\rho}}, \quad \lambda_2 = v, \quad \lambda_3 = v + \sqrt{\frac{\gamma p}{\rho}}$$

- The first thing to note is that provided $\rho > 0$ and $p > 0$, the system is hyperbolic - the first **must** be true, the second is true for an ideal gas (which we have assumed is true)
- The second thing to note is that the speed of sound for an ideal gas is given by

$$c_s = \sqrt{\frac{\gamma p}{\rho}}$$

- We have shown the Euler equations are hyperbolic **if** we have an ideal gas - can we do better than that (without having to resort to messy undefined derivatives)?

The speed of sound

- Yes (sort of), but first it is worth taking a closer look at the speed of sound
- This is a quantity defined by

$$c_s = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s}$$

  where s is entropy
- This would be really easy to compute if you have an equation of state defined by $p = p(\rho, s)$
- Unfortunately, for many materials, entropy is not a natural quantity to use - what is the connection between $p = p(\rho, s)$ and $p = p(\rho, \varepsilon)$?

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- Unfortunately, for many materials, entropy is not a natural quantity to use - what is the connection between $p = p(\rho, s)$ and $p = p(\rho, \varepsilon)$?
- Recall the second law of thermodynamics, which can be written

$$T\mathrm{d}s = \mathrm{d}\varepsilon + p\mathrm{d}\nu$$

  where $\nu = 1/\rho$ is specific volume
- This is why an equation of state can be written in many different, but thermodynamically complete, ways

The speed of sound

- We are going to look for a speed of sound from an equation in the form $\varepsilon = \varepsilon(p, \rho) = \varepsilon(p(\rho, s), \rho)$
- This will give us a relationship in terms of pressure and density only (which we have previously found in the eigenvalues for the Euler equations)
- How do we get an expression for the speed of sound from this?

$$\left.\frac{\partial \varepsilon}{\partial \rho}\right|_s = \left.\frac{\partial \varepsilon}{\partial p}\right|_\rho \left.\frac{\partial p}{\partial \rho}\right|_s + \left.\frac{\partial \varepsilon}{\partial \rho}\right|_p$$

- The only thing we can't compute directly is the first term, this can be rewritten using the second law of thermodynamics

$$T\mathrm{d}s = \mathrm{d}\varepsilon + p\mathrm{d}\nu$$

- Evaluating this term at constant entropy means $\mathrm{d}s = 0$, and also note $\mathrm{d}\nu = -\mathrm{d}\rho/\rho^2$

The speed of sound

- Piecing this together we get:

$$\mathrm{d}\varepsilon = \frac{p}{\rho^2}\mathrm{d}\rho \qquad \Longrightarrow \qquad \left.\frac{\partial \varepsilon}{\partial \rho}\right|_s = \frac{p}{\rho^2}$$

- This gives

$$\frac{p}{\rho^2} = \left.\frac{\partial \varepsilon}{\partial p}\right|_\rho \left.\frac{\partial p}{\partial \rho}\right|_s + \left.\frac{\partial \varepsilon}{\partial \rho}\right|_p$$

or

$$c_s = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} = \sqrt{\frac{p}{\rho^2 \frac{\partial \varepsilon}{\partial p}} - \frac{\frac{\partial \varepsilon}{\partial \rho}}{\frac{\partial \varepsilon}{\partial p}}}$$

- We can now work out the speed of sound for any equation of state given by $\varepsilon = \varepsilon(p, \rho)$
- Why is this important?

Primitive variable form

- The conservation form of the Euler equations uses three **conserved variables**, density, momentum and energy - these are not necessarily the most natural variables to work with
- However, we have already mentioned several, more natural variables, which are commonly measured, such as velocity and pressure
- This could be useful in a primitive variable form of the Euler equations
- Of course, we still need three independent variables, and we know that density (or mass) is very easy to measure
- There is nothing wrong with a variable being both a conserved **and** a primitive variable
- Now, with $\mathbf{w} = (\rho, v, p)^T$, we can aim to write the Euler equations in primitive variable form

$$\frac{\partial \mathbf{w}}{\partial t} + B\frac{\partial \mathbf{w}}{\partial x} = 0$$

Primitive variable form for the Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho v\\ \rho v^2+p\\ (E+p)\,v\end{pmatrix}=0$$

- We shall attempt to write the Euler equations in terms of density, velocity and pressure - whilst we could do this by computing the matrix W, it is not the easiest way in this case
- Note, for the first equation, the conservation of mass, this is trivial

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0$$

Primitive variable form for the Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho v\\ \rho v^2+p\\ (E+p)\,v\end{pmatrix}=0$$

- For the second equation, this is also straightforward, if we are careful with our product rule

$$\rho\frac{\partial v}{\partial t}+v\frac{\partial \rho}{\partial t}+v\frac{\partial}{\partial x}(\rho v)+\rho v\frac{\partial v}{\partial x}+\frac{\partial p}{\partial x}=0$$

- Two of these terms are simply the conservation of mass multiplied by velocity, we can remove these, and divide through by density giving

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$

Primitive variable form for the Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho v\\ \rho v^2+p\\ (E+p)\,v\end{pmatrix}=0$$

- The energy equation requires more work, first we need to expand the total energy out into primitive variables

$$\frac{\partial}{\partial t}\left(\rho\varepsilon+\frac{1}{2}\rho v^2\right)+\frac{\partial}{\partial x}\left(\rho\varepsilon v+\frac{1}{2}\rho v^3+pv\right)=0$$

- Again we apply some careful product rule derivatives

$$\varepsilon\frac{\partial\rho}{\partial t}+\varepsilon\frac{\partial}{\partial x}(\rho v)+\frac{1}{2}v^2\frac{\partial\rho}{\partial t}+\frac{1}{2}v^2\frac{\partial}{\partial x}(\rho v)+\rho\frac{\partial\varepsilon}{\partial t}+\rho v\frac{\partial\varepsilon}{\partial x}$$
$$+\rho v\frac{\partial v}{\partial t}+\rho v^2\frac{\partial v}{\partial x}+v\frac{\partial p}{\partial x}+p\frac{\partial v}{\partial x}=0$$

- Here, we can remove the conservation of mass terms **twice**

Primitive variable form for the Euler equations

- This simplifies things to

$$\rho\frac{\partial \varepsilon}{\partial t} + \rho v\frac{\partial \varepsilon}{\partial x} + \rho v\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right) + p\frac{\partial v}{\partial x} = 0$$

- We can now remove the conservation of momentum equation, giving us just three terms
- We now need to do something about the specific internal energy terms, here we note that by the chain rule, we have

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial \varepsilon}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial \varepsilon}{\partial \rho}\frac{\partial \rho}{\partial t}$$

- Therefore, our equation now becomes

$$\rho\left(\frac{\partial \varepsilon}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial \varepsilon}{\partial \rho}\frac{\partial \rho}{\partial t}\right) + \rho v\left(\frac{\partial \varepsilon}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \varepsilon}{\partial \rho}\frac{\partial \rho}{\partial x}\right) + p\frac{\partial v}{\partial x} = 0$$

Primitive variable form for the Euler equations

- We now at least have pressure derivatives in our equation, yet more work is needed though; we write the equation as

$$\frac{\partial \varepsilon}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{\partial \varepsilon}{\partial p}\frac{\partial p}{\partial t}+v\frac{\partial \varepsilon}{\partial p}\frac{\partial p}{\partial x}+\frac{p}{\rho}\frac{\partial v}{\partial x}=0$$

- Using the conservation of mass, we can rewrite this as

$$\frac{\partial p}{\partial t}+\frac{p}{\rho\frac{\partial \varepsilon}{\partial p}}\frac{\partial v}{\partial x}-\frac{\rho\frac{\partial \varepsilon}{\partial \rho}}{\frac{\partial \varepsilon}{\partial p}}\frac{\partial v}{\partial x}+v\frac{\partial p}{\partial x}=0$$

- Finally, a bit of factorisation gives us

$$\frac{\partial p}{\partial t}+\rho\left(\frac{p}{\rho^2\frac{\partial \varepsilon}{\partial p}}-\frac{\frac{\partial \varepsilon}{\partial \rho}}{\frac{\partial \varepsilon}{\partial p}}\right)\frac{\partial v}{\partial x}+v\frac{\partial p}{\partial x}=\frac{\partial p}{\partial t}+\rho c_s^2\frac{\partial v}{\partial x}+v\frac{\partial p}{\partial x}=0$$

Hyperbolicity revisited

- We now have a system of equations in primitive form

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ v\\ p\end{pmatrix}+\begin{pmatrix}v & \rho & 0\\ 0 & v & \frac{1}{\rho}\\ 0 & \rho c_s^2 & v\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ v\\ p\end{pmatrix}=0$$

- This took some effort to get here, but we did it for a **general** equation of state
- In this form, the eigenvalues are now straightforward to calculate, again we get

$$\lambda_1 = v - c_s, \quad \lambda_2 = v, \quad \lambda_3 = v + c_s$$

- Because this matrix (B) and the Jacobian $(\partial\mathbf{f}/\partial\mathbf{u})$ are similar matrices, we see that the Euler equations have three real eigenvalues, the system is hyperbolic

Characteristics of systems of equations

- As with our previous equations, now we know we have a hyperbolic system, we can compute characteristic variables

$$\frac{\partial \boldsymbol{\mathcal{V}}}{\partial t} + \Lambda(\boldsymbol{\mathcal{V}}) \frac{\partial \boldsymbol{\mathcal{V}}}{\partial x} = 0$$

- Recall the definition of the characteristic variables as

$$(C_{ij})^{-1} = \frac{\partial \mathcal{V}_j}{\partial w_i}$$

- When we have more complex systems, with unknown functions (such as c_s), integrating this expression might not give anything useful
- Instead, it is more common to consider the **jumps** in characteristic variables

$$\mathrm{d}\boldsymbol{\mathcal{V}} = Q^{-1} \mathrm{d}\mathbf{u}$$

- This relates the change in a characteristic variable as a function of the change in a conserved (or primitive) variable

Characteristic variables for the Euler equations

- For simplicity, we use the primitive variable form - for this, we get matrices of eigenvectors:

$$C = \begin{pmatrix} 1 & \frac{\rho}{2c_s} & -\frac{\rho}{2c_s} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\rho c_s}{2} & -\frac{\rho c_s}{2} \end{pmatrix} \qquad C^{-1} \begin{pmatrix} 1 & 0 & -\frac{1}{c_s^2} \\ 0 & 1 & \frac{1}{\rho c_s} \\ 0 & 1 & -\frac{1}{\rho c_s} \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} v & 0 & 0 \\ 0 & v + c_s & 0 \\ 0 & 0 & v - c_s \end{pmatrix}$$

- Our characteristic variables are

$$\mathrm{d}\boldsymbol{\nu} = \begin{pmatrix} \mathrm{d}v_0 \\ \mathrm{d}v_+ \\ \mathrm{d}v_- \end{pmatrix} = \begin{pmatrix} \mathrm{d}\rho - \frac{\mathrm{d}p}{c_s^2} \\ \mathrm{d}v + \frac{\mathrm{d}p}{\rho c_s} \\ \mathrm{d}v - \frac{\mathrm{d}p}{\rho c_s} \end{pmatrix} \qquad \text{along} \qquad \begin{array}{c} \mathrm{d}x = v\mathrm{d}t \\ \mathrm{d}x = (v + c_s)\,\mathrm{d}t \\ \mathrm{d}x = (v - c_s)\,\mathrm{d}t \end{array}$$

- We introduce the subscripts 0, $+$ and $-$ to identify the three eigenvalues

Using characteristic variables

- We don't get quite so much information about the solution of the Euler equations from the characteristic variables (that is why we want numerical solutions!)
- They do have important uses through - they still tell us about **wave speeds**, and the fact that there are three independent eigenvalues tell us that solutions to the Euler equations comprise **three waves**
- They can be used to identify **linearised solutions** to Riemann problems (more on this in a future course)
- Also, for an ideal gas, entropy is defined by

$$s = c_V \ln p - c_p \ln \rho + \text{const}$$

  where we have specific heats at constant volume and pressure, related to the equation of state through $\gamma = c_p/c_V$
- This gives

$$\mathrm{d}s = c_V \frac{1}{p}\mathrm{d}p - c_p \frac{1}{\rho}\mathrm{d}\rho$$

Considering constant entropy

$$\mathrm{d}s = c_V \frac{1}{p}\mathrm{d}p - c_p \frac{1}{\rho}\mathrm{d}\rho = 0$$

- If we rearrange this equation, we get

$$-\frac{\rho}{\gamma p}\mathrm{d}p + \mathrm{d}\rho = \mathrm{d}\rho - \frac{1}{c_s^2}\mathrm{d}p = 0$$

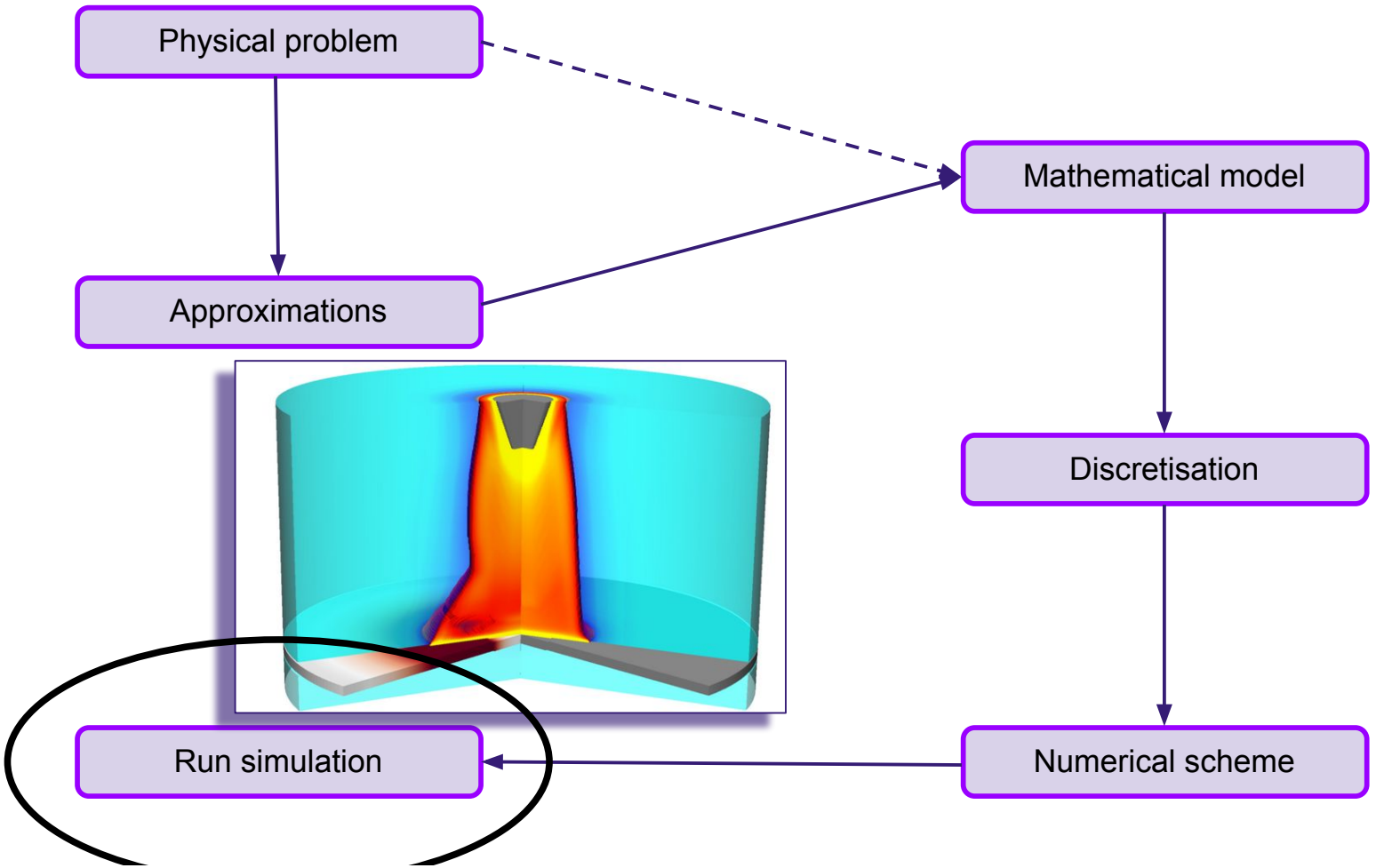
- In other words, this relationship, which is the characteristic for $\lambda_0 = v$, is equivalent to

$$\mathrm{d}s = 0$$

- Therefore, one of the characteristic variables is **entropy**, $v_0 = s$
- As a result, you will sometimes see waves travelling with speed v referred to as **entropy waves**
- The other wave types, which have speeds related to the speed of sound, (characteristics move with $v \pm c_s$), are sometimes called **acoustic waves**

Outline

Numerical method for the Euler equations



FORCE for the Euler equations

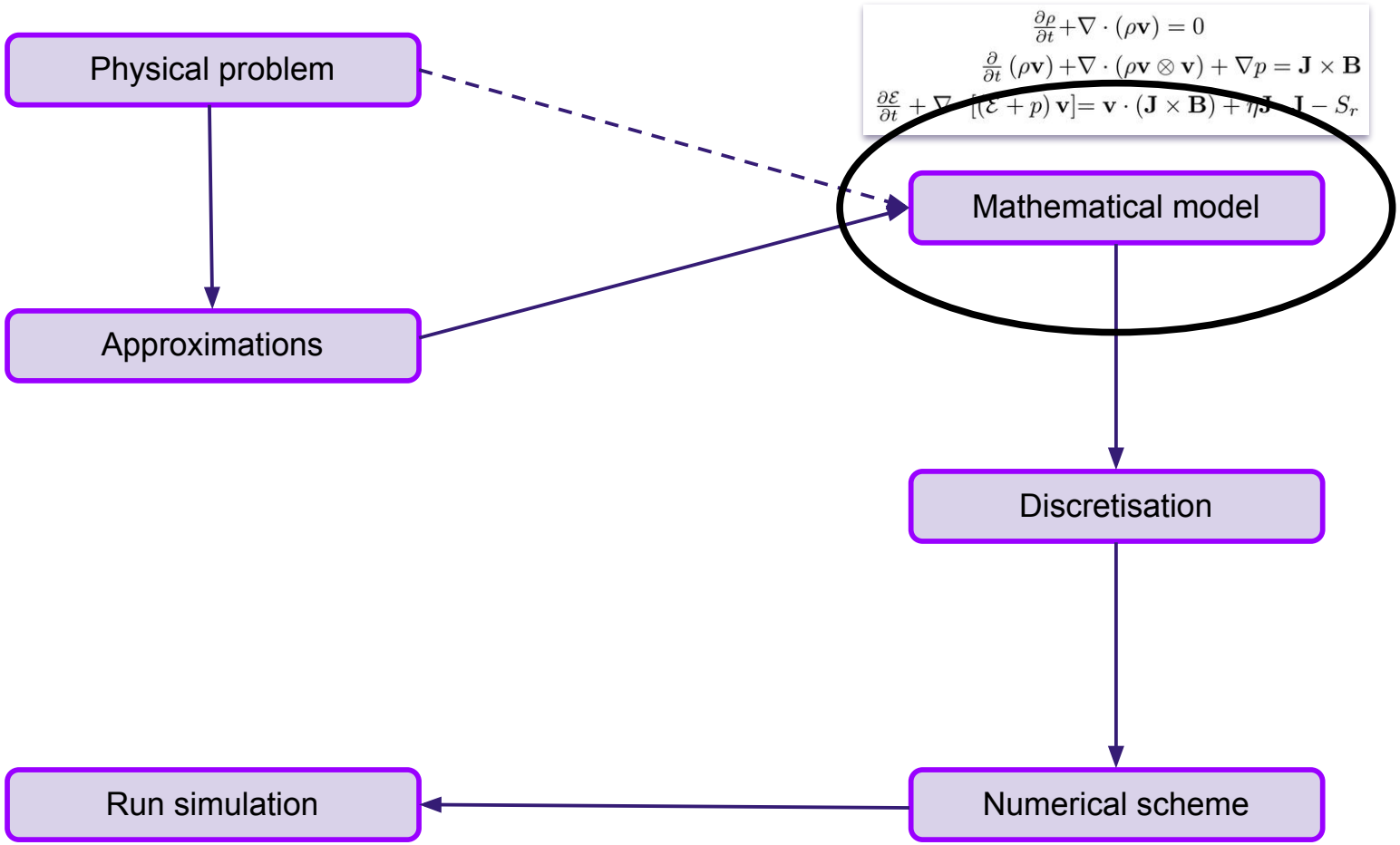
$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{f}_{i+1/2}^{\mathrm{FORCE}} - \mathbf{f}_{i-1/2}^{\mathrm{FORCE}}\right)$$

- Recall the FORCE scheme introduced for solving non-linear scalar PDEs
- The form we introduced does not change for systems of PDEs
- Extra consideration needs to be given to storing data; an `std::vector` of `std::array` objects is a sensible choice
- The ability to convert between conserved and primitive variables will also be required
- The time step will need to be correctly computed considering all waves - what is the maximum wave speed?

$$a_{\max} = |v| + c_s$$

Outline

The Riemann problem

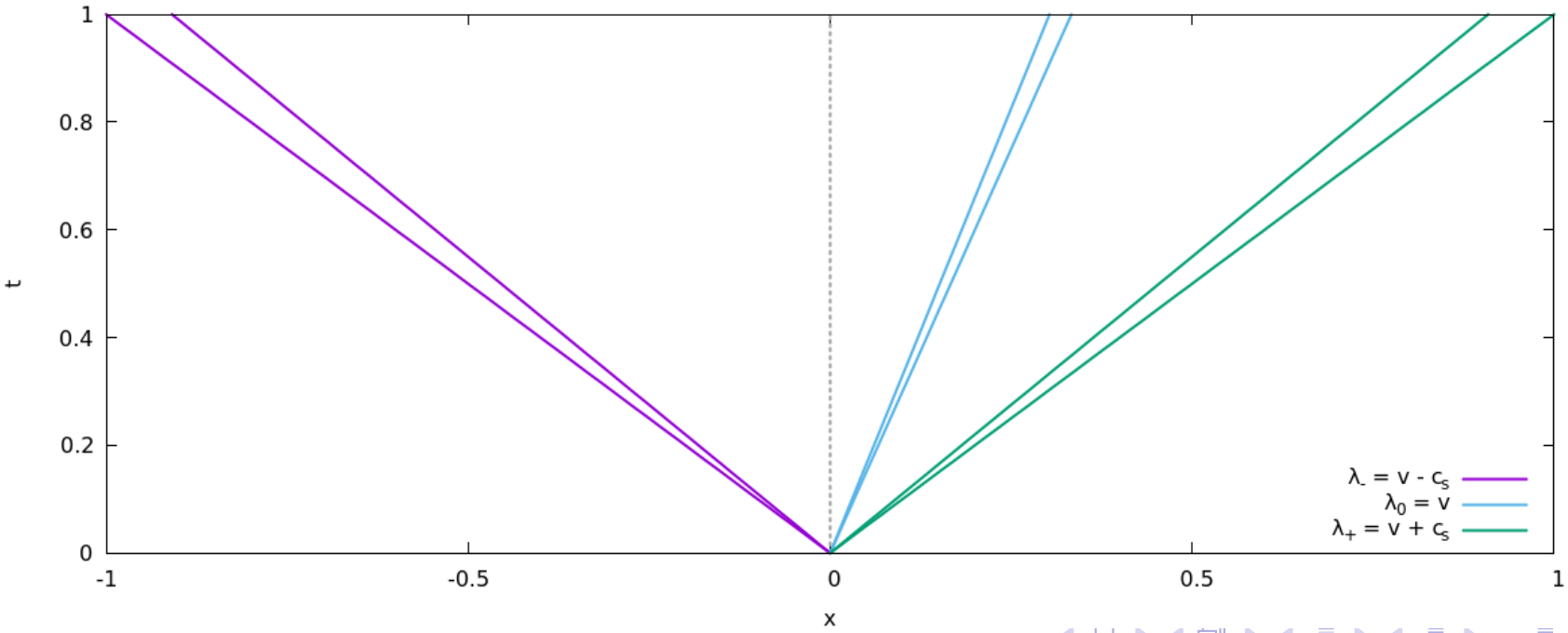


The Riemann problem for the Euler equations

- As we stated before, the three independent eigenvalues of the Euler equations mean that we expect to see three waves in the solution
- What do we really mean by this?
- We can generally only consider this distinct wave behaviour for Riemann problems (or piecewise continuous initial data)
- In this case, at each point in space (or time) we have three characteristics
- Each of these moves with a slope associated with one of the eigenvalues
- We can try to visualise this on an x-t diagram, but it gets somewhat messy

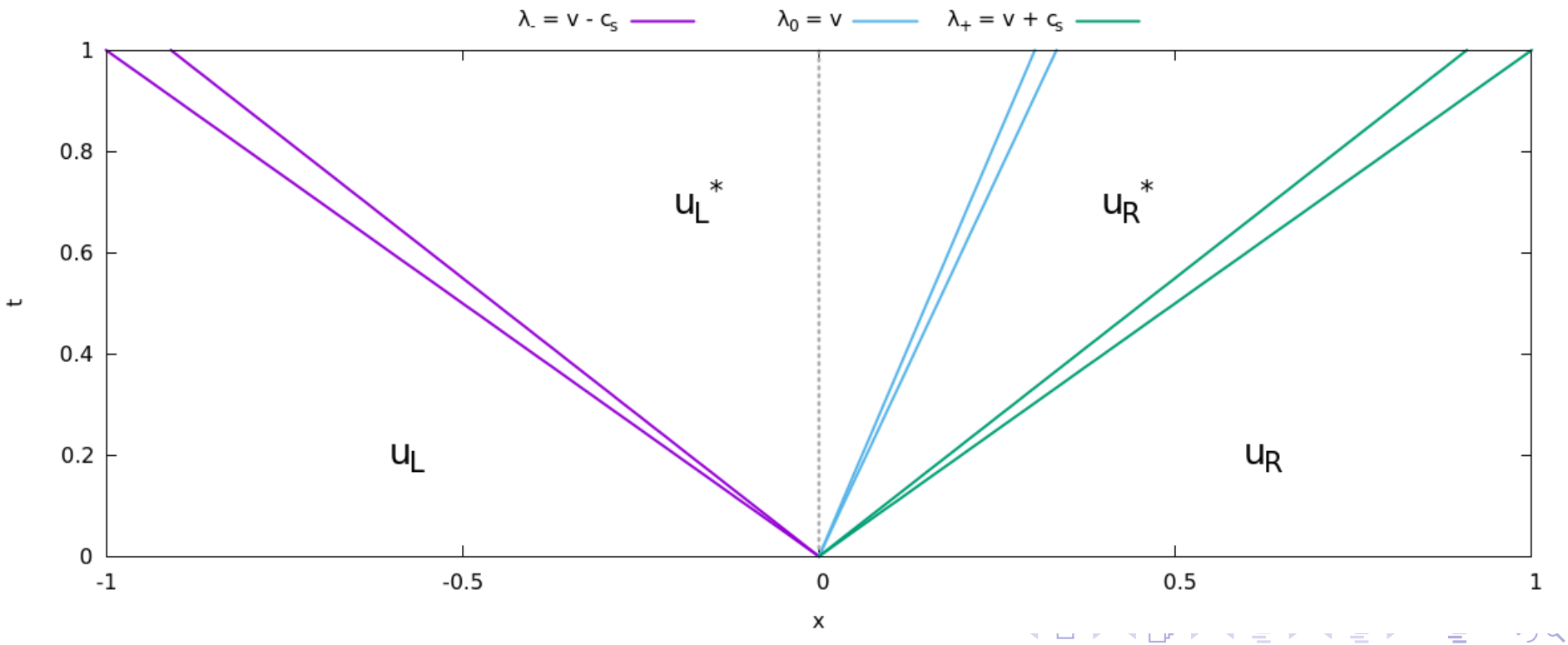
The three-wave solution

- Instead, it is better to start our visualisations with the lines of the features, or **waves**, that emerge from the Riemann problem
- At each of these features, the material properties will change, e.g. across a shock wave
- We don't know yet what will happen - these double-lines are a convention for generic change



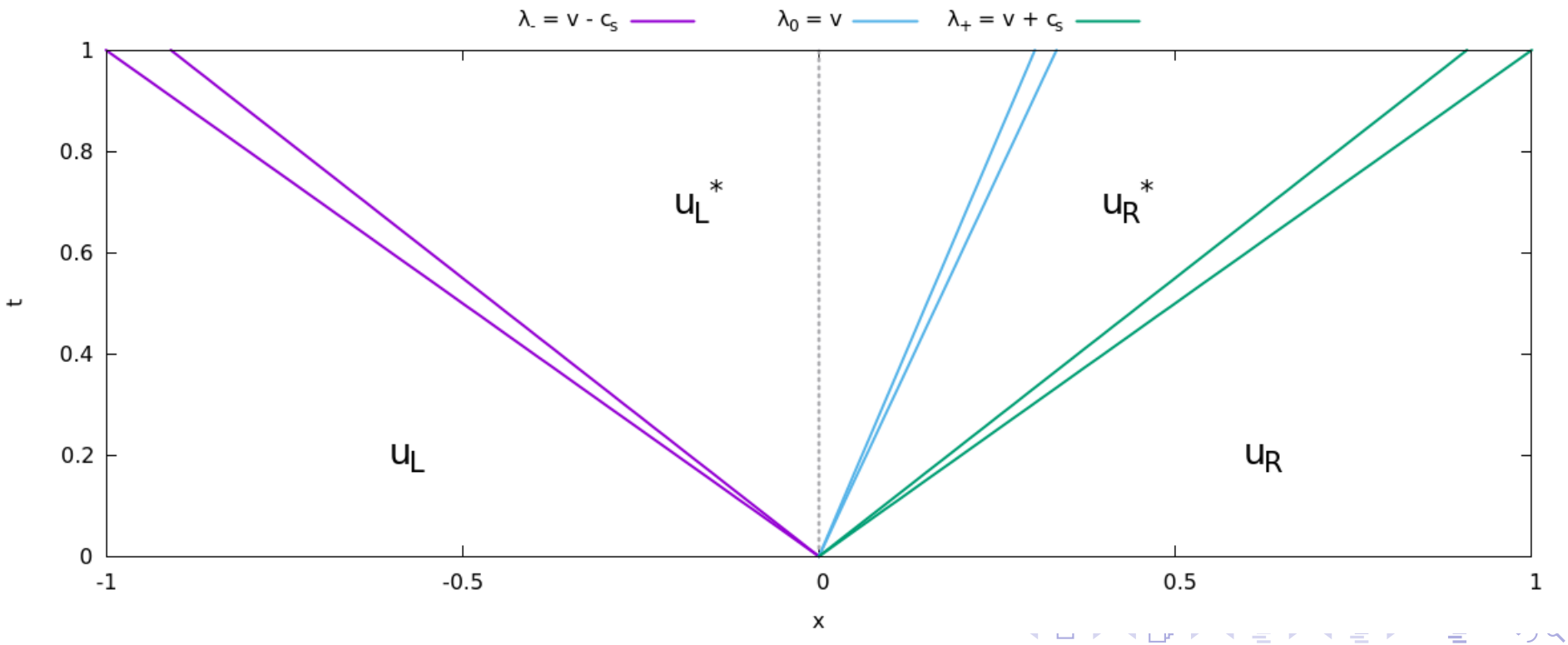
Structure of a Riemann solution

- We shall shortly deal with identifying wave types
- Between the waves, We have four states, two initial data, $\mathbf{u}_L$ and $\mathbf{u}_R$ and two **intermediate** or **star** states, $\mathbf{u}_L^*$ and $\mathbf{u}_R^*$
- We know that the initial states are constant, but what about the intermediate ones?



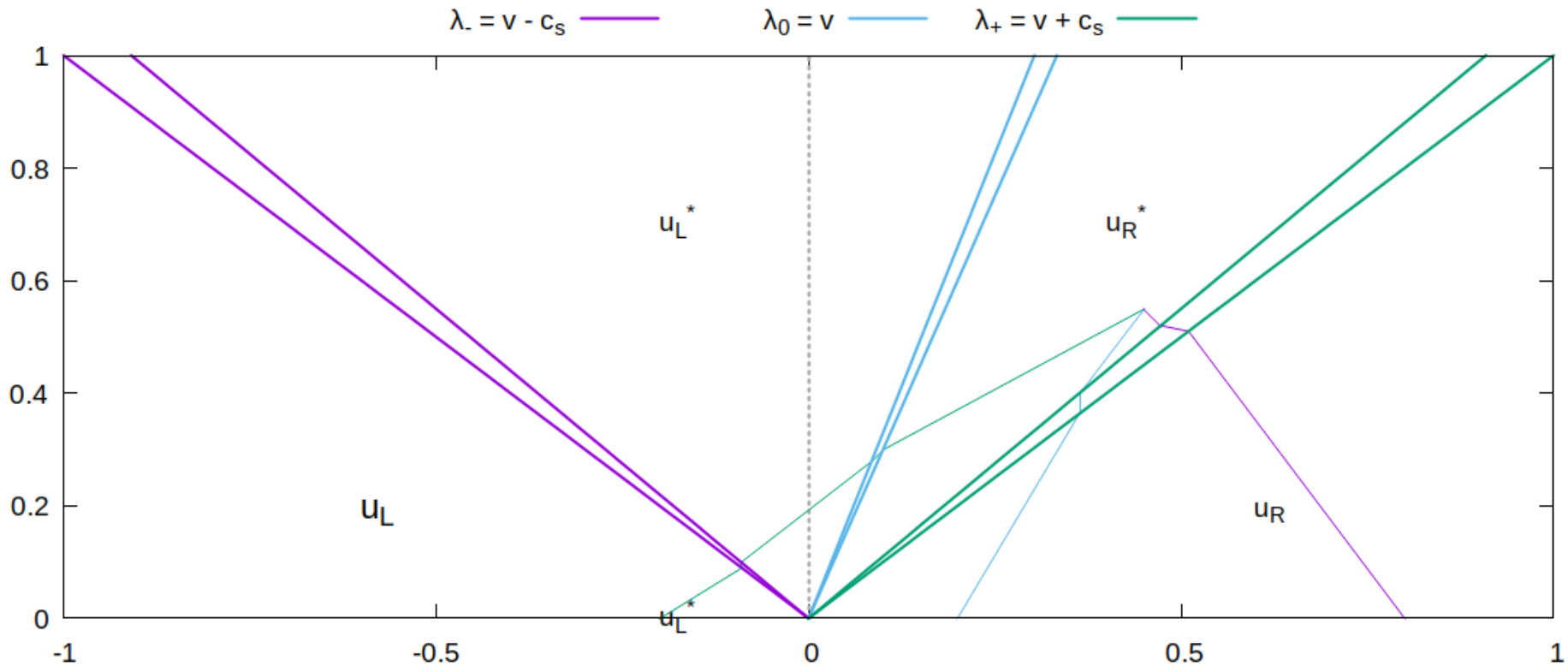
Structure of a Riemann solution

- We know the order of the waves, $\lambda_1 \leq \lambda_2 \leq \lambda_3$ or $\lambda_- \leq \lambda_0 \leq \lambda_+$
- We can obtain a non-rigorous argument as to what the intermediate states are by tracing back characteristics to coming from either $\mathbf{u}_L$ or $\mathbf{u}_R$
- All points in $\mathbf{u}_L^*$ have the same starting point (and similarly for $\mathbf{u}_R^*$), hence they must be constant



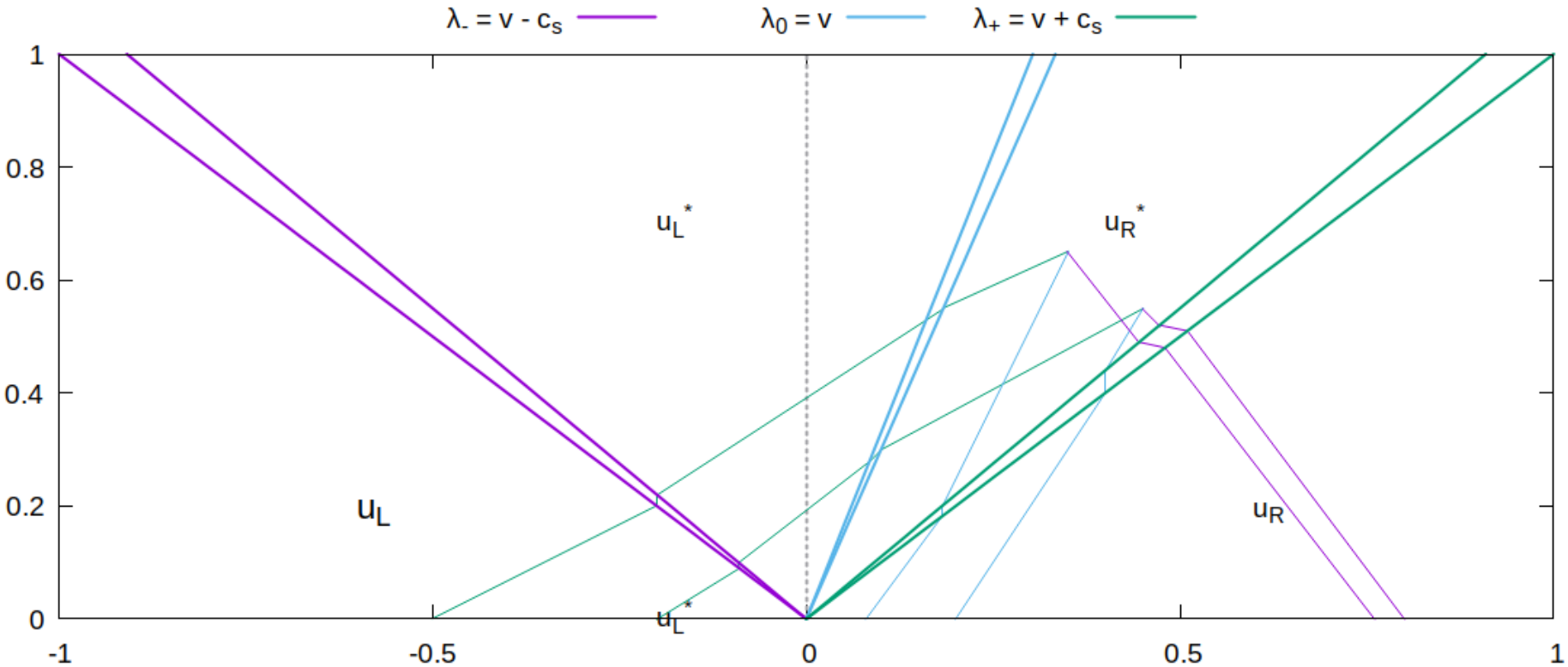
Structure of a Riemann solution

- For example, we can start with a point somewhere in $\mathbf{u}_R^*$
- Three characteristics lead to this point, one associated with each eigenvalue
- If we 'follow them back', two of them lead back to $\mathbf{u}_R$ and one to $\mathbf{u}_L$ - we may need to work out what happens when they reach one of the three features



Structure of a Riemann solution

- If we consider a second point in $\mathbf{u}_R^*$, and trace back the characteristics, we find the same thing
- Effectively, all states in $\mathbf{u}_R^*$ come from one characteristic in $\mathbf{u}_L$ and two in $\mathbf{u}_R$
- Because $\mathbf{u}_L$ and $\mathbf{u}_R$ are constant states, so are the intermediate states



Euler equation wave types

- We now know that we have four constant states, separated by three waves
- If we understand how the characteristics behave around each of the waves, we might find out a bit more
- Each wave is associated with an eigenvalue (and corresponding right-and left-eigenvector)
- Characteristics associated with each eigenvalue exist everywhere - they form a **characteristic field**
- For a given eigenvalue, λ_i, two classes of characteristic field might exist:
  1. **Linearly degenerate**:
  $$\frac{\partial \lambda_i}{\partial \mathbf{u}} \cdot \mathbf{r}^i = 0$$
  2. **Genuinely nonlinear**:
  $$\frac{\partial \lambda_i}{\partial \mathbf{u}} \cdot \mathbf{r}^i = 0$$

Characteristic field behaviour

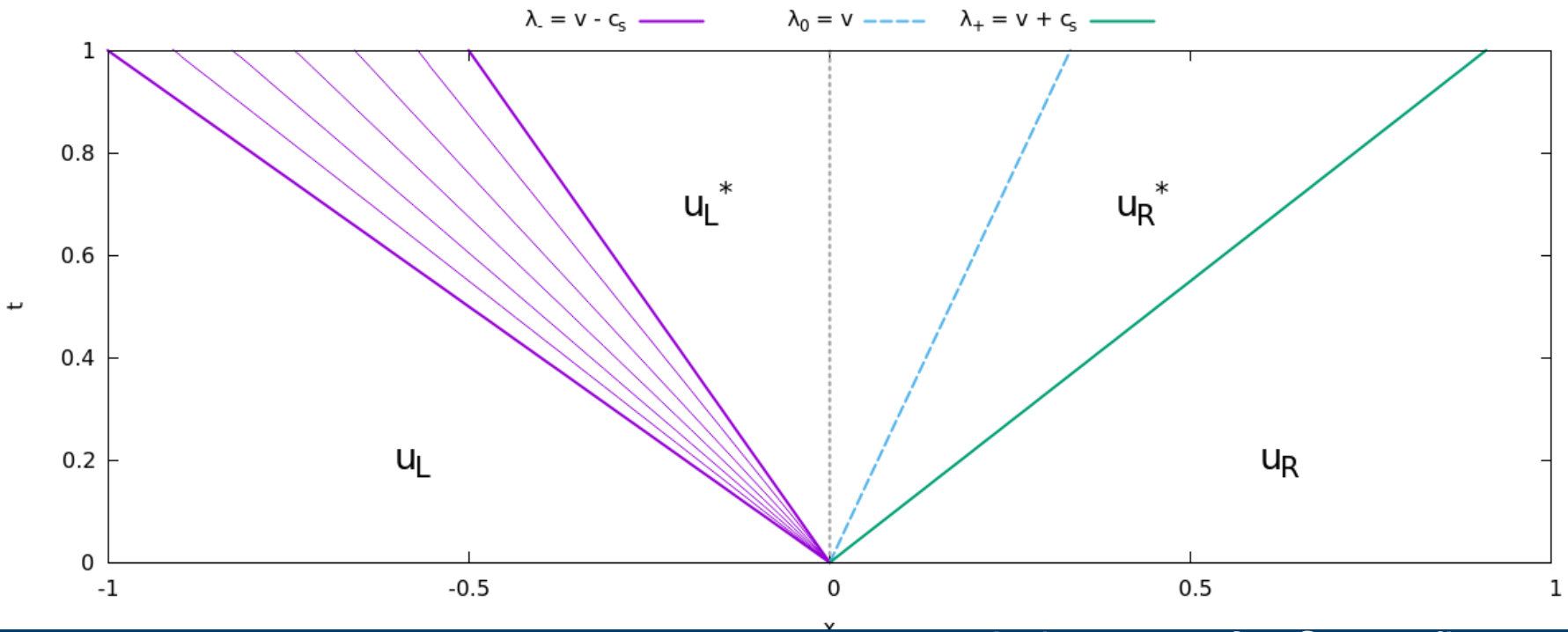
- For the Euler equations, the characteristic field associated with $\lambda_0 = v$ is linearly degenerate
- The other two are genuinely nonlinear
- When visualising this; recall that the set of eigenvectors forms a coordinate system in (ρ,v,p) space
- Also, $\frac{\partial \lambda_i}{\partial \mathbf{u}}$ is parallel to λ_i, also in (ρ,v,p) space
- $\frac{\partial \lambda_i}{\partial \mathbf{u}} \cdot \mathbf{r}^i = 0$ then means that the characteristic field associated with λ_i is **parallel** to λ_i
- In the Euler equations, this implies that the velocity in each of the intermediate regions is parallel to the wave associated with $\lambda_0 = v$
- And, it then follows that $v_L^* = v_R^*$

Structure of a Riemann solution

- Both of the waves associated with $\lambda_\pm$ can either a **shock** or a **rarefaction**
- Until we know more about the intermediate states, we do not know which of the wave types each wave is
- We need to know whether the characteristics are converging or diverging (but we know they are not parallel)
- The wave associated with λ_0 is a **contact discontinuity**
- Despite the increased complexity resulting from our system of equations, each wave has been seen before
- Burgers' equation supports shocks and rarefactions, and the advection equation (with discontinuous initial data) was an example of a contact discontinuity (though we did not call it that at the time)
- Here, we give an overview of these waves, full derivations are in all the recommended reading

Euler equation wave types - example

- Here we have an x-t diagram for a Riemann problem solution showing all three wave types
- Rarefactions a shown by the 'fan' structure, bounded by $(v - c_s)_L$ and $(v - c_s)^*_L$ or $(v + c_s)_R$ and $(v + c_s)^*_R$
- Shock waves are a solid line with slope $1/S$ (the shock speed), and contact discontinuities a dashed line with slope $1/v$



Contact discontinuities for the Euler equations

- Contact discontinuities are characterised by a jump in density, whilst pressure and velocity remain continuous

$$v_L^* = v_R^* = v^*, \qquad p_L^* = p_R^* = p^*$$

- We can examine where this behaviour comes from by considering the Euler equations
- We have already seen that parallel characteristics implies that v must be constant across a contact discontinuity
- However, if v is constant, the momentum evolution equation simplifies (note, we remove the conservation of mass to get this):

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0 \qquad \rightarrow \frac{\partial p}{\partial x} = 0$$

- In other words, pressure must also be constant across a contact discontinuity

Contact discontinuities for the Euler equations

$$v_L^* = v_R^* = v^*, \qquad p_L^* = p_R^* = p^*$$

- We note that there is no such simplification in the other equations; density and energy can take any values, i.e. they will jump across the contact discontinuity
- Since velocity is constant across a contact discontinuity, there is no mass transfer across these waves
- We note that since the equation of state can be written $T = T(\rho, p)$, a contact discontinuity represents two states at different temperatures
- Over longer time scales, we would see **diffusion** processes here; these processes are neglected in the inviscid Euler equations we consider - a valid assumption over short time scales

Rarefactions for the Euler equations

- Across a rarefaction, entropy is constant
- For an ideal gas, this implies $p \propto \rho^\gamma$
- We can also derive that for a rarefaction associated with $\lambda_\pm$, we have

$$\lambda_- : \quad v_L + \frac{2c_{s,L}}{\gamma - 1} = v_* + \frac{2c^*_{s,L}}{\gamma - 1} \qquad \lambda_+ : \quad v_L - \frac{2c_{s,R}}{\gamma - 1} = v_* - \frac{2c^*_{s,R}}{\gamma - 1}$$

- Toro's book contains a derivation of these relationships
- Here, we have two equations, and six quantities, four variables must be specified to fully describe a rarefaction, e.g. the entirety of the $\mathbf{u}_K$ state and p^*

$$\rho^*_K = \rho_K \left(\frac{p_*}{p_K}\right)^{1/\gamma}, \qquad v^* = v_K \pm \frac{2c_{s,K}}{\gamma - 1}\left[\left(\frac{p_*}{p_K}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right]$$

- Here, $K = L$ for λ_- and $K = R$ for λ_+

Shocks for the Euler equations

- Recall that shock waves are governed by jump conditions

$$\mathbf{f}(\mathbf{u}_K) - \mathbf{f}(\mathbf{u}_K^*) = S\left(\mathbf{u}_K - \mathbf{u}_K^*\right)$$

- This is three equations for seven quantities (plus an equation of state); the shock speed is an additional unknown quantity
- Again, we need to specify four variables - we again assume that we have an ideal gas, and that we know $\mathbf{u}_K$ state and p^*
- Manipulation of the jump conditions, again for an ideal gas, gives us

$$\rho_K^* = \rho_K \frac{\frac{p^*}{p_K} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p^*}{p_K} + 1} \qquad S_K = v_K \pm c_{s,K}\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\frac{p^*}{p_K} + \frac{\gamma-1}{2\gamma}}$$

$$v^* = \left(1 - \frac{\rho_K}{\rho_K^*}\right) S_K + v_K \frac{\rho_K}{\rho_K^*}$$

- Not a short derivation - again see Toro for details

What do these relationships tell us?

- If we set up a Riemann problem, we will know the two initial states, $\mathbf{u}_K$
- We have 4 to 6 unknown quantities, ρ_K^*, v^*, p^* and S_K^* (only needed if wave K is a shock)
- We also have 4 to 6 corresponding relationships - if a wave is a rarefaction, 2 for a rarefaction, 3 for a shock
- We always have a closed system - we can solve the Riemann problem completely
- For the Euler equations, with an ideal gas, it is possible to solve the Riemann problem **exactly**
- This means we can use Riemann problem-based methods to solve the Euler equations (and avoid the smearing associated with the centred methods)
- Additionally, exact solutions are essential for **validating** our numerical method - especially since behaviour is not so intuitive when we have three waves

Solving the Riemann problem for the Euler equations

- For both shock waves and rarefactions, we gave a relationship for the intermediate velocity, v^*

$$v^*_{\text{shock}} = \left(1 - \frac{\rho_K}{\rho^*_K}\right) S_K + v_K \frac{\rho_K}{\rho^*_K}, \qquad v^*_{\text{raref.}} = v_K \pm \frac{c_{s,K}}{\gamma - 1}\left[\left(\frac{p_*}{p_K}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right]$$

- In both cases, we have $v^* = g\left(p^*, \mathbf{u}_K\right)$ (recall S_K was also defined through p^* and $\mathbf{u}_K$ only)
- We know that computing v^* from either the left or the right states **must** give the same intermediate state
- Following Toro's notation, we can write this

$$f(p^*, \mathbf{u}_L, \mathbf{u}_R) = g\left(p^*, \mathbf{u}_R\right) - g\left(p^*, \mathbf{u}_L\right) = f_R\left(p^*, \mathbf{u}_R\right) + f_L\left(p^*, \mathbf{u}_L\right) + \Delta v = 0,$$

- Here, we have $\Delta v = v_R - v_L$
- This is an **implicit** equation for p^* - in general we cannot write an explicit relationship, but methods exist to give us the exact solution

The Riemann problem solution

$$f(p^*, \mathbf{u}_L, \mathbf{u}_R) = g(p^*, \mathbf{u}_R) - g(p^*, \mathbf{u}_L) = f_R(p^*, \mathbf{u}_R) + f_L(p^*, \mathbf{u}_L) + \Delta v = 0,$$

- The reason for writing the solution in the form above is that a single function $f_K(p^*, \mathbf{u}_K)$ can be used for both left and right states

$$f_K(p^*, \mathbf{u}_K) = \begin{cases} (p - p_K)\sqrt{\frac{A_K}{p^* + B_K}}, & \text{shock} \\ \frac{2c_{s,K}}{\gamma - 1}\left[\left(\frac{p^*}{p_K}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right], & \text{rarefaction} \end{cases} \qquad A_K = \frac{2}{(\gamma+1)\rho_K}, \quad B_K = \frac{\gamma-1}{\gamma+1}$$

- How do we know if we have a shock or a rarefaction?
- We know that waves originate from the initial discontinuity, and that a shock wave is a compression (pressure must be higher in the star state) whilst a rarefaction is an expansion (pressure must be lower in the star state)

Finding the intermediate pressure

$$f_K\left(p^*, \mathbf{u}_K\right) = \begin{cases} \left(p - p_K\right)\sqrt{\frac{A_K}{p^* + B_K}}, & p* > p_K \\ \frac{2c_{s,K}}{\gamma - 1}\left[\left(\frac{p^*}{p_K}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right], & p^* < p_K \end{cases} \qquad A_K = \frac{2}{(\gamma+1)\rho_K} \qquad B_K = \frac{\gamma-1}{\gamma+1}$$

- If we don't know p^*, does this help us solve the equations?
- As we can't write an equation for p^* explicitly, we need to use an iterative root-finding technique (e.g. Newton-Raphson or bisection) - this involves making a guess for p^* and using the root-finding technique to update this guess
- For each guess, we can compare p^* to p_L and p_R, and evaluate f_L and f_R appropriately
- For fast solutions, a Newton-Raphson method is recommended,

$$p^*_{(m)} = p^*_{(m-1)} - \frac{f(p^*_{(m-1)}, \mathbf{u}_L, \mathbf{u}_R)}{f'(p^*_{(m-1)}, \mathbf{u}_L, \mathbf{u}_R)}$$

- Here, $p^*_{(m)}$ is the m^{th} guess for p^*, and the derivative f' is with respect to p^*

Completing the solution

- Once we have p^*, computing v^*, ρ_K^* and S_K (where necessary) is straightforward - we note that v^* can **also** be computed through

$$v^* = \frac{1}{2}\left(v_L + v_R\right) + \frac{1}{2}\left(f_R\left(p^*, \mathbf{u}_R\right) - f_L\left(p^*, \mathbf{u}_L\right)\right)$$

- The only thing missing is what is the state **inside** any rarefactions?
- The results that gave us the intermediate state across a rarefaction fan can also be used to give us the interior state (see Toro)

$$\rho_{K,\text{raref.}} = \rho_K \left[\frac{2}{\gamma+1} \mp \frac{\gamma-1}{(\gamma+1)\,c_{s,K}}\left(v_K - \frac{x-x_0}{t-t_0}\right)\right]^{2/(\gamma-1)}$$

$$v_{K,\text{raref.}} = \frac{2}{\gamma+1}\left[\mp c_{s,K} + \frac{\gamma-1}{2}v_K + \frac{x-x_0}{t-t_0}\right]$$

$$p_{K,\text{raref.}} = p_K \left[\frac{2}{\gamma+1} \mp \frac{\gamma-1}{(\gamma+1)\,c_{s,K}}\left(v_K - \frac{x-x_0}{t-t_0}\right)\right]^{2\gamma/(\gamma-1)}$$

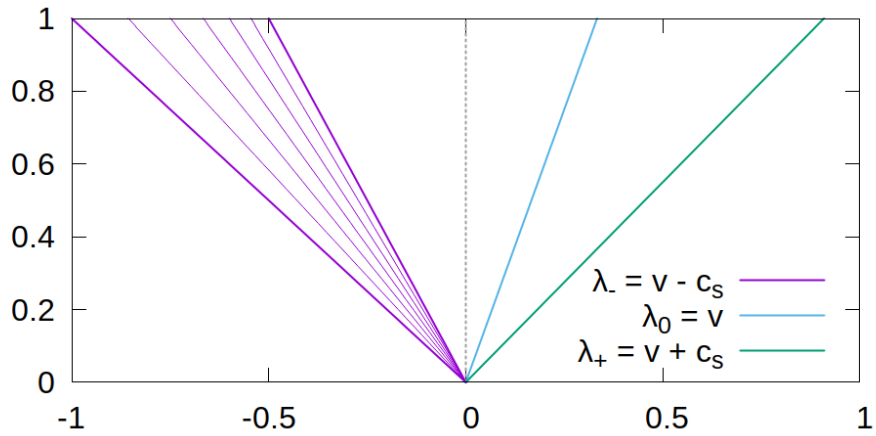
Godunov's method for the Euler equations

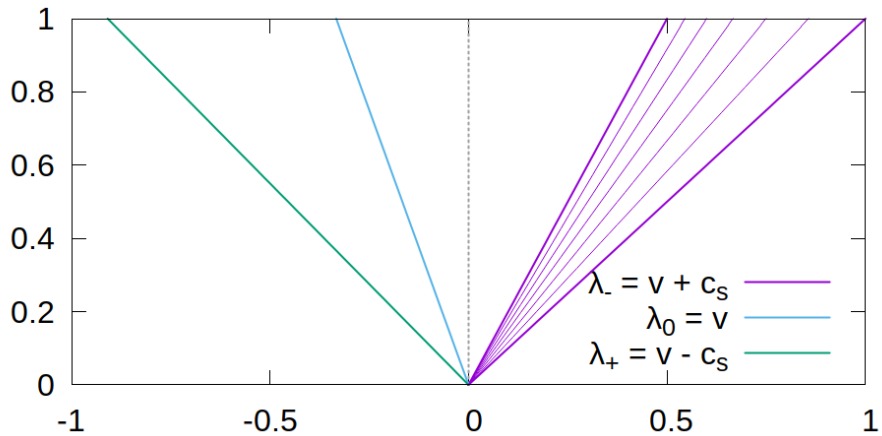
$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{f}(\mathbf{u}_{i+1/2}) - \mathbf{f}(\mathbf{u}_{i-1/2})\right)$$

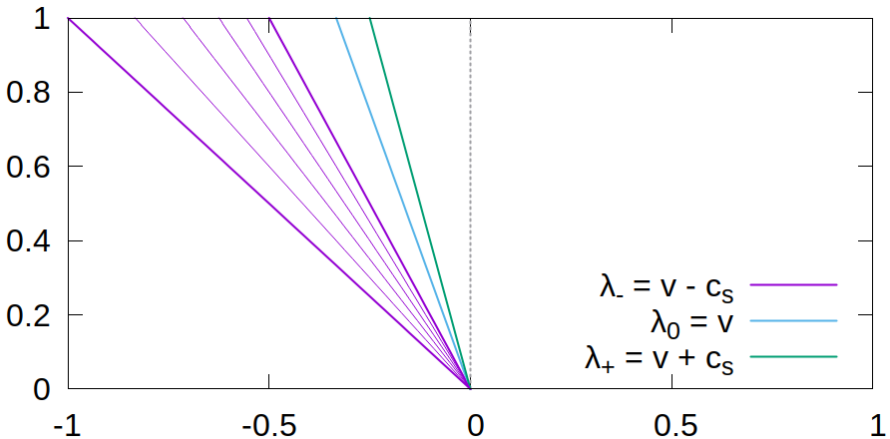
- We now have enough information to apply Godunov's method to solve the Euler equations, with $\mathbf{u}_{i+1/2}$ being the solution to the Riemann problem between $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$ along $x = x_{i+1/2}$
- This could come from any of the four states, or from the solution in the middle of a rarefaction - by identifying the wave speeds, and at least some of the information about the intermediate states, we know which values we need
- How could the solution along $x_{i+1/2}$ be one of the initial states, $\mathbf{u}_i$ or $\mathbf{u}_{i+1}$?
- To use Godunov's method for the Euler equations, we need to solve a root-finding problem at every cell interface - is this a bit computationally wasteful (compared to, e.g. a centred scheme)?

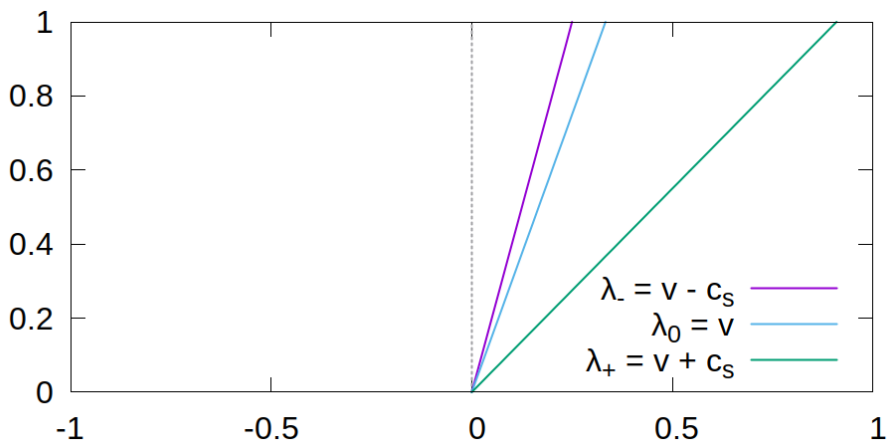
A brief note on Riemann problem solutions

- It is standard to show the three nicely-spaced waves in illustrating Riemann problem solutions - but these could be solutions









A brief note on Riemann problem solutions

- It is standard to show the three nicely-spaced waves in illustrating Riemann problem solutions - as could these