Computational Continuum Modelling
Practical 3: Finite volume methods for scalar equations

This practical aims to take the finite difference advection codes you have been working on and convert them to a finite volume code, capable of solving either the advection equation or Burgers' equation. We shall stick to first order methods at the moment, to avoid oscillatory behaviour.

The aim is to develop a code to solve

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^n - f_{i-1/2}^n\right)$$

where $f_{i\pm1/2}$ is a finite volume flux which can be computed by one of three methods:

1. Lax-Friedrichs scheme (in conservation form)
2. FORCE
3. Godunov's method

The key goal is to be able to solve Burgers' equation, but you should first test each of the schemes for the advection equation ($a = 1$ is an obvious choice, but use anything you like). When implementing a new method, it is always sensible to check something you have already got working elsewhere continues to work.

When solving Burgers' equation, the periodic boundaries we have worked with so far are not the best choice. **Transmissive boundaries** are most appropriate here - it does not make sense for a shock wave to wrap around a boundary. Implementing these is straightforward, in a domain $i \in [0 : N]$:

$$u_{-1} = u_0, \qquad u_{N+1} = u_N$$

As you progress with these tests, you will want to output more data than just the initial and final states. Modify your code to output, for example, five times over the course of the simulation. Plotting all five times on a single plot can give a good feel for how the simulation is evolving.

Exercises:

There are four tests which should be implemented with these methods:

1. The shock wave Riemann problem example from lectures
$$u_0(x) = \begin{cases} 2 & x < 0.5 \\ 1 & x > 0.5 \end{cases}$$
This test should be on a domain $x \in [0, 1]$ and run to final time $t = 0.2$.

2. The rarefaction Riemann problem example from lectures
$$u_0(x) = \begin{cases} 1 & x < 0.5 \\ 2 & x > 0.5 \end{cases}$$
Again this test should be on a domain $x \in [0, 1]$ and run to final time $t = 0.2$

3. Another test from Toro's book
$$u_0(x) = \begin{cases} -\frac{1}{2}, & x \leq \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1 \\ 0 & x > 1 \end{cases}.$$
This test should be on a domain $x \in [0 : 1.5]$, and have a final time $t = 1.5$. The results in Toro's book use 75 cells and $C_{\mathrm{CFL}} = 0.8$, but it will also be worth trying higher numbers of cells.

4. The cosine curve seen in the lectures
$$u_0(x) = \cos(2\pi x)$$
with a domain $x \in [0 : 1]$.

   In this case, the choice of final time is up to you - the x-t diagram of the characteristics from the lectures might give you some idea though. Do you want a characteristic that has come from outside of the domain to interact with your wave structure?