Lab1: 1D Continuous Convolution

Convolution is a fundamental operation in signal processing and computer vision. It describes how the shape of one function is modified by another and is the mathematical model behind many filtering operations (blurring, sharpening, edge detection, etc.). In computer vision, discrete versions of convolution are applied to images using small kernels (filters) to detect features such as edges, corners, and textures, and the same idea is used at scale in convolutional neural networks (CNNs) to learn visual features from data.

In this lab, we will practice computing continuous-time convolutions by hand using simple piecewise functions. We will focus on:

- Rectangular (box) functions,
- Triangular functions.

Throughout, we use the following definition of convolution:

$$g(x) = (f * h)(x) = \int_{-\infty}^{+\infty} f(t)\, h(x-t)\, dt.$$

We will use the following basic functions.

Rectangle (centered at 0, width 2).

$$\square(t) = \begin{cases} 1, & -1 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Rectangle (right-shifted, width 1).

$$\square_+(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Triangle (centered at 0, support $[-1,1]$).

$$\triangle(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 1: Convolution of Two Rectangles

In this exercise, we convolve two identical rectangles:

$$f(t) = \square(t), \qquad h(t) = \square(t).$$

1. Write down the mathematical definition of the convolution $g(x) = (f * h)(x)$ using f and h:
$$g(x) = \int_{-\infty}^{+\infty} \dots \, dt.$$

2. For a general x, write down:
   - the support (interval where the function is nonzero) of $f(t)$,
   - the support of $h(x - t)$.

3. For the convolution $g(x) = (f * h)(x)$, **draw** and **compute** the convolution by identifying the ranges of x corresponding to:
   - no overlap,
   - first partial overlap,
   - full overlap,
   - second partial overlap,
   - end of overlap.

   For each range of x:
   - draw a representative example of $f(t)$ and $h(x - t)$ on the t-axis,
   - determine the overlap interval(s) in t,
   - set up and evaluate the integral(s) that define $g(x)$ on that range.

4. Combine all cases into a single piecewise expression for $g(x)$ valid for all x.

Exercise 2: Convolution of Triangle with Rectangle

In this exercise we convolve a triangle with a rectangle:

$$f(t) = \triangle(t), \qquad h(t) = \square(t).$$

Tasks: Repeat the same steps as in **Exercise 1** (Questions 1–4), now using the functions above:

- write the convolution definition $g(x) = (f * h)(x)$,
- find and sketch the supports of $f(t)$ and $h(x - t)$ for a general x,
- for each overlap case (no / partial / full / partial / end), draw, determine the overlap interval(s) in t, and compute $g(x)$,
- derive the final piecewise expression of $g(x)$.

Exercise 3 (Homework): Mixed Rectangles

In this exercise, we convolve two *different* rectangles:

$$f(t) = \square(t) = \begin{cases} 1, & -1 \leq t \leq 1, \\ 0, & \text{otherwise,} \end{cases} \qquad h(t) = \square_+(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Tasks: Repeat the same steps as in **Exercise 1** (Questions 1–4), now using the functions above:

- write the convolution definition $g(x) = (f * h)(x)$,
- find and sketch the supports of $f(t)$ and $h(x - t)$ for a general x,
- for each overlap case (no / partial / full / partial / end), draw, determine the overlap interval(s) in t, and compute $g(x)$,
- derive the final piecewise expression of $g(x)$.