A New Dynamic Reference Point Specification Mechanism in hypervolume-based EMOA based on weak convergence detection

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***Abstract*—In the field of indicator-based evolutionary multi-objective optimization algorithms(indicator-based EMOAs), hypervolume is a popular indicator, which is the only pareto-compliant indicator up to now. But recently, a paper shows that the position of reference point will influence the diversity of the final solutions of hypervolume-based EMOAs when applying to the inverted-triangular pareto front problems, by influencing the hypervolume contribution of the external solutions. In this paper, we state this phenomenon and introduce the reference point specification with the dynamic mechanism in terms of necessity. Then we propose a new dynamic mechanism based on weak convergence detection. In our approach, the simple Least Squares and the information of nadir point are used to detect the convergence. After that, we examine the difference between this new dynamic mechanism with a state-of-the-art linearly decrease mechanism.**

***Keywords*—reference point adaptation; indicator-based algorithm; hypervolume; evolutionary multi-objective optimization; behavior; dynamic mechanism; convergence detection**

I. Introduction

In the field of evolutionary multi-objective optimization algorithms (EMOAs), several researchers focus on various indicators including hypervolume [1], R2 [2], ϵ_+ indicator [3] and IGD [4]. The indicators are designed for different purposes, and as a matter of fact, have their own strong points and drawbacks. Different from IGD, hypervolume does not need the pre-knowledge of the shape of the pareto front and is the only pareto-compliant indicator up to now [5]. But with a heavy computation load of hypervolume computation [6], the hypervolume-based algorithms get poor performances in running time when dealing with problems which have more than three objectives, which is so-called Many-Objective Optimization Problems(MaOPs).

SMS-EMOA [7] is a basic hypervolume-based algorithm. The hypervolume contribution is used to determine which solution to be discarded in the algorithm. After proposing the SMS-EMOA, in order to reduce the heavy computation cost of hypervolume computation, many new indicators or new methods have been proposed to estimate the hypervolume. For example, HypE use a Monte Carlo simulation technology to estimate the effect of hypervolume [8]; R2 indicator bases on a standard weighted Tchebycheff function [2] and a new R2 is proposed by Shang et al [9]. Recently, an improved SMS-EMOA with adaptive resource allocation has been proposed to reduce the number of hypervolume calculations [10]; In 2015, a simple and fast version of SMS-EMOA [7], so-called FV-EMOA, has been proposed [11]. In order to further reduce the bottleneck of high time complexity for calculating the hypervolume contributions, the FV-EMOA considers the fact that the hypervolume contribution of a single solution is only associated with partial solutions rather than the whole solution set [11]. Based on this point, the FV-MOEA reduces the computational cost greatly.

The specification of the reference point is one of the important but easy to be ignored parts in hypervolume computation, for that the effect on many problems including DTLZ series [12] are not so significant. But it has been reported that the position of reference point strongly influences the value of hypervolume contribution of the external solutions on the problems with inverted-shape pareto fronts [13]–[15]. A suitable set of reference point position for flat pareto fronts has been sufficiently investigated in Hisao et al. [14]. And a dynamic reference point specification mechanism has also been proposed in Hisao et al. [16]. Another strategy proposed is to use two reference points in hypervolume-based EMOA [17]. In this paper, the dynamic mechanism is stated in two different stages of the algorithm running process. At the early stage, for a better searching behavior of the algorithm, the reference point should be set a little worse than the suggested position at the beginning. Then with a dynamic mechanism, the reference point is gradually decreasing to the suggested position following the iteration.

After that, we propose a new dynamic reference point specification mechanism with a weak convergence detection in hypervolume-based EMOA. The simple least squares and the logarithm value of nadir point as a convergence indicator are used in this weak convergence detection. Given w_l generations as a window, if the slope of linear regression is below one threshold, we report the convergence. The comparison of two

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dynamic mechanisms and simple reference point specification(without dynamic mechanism) is represented in the experiment section. On some specific problems, for example the multi-objective distance minimization problems [18], our weak convergence detection mechanism outperforms the linearly decrease mechanism.

The remainder of this paper is organized as follows. Firstly, we briefly introduce the basic idea of reference point specification by a simple example in Section II. Then, we explore the details of the reference point specification mechanism in two stages of the algorithm, which elicits the necessity of using dynamic mechanism in Section III. After that, we state the details of the dynamic reference point specification mechanism and introduce a proposed linearly decrease mechanism in Section IV. Then the details of the new dynamic mechanism with a weak convergence detection proposed in this paper is in Section V. And we report our computational experiments of SMS-EMOA with several triangular and inverted-triangular problems in Section VI. Our experiments are performed on 10-dimension problems for clearly comparing of two dynamic mechanisms and simple reference point adaptation (without dynamic mechanism). Finally, the conclusion is shown in Section VII.

II. Reference Point Specification in Hypervolume-based EMOA

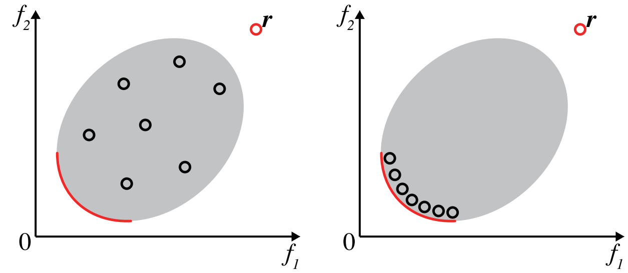
When hypervolume(HV) is used in indicator-based algorithms, one important thing to be considered is how to specify the reference point. Before calculating the HV values, a reference point needs to be chosen in advance. However, it is not suggested that the reference point is set only once at the beginning. This may cause a very far away reference point from solutions for those problems with a very large feasible space(As shown in Fig. 1), as the solutions set is gradually converging to the pareto front during the iteration of the algorithm process.

There is a big problem when applying this strategy to some problems with specific pareto front shape, for example, the inverted-DTLZ1 problem with an inverted-triangular pareto front in 3 dimensions, that many solutions in the final solutions set will distribute at the boundary of the pareto front (Fig. 2a comparing with Fig. 2b) [14]–[16]. Although it has no effect on the distributions of solutions set in problems with triangular pareto front in 3 dimensions (Fig. 2c comparing with Fig. 2d), it is necessary using a good reference point specification method during the algorithm progress. And the reason is illustrated detailedly in Hisao et al [15].

In the original SMS-EMOA [7] paper, the reference point is specified as the vector of the estimated nadir point increased by $\mathbf{1.0}$ (in two-dimension, a sufficiently large reference point is chosen). But in current implementation, the following mechanism is always used:

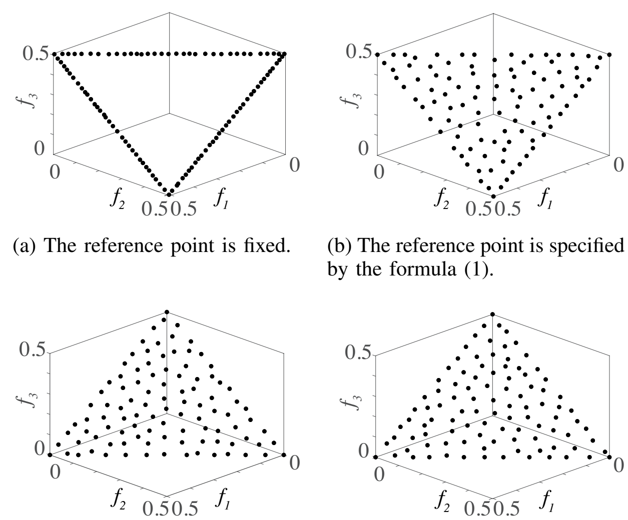
$$\boldsymbol{R} = r * \boldsymbol{N}, \boldsymbol{R}, \boldsymbol{N} \in \mathbb{R}^m. \tag{1}$$

Note that $\boldsymbol{R}$ is the reference point in each generation, and $\boldsymbol{N}$ is the estimated nadir point of the last front of the current population. r is specified as 1.1 in PlatEMO [19], a well-known EMO framework. The specification of reference point is one of the important parts in the hypervolume-based algorithm implementation. In the source code of PlatEMO, the values of r in hypervolume-based algorithms are chosen



(a) The initial generation. (b) The final generation.

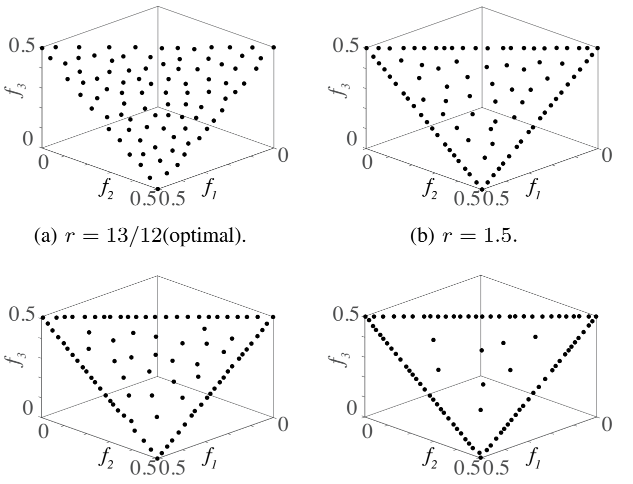
Fig. 1: The reference point is set with a large feasible space. pareto front can be far away from reference point. The gray region shows the feasible region and the red arc is the corresponding pareto front. The red circle r is the reference point calculated by the initial solutions in (1a) which is randomly generated. After some generations, the current solutions reach to the five black circles in (1b), which is far away from the reference point.



(a) The reference point is fixed. (b) The reference point is specified by the formula (1).

(c) The reference point is fixed. (d) The reference point is specified by the formula (1).

Fig. 2: The final distribution of solutions set in the inverted-DTLZ1(2a and 2b) and the DTLZ1 problem(2c and 2d). The algorithm is SMS-EMOA with population size = 100, evaluation number = 20000. 2a and 2c: the reference point is calculated only once at the initial step; 2b and 2d: the reference point is specified by the formula (1) with $r = 1.1$. All the solutions in the final distribution are at the boundary of the pareto front in (2a), which shows the bad effect of a faraway reference point on the final distribution of inverted-triangular problems. This bad effect can not be observed on triangular problems in (2c).



(a) $r = 13/12$(optimal). (b) $r = 1.5$.

(c) $r = 2$. (d) $r = 5$.

Fig. 3: The final distribution of solutions set in the inverted-DTLZ1 problem. The algorithm is SMS-EMOA with population size $= 91(H = 12)$ and evaluation number $= 20000$. $r = 13/12$ is the optimal setting and we observed a evenly distribution in (3a). As the increasing of r, solutions are more likely to be at the boundary (3b-3d).

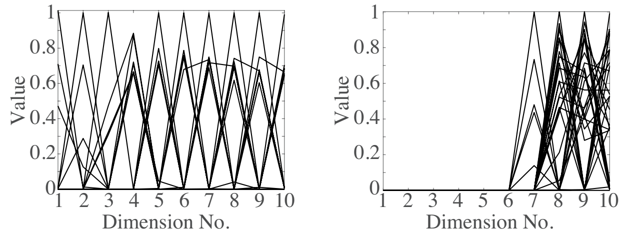
as follow: 1.1(SMS-EMOA [7]) and 1.2(HypE [8]). When the solutions in the current population are obtained, we use hypervolume as the indicator to evaluate the performance of the solutions set. Then the reference point used to calculate the hypervolume is calculated by the formula above.

But actually, fixing the r value to 1.1 is not suggested, especially on problems with the inverted-triangular pareto front [13]. The research on specifying the value of r is limited, for the reason that, the effect of the location of the reference point on the pareto front is not fatal on some benchmark problems, for example, the problems with triangular pareto front.

In Hisao et al. [13]–[15], the suggested value of r is investigated on flat (not concave or convex) pareto front problems. Specifically, on inverted-triangular problems(inverted-DTLZ1 [15] etc.), in order to have same hypervolume contribution, the interval between two boundary solutions should be the same as that between two inner solutions. In hisao et al. [13], the suggested value of r is:

$$r = 1 + \frac{1}{H}, \tag{2}$$

as shown in Fig. 3. H is the number of solutions intervals in 2-dimension and the number of intervals at each boundary of pareto front in many-dimension. In Fig. 3a, $r = 13/12(H = 12)$ is the optimal setting for a 91-individual 3-dimensional inverted-DTLZ1 problem and a optimal distribution is observed. In Fig. 3b - Fig. 3d, the inner solutions are decreased and move to the boundary of pareto front as the increasing of r.



(a) $r = 10$. (b) $r = 2$(optimal).

Fig. 4: The final distribution of SMS-EMOA on 10-dimension DTLZ2 problems. The population number is $30(H = 1)$, total evaluation number is 10000. The first 6 dimensionalities are all 0 in 4b.

III. Exploration on Reference Point Specification Mechanism in Detail

Basically, the process of Evolutionary Multi-objective Optimization Algorithm can be separated into two stages:

1) Early Stage: In this stage, all the solutions are far away from pareto front. The main task is to converge the solutions to pareto front. We also call this stage the convergence stage.

2) Final Stage: In this stage, all the solutions are in or near the pareto front. So the main task is to make the distribution of solutions more evenly in the pareto front. We also call this stage the diversity stage.

A. Specify the Value of r for Better Searching Behavior

In the early stage, solutions set is not close to the pareto front, which makes the estimated ideal and nadir points far away from true ideal and nadir points, for the reason that they are calculated by current solutions in each generation. And considering the problems with 10 objectives, if the external solutions have the same hypervolume contribution with the inner solutions, the exploration of solutions will be poor. As a result, some dimensionalities of solutions will be missing and the breadth-diversity [20] will be poor. A example is given in Fig. 4. The first 6 dimensionalities are all 0 for the solutions obtained by SMS-EMOA on 10-dimension DTLZ2 problems(Fig. 4b). In [16], a larger value of r than the $1+1/H$ is suggested in the early stage.

B. Specify the Value of r for Optimal Distribution

When algorithm reaches to the final stage, all solutions are near the pareto front, r should be specified as $1 + 1/H$, as in formula 2, in order to get a optimal solutions distribution. That is, $r(T) = 1 + 1/H$. On inverted-triangular problems, the distribution of solutions on pareto front strongly depends on the value of r (as shown in Fig. 3).

The sensitivity of the value of r on the solutions distribution is also observed on some real-world problems, for example, distance minimization problems. This observation potential shows the usefulness of the dynamical reference point specification [16].

IV. Dynamic Reference Point Specification Mechanism

For different purposes in the two stages, the r should be treated differently [16]. Not only the reference point but also the value of r needs to be adapted in each iteration of the algorithm. This is called dynamical reference point specification. Based on the formula (1), we define the dynamic reference point specification as:

$$\boldsymbol{R} = r(t) * \boldsymbol{N}, \boldsymbol{R}, \boldsymbol{N} \in \mathbb{R}^m, t = 0, 1, \ldots, T, \quad (3)$$

where T is the total number of generations, and $r = r(t)$ is a function of the current generation t. The value of r is adapted during the process of the algorithm.

A. The Proposed Linearly Decrease Mechanism

Based on the theory above, r is suggested to be specified dynamically at different stages of the algorithm(at early stage, a larger r is chosen; at final stage, $r = 1 + 1/H$ is chosen). But unfortunately, there is no best mechanism on how to specify the value of r dynamically outperforming the others in all problems and all experiment settings. One mechanism may be the best when working on some specific experiment conditions, but may not be good on other conditions.

In [16], a linearly decrease mechanism has been proposed:

$$r(t) = r_{Initial}\frac{(T-t)}{T} + (1 + 1/H)\frac{t}{T}, t = 0, 1, \ldots, T, \quad (4)$$

where T is the total number of generations, and $r_{Initial}$ is the initial value of r, which is larger than $1 + 1/H$. It is a simple and practical mechanism. In (4), the value of r starts from $r_{Initial}$, then gradually decreases to the suggested value in a linearly decrease process.

In the next section, we will propose another good dynamic mechanism based on weak convergence detection criterion outperforms simple linearly decrease mechanism on some specific problems.

V. A New Dynamic Reference Point Specification Mechanism

In this section, we will introduce a new mechanism that uses a weak convergence detection criterion to decide whether to change the value of r from $r_{Initial}$ to $1 + 1/H$.

As we have explained before, a larger r is suggested at the initial stage of the algorithms. But for good diversity at the final stage, it is needed to set r to its optimal value $(1+1/H)$. For this purpose, we detect whether the algorithm is converged or not. If solutions are all close to the pareto front, we change the value of r to $r_{Optimal}$; otherwise, we set value of r to $r_{Initial}$. The mechanism is also shown below:

$$r(t) = \begin{cases} r_{Initial}, t < t_{Convergent} \\ 1 + 1/H, t \geq t_{Convergent} \end{cases} \quad t = 0, 1, \ldots, T. \quad (5)$$

$r(t)$ equals to $r_{Initial}$ before reaching to the convergent generation $t_{Convergent}$, and changes to $1+1/H$ after $t_{Convergent}$. The $t_{Convergent}$ is determined by a weak convergence detection criterion.

A. Weak Convergence Detection

Consider many convergence detection papers [21]–[27], various indicators including convergence detection indicators are using to detect the stagnation. They focus on the accuracy of convergence, which is not the purpose in our approach for the reason that after algorithm convergent, we still need some generations in order to get even distribution of solutions set. We summarize our weak convergence detection criterions as follow:

1) inaccuracy: It is not necessary to have an accurate convergence detection. The convergence can be reported if current solutions are close to the pareto front. In other words, the estimated ideal and nadir points based on the current solutions are close to the true ideal and nadir points.

2) saving time: We should not spend too much time in convergence detection for the reason that the state-of-the-art indicator-based algorithms such as SMS-EMOA and HypE, are time-consuming when the dimension is very high.

We are discussing the effect of reference points in calculating hypervolume. It seems to be a good idea to use progress indicator hypervolume as our convergence detection indicator, for that, we have calculated hypervolume in each generation in the hypervolume-based evolutionary multi-objective optimization algorithm. But during the process of algorithm, the reference point is calculated by (1) in each generation. As a result, we can not just simply compare hypervolume calculated in algorithm among different generations.

We are trying on some other good indicators satisfying our convergence detection criterions. Fig. 5 shows the change of hypervolume(HV) and nadir point on the SMS-EMOA algorithm with the 3-dimensional inverted-DTLZ1 problem. When current solutions are close to the pareto front, the estimated nadir point of current solutions is close to the true nadir points. This implies that the estimated nadir point can be a good indicator of our purpose. But for some problems with the large feasible region, the moving distance of the estimated nadir point in the early generations is larger than that near convergence. So we consider the logarithm. And considering the possible bad variance of the indicator, finally, we use the best logarithmic nadir point so-far as our indicator. more specifically, for a minimization problem, we consider the indicator I as follows:

$$\begin{aligned} &\boldsymbol{N_t} = [f_{t1}, f_{t2}, \ldots, f_{tm}]^\top \in \mathbb{R}^m, \\ &I_0 = \frac{1}{m}\sum_{i=1}^{m} lnf_{0i}, \\ &I_t = min(I_{t-1}, \frac{1}{m}\sum_{i=1}^{m} lnf_{ti}), t = 1, 2, \ldots, T, \end{aligned} \quad (6)$$

where T is the total number of generations, $\boldsymbol{N_t}$ is the estimated nadir point at the t^{th} generation with m objectives: $f_{t1}, f_{t2}, \ldots, f_{tm}$. I_0 is the initial indicator calculated by the initial population. And I_t is the minimum value before the tth generation (including the tth generation).

After chosen the indicator, the next step is to detect the

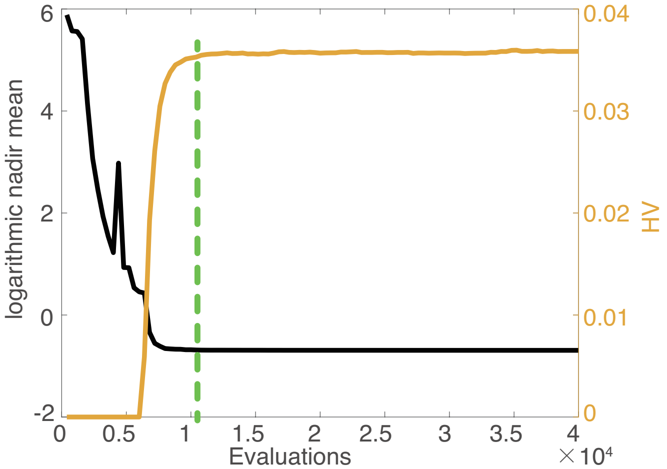


Fig. 5: Example of nadir point and hypervolume(HV) value on inverted-DTLZ1 3-dimension problem with SMS-EMOA algorithm. The yellow curve is the change of hypervolume while the black curve is the change of the logarithmic nadir point. The green dotted line shows the convergence detection.

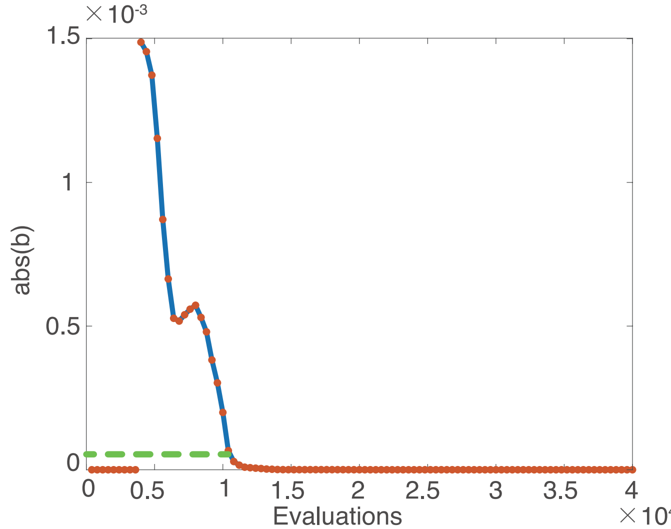


Fig. 6: Example of $|b|$ on inverted-DTLZ1 3-dimension problem (window size = 4000 evaluations with population size = 100). The green dotted line shows the threshold.

stagnation of the indicator. We use a basic linear regression method called Simple Least Squares [28] with a simple least squares convergence detection strategy introduced in [22]. If the absolute value of the slope of the indicator is below a threshold, the convergence is reported. Briefly speaking, for a simple linear regression $I = a + bt$, the intercept a and slope b of the tth generation can be calculated with the following matrix-based formula:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum t_i^2 & \sum t_i \\ \sum t_i & w\_l \end{bmatrix}^{-1} * \begin{bmatrix} \sum t_i * I_{t_i} \\ \sum I_{t_i} \end{bmatrix} \tag{7}$$

where w_l is the length of the chosen window and t_i is the evaluated number in the chosen window. The value of slope b is shown in Fig. 6 (Note that the value in the first w_l evaluations is 0, and we should not consider the first w_l evaluations).

With the above formula (7), the convergence detection criterion is defined as:

$$convergence = |b| < thres \tag{8}$$

The chosen of the $thres$ value is not so important as the report ahead or delay is not fatal to the algorithm or to the final solutions set. We choose the $thres$ value as 10^{-5} after some experimental computation with the window size $w\_l = 4000$ evaluations. If we do not change the window size, this threshold can be applied to other problems or indicator-based algorithms because of a weak convergence detection purpose.

VI. Computational Experiments

To clearly represent the superiority of dynamically reference point specification and to ease the comparison process of two different dynamic mechanisms (linearly decrease mechanism and weak convergence detection mechanism), the mechanisms are going to be tested with the state-of-the-art algorithm SMS-EMOA [7]. The problems include: DTLZ series [12], WFG series [29], their minus-versions [30] and a multi-objective distance minimization problem(MPDMP) [18]. To show the performance on many-objective, we tested in 10 dimensions. All the code in this section is implemented in PlatEMO framework [19] with the following additional settings:

Population size: 30($H = 1$),
Total evaluation number: 100,000 solution evaluations,
Crossover: simulated binary (probability: 1.0),
Mutation: polynormal (probability: $1/D$),
Number of decision variables D:
  14 (DTLZ1 and minus-DTLZ1),
  2 (MPDMP),
  19 (other problems),
Distribution index in Crossover and Mutation: 20,
Number of runs: 20 runs.

A. Computational Results

Each experiment has been run for 20 times independently. And the population size is 30, $r_{Initial} = 10$. The detailed algorithms are as following: SMS-EMOA-10(the SMS-EMOA [7] algorithm with $r = 10$), SMS-EMOA-Opt(the SMS-EMOA algorithm with $r = 1 + 1/H$), SMS-EMOA-LD(the SMS-EMOA algorithm with a linearly decrease mechanism), and SMS-EMOA-CD(the SMS-EMOA algorithm with a weak convergence detection mechanism, proposed in this paper). We obtain the hypervolume results after $100,000$ evaluations for each algorithm. The computational results of the hypervolume metric is shown in TABLE I and TABLE II. The best result in each row is highlighted in bold, and relatively, the worst result is shaded. The basic Wilcoxon signed-rank sum test is used in order to show the statistical significance for the algorithm comparing with SMS-EMOA-CD proposed in this

TABLE I: HV mean and standard deviation over 20 independent runs for DTLZ and WFG problems.

Problem	M	D	SMS-EMOA-10	SMS-EMOA-Opt	SMS-EMOA-LD	SMS-EMOA-CD
DTLZ1	10	14	6.8448e-1 (4.61e-1) ≈	2.0261e-1 (2.23e-1) −	**8.8369e-1 (2.00e-1) ≈**	6.9062e-1 (3.95e-1)
DTLZ2	10	19	**1.0234e+3 (4.72e-1) ≈**	9.9315e+2 (3.94e+1) −	1.0234e+3 (7.28e-1) ≈	1.0184e+3 (1.85e+1)
DTLZ3	10	19	0.0000e+0 (0.00e+0) ≈	**1.9068e+1 (8.53e+1) ≈**	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0)
DTLZ4	10	19	**8.0029e+2 (2.21e+2) ≈**	5.2691e+2 (1.79e+2) −	6.6262e+2 (2.17e+2) ≈	7.2840e+2 (2.26e+2)
WFG1	10	19	2.2025e+12 (2.55e+11) ≈	2.2597e+12 (2.64e+11) ≈	2.2005e+12 (1.81e+11) ≈	**2.2601e+12 (3.42e+11)**
WFG2	10	19	**3.3604e+12 (2.92e+10) ≈**	3.3493e+12 (2.39e+10) +	3.3487e+12 (7.64e+10) ≈	3.3420e+12 (8.46e+10)
WFG3	10	19	4.7217e-3 (1.04e-2) −	**2.5310e-2 (3.07e-2) ≈**	1.7230e-2 (2.35e-2) ≈	2.3443e-2 (2.84e-2)
WFG4	10	19	3.7667e+12 (1.77e+10) ≈	3.5602e+12 (9.48e+10) −	3.7585e+12 (3.74e+10) ≈	**3.7737e+12 (1.63e+10)**
WFG5	10	19	3.6535e+12 (7.68e+9) ≈	3.5027e+12 (1.16e+11) −	**3.6555e+12 (8.64e+9) ≈**	3.6523e+12 (1.93e+10)
WFG6	10	19	3.6082e+12 (7.78e+10) ≈	3.5432e+12 (6.24e+10) −	3.5829e+12 (5.75e+10) ≈	**3.6099e+12 (4.49e+10)**
WFG7	10	19	**3.7967e+12 (7.58e+9) ≈**	3.7326e+12 (5.73e+10) −	3.7959e+12 (1.35e+10) ≈	3.7922e+12 (1.27e+10)
WFG8	10	19	3.7512e+12 (1.51e+10) ≈	3.7087e+12 (3.64e+10) −	**3.7613e+12 (1.13e+10) +**	3.7524e+12 (1.23e+10)
WFG9	10	19	3.5727e+12 (1.85e+11) ≈	3.3072e+12 (2.80e+11) −	**3.6302e+12 (1.57e+11) +**	3.5146e+12 (2.31e+11)
$+/-/\approx$			0/1/12	1/9/3	2/0/11	

TABLE II: HV mean and standard deviation over 20 independent runs for corresponding minus-versions and MPMP.

Problem	M	D	SMS-EMOA-10	SMS-EMOA-Opt	SMS-EMOA-LD	SMS-EMOA-CD
minus-DTLZ1	10	14	4.3889e+28 (5.56e+26) ≈	**4.4237e+28 (5.99e+26) ≈**	4.4120e+28 (4.79e+26) ≈	4.3872e+28 (7.72e+26)
minus-DTLZ2	10	19	1.2299e+7 (1.38e+5) −	1.4737e+7 (1.55e+5) −	**1.4891e+7 (1.11e+5) ≈**	1.4844e+7 (1.45e+5)
minus-DTLZ3	10	19	1.1227e+35 (3.16e+33) −	1.3385e+35 (3.41e+33) ≈	**1.3702e+35 (3.42e+33) ≈**	1.3582e+35 (3.78e+33)
minus-DTLZ4	10	19	1.1638e+7 (2.11e+5) −	**1.4899e+7 (1.20e+5) +**	1.4895e+7 (1.27e+5) +	1.3585e+7 (1.57e+6)
minus-WFG1	10	19	4.4808e+10 (5.25e+8) ≈	**4.4894e+10 (4.81e+8) ≈**	4.4890e+10 (4.35e+8) ≈	4.4705e+10 (5.60e+8)
minus-WFG2	10	19	6.6225e+10 (1.05e+8) −	6.7628e+10 (7.51e+8) ≈	6.7870e+10 (3.31e+7) −	**6.7907e+10 (1.60e+7)**
minus-WFG3	10	19	6.3620e+10 (5.49e+8) ≈	**6.4321e+10 (2.29e+8) +**	6.3642e+10 (4.93e+8) ≈	6.3799e+10 (5.87e+8)
minus-WFG4	10	19	1.6478e+11 (3.22e+9) −	**1.9896e+11 (2.20e+9) ≈**	1.9752e+11 (2.39e+9) ≈	1.9260e+11 (1.34e+10)
minus-WFG5	10	19	1.6534e+11 (2.09e+9) −	1.9817e+11 (1.82e+9) ≈	**1.9897e+11 (1.93e+9) ≈**	1.9819e+11 (1.40e+9)
minus-WFG6	10	19	1.6532e+11 (2.11e+9) −	1.9865e+11 (1.83e+9) ≈	**1.9989e+11 (2.03e+9) ≈**	1.9960e+11 (1.94e+9)
minus-WFG7	10	19	1.6534e+11 (2.88e+9) −	1.9870e+11 (1.30e+9) −	**1.9978e+11 (2.21e+9) ≈**	1.9937e+11 (2.97e+9)
minus-WFG8	10	19	1.6808e+11 (1.46e+9) −	1.9932e+11 (1.56e+9) −	**2.0220e+11 (1.71e+9) +**	2.0065e+11 (1.77e+9)
minus-WFG9	10	19	1.6415e+11 (3.16e+9) −	1.9777e+11 (2.21e+9) ≈	1.9557e+11 (2.17e+9) −	**1.9852e+11 (1.86e+9)**
MPDMP	10	2	1.4848e+5 (2.50e+2) −	1.5051e+5 (3.10e+2) −	1.5017e+5 (1.98e+2) −	**1.5097e+5 (1.64e+2)**
$+/-/\approx$			0/11/3	2/4/8	2/3/9	

paper. The three symbols "+", "−", "≈" mean significantly better, significantly worse and no significant difference.

In the basic test problems(DTLZ1-4, WFG1-9), SMS-EMOA-Opt performs the worst (9 out of 13 significantly worse than SMS-EMOA-CD) among the algorithms in TABLE I. This phenomenon clearly show the bad searching behavior when applying $r = 1 + 1/H$ all the time, which is discussed in Section III. As for SMS-EMOA-10, we can not tell the differences with SMS-EMOA-CD, that the Wilcoxon signed-rank sum tests show almost all the results except one are "≈". This is because that the influence of reference point on the triangular pareto front problems is small as discussed in [15]. 2 better results from SMS-EMOA-LD obtained indicate that SMS-EMOA-LD is slightly better than SMS-EMOA-CD on triangular pareto front problems.

In the results of inverted-triangular pareto front problems (minus-DTLZ1-4, minus-WFG1-9 and MPDMP), SMS-EMOA-10 performs the worst(12 out of 14) in all experiments in average, and the Wilcoxon signed-rank sum tests show that almost all the results from SMS-EMOA-10 are significantly worse than SMS-EMOA-CD (11 out of 14 worse and no better results). The reason is that the r is 10 all the process of SMS-EMOA-10 algorithm and many solutions are at the boundary of pareto front. Comparing with SMS-EMOA-CD, SMS-EMOA-Opt(2 better but 4 worse) and SMS-EMOA-LD(2 better but 3 worse) are slightly worse on inverted-triangular pareto front problems. The result of MPDMP shows that SMS-EMOA-CD is better among the four algorithms on some specific problems.

So we plot the runtime hypervolume graph of 4 mechanisms (as shown in Fig. 7) on the 10-dimensional MPDMP problem. In Fig. 7, the hypervolume of SMS-EMOA-LD (the red curve) is gradually increased and finally reaches to the same level as SMS-EMOA-Opt, as the value of r is gradually decrease to $1 + 1/H$. The hypervolume of SMS-EMOA-CD (the yellow curve) firstly reaches to a stable level similar to SMS-EMOA-10, for that their values of r are both 10 before 4,500 evaluations. Then after reporting the convergence detection at about 4,500 evaluations, the value of r in SMS-EMOA-CD reaches to the optimal value $1 + 1/H$ and the hypervolume increases. The reason are that the boundary solutions decrease, and on the other hand, the inner solutions increase. Finally, the hypervolume of SMS-EMOA-CD is better than SMS-EMOA-Opt and SMS-EMOA-LD.

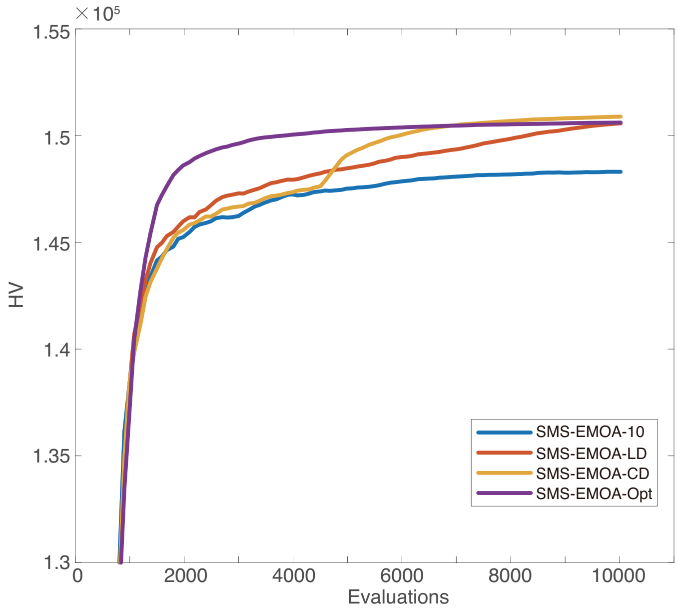


Fig. 7: The comparison of 4 mechanisms on hypervolume graph of solutions in each evaluation. The value of each point is the average of 20 individual runs in each evaluation and the problem is MPDMP as shown in TABLE I.

VII. Conclusions

In this paper, we emphasize the importance of reference point specification in indicator-based EMOA by a simple example. Without a good reference point specification mechanism, the diversity of the final solutions will be poor on inverted-shape pareto front problems. This phenomenon will not be observed on the triangular pareto front problems when reference point is worse than nadir point. Then we state the dynamic reference point specification mechanism with the illustration by two aspects:

1) Better Searching Behavior: In the early stage, the solutions may not be close to the true pareto front. A larger value of r can achieve a better searching behavior. We give an example of the DTLZ2 problem.

2) Optimal Distribution: Considering the flat triangular problems, the optimal setting of reference point is $r = 1 + 1/H$.

We summarize the whole process of dynamic reference point specification mechanism as that, the value of r should be specified larger than $1+1/H$ at first and be equal to $1+1/H$ in the end, as in formula 3-5.

After that, a new dynamic reference point adaptation mechanism is proposed in this paper. A weak convergence detection mechanism is used. We apply our new dynamic mechanism on many test problems containing triangular, inverted-triangular pareto fronts problems. The results show that SMS-EMOA with $r = 1 + 1/H$ performs the worst on the problems with triangular pareto front, and SMS-EMOA with $r = 10$ performs the worst on the problems with inverted-triangular pareto fronts, which hint the superiority of dynamic mechanism. SMS-EMOA with weak convergence detection mechanism performs better or worse than SMS-EMOA with linearly decrease mechanism.

We compare our new mechanism with the proposed linearly decrease mechanism and find that on some problems, specifically the multi-objective distance minimization problems, our weak convergence detection mechanism outperforms the linear decrease mechanism.

In the future, we plan to further investigate the behavior of our new mechanism. A larger dimensionality of MaOPs should be considered and the problems with different pareto front shapes should be tested and analyzed detailedly. Our weak convergence detection mechanism can also be further improved.

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