Weak Convergence Detection-based Dynamic Reference Point Specification in SMS-EMOA

Weiduo Liao, Ke Shang, Lie Meng Pang, Hisao Ishibuchi
Shenzhen Key Laboratory of Computational Intelligence,
University Key Laboratory of Evolving Intelligent Systems of Guangdong Province,
Department of Computer Science and Engineering,
Southern University of Science and Technology (SUSTech), Shenzhen, China
Email: 11849249@mail.sustech.edu.cn; kshang@foxmail.com; panglm@sustech.edu.cn; hisao@sustech.edu.cn

***Abstract*—In evolutionary multi-objective optimization (EMO) field, the hypervolume (HV) indicator is one of the most popular performance indicators. It is not only used for performance evaluation of EMO algorithms (EMOAs) but also adopted in EMOAs for selection (e.g., SMS-EMOA). However, the specification of the reference point has a large effect on the performance of SMS-EMOA. Thus, the reference point specification should be carefully treated in SMS-EMOA. In this paper, the importance of the dynamic reference point specification in SMS-EMOA is explained first, then a new dynamic reference point specification mechanism based on weak convergence detection is introduced for SMS-EMOA. Experimental comparisons are conducted among SMS-EMOA with the proposed mechanism, a linearly decreasing mechanism and two static mechanisms. Our results demonstrate the effectiveness of the proposed dynamic reference point specification mechanism.**

***Keywords*—reference point specification; SMS-EMOA; hypervolume; evolutionary multi-objective optimization; dynamic mechanism; convergence detection**

I. Introduction

In the field of Evolutionary Multi-objective Optimization Algorithms (EMOAs), one of the active research areas is the development of performance indicators. Over the years, various indicators have been proposed. These indicators include hypervolume (HV) [1], R2 [2], ϵ_+ indicator [3], IGD [4] etc. They are designed for different purposes and each of them has its own advantages and disadvantages. Different from IGD, HV does not need the pre-knowledge of the shape of the Pareto front (PF) and is the only Pareto-compliant indicator up to now [5]. However, due to the heavy computation load of HV computation [6], the HV-based algorithms are inefficient when dealing with Many-Objective Optimization Problems (MaOPs) which have more than three objectives.

SMS-EMOA [7] is a classical HV-based algorithm. The HV contribution is used to determine which solution to be discarded in the population. To reduce the heavy computation load of HV computation, many new indicators or new methods have been proposed to estimate the HV. For example, HypE uses a Monte Carlo simulation technology to estimate the HV [8]; R2 indicator estimates the HV by a standard weighted Tchebycheff function [2]; an improved new R2 is proposed by Shang et al. [9] to approximate the HV. In 2014, a simple and fast version of SMS-EMOA, known as FV-EMOA, has been proposed [10] to further increase the efficiency. FV-EMOA considers the fact that the HV contribution of a solution is only determined by partial solutions rather than the whole solution set [10]. Recently, an improved SMS-EMOA with adaptive resource allocation has been proposed to reduce the number of HV calculations [11].

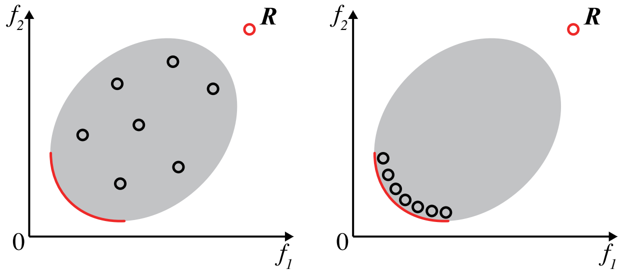
The specification of the reference point is one of the important but easy to be ignored issues in HV computation. It has been reported that the position of the reference point strongly influences the hypervolume optimal distributions on the inverted-triangular PFs [12]–[14]. Thus, a reference point specification method is proposed in [13] for fair HV computation. A dynamic reference point specification mechanism has also been proposed in [15]. Another proposed strategy is to use two reference points in HV-based EMOA [16].

In this paper, we analyzed the reference point specification during the algorithm process. At the early stage of the algorithm process, the reference point should be set far away from the estimated nadir point to enhance the searching diversity of the algorithm. In the final stage of the algorithm process, the reference point should be specified properly to achieve a uniformly distributed solution set on the PF.

Based on our analysis, we propose a new dynamic reference point specification mechanism based on weak convergence detection in SMS-EMOA. In this weak convergence detection mechanism, we use the logarithm nadir point as the convergence indicator and use the Simple Least Squares [17] to detect the convergence. The comparison among two dynamic mechanisms and the simple reference point specification (without dynamic mechanism) is presented in the experiment section. On some specific problems (e.g., the multi-objective distance minimization problems [18]), our weak convergence detection mechanism outperforms the linearly decreasing mechanism.

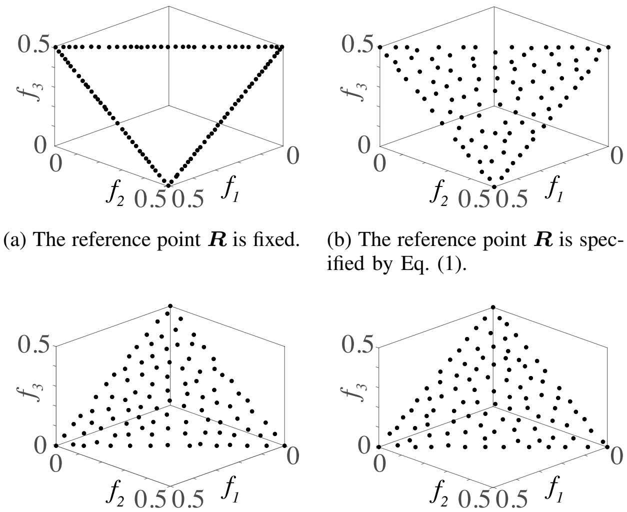
The remainder of this paper is organized as follows. Section II introduce the original reference point specification in SMS-EMOA and the one proposed in [12]. Then, we explore the

This work was supported by National Natural Science Foundation of China (Grant No. 61876075), the Program for Guangdong Introducing Innovative and Enterpreneurial Teams (Grant No. 2017ZT07X386), Shenzhen Peacock Plan (Grant No. KQTD2016112514355531), the Science and Technology Innovation Committee Foundation of Shenzhen (Grant No. ZDSYS201703031748284), and the Program for University Key Laboratory of Guangdong Province (Grant No. 2017KSYS008). Corresponding Author: Hisao Ishibuchi (hisao@sustc.edu.cn)



(a) The initial generation. (b) The final generation.

Fig. 1: The reference point is set with a large feasible region in the objective space. The PF can be far away from the reference point. The gray region shows the feasible region and the red arc is the corresponding PF. The red circle $\boldsymbol{R}$ is the reference point calculated by the initial solutions in (a) which are randomly generated. After some generations, the current solutions reach the seven black circles in (b), which is far away from the reference point.



(a) The reference point $\boldsymbol{R}$ is fixed. (b) The reference point $\boldsymbol{R}$ is specified by Eq. (1).

(c) The reference point $\boldsymbol{R}$ is fixed. (d) The reference point $\boldsymbol{R}$ is specified by Eq. (1).

Fig. 2: The final distribution of the solution set on inverted-DTLZ1((a) and (b)) and DTLZ1 ((c) and (d)). The algorithm is SMS-EMOA with population size $\mu = 100$ and total evaluation number $= 20,000$. (a) and (c): the reference point is calculated only once at the initial step; (b) and (d): the reference point is specified by Eq. (1) with $r = 1.1$. All the solutions in the final distribution are on the boundary of the PF in (a), which shows the bad effect of a faraway reference point on the final distribution of inverted-triangular problems. This bad effect can not be observed on triangular problems in (c).

details of the reference point specification mechanism in two stages of the algorithm, which elicit the necessity of using the dynamic reference point specification mechanism. After that, we state the details of the dynamic reference point specification mechanism and introduce a linearly decreasing mechanism in Section III. The details of the new dynamic mechanism with weak convergence detection proposed in this paper are also presented in Section III. The experiments comparing SMS-EMOA with different reference point specification mechanisms on 10-objective triangular and inverted-triangular problems are presented in Section IV. Finally, the conclusion is drawn in Section V.

II. Reference Point Specification in SMS-EMOA

When HV is used in SMS-EMOA, one important thing to be considered is how to specify the reference point. Before calculating the HV value, a reference point needs to be specified in advance. However, it is not suggested that the reference point is set only once at the beginning of the algorithm process. This may cause a very faraway reference point for those problems with a very large feasible region in the objective space (as illustrated in Fig. 1), as the solution set is gradually converging to the PF.

There is an issue when applying this strategy to some problems with special PF shapes. For example, the inverted-DTLZ1 [19] has an inverted-triangular PF, many solutions in the final solution set will distribute on the boundary of the PF (as shown in Fig. 2a) [13]–[15]. Although the position of the reference point does not affect the distributions of the solution set on triangular PF (as shown in Fig. 2c and 2d), it is necessary to use a proper reference point specification during the algorithm run. The reason is explained in detail in [14].

A. Original Reference Point Specification

In the original paper of SMS-EMOA [7], the reference point is specified as the estimated nadir point increased by 1.0 (in two-objective case, a sufficiently large reference point is chosen). However in the current implementation of SMS-EMOA in PlatEMO [20], the reference point is specified as:

$$\boldsymbol{R} = r \cdot \boldsymbol{N}, \tag{1}$$

where $\boldsymbol{R} \in \mathbb{R}^m$ is the reference point in each generation, and $\boldsymbol{N} \in \mathbb{R}^m$ is the estimated nadir point of the last front of the current population. r is specified as 1.1 in PlatEMO. In the source code of PlatEMO, the values of r in HV-based algorithms are specified as follow: 1.1 (SMS-EMOA [7]) and 1.2 (HypE [8]). In the process of SMS-EMOA, we use the HV contribution to evaluate each solution. The reference point used to calculate the HV is specified as in Eq. (1).

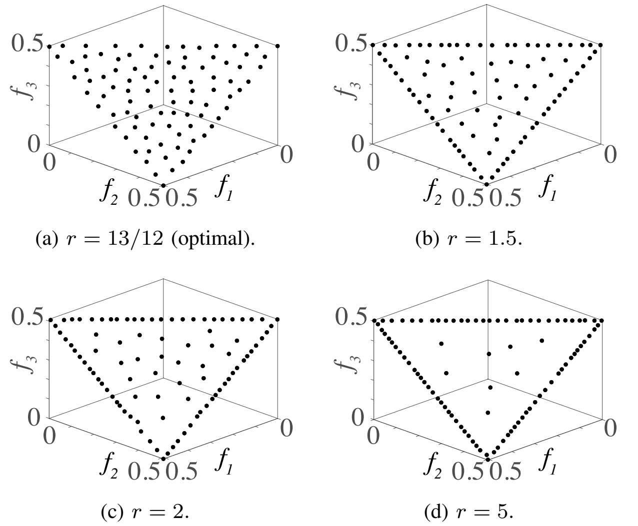
However, fixing the r value to 1.1 is not recommended, especially on problems with the inverted-triangular PF [12]. The research on specifying the value of r is limited, as the effect of the location of the reference point on the PF is not fatal on some benchmark problems (e.g., triangular PF problems).

B. Reference Point Specification Proposed in [12]

In [12]–[14], the suggested value of r is investigated on linear PF problems. Specifically, on inverted-triangular problems (e.g., inverted-DTLZ1 [14]), the suggested value of r is:

$$r = 1 + \frac{1}{H}, \tag{2}$$

where H is a parameter used in MOEA/D [21] for generating uniformly distributed weight vectors [15]. Given the population size μ and the objective number m, the value of H can



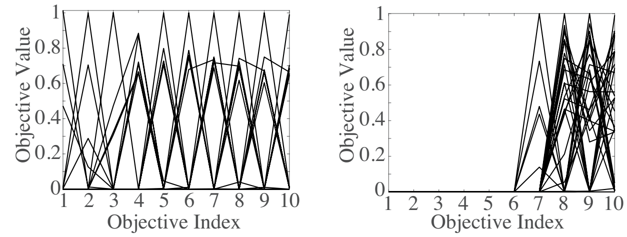
(a) $r = 13/12$ (optimal).

(b) $r = 1.5$.

(c) $r = 2$.

(d) $r = 5$.

Fig. 3: The final distribution of the solution set on inverted-DTLZ1. The algorithm is SMS-EMOA with population size $\mu = 91$ ($H = 12$) and total evaluation number $= 20,000$. $r = 13/12$ is the optimal setting and we observed a uniform distribution in (a). With the increase of r, more and more solutions move to the boundary of the PF ((b)-(d)).



(a) $r = 10$.

(b) $r = 2$ (optimal).

Fig. 4: The final solution distribution of SMS-EMOA on the 10-objective DTLZ2 with different settings of r. The population size $\mu = 30$ ($H = 1$), the total evaluation number is 10000.

be calculated as follows:

$$C_{m-1}^{H+m-1} \leq \mu < C_{m-1}^{H+m}. \quad (3)$$

In Fig. 3a, $r = 13/12 (H = 12)$ is the optimal setting for inverted-DTLZ1 with $\mu = 91$ and $m = 3$, and the solution set is uniformly distributed on the PF. In Fig. 3b - 3d, the number of the inner solutions is decreased and many solutions move to the boundary of the PF with the increase of r.

Basically, the process of EMOAs can be divided into two stages:

1) Early Stage: In this stage, all the solutions are far away from the PF. The main task is to converge the solutions to the PF. We also call this stage the convergence stage.

2) Final Stage: In this stage, all the solutions are inside or near the PF. So the main task is to distribute the solutions more evenly on the PF. We also call this stage the diversity stage.

C. Specify the Value of r for Better Searching Behavior

Fig. 4 shows the final solution distribution of SMS-EMOA on the 10-objective DTLZ2 with different settings of r. It can be observed that when r is set to 2, the solutions for the first six dimensions in Fig. 4b are all 0. This phenomenon shows that the breadth-diversity [22] in Fig. 4b is worse than that in Fig. 4a, although $r = 2$ is the suggested optimal value for r in this experimental setting.

If we set $r = 1 + 1/H$ at the early stage of the algorithm, the exploration of solutions will be poor. So, a larger value of r than $1 + 1/H$ is suggested in the early stage [15].

D. Specify the Value of r for Uniform Distribution

When the algorithm reaches the final stage, all solutions are near the PF. To get a uniform solution distribution, r should be specified as $1 + 1/H$, as in Eq. (2). On inverted-triangular problems, the distribution of solutions on the PF strongly depends on the value of r (as shown in Fig. 3).

The sensitivity of the r value on the solutions distribution is also observed in some real-world problems, for example, distance minimization problems. This observation shows the potential use of the dynamic reference point specification [15].

III. Dynamic Reference Point Specification Mechanism

For different purposes in the two stages, the value of r should be treated differently [15]. Not only the reference point but also the value of r needs to be adapted in each iteration of the algorithm. This is called dynamic reference point specification. Based on Eq. (1), we define the dynamic reference point specification as:

$$\boldsymbol{R} = r(t) \cdot \boldsymbol{N}, t = 0, 1, \ldots, T, \quad (4)$$

where T is the total number of generations, and $r = r(t)$ is a function of the current generation t. The value of r is adapted during the process of the algorithm.

A. Linearly Decreasing Mechanism Proposed in [15]

Based on the description above, r is suggested to be specified dynamically at different stages of the algorithm (at the early stage, a larger r is specified; at the final stage, $r = 1 + 1/H$ is specified).

In [15], a linearly decreasing mechanism has been proposed:

$$r(t) = r_{Initial}\frac{(T-t)}{T} + (1 + 1/H)\frac{t}{T}, t = 0, 1, \ldots, T, \quad (5)$$

where T is the total number of generations, and $r_{Initial}$ is the initial value of r, which is larger than $1 + 1/H$. It is a simple and practical mechanism. In Eq. (5), the value of r starts from $r_{Initial}$, then gradually decreases to the suggested value in a linearly decreasing manner.

In the next section, another dynamic mechanism based on weak convergence detection criterion is proposed. We show that it outperforms the simple linearly decreasing mechanism on some specific problems.

B. A New Dynamic Reference Point Specification Mechanism

In this section, we will introduce a new mechanism that uses weak convergence detection criterion to decide when to change the value of r from $r_{Initial}$ to $1+1/H$.

As we have explained before, a larger r is suggested at the early stage of the algorithms. But for good diversity at the final stage, it is needed to set r to its suggested value $1+1/H$. For this purpose, it is necessary to detect whether the algorithm is converged. If solutions are all close to the PF, we change the value of r to $1+1/H$; otherwise, we set the value of r to $r_{Initial}$. The mechanism is shown below:

$$r(t)=\begin{cases} r_{Initial}, t<t_{Converged} \\ 1+1/H, t\geq t_{Converged} \end{cases} \quad t=0,1,\ldots,T. \tag{6}$$

$r(t)$ equals to $r_{Initial}$ before reaching the converged generation $t_{Converged}$, and changes to $1+1/H$ after $t_{Converged}$. $t_{Converged}$ is determined by weak convergence detection criterion.

Various indicators including convergence detection indicators which are used to detect the stagnation of the algorithm have been proposed in the literature [23]–[29]. They focus on accurately detecting the convergence, which is not the purpose in our approach. After algorithm converged, we still need some generations in order to get a uniform distribution of solution set. We summarize our weak convergence detection criterion as follow:

1) Inaccuracy: It is not necessary to have an accurate convergence detection. The convergence can be reported if the current solutions are close to the PF. In other words, the estimated ideal and nadir points based on the current solutions are close to the true ideal and nadir points.

2) Time-saving: We should not spend too much time on convergence detection. The state-of-the-art HV-based algorithms (e.g., SMS-EMOA and HypE) are time-consuming when the dimension of the objective space is very high.

HV is not a good choice as our weak convergence detection indicator. The reason is that during the process of the algorithm, the reference point is calculated by Eq. (1), which means that they are different among generations. So, we can not simply compare HV calculated in the algorithm among different generations.

We should consider other indicators that satisfying our convergence detection criterion. In the process of the algorithm, the HV of the current solution set increases while the estimated nadir point of the current solution set is gradually approaching the PF. The estimated nadir point can be a good choice for our purpose. More specifically, for a minimization problem, we consider the indicator I as follows:

$$\begin{aligned} &\boldsymbol{N_t}=[f_{t1},f_{t2},\ldots,f_{tm}]^\top\in\mathbb{R}^m, \\ &I_0=\frac{1}{m}\sum_{i=1}^{m} lnf_{0i}, \\ &I_t=min(I_{t-1},\frac{1}{m}\sum_{i=1}^{m} lnf_{ti}),t=1,2,\ldots,T, \end{aligned} \tag{7}$$

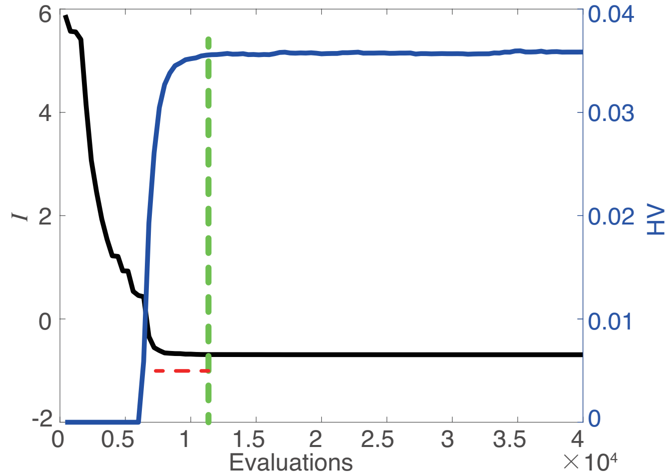


Fig. 5: Example of nadir point and HV value on the 3-objective inverted-DTLZ1 with SMS-EMOA. The blue curve is the change of HV while the black curve is the change of the indicator I: best logarithmic nadir point so-far. The green dotted line shows the evaluation where the convergence is detected and the red dotted line shows the considered window for evaluated number $t_{Converged}$.

where T is the total number of generations, $\boldsymbol{N_t}$ is the estimated nadir point at the t^{th} generation with m elements: $f_{t1},f_{t2},\ldots,f_{tm}$. I_0 is the average value of logarithmic nadir point calculated by the initial population. And I_t is the minimum value of I before the t^{th} generation (including the t^{th} generation). Fig. 5 shows the change of HV and indicator I of SMS-EMOA on 3-objective inverted-DTLZ1. When current solutions are close to the PF, the estimated nadir point of the current solutions is close to the true nadir point.

After choosing the indicator, the next step is to detect the stagnation of the indicator. We use a basic linear regression method called Simple Least Squares [17] with a simple least squares convergence detection strategy introduced in [24]. If the absolute value of the slope of the linear regression is below a threshold, the convergence is reported. Briefly speaking, considering a simple linear regression model $I(t)=a+bt$, the intercept a and slope b of the t^{th} generation can be calculated with the following matrix-based formula:

$$\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} \Sigma t_i^2 & \Sigma t_i \\ \Sigma t_i & w\_l \end{bmatrix}^{-1} * \begin{bmatrix} \Sigma t_i * I_{t_i} \\ \Sigma I_{t_i} \end{bmatrix}, \tag{8}$$

where w_l is the length of the chosen window (in the example of Fig. 5, the chosen window for $t_{Converged}$ is represented with the red dotted line), and t_i is the evaluation number in the chosen window, $t_i\in(t',t], t-t'=w_l$. The absolute value of slope b is shown in Fig. 6 (Note that in the first w_l evaluations is 0, and we should not consider the first w_l evaluations).

Using Eq. (8), we report the convergence of the algorithm if the following condition holds:

$$|b|<thres. \tag{9}$$

The value of $t_{Converged}$ equals to t (i.e., the current evaluation

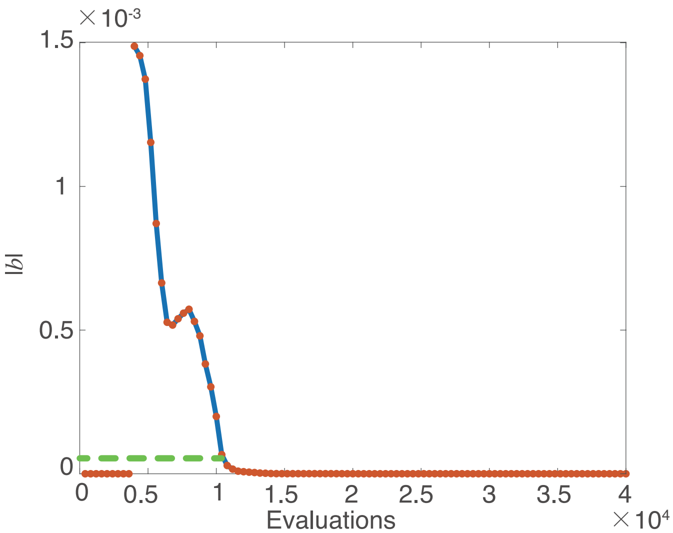


Fig. 6: Example of $|b|$ on the 3-objective inverted-DTLZ1 (window size $w\_l = 4000$ and population size $\mu = 100$). The green dotted line shows the threshold.

number) when the convergence is detected. The whole process of weak convergence detection is also described in Algorithm 1. If Algorithm 1 returns True, we report the convergence, vice versa. The choice of the $thres$ value is trivial as the report ahead or delay is not fatal to the algorithm or the final solution set. We choose $thres$ value as 10^{-5} after some experimental computation with the window size $w\_l = 4000$.

Algorithm 1: Weak Convergence Detection

Input:
 w_l, // Window size.
 $\boldsymbol{S_t}$, // Solution set when evaluaiton number is t.
 $\boldsymbol{I} = \{I_0, I_1, \ldots, I_{t-1}\}$, // Stored indicator values.
 $thres$, // The chosen threshold for slope.
Output: Return True if converged, otherwise return False.
 Calculate nadir point $\boldsymbol{N_t}$ of $\boldsymbol{S_t}$;
 Calculate indicator I_t; // By Eq. (7).
 $\boldsymbol{I} \leftarrow \boldsymbol{I} \cup \{I_t\}$;
 if $t \geq w\_l$ **then**
  Calculate b; // By Eq. (8).
  if $|b| < thres$ **then**
   Return True; // Converged.
  end if
 end if
 Return False; // Not converged.

IV. Computational Experiments

A. Experimental Settings

In this section, the two different dynamic reference point specification mechanism (i.e., the linearly decreasing mechanism and the weak convergence detection mechanism) are tested with the state-of-the-art algorithm SMS-EMOA [7]. The DTLZ test suite [30], WFG test suite [31], their minus-versions [32], and Multi-Point Distance Minimization Problem (MPDMP) [18] are used in this experiment. We consider the problems with 10 objectives. All the code in this section is implemented in PlatEMO framework [20] with the following settings:

Population size: 30 ($H = 1$),
Total evaluation number: 100,000 solution evaluations,
Initial value of r ($r_{Initial}$): 10,
Crossover: Simulated binary crossover,
Crossover probability: 1.0,
Mutation: Polynomial mutation,
Mutation probability: 1/D,
Distribution index of Crossover and Mutation: 20,
Number of decision variables D:
 14 (DTLZ1 and minus-DTLZ1),
 2 (MPDMP),
 19 (other problems),
Number of runs: 20 runs.

B. Computational Results

In our experiments, four versions of SMS-EMOA algorithm with different reference point specification mechanisms are considered. These algorithms are named as SMS-EMOA-10, SMS-EMOA-Opt, SMS-EMOA-LD, and SMS-EMOA-CD. For SMS-EMOA-10, the value of r is set to 10. The value of r for SMS-EMOA-Opt is set to $r = 1 + 1/H$. SMS-EMOA with the linearly decreasing mechanism and with the weak convergence detection mechanism are referred to as SMS-EMOA-LD and SMS-EMOA-CD, respectively. The HV values after 100,000 evaluations are obtained for each algorithm. Table I and Table II show the computational results for HV metrics. The best result in each row is highlighted in bold, and the worst result is shaded. The Wilcoxon rank sum test is used to show the statistical significance for SMS-EMOA-10, SMS-EMOS-Opt, SMS-EMOA-LD in comparison to our proposed SMS-EMOA-CD. The three symbols "+", "−", "≈" mean significantly better, significantly worse and no significant difference.

In Table I, the results show that SMS-EMOA-Opt performs the worst (9 out of 13 significantly worse than SMS-EMOA-CD) among the algorithms. This result shows that when applying $r = 1 + 1/H$ all the time, bad searching behavior is obtained. We can not tell the differences between SMS-EMOA-10 and SMS-EMOA-CD, as the Wilcoxon rank sum tests show almost all the results are "≈". This is probably because of the small influence of the reference point on the triangular PF problems. As for SMS-EMOA-LD, two better results are obtained when compared to SMS-EMOA-CD. This indicates that SMS-EMOA-LD is slightly better than SMS-EMOA-CD on triangular PF problems.

Table II shows the results obtained for inverted-triangular PF problems (i.e., minus-DTLZ1-4, minus-WFG1-9, and MPDMP). The results show that SMS-EMOA-10 performs the worst (12 out of 14) on most of the inverted-triangular PF problems, and the Wilcoxon rank sum tests show that

TABLE I: MEAN HV AND STANDARD DEVIATION OVER 20 INDEPENDENT RUNS FOR DTLZ AND WFG TEST PROBLEMS.

Problem	M	D	SMS-EMOA-10	SMS-EMOA-Opt	SMS-EMOA-LD	SMS-EMOA-CD
DTLZ1	10	14	6.8448e-1 (4.61e-1) ≈	2.0261e-1 (2.23e-1) −	**8.8369e-1 (2.00e-1) ≈**	6.9062e-1 (3.95e-1)
DTLZ2	10	19	**1.0234e+3 (4.72e-1) ≈**	9.9315e+2 (3.94e+1) −	1.0234e+3 (7.28e-1) ≈	1.0184e+3 (1.85e+1)
DTLZ3	10	19	0.0000e+0 (0.00e+0) ≈	**1.9068e+1 (8.53e+1) ≈**	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0)
DTLZ4	10	19	**8.0029e+2 (2.21e+2) ≈**	5.2691e+2 (1.79e+2) −	6.6262e+2 (2.17e+2) ≈	7.2840e+2 (2.26e+2)
WFG1	10	19	2.2025e+12 (2.55e+11) ≈	2.2597e+12 (2.64e+11) ≈	2.2005e+12 (1.81e+11) ≈	**2.2601e+12 (3.42e+11)**
WFG2	10	19	**3.3604e+12 (2.92e+10) ≈**	3.3493e+12 (2.39e+10) +	3.3487e+12 (7.64e+10) ≈	3.3420e+12 (8.46e+10)
WFG3	10	19	4.7217e-3 (1.04e-2) −	**2.5310e-2 (3.07e-2) ≈**	1.7230e-2 (2.35e-2) ≈	2.3443e-2 (2.84e-2)
WFG4	10	19	3.7667e+12 (1.77e+10) ≈	3.5602e+12 (9.48e+10) −	3.7585e+12 (3.74e+10) ≈	**3.7737e+12 (1.63e+10)**
WFG5	10	19	3.6535e+12 (7.68e+9) ≈	3.5027e+12 (1.16e+11) −	**3.6555e+12 (8.64e+9) ≈**	3.6523e+12 (1.93e+10)
WFG6	10	19	3.6082e+12 (7.78e+10) ≈	3.5432e+12 (6.24e+10) −	3.5829e+12 (5.75e+10) ≈	**3.6099e+12 (4.49e+10)**
WFG7	10	19	**3.7967e+12 (7.58e+9) ≈**	3.7326e+12 (5.73e+10) −	3.7959e+12 (1.35e+10) ≈	3.7922e+12 (1.27e+10)
WFG8	10	19	3.7512e+12 (1.51e+10) ≈	3.7087e+12 (3.64e+10) −	**3.7613e+12 (1.13e+10) +**	3.7524e+12 (1.23e+10)
WFG9	10	19	3.5727e+12 (1.85e+11) ≈	3.3072e+12 (2.80e+11) −	**3.6302e+12 (1.57e+11) +**	3.5146e+12 (2.31e+11)
+/ − / ≈			0/1/12	1/9/3	2/0/11	

TABLE II: MEAN HV AND STANDARD DEVIATION OVER 20 INDEPENDENT RUNS FOR MINUS-DTLZ, MINUS-WFG AND MPDMP.

Problem	M	D	SMS-EMOA-10	SMS-EMOA-Opt	SMS-EMOA-LD	SMS-EMOA-CD
minus-DTLZ1	10	14	4.3889e+28 (5.56e+26) ≈	**4.4237e+28 (5.99e+26) ≈**	4.4120e+28 (4.79e+26) ≈	4.3872e+28 (7.72e+26)
minus-DTLZ2	10	19	1.2299e+7 (1.38e+5) −	1.4737e+7 (1.55e+5) −	**1.4891e+7 (1.11e+5) ≈**	1.4844e+7 (1.45e+5)
minus-DTLZ3	10	19	1.1227e+35 (3.16e+33) −	1.3385e+35 (3.41e+33) ≈	**1.3702e+35 (3.42e+33) ≈**	1.3582e+35 (3.78e+33)
minus-DTLZ4	10	19	1.1638e+7 (2.11e+5) −	**1.4899e+7 (1.20e+5) +**	1.4895e+7 (1.27e+5) +	1.3585e+7 (1.57e+6)
minus-WFG1	10	19	4.4808e+10 (5.25e+8) ≈	**4.4894e+10 (4.81e+8) ≈**	4.4890e+10 (4.35e+8) ≈	4.4705e+10 (5.60e+8)
minus-WFG2	10	19	6.6225e+10 (1.05e+8) −	6.7628e+10 (7.51e+8) ≈	6.7870e+10 (3.31e+7) −	**6.7907e+10 (1.60e+7)**
minus-WFG3	10	19	6.3620e+10 (5.49e+8) ≈	**6.4321e+10 (2.29e+8) +**	6.3642e+10 (4.93e+8) ≈	6.3799e+10 (5.87e+8)
minus-WFG4	10	19	1.6478e+11 (3.22e+9) −	**1.9896e+11 (2.20e+9) ≈**	1.9752e+11 (2.39e+9) ≈	1.9260e+11 (1.34e+10)
minus-WFG5	10	19	1.6534e+11 (2.09e+9) −	1.9817e+11 (1.82e+9) ≈	**1.9897e+11 (1.93e+9) ≈**	1.9819e+11 (1.40e+9)
minus-WFG6	10	19	1.6532e+11 (2.11e+9) −	1.9865e+11 (1.83e+9) ≈	**1.9989e+11 (2.03e+9) ≈**	1.9960e+11 (1.94e+9)
minus-WFG7	10	19	1.6534e+11 (2.88e+9) −	1.9870e+11 (1.30e+9) −	**1.9978e+11 (2.21e+9) ≈**	1.9937e+11 (2.97e+9)
minus-WFG8	10	19	1.6808e+11 (1.46e+9) −	1.9932e+11 (1.56e+9) −	**2.0220e+11 (1.71e+9) +**	2.0065e+11 (1.77e+9)
minus-WFG9	10	19	1.6415e+11 (3.16e+9) −	1.9777e+11 (2.21e+9) ≈	1.9557e+11 (2.17e+9) −	**1.9852e+11 (1.86e+9)**
MPDMP	10	2	1.4848e+5 (2.50e+2) −	1.5051e+5 (3.10e+2) −	1.5017e+5 (1.98e+2) −	**1.5097e+5 (1.64e+2)**
+/ − / ≈			0/11/3	2/4/8	2/3/9	

almost all the results from SMS-EMOA-10 are significantly worse than SMS-EMOA-CD (11 out of 14 worse and no better results). This result can be explained by the setting of r. Since the value of r is set to 10 for the whole process of SMS-EMOA-10, many solutions are on the boundary of PF. As compared to SMS-EMOA-CD, the performances of SMS-EMOA-Opt (2 better but 4 worse results) and SMS-EMOA-LD (2 better but 3 worse results) are slightly worse on inverted-triangular PF problems. The results of MPDMP shows that SMS-EMOA-CD is the best among the four algorithms.

Fig. 7 shows the plot of HV for the four mechanisms on the 10-dimensional MPDMP problem. In Fig. 7, the HV of SMS-EMOA-LD (the red curve) gradually increases and finally reaches the same level as SMS-EMOA-Opt, as the value of r is gradually decreased to $1+1/H$. The HV of SMS-EMOA-CD (the yellow curve) firstly reaches a stable level similar to SMS-EMOA-10, for that their values of r are both 10 before 4,500 evaluations. The convergence detection is reported at about 4,500 evaluations for SMS-EMOA-CD. Then, the values of r in SMS-EMOA-CD reaches the optimal value of $1+1/H$ and the HV increases. The reason is due to the decrease of the boundary solutions, and on the other hand, the inner solutions increase. Finally, the HV of SMS-EMOA-CD is better than SMS-EMOA-Opt and SMS-EMOA-LD.

V. Conclusions

In this paper, we emphasize the importance of reference point specification in SMS-EMOA by a simple example. We have demonstrated that without a good reference point specification mechanism, poor diversity of the final solutions on inverted-shape PF problems will be obtained. This phenomenon is hardly observed on the triangular PF problems when the reference point is worse than the nadir point. We introduced the dynamic reference point specification mechanism with the illustration by two aspects:

1) Better Searching Behavior: In the early stage, the solutions may be far away from the true PF. A larger value of r can achieve better searching behavior. We give an example of DTLZ2.

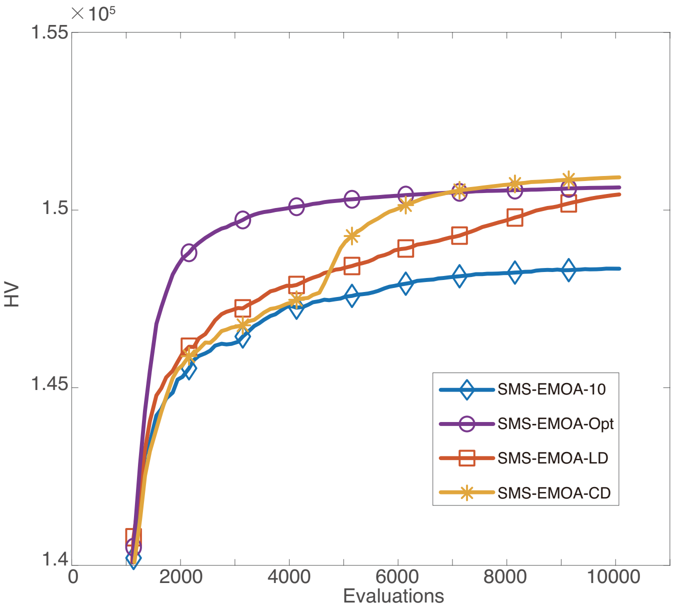


Fig. 7: The comparison of four mechanisms. The value of each point is the average of 20 independent runs in each evaluation and the problem is the 10-objective MPDMP in TABLE II.

2) Uniform Distribution: Considering the linear triangular problems, the optimal setting of the reference point is $r = 1 + 1/H$.

We summarize the basic idea of the dynamic reference point specification mechanism as follows: the value of r should be specified larger than $1 + 1/H$ at the initial stage and equal to $1/1 + H$ in the final stage, as in Eqs. (4)-(6).

Besides that, we have proposed a new dynamic reference point specification mechanism in this paper. A weak convergence detection mechanism is used in the proposed method. The new dynamic reference point specification mechanism is tested on SMS-EMOA algorithm and compared with other SMS-EMOAs under different settings of r. The results show that SMS-EMOA with $r = 1 + 1/H$ performs the worst on the triangular PF problems, and SMS-EMOA with $r = 10$ performs the worst on the inverted-triangular PF problems. This gives us a hint on the necessary of the dynamic mechanism. We have also compared our proposed mechanism with the linearly decreasing mechanism. The results show that our new mechanism outperforms the linearly decreasing mechanism on some test problems, specifically on the MPDMP problem.

In the future, we plan to investigate the behavior of our new mechanism and further improve it. Besides that, the problems with different PF shapes will be tested and analyzed.

References

[1] E. Zitzler and L. Thiele, "Multiobjective optimization using evolutionary algorithms—A comparative case study," in *Parallel Problem Solving from Nature, V*. Berlin, Germany: Springer-Verlag, 1998, pp. 292–301.

[2] M. P. Hansen and A. Jaszkiewicz, *Evaluating the quality of approximations to the non-dominated set*. IMM, Department of Mathematical Modelling, Technical Universityof Denmark, 1994.

[3] E. Zitzler, L. Thiele, M. Laumanns, C. M. Fonseca, and V. G. da Fonseca, "Performance assessment of multiobjective optimizers: An analysis and review," *Trans. Evol. Comp*, vol. 7, no. 2, pp. 117–132, Apr. 2003. [Online]. Available: http://dx.doi.org/10.1109/TEVC.2003.810758

[4] C. A. C. Coello and M. R. Sierra, "A study of the parallelization of a coevolutionary multi-objective evolutionary algorithm," in *Mexican International Conference on Artificial Intelligence*. Springer, 2004, pp. 688–697.

[5] E. Zitzler, D. Brockhoff, and L. Thiele, "The hypervolume indicator revisited: On the design of pareto-compliant indicators via weighted integration," in *International Conference on Evolutionary Multi-Criterion Optimization*. Springer, 2007, pp. 862–876.

[6] N. Beume, C. M. Fonseca, M. Lopez-Ibanez, L. Paquete, and J. Vahrenhold, "On the complexity of computing the hypervolume indicator," *IEEE Transactions on Evolutionary Computation*, vol. 13, no. 5, pp. 1075–1082, 2009.

[7] N. Beume, B. Naujoks, and M. Emmerich, "SMS-EMOA: Multiobjective selection based on dominated hypervolume," *European Journal of Operational Research*, vol. 181, no. 3, pp. 1653–1669, 2007.

[8] J. Bader and E. Zitzler, "HypE: An algorithm for fast hypervolume-based many-objective optimization," *Evolutionary Computation*, vol. 19, no. 1, pp. 45–76, 2011.

[9] K. Shang, H. Ishibuchi, M.-L. Zhang, and Y. Liu, "A new R2 indicator for better hypervolume approximation," in *Proceedings of the Genetic and Evolutionary Computation Conference*, ser. GECCO '18. New York, NY, USA: ACM, 2018, pp. 745–752. [Online]. Available: http://doi.acm.org/10.1145/3205455.3205543

[10] S. Jiang, J. Zhang, Y.-S. Ong, A. N. Zhang, and P. S. Tan, "A simple and fast hypervolume indicator-based multiobjective evolutionary algorithm," *IEEE Transactions on Cybernetics*, vol. 45, no. 10, pp. 2202–2213, 2014.

[11] A. Menchaca-Méndez, E. Montero, and S. Zapotecas-Martínez, "An improved s-metric selection evolutionary multi-objective algorithm with adaptive resource allocation," *IEEE Access*, vol. 6, pp. 63 382–63 401, 2018.

[12] H. Ishibuchi, R. Imada, S. Yu, and Y. Nojima, "How to specify a reference point in hypervolume calculation for fair performance comparison," *Evolutionary Computation*, vol. 26, no. 3, pp. 1–29, 2018.

[13] H. Ishibuchi, R. Imada, Y. Setoguchi, and Y. Nojima, "Reference point specification in hypervolume calculation for fair comparison and efficient search," in *Proceedings of the Genetic and Evolutionary Computation Conference*. ACM, 2017, pp. 585–592.

[14] ——, "Hypervolume subset selection for triangular and inverted triangular pareto fronts of three-objective problems," in *Proceedings of the 14th ACM/SIGEVO Conference on Foundations of Genetic Algorithms*. ACM, 2017, pp. 95–110.

[15] H. Ishibuchi, R. Imada, N. Masuyama, and Y. Nojima, "Dynamic specification of a reference point for hypervolume calculation in SMS-EMOA," in *2018 IEEE Congress on Evolutionary Computation (CEC)*. IEEE, 2018, pp. 1–8.

[16] ——, "Use of two reference points in hypervolume-based evolutionary multiobjective optimization algorithms," in *International Conference on Parallel Problem Solving from Nature*. Springer, 2018, pp. 384–396.

[17] W. Ericson, "Introductory probability and statistical applications," *Technometrics*, vol. 8, no. 4, pp. 720–722, 1970.

[18] H. Ishibuchi, K. Doi, and Y. Nojima, "On the effect of normalization in MOEA/D for multi-objective and many-objective optimization," *Complex & Intelligent Systems*, vol. 3, no. 4, pp. 279–294, 2017.

[19] H. Jain and K. Deb, "An evolutionary many-objective optimization algorithm using reference-point based nondominated sorting approach, part II: handling constraints and extending to an adaptive approach," *IEEE Transactions on Evolutionary Computation*, vol. 18, no. 4, pp. 602–622, 2013.

[20] T. Ye, C. Ran, X. Zhang, and Y. Jin, "PlatEMO: A MATLAB platform for evolutionary multi-objective optimization [educational forum]," *IEEE Computational Intelligence Magazine*, vol. 12, no. 4, pp. 73–87, 2017.

[21] Q. Zhang and H. Li, "MOEA/D: A multiobjective evolutionary algorithm based on decomposition," *IEEE Transactions on evolutionary computation*, vol. 11, no. 6, pp. 712–731, 2007.

[22] Z. Wang, Y.-S. Ong, J. Sun, A. Gupta, and Q. Zhang, "A generator for multiobjective test problems with difficult-to-approximate pareto front boundaries," *IEEE Transactions on Evolutionary Computation*, 2018.

[23] H. Trautmann, U. Ligges, J. Mehnen, and M. Preuss, "A convergence criterion for multiobjective evolutionary algorithms based on systematic

statistical testing," in *International Conference on Parallel Problem Solving from Nature*. Springer, 2008, pp. 825–836.

[24] J. L. Guerrero, L. Martí, A. Berlanga, J. García, and J. M. Molina, "Introducing a robust and efficient stopping criterion for MOEAs," in *IEEE Congress on Evolutionary Computation*. IEEE, 2010, pp. 1–8.

[25] T. Wagner, H. Trautmann, and B. Naujoks, "OCD: Online convergence detection for evolutionary multi-objective algorithms based on statistical testing," in *International Conference on Evolutionary Multi-Criterion Optimization*. Springer, 2009, pp. 198–215.

[26] H. Trautmann, T. Wagner, B. Naujoks, M. Preuss, and J. Mehnen, "Statistical methods for convergence detection of multi-objective evolutionary algorithms," *Evolutionary Computation*, vol. 17, no. 4, pp. 493–509, 2009.

[27] K. Deb and S. Jain, "Running performance metrics for evolutionary multi-objective optimization," Tech. Rep., 2002.

[28] O. Rudenko and M. Schoenauer, "A steady performance stopping criterion for pareto-based evolutionary algorithms," in *6th International Multi-Objective Programming and Goal Programming Conference*, 2004.

[29] T. Wagner, H. Trautmann, and L. Martí, "A taxonomy of online stopping criteria for multi-objective evolutionary algorithms," in *International Conference on Evolutionary Multi-Criterion Optimization*. Springer, 2011, pp. 16–30.

[30] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler, "Scalable multi-objective optimization test problems," in *Proceedings of the 2002 Congress on Evolutionary Computation. CEC'02 (Cat. No. 02TH8600)*, vol. 1. IEEE, 2002, pp. 825–830.

[31] S. Huband, P. Hingston, L. Barone, and L. While, "A review of multiobjective test problems and a scalable test problem toolkit," *IEEE Transactions on Evolutionary Computation*, vol. 10, no. 5, pp. 477–506, 2006.

[32] H. Ishibuchi, Y. Setoguchi, H. Masuda, and Y. Nojima, "Performance of decomposition-based many-objective algorithms strongly depends on pareto front shapes," *IEEE Transactions on Evolutionary Computation*, vol. 21, no. 2, pp. 169–190, 2016.