Weak Convergence Detection based Dynamic Reference Point Specification in SMS-EMOA

Weiduo Liao, Ke Shang, Hisao Ishibuchi
Department of Computer Science and Engineering
Southern University of Science and Technology, Shenzhen, China
Email: 11849249@mail.sustech.edu.cn; kshang@foxmail.com; hisao@sustech.edu.cn

***Abstract*—In evolutonary multi-objective optimization (EMO) field, the hypervolume (HV) indicator is one of the most popular performance indicators. It is not only used for performance evaluation of EMO algorithms (EMOAs), but also adopted in EMOAs for selection (e.g., SMS-EMOA). However, the specification of the reference point has a large effect on the performance of SMS-EMOA. Thus, the reference point speficiation should be carefully treated in SMS-EMOA. In this paper, the importance of the dynamic reference point speficication in SMS-EMOA is explained first, then a new dynamic reference point specification mechanism based on weak convergence detection is introduced for SMS-EMOA. Experimental comparisons are conducted among SMS-EMOA with the proposed mechanism, a linear decreasing mechanism and two static mechanisms. The results demonstrate the effectiveness of the proposed mechanism.**

***Keywords*—reference point adaptation; indicator-based algorithm; hypervolume; evolutionary multi-objective optimization; behavior; dynamic mechanism; convergence detection**

I. Introduction

In the field of evolutionary multi-objective optimization algorithms (EMOAs), many researchers focus on various indicators including Hypervolume(HV) [1], R2 [2], ϵ_+ indicator [3] and IGD [4]. The indicators are designed for different purposes, and have their own strong points and drawbacks. Different from IGD, HV does not need the pre-knowledge of the shape of the Pareto front(PF) and is the only pareto-compliant indicator up to now [5]. But due to the heavy computation load of HV computation [6], the HV-based algorithms get poor performances in terms of running time when dealing with problems which have more than three objectives, which is known as Many-Objective Optimization Problems (MaOPs).

SMS-EMOA [7] is a basic HV-based algorithm. The HV contribution is used to determine which solution to be discarded in the algorithm. In order to reduce the heavy computation cost of HV computation, many new indicators or new methods have been proposed to estimate the HV. For example, HypE [8] uses a Monte Carlo simulation technology to estimate the effect of HV; R2 [2] indicator is based on a standard weighted Tchebycheff function; And a new R2 [9] is proposed by Shang et al. Recently, an improved SMS-EMOA with adaptive resource allocation has been proposed to reduce the number of HV calculations [10]. In 2015, a simple and fast version of SMS-EMOA [7], so-called FV-EMOA, has been proposed [11]. In order to further reduce the bottleneck of high time complexity for calculating the HV contributions, the FV-EMOA considers the fact that the HV contribution of a single solution is only associated with partial solutions rather than the whole solution set [11]. Based on this point, the FV-MOEA reduces the computational cost greatly.

The specification of the reference point is one of the important but easy to be ignored parts in HV computation, for that the effect on many problems including DTLZ [12] are not so significant. But it has been reported that the position of reference point strongly influences the value of HV contribution of the external solutions on the problems with inverted-shape PFs [13]–[15]. A suitable set of reference point position for linear PFs has been sufficiently investigated in [14]. A dynamic reference point specification mechanism has also been proposed in [16]. Another proposed strategy is to use two reference points in HV-based EMOA [17]. In this paper, the dynamic mechanism is stated in two different stages of the algorithm running process. At the early stage, for a better searching behavior of the algorithm, the reference point should be set slightly worse than the suggested position at the beginning. Then with a dynamic mechanism, the reference point is gradually decreasing to the suggested position following the iteration.

After that, we propose a new dynamic reference point specification mechanism with a weak convergence detection in HV-based EMOA. In this weak convergence detection mechanism, we use the logarithm nadir point as the convergence indicator and use the simple Least Squares [18] to detect the convergence. Given w_l generations as a window, if the slope of linear regression is below one threshold, we report the convergence. The comparison of two dynamic mechanisms and simple reference point specification(without dynamic mechanism) is represented in the experiment section. On some specific problems, for example the multi-objective distance minimization problems [19], our weak convergence detection mechanism outperforms the linearly decreasing mechanism.

The remainder of this paper is organized as follows. Firstly in Section II, we introduce reference point specification in SMS-EMOA with the original one and the one proposed in

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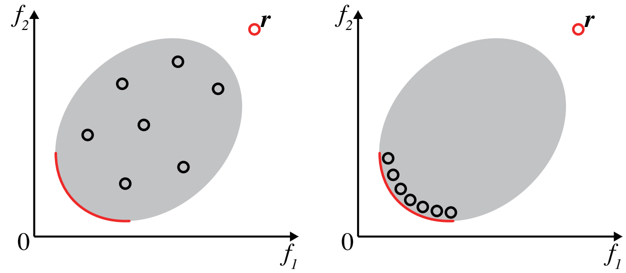
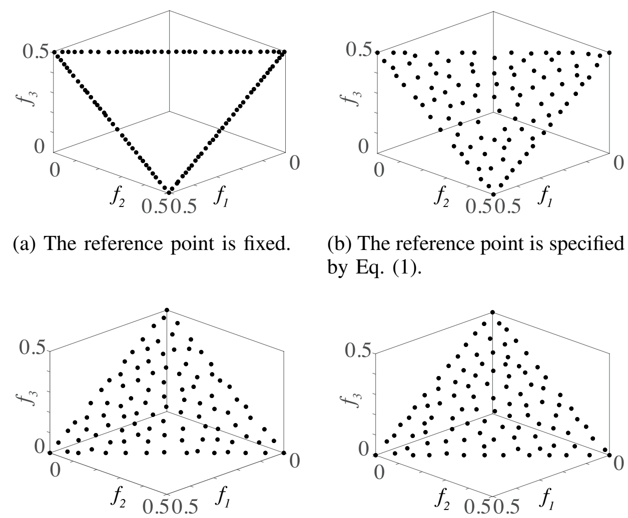


(a) The initial generation. (b) The final generation.

Fig. 1: The reference point is set with a large feasible space. The PF can be far away from reference point. The gray region shows the feasible region and the red arc is the corresponding PF. The red circle r is the reference point calculated by the initial solutions in (1a) which is randomly generated. After some generations, the current solutions reach to the five black circles in (1b), which is far away from the reference point.



(a) The reference point is fixed. (b) The reference point is specified by Eq. (1).

(c) The reference point is fixed. (d) The reference point is specified by Eq. (1).

Fig. 2: The final distribution of solution set in the inverted-DTLZ1(2a and 2b) and the DTLZ1 problem(2c and 2d). The algorithm is SMS-EMOA with population size $= 100$, evaluation number $= 20000$. 2a and 2c: the reference point is calculated only once at the initial step; 2b and 2d: the reference point is specified by the Eq. (1) with $r = 1.1$. All the solutions in the final distribution are on the boundary of the PF in (2a), which shows the bad effect of a faraway reference point on the final distribution of inverted-triangular problems. This bad effect can not be observed on triangular problems in (2c).

[13], then explore the details of the reference point specification mechanism in two stages of the algorithm, which elicits the necessity of using dynamic mechanism. After that, we state the details of the dynamic reference point specification mechanism and introduce a linearly decreasing mechanism in Section III. The details of the new dynamic mechanism with a weak convergence detection proposed in this paper is also presented in Section III. We report our computational experiments of SMS-EMOA with several triangular and inverted-triangular problems in Section IV. Our experiments are performed on 10-objecitve problems for clearly comparing of two dynamic mechanisms and simple reference point specification (without dynamic mechanism). Finally, the conclusion is shown in Section V.

II. Reference Point Specification in SMS-EMOA

When HV is used in SMS-EMOA algorithms, one important thing to be considered is how to specify the reference point. Before calculating the HV values, a reference point needs to be specified in advance. However, it is not suggested that the reference point is set only once at the beginning. This may cause a very far away reference point from solutions for those problems with a very large feasible space (as shown in Fig. 1), as the solution set is gradually converging to the PF with the process of the algorithm.

There is a big problem when applying this strategy to some problems with special PF shape. For example, the inverted-DTLZ1 [20] has an inverted-triangular PF, so many solutions in the final solution set will distribute on the boundary of the PF (Fig. 2a) [14]–[16]. Although it has no effect on the distributions of solution set in triangular PF problems (Fig. 2c comparing with Fig. 2d), it is necessary to use a good reference point specification method during the algorithm progress. The reason is explained in detail in [15].

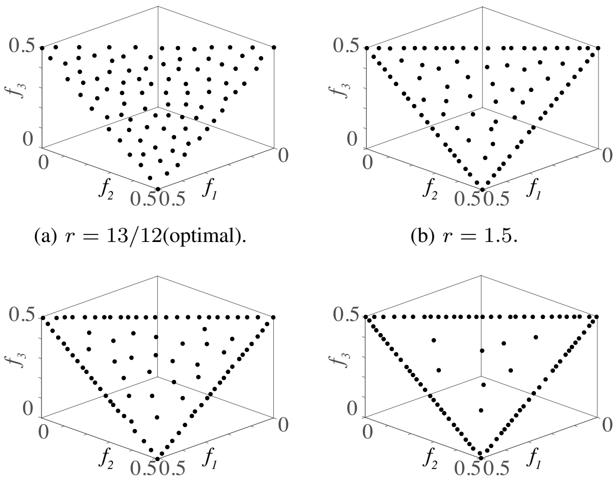
A. Original Reference Point Specification

In the original paper of SMS-EMOA [7], the reference point is specified as the estimated nadir point increased by $\mathbf{1.0}$ (in two-dimension, a sufficiently large reference point is chosen). But in current implementation, the following mechanism is always used:

$$\boldsymbol{R} = r * \boldsymbol{N}, \tag{1}$$

where $\boldsymbol{R} \in \mathbb{R}^m$ is the reference point in each generation, and $\boldsymbol{N} \in \mathbb{R}^m$ is the estimated nadir point of the last front of the current population. r is specified as 1.1 in PlatEMO [21], a well-known EMO framework. The specification of reference point is also one of the important parts in the HV-based algorithm implementation. In the source code of PlatEMO, the values of r in HV-based algorithms are chosen as follow: 1.1(SMS-EMOA [7]) and 1.2(HypE [8]). When the solutions in the current population are obtained, we use HV as the indicator to evaluate the performance of the solution set. Then the reference point used to calculate the HV is calculated by the formula above.

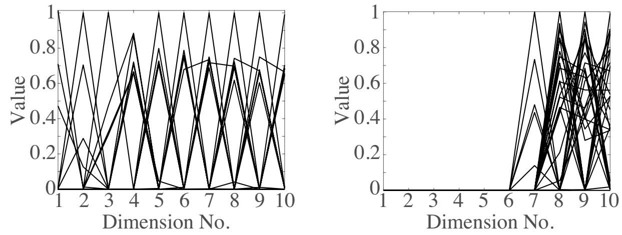
However, fixing the r value to 1.1 is not recommended, especially on problems with the inverted-triangular PF [13]. The research on specifying the value of r is limited, for the reason that, the effect of the location of the reference point on the PF is not fatal on some benchmark problems, for example,



(a) $r = 13/12$(optimal). (b) $r = 1.5$.

(c) $r = 2$. (d) $r = 5$.

Fig. 3: The final distribution of solution set on the inverted-DTLZ1 problem. The algorithm is SMS-EMOA with population size $= 91(H = 12)$ and evaluation number $= 20000$. $r = 13/12$ is the optimal setting and we observed a evenly distribution in (3a). As the increasing of r, solutions are more likely to be on the boundary (3b-3d).



(a) $r = 10$. (b) $r = 2$(optimal).

Fig. 4: The final distribution of SMS-EMOA on 10-objective DTLZ2 problems with different settings of r. The population number is $30(H = 1)$, total evaluation number is 10000.

the problems with triangular PF.

B. Reference Point Specification Proposed in [13]

In [13]–[15], the suggested value of r is investigated on linear PF problems. Specifically, on inverted-triangular problems(inverted-DTLZ1 [15] etc.), in order to have same HV contribution, the interval between two boundary solutions should be the same as that between two inner solutions. In [13], the suggested value of r is:

$$r = 1 + \frac{1}{H}. \tag{2}$$

H is the number of solutions intervals in 2-dimension and the number of intervals on each boundary of PF in many-dimension. The idea is inspired from MOEA/D [22], in which the H is used for generating uniformly distributed weight vectors [16]. Given the population size μ and dimensionality m, the value of H can be calculated by the following formula:

$$C_{m-1}^{H+m-1} \leq \mu < C_{m-1}^{H+m}. \tag{3}$$

In Fig. 3a, $r = 13/12(H = 12)$ is the optimal setting for a 91-individual 3-dimensional inverted-DTLZ1 problem and a uniform distribution is observed. In Fig. 3b - Fig. 3d, the inner solutions are decreased and move to the boundary of PF with the increasing of r.

Basically, the process of EMOAs can be separated into two stages:

1) Early Stage: In this stage, all the solutions are far away from the PF. The main task is to converge the solutions to the PF. We also call this stage the convergence stage.

2) Final Stage: In this stage, all the solutions are inside or near the PF. So the main task is to make the distribution of solutions more evenly in the PF. We also call this stage the diversity stage.

C. Specify the Value of r for Better Searching Behavior

In the early stage, solution set is not close to the PF, which makes the estimated ideal and nadir points far away from true ideal and nadir points, for the reason that they are calculated by current solutions in each generation. And considering the problems with 10 objectives, if the external solutions have the same HV contribution with the inner solutions, the exploration of solutions will be poor. As a result, some dimensionalities of solutions will be missing and the breadth-diversity [23] will be poor. An example is given in Fig. 4. The first 6 dimensionalities are all 0 for the solutions obtained by SMS-EMOA on 10-objective DTLZ2 problems(Fig. 4b). In [16], a larger value of r than $1 + 1/H$ is suggested in the early stage.

D. Specify the Value of r for Uniform Distribution

When algorithm reaches the final stage, all solutions are near the PF, r should be specified as $1 + 1/H$, as in Eq. (2), in order to get an uniform solutions distribution. That is, $r(T) = 1 + 1/H$. On inverted-triangular problems, the distribution of solutions on PF strongly depends on the value of r (as shown in Fig. 3).

The sensitivity of the value of r on the solutions distribution is also observed on some real-world problems, for example, distance minimization problems. This observation shows the potential use of the dynamical reference point specification [16].

III. Dynamic Reference Point Specification Mechanism

For different purposes in the two stages, the r should be treated differently [16]. Not only the reference point but also the value of r needs to be adapted in each iteration of the algorithm. This is called dynamical reference point specification. Based on the Eq. (1), we define the dynamic reference point specification as:

$$\boldsymbol{R} = r(t) * \boldsymbol{N}, t = 0, 1, \ldots, T, \tag{4}$$

where T is the total number of generations, and $r = r(t)$ is a function of the current generation t. The value of r is adapted during the process of the algorithm.

A. Linearly Decreasing Mechanism Proposed in [16]

Based on the theory above, r is suggested to be specified dynamically at different stages of the algorithm(at early stage, a larger r is chosen; at final stage, $r = 1+1/H$ is chosen). But unfortunately, there is no best mechanism on how to specify the value of r dynamically that outperforming the others in all problems and all experiment settings. One mechanism may be the best when working on some specific experiment conditions, but may not be good on other conditions.

In [16], a linearly decreasing mechanism has been proposed:

$$r(t) = r_{Initial}\frac{(T-t)}{T} + (1+1/H)\frac{t}{T}, t = 0, 1, \ldots, T, \quad (5)$$

where T is the total number of generations, and $r_{Initial}$ is the initial value of r, which is larger than $1+1/H$. It is a simple and practical mechanism. In (5), the value of r starts from $r_{Initial}$, then gradually decreases to the suggested value in a linearly decreasing process.

In the next section, another dynamic mechanism based on weak convergence detection criterion is proposed. We show that it outperforms simple linearly decreasing mechanism on some specific problems.

B. A New Dynamic Reference Point Specification Mechanism

In this section, we will introduce a new mechanism that uses a weak convergence detection criterion to decide whether to change the value of r from $r_{Initial}$ to $1+1/H$.

As we have explained before, a larger r is suggested at the early stage of the algorithms. But for good diversity at the final stage, it is needed to set r to its optimal value $(1+1/H)$. For this purpose, it is necessary to detect whether the algorithm is converged. If solutions are all close to the PF, we change the value of r to $r_{Optimal}$; otherwise, we set value of r to $r_{Initial}$. The mechanism is shown below:

$$r(t) = \begin{cases} r_{Initial}, t < t_{Converged} \\ 1+1/H, t \geq t_{Converged} \end{cases} \quad t = 0, 1, \ldots, T. \quad (6)$$

$r(t)$ equals to $r_{Initial}$ before reaching the converged generation $t_{Converged}$, and changes to $1+1/H$ after $t_{Converged}$. $t_{Converged}$ is determined by a weak convergence detection criterion.

Various indicators including convergence detection indicators that are used to detect the stagnation have been proposed in the literature [24]–[30]. They focus on the accuracy of convergence, which is not the purpose in our approach for the reason that after algorithm converged, we still need some generations in order to get even distribution of solution set. We summarize our weak convergence detection criterion as follow:

1) Inaccuracy: It is not necessary to have an accurate convergence detection. The convergence can be reported if current solutions are close to the PF. In other words, the estimated ideal and nadir points based on the current solutions are close to the true ideal and nadir points.

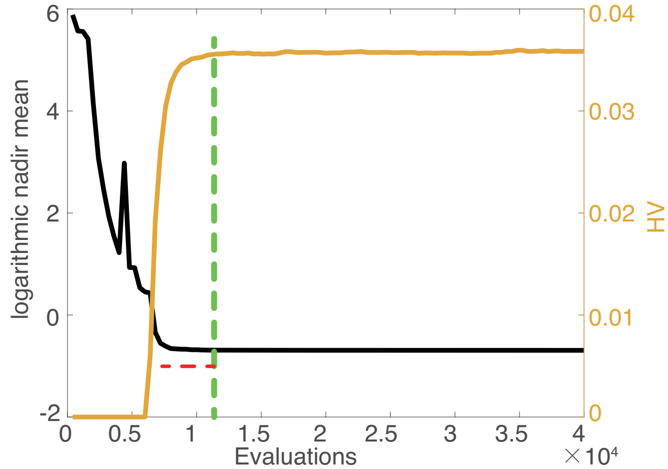


Fig. 5: Example of nadir point and HV value on 3-objective inverted-DTLZ1 problem with SMS-EMOA algorithm. The yellow curve is the change of HV while the black curve is the change of the logarithmic nadir point. The green dotted line shows the convergence detection and the red dotted line shows the considered window for evaluated number $t_{Converged}$.

2) Saving time: We should not spend too much time in convergence detection for the reason that the state-of-the-art HV-based algorithms such as SMS-EMOA and HypE, are time-consuming when the dimension is very high.

We are discussing the effect of reference points in calculating HV. It seems to be a good idea to use HV as our convergence detection indicator as we have calculated HV in each generation in SMS-EMOA. But during the process of algorithm, the reference point is calculated by Eq. (1), which means that they are different among generations. So, we can not simply compare HV calculated in algorithm among different generations.

We are trying on some other good indicators satisfying our convergence detection criterions. Fig. 5 shows the change of HV and nadir point on SMS-EMOA with 3-objective inverted-DTLZ1 problem. When current solutions are close to the PF, the estimated nadir point of current solutions is close to the true nadir points. This implies that the estimated nadir point can be a good indicator for our purpose. But for some problems with a large feasible region, the moving distance of the estimated nadir point in the early generations is larger than that near convergence. So we consider the logarithm. And considering the possible bad variance of the indicator, finally, we use the best logarithmic nadir point so-far as our indicator. More specifically, for a minimization problem, we consider the indicator I as follows:

$$\begin{aligned} \boldsymbol{N_t} &= [f_{t1}, f_{t2}, \ldots, f_{tm}]^\top \in \mathbb{R}^m, \\ I_0 &= \frac{1}{m}\sum_{i=1}^{m} lnf_{0i}, \\ I_t &= min(I_{t-1}, \frac{1}{m}\sum_{i=1}^{m} lnf_{ti}), t = 1, 2, \ldots, T, \end{aligned} \quad (7)$$

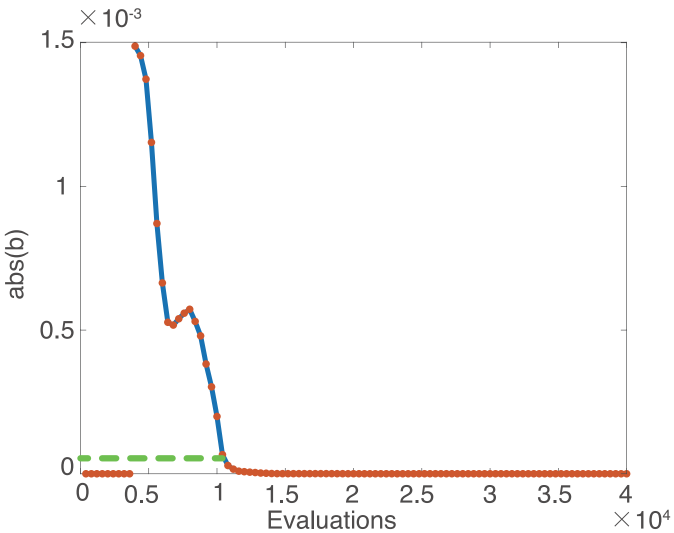


Fig. 6: Example of $|b|$ on inverted-DTLZ1 3-objective problem (window size = 4000 evaluations with population size = 100). The green dotted line shows the threshold.

where T is the total number of generations, $\boldsymbol{N_t}$ is the estimated nadir point at the t^{th} generation with m objectives: $f_{t1}, f_{t2}, \ldots, f_{tm}$. I_0 is the initial indicator calculated by the initial population. And I_t is the minimum value before the t^{th} generation (including the t^{th} generation).

After choosing the indicator, the next step is to detect the stagnation of the indicator. We use a basic linear regression method called Simple Least Squares [18] with a simple least squares convergence detection strategy introduced in [25]. If the absolute value of the slope of the indicator is below a threshold, the convergence is reported. Briefly speaking, for a simple linear regression $I(t) = a+bt$, the intercept a and slope b of the t^{th} generation can be calculated with the following matrix-based formula:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma t_i^2 & \Sigma t_i \\ \Sigma t_i & w\_l \end{bmatrix}^{-1} * \begin{bmatrix} \Sigma t_i * I_{t_i} \\ \Sigma I_{t_i} \end{bmatrix}, \tag{8}$$

where w_l is the length of the chosen window (in the example of Fig. 5, the chosen window for $t_{Converged}$ is represented with the red dotted line), and t_i is the evaluated number in the chosen window, which holds the following condition: $t_i \in (t', t], t - t' = w_l$. The value of slope b is shown in Fig. 6 (Note that the value in the first w_l evaluations is 0, and we should not consider the first w_l evaluations).

With the above Eq. (8), we report the convergence if the following formula holds:

$$|b| < thres. \tag{9}$$

The value of $t_{Converged}$ equals to t (i.e., the current evaluation number) when convergence is detected. The whole process of weak convergence detection is also clearly described in Algorithm 1. If Algorithm 1 return True, we report the convergence, vice versa. The choice of the $thres$ value is trivial as the report ahead or delay is not fatal to the algorithm or to the final solution set. We choose $thres$ value as 10^{-5} after some experimental computation with the window size $w\_l = 4000$. If we do not change the window size, this threshold can be applied to other problems or other HV-based algorithms because of a weak convergence detection purpose.

Algorithm 1: Weak Convergence Detection

Input:
w_l, // Window size.
$\boldsymbol{S_t}$, // Solution set when evaluaiton number is t.
$\boldsymbol{I} = \{I_0, I_1, \ldots, I_{t-1}\}$, // Stored indicator values.
$thres$, // The chosen threshold for slope.
Output: Return True if converged, otherwise return False.
Calculate nadir point $\boldsymbol{N_t}$ of $\boldsymbol{S_t}$;
Calculate indicator I_t; // By Eq. (7).
$\boldsymbol{I} \leftarrow \boldsymbol{I} \cup \{I_t\}$;
if $t \geq w\_l$ **then**
  Calculate b; // By Eq. (8).
  if $|b| < thres$ **then**
    Return True; // Converged.
  end if
end if
Return False; // Not converged.

IV. Computational Experiments

A. Experimental Settings

To clearly represent the superiority of dynamically reference point specification and to ease the comparison process of two different dynamic mechanisms (linearly decreasing mechanism and weak convergence detection mechanism), the mechanisms are tested with the state-of-the-art algorithm SMS-EMOA [7]. The test problems include: DTLZ [12], WFG [31], their minus-versions [32] and a multi-objective distance minimization problem(MPDMP) [19]. We test all the problems with 10 objectives. All the code in this section is implemented in PlatEMO framework [21] with the following settings:

Population size: 30 ($H = 1$),
Total evaluation number: 100,000 solution evaluations,
Initial value of r ($r_{Initial}$): 10.
Crossover: simulated binary crossover (probability: 1.0),
Mutation: polynomial (probability: $1/D$),
Number of decision variables D:
  14 (DTLZ1 and minus-DTLZ1),
  2 (MPDMP),
  19 (other problems),
Distribution index in Crossover and Mutation: 20,
Number of runs: 20 runs.

B. Computational Results

The detailed algorithms are as follows: SMS-EMOA-10 (SMS-EMOA [7] with $r = 10$), SMS-EMOA-Opt (SMS-EMOA with $r = 1 + 1/H$), SMS-EMOA-LD (SMS-EMOA with a linearly decreasing mechanism), and SMS-EMOA-CD

TABLE I: HV mean and standard deviation over 20 independent runs for DTLZ and WFG problems.

Problem	M	D	SMS-EMOA-10	SMS-EMOA-Opt	SMS-EMOA-LD	SMS-EMOA-CD
DTLZ1	10	14	6.8448e-1 (4.61e-1) ≈	2.0261e-1 (2.23e-1) −	**8.8369e-1 (2.00e-1) ≈**	6.9062e-1 (3.95e-1)
DTLZ2	10	19	**1.0234e+3 (4.72e-1) ≈**	9.9315e+2 (3.94e+1) −	1.0234e+3 (7.28e-1) ≈	1.0184e+3 (1.85e+1)
DTLZ3	10	19	0.0000e+0 (0.00e+0) ≈	**1.9068e+1 (8.53e+1) ≈**	0.0000e+0 (0.00e+0) ≈	0.0000e+0 (0.00e+0)
DTLZ4	10	19	**8.0029e+2 (2.21e+2) ≈**	5.2691e+2 (1.79e+2) −	6.6262e+2 (2.17e+2) ≈	7.2840e+2 (2.26e+2)
WFG1	10	19	2.2025e+12 (2.55e+11) ≈	2.2597e+12 (2.64e+11) ≈	2.2005e+12 (1.81e+11) ≈	**2.2601e+12 (3.42e+11)**
WFG2	10	19	**3.3604e+12 (2.92e+10) ≈**	3.3493e+12 (2.39e+10) +	3.3487e+12 (7.64e+10) ≈	3.3420e+12 (8.46e+10)
WFG3	10	19	4.7217e-3 (1.04e-2) −	**2.5310e-2 (3.07e-2) ≈**	1.7230e-2 (2.35e-2) ≈	2.3443e-2 (2.84e-2)
WFG4	10	19	3.7667e+12 (1.77e+10) ≈	3.5602e+12 (9.48e+10) −	3.7585e+12 (3.74e+10) ≈	**3.7737e+12 (1.63e+10)**
WFG5	10	19	3.6535e+12 (7.68e+9) ≈	3.5027e+12 (1.16e+11) −	**3.6555e+12 (8.64e+9) ≈**	3.6523e+12 (1.93e+10)
WFG6	10	19	3.6082e+12 (7.78e+10) ≈	3.5432e+12 (6.24e+10) −	3.5829e+12 (5.75e+10) ≈	**3.6099e+12 (4.49e+10)**
WFG7	10	19	**3.7967e+12 (7.58e+9) ≈**	3.7326e+12 (5.73e+10) −	3.7959e+12 (1.35e+10) ≈	3.7922e+12 (1.27e+10)
WFG8	10	19	3.7512e+12 (1.51e+10) ≈	3.7087e+12 (3.64e+10) −	**3.7613e+12 (1.13e+10) +**	3.7524e+12 (1.23e+10)
WFG9	10	19	3.5727e+12 (1.85e+11) ≈	3.3072e+12 (2.80e+11) −	**3.6302e+12 (1.57e+11) +**	3.5146e+12 (2.31e+11)
$+/-/\approx$			0/1/12	1/9/3	2/0/11	

TABLE II: HV mean and standard deviation over 20 independent runs for minus-DTLZ, minus-WFG and MPMP.

Problem	M	D	SMS-EMOA-10	SMS-EMOA-Opt	SMS-EMOA-LD	SMS-EMOA-CD
minus-DTLZ1	10	14	4.3889e+28 (5.56e+26) ≈	**4.4237e+28 (5.99e+26) ≈**	4.4120e+28 (4.79e+26) ≈	4.3872e+28 (7.72e+26)
minus-DTLZ2	10	19	1.2299e+7 (1.38e+5) −	1.4737e+7 (1.55e+5) −	**1.4891e+7 (1.11e+5) ≈**	1.4844e+7 (1.45e+5)
minus-DTLZ3	10	19	1.1227e+35 (3.16e+33) −	1.3385e+35 (3.41e+33) ≈	**1.3702e+35 (3.42e+33) ≈**	1.3582e+35 (3.78e+33)
minus-DTLZ4	10	19	1.1638e+7 (2.11e+5) −	**1.4899e+7 (1.20e+5) +**	1.4895e+7 (1.27e+5) +	1.3585e+7 (1.57e+6)
minus-WFG1	10	19	4.4808e+10 (5.25e+8) ≈	**4.4894e+10 (4.81e+8) ≈**	4.4890e+10 (4.35e+8) ≈	4.4705e+10 (5.60e+8)
minus-WFG2	10	19	6.6225e+10 (1.05e+8) −	6.7628e+10 (7.51e+8) ≈	6.7870e+10 (3.31e+7) −	**6.7907e+10 (1.60e+7)**
minus-WFG3	10	19	6.3620e+10 (5.49e+8) ≈	**6.4321e+10 (2.29e+8) +**	6.3642e+10 (4.93e+8) ≈	6.3799e+10 (5.87e+8)
minus-WFG4	10	19	1.6478e+11 (3.22e+9) −	**1.9896e+11 (2.20e+9) ≈**	1.9752e+11 (2.39e+9) ≈	1.9260e+11 (1.34e+10)
minus-WFG5	10	19	1.6534e+11 (2.09e+9) −	1.9817e+11 (1.82e+9) ≈	**1.9897e+11 (1.93e+9) ≈**	1.9819e+11 (1.40e+9)
minus-WFG6	10	19	1.6532e+11 (2.11e+9) −	1.9865e+11 (1.83e+9) ≈	**1.9989e+11 (2.03e+9) ≈**	1.9960e+11 (1.94e+9)
minus-WFG7	10	19	1.6534e+11 (2.88e+9) −	1.9870e+11 (1.30e+9) −	**1.9978e+11 (2.21e+9) ≈**	1.9937e+11 (2.97e+9)
minus-WFG8	10	19	1.6808e+11 (1.46e+9) −	1.9932e+11 (1.56e+9) −	**2.0220e+11 (1.71e+9) +**	2.0065e+11 (1.77e+9)
minus-WFG9	10	19	1.6415e+11 (3.16e+9) −	1.9777e+11 (2.21e+9) ≈	1.9557e+11 (2.17e+9) −	**1.9852e+11 (1.86e+9)**
MPDMP	10	2	1.4848e+5 (2.50e+2) −	1.5051e+5 (3.10e+2) −	1.5017e+5 (1.98e+2) −	**1.5097e+5 (1.64e+2)**
$+/-/\approx$			0/11/3	2/4/8	2/3/9	

(SMS-EMOA with a weak convergence detection mechanism, proposed in this paper). We obtain the HV results after $100,000$ evaluations for each algorithm. The computational results of the HV metric is shown in TABLE I and TABLE II. The best result in each row is highlighted in bold, and the worst result is shaded. The basic Wilcoxon signed-rank sum test is used in order to show the statistical significance for the algorithm comparing with SMS-EMOA-CD proposed in this paper. The three symbols "+", "−", "≈" mean significantly better, significantly worse and no significant difference.

In the basic test problems(DTLZ1-4, WFG1-9), SMS-EMOA-Opt performs the worst (9 out of 13 significantly worse than SMS-EMOA-CD) among the algorithms in TABLE I. This phenomenon clearly show the bad searching behavior when applying $r = 1 + 1/H$ all the time, which is discussed in Section III. As for SMS-EMOA-10, we can not tell the differences with SMS-EMOA-CD, that the Wilcoxon signed-rank sum tests show almost all the results except one are "≈". This is because of the small influence of reference point on the triangular PF problems. 2 better results from SMS-EMOA-LD indicate that SMS-EMOA-LD is slightly better than SMS-EMOA-CD on triangular PF problems.

In the results of inverted-triangular PF problems (minus-DTLZ1-4, minus-WFG1-9 and MPDMP), SMS-EMOA-10 performs the worst(12 out of 14) in all experiments in average, and the Wilcoxon signed-rank sum tests show that almost all the results from SMS-EMOA-10 are significantly worse than SMS-EMOA-CD (11 out of 14 worse and no better results). The reason is that the r is set to 10 for the whole process of SMS-EMOA-10 algorithm and many solutions are on the boundary of PF. Comparing with SMS-EMOA-CD, SMS-EMOA-Opt(2 better but 4 worse) and SMS-EMOA-LD(2 better but 3 worse) are slightly worse on inverted-triangular PF problems. The result of MPDMP shows that SMS-EMOA-CD is the best among the four algorithms on some specific problems.

We plot the HV graph of 4 mechanisms (as shown in Fig. 7) on the 10-dimensional MPDMP problem. In Fig. 7, the HV of SMS-EMOA-LD (the red curve) is gradually increased and finally reaches the same level as SMS-EMOA-Opt, as the value of r is gradually decrease to $1 + 1/H$. The HV of SMS-EMOA-CD (the yellow curve) firstly reaches a stable level similar to SMS-EMOA-10, for that their values of r are both 10 before 4,500 evaluations. Then after reporting the convergence detection at about 4,500 evaluations, the value of r in SMS-EMOA-CD reaches the optimal value $1 + 1/H$ and the HV increases. The reason is because the boundary solutions decrease, and on the other hand, the inner solutions

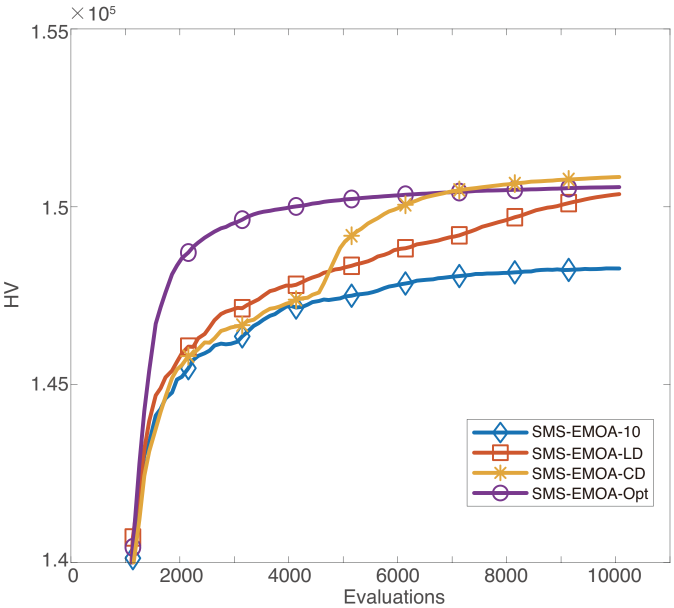


Fig. 7: The comparison of four mechanisms. The value of each point is the average of 20 independent runs in each evaluation and the problem is MPDMP.

increase. Finally, the HV of SMS-EMOA-CD is better than SMS-EMOA-Opt and SMS-EMOA-LD.

V. Conclusions

In this paper, we emphasize the importance of reference point specification in SMS-EMOA by a simple example. Without a good reference point specification mechanism, the diversity of the final solutions will be poor on inverted-shape PF problems. This phenomenon will not be observed on the triangular PF problems when the reference point is worse than the nadir point. Then we state the dynamic reference point specification mechanism with the illustration by two aspects:

1) Better Searching Behavior: In the early stage, the solutions may not be close to the true PF. A larger value of r can achieve a better searching behavior. We give an example of the DTLZ2 problem.

2) Uniform Distribution: Considering the linear triangular problems, the optimal setting of reference point is $r = 1 + 1/H$.

We summarize the whole process of dynamic reference point specification mechanism as that, the value of r should be specified larger than $1+1/H$ at first and be equal to $1+1/H$ in the end, as in Eqs. (4)-(6).

After that, a new dynamic reference point adaptation mechanism is proposed in this paper. A weak convergence detection mechanism is used. We apply our new dynamic mechanism on different test problems including the triangular and the inverted-triangular PFs problems. The results show that SMS-EMOA with $r = 1+1/H$ performs the worst on the problems with triangular PF, and SMS-EMOA with $r = 10$ performs the worst on the problems with inverted-triangular PFs, which hint the superiority of dynamic mechanism. SMS-EMOA with weak convergence detection mechanism performs better or worse than SMS-EMOA with linearly decrease mechanism.

We compare our new mechanism with the proposed linearly decreasing mechanism and find that on some problems, specifically the multi-objective distance minimization problems, our weak convergence detection mechanism outperforms the linear decreasing mechanism.

In the future, we plan to further investigate the behavior of our new mechanism. A larger dimensionality of MaOPs should be considered and the problems with different PF shapes should be tested and analyzed detailedly. Our weak convergence detection mechanism can also be further improved.

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