

# Weak Convergence Detection based Dynamic Reference Point Specification in SMS-EMOA

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**Abstract**—In evolutionary multi-objective optimization (EMO) field, the hypervolume (HV) indicator is one of the most popular performance indicators. It is not only used for performance evaluation of EMO algorithms (EMOAs) but also adopted in EMOAs for selection (e.g., SMS-EMOA). However, the specification of the reference point has a large effect on the performance of SMS-EMOA. Thus, the reference point specification should be carefully treated in SMS-EMOA. In this paper, the importance of the dynamic reference point specification in SMS-EMOA is explained first, then a new dynamic reference point specification mechanism based on weak convergence detection is introduced for SMS-EMOA. Experimental comparisons are conducted among SMS-EMOA with the proposed mechanism, a linearly decreasing mechanism and two static mechanisms. Our results demonstrate the effectiveness of the proposed mechanism.

**Keywords**—reference point specification; SMS-EMOA; hypervolume; evolutionary multi-objective optimization; dynamic mechanism; convergence detection

## I. INTRODUCTION

In the field of Evolutionary Multi-objective Optimization Algorithms (EMOAs), one of the active research areas is the development of performance indicators. Over the years, various indicators have been proposed. These indicators include hypervolume (HV) [1], R2 [2],  $\epsilon_+$  indicator [3] and IGD [4]. They are designed for different purposes and each of them have their own pros and cons. Different from IGD, HV does not need the pre-knowledge of the shape of the Pareto front (PF) and is the only Pareto-compliant indicator up to now [5]. However due to the heavy computation load of HV computation [6], the HV-based algorithms are inefficient when dealing with Many-Objective Optimization Problems (MaOPs) which have more than three objectives.

SMS-EMOA [7] is a classical HV-based algorithm. The HV contribution is used to determine which solution to be discarded in the algorithm. To reduce the heavy computation load of HV computation, many new indicators or new methods have been proposed to estimate the HV. For example, Hype

uses a Monte Carlo simulation technology to estimate the HV [8]; R2 indicator estimates the HV by a standard weighted Tchebycheff function [2]; an improved new R2 is proposed by Shang et al. [9] to approximate the HV. Recently, an improved SMS-EMOA with adaptive resource allocation has been proposed to reduce the number of HV calculations [10]. In 2015, a simple and fast version of SMS-EMOA [7], so-called FV-EMOA, has been proposed [11] to further increase the efficiency. FV-EMOA considers the fact that the HV contribution of a solution is only determined by partial solutions rather than the whole solution set [11].

The specification of the reference point is one of the important but easy to be ignored parts in HV computation. It has been reported that the position of reference point strongly influences the hypervolume optimal distributions on the inverted-triangular PFs [12]–[14]. A reference point specification method is proposed in [13] for fair HV computation. A dynamic reference point specification mechanism has also been proposed in [15]. Another proposed strategy is to use two reference points in HV-based EMOA [16]. In this paper, the dynamic mechanism is stated based on two different stages of the algorithm process. At the early stage, for a better searching behavior of the algorithm, the reference point should be set far from the estimated nadir point. Then with a dynamic mechanism, the reference point is gradually decreasing to the suggested position following the iteration.

After that, we propose a new dynamic reference point specification mechanism based on a weak convergence detection in SMS-EMOA. In this weak convergence detection mechanism, we use the logarithm nadir point as the convergence indicator and use the simple Least Squares [17] to detect the convergence. Given  $w\_l$  generations as a window, if the slope of linear regression is below one threshold, we report the convergence. The comparison among two dynamic mechanisms and the simple reference point specification (without dynamic mechanism) is represented in the experiment section. On some specific problems, for example the multi-objective distance minimization problems [18], our weak convergence detection mechanism outperforms the linearly decreasing mechanism.

The remainder of this paper is organized as follows. Firstly in Section II, we introduce reference point specification in SMS-EMOA with the original one and the one proposed in

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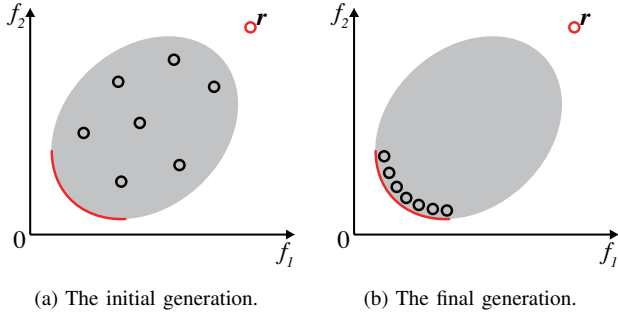


Fig. 1: The reference point is set with a large feasible space. The PF can be far away from the reference point. The gray region shows the feasible region and the red arc is the corresponding PF. The red circle  $r$  is the reference point calculated by the initial solutions in (1a) which are randomly generated. After some generations, the current solutions reach the seven black circles in (1b), which is far away from the reference point.

[12], then explore the details of the reference point specification mechanism in two stages of the algorithm, which elicits the necessity of using the dynamic mechanism. After that, we state the details of the dynamic reference point specification mechanism and introduce a linearly decreasing mechanism in Section III. The details of the new dynamic mechanism with a weak convergence detection proposed in this paper are also presented in Section III. We report our computational experiments of SMS-EMOA with several triangular and inverted-triangular problems in Section IV. Our experiments are performed on 10-objective problems for clearly comparing two dynamic mechanisms and the simple reference point specification (without dynamic mechanism). Finally, the conclusion is shown in Section V.

## II. REFERENCE POINT SPECIFICATION IN SMS-EMOA

When HV is used in SMS-EMOA, one important thing to be considered is how to specify the reference point. Before calculating the HV value, a reference point needs to be specified in advance. However, it is not suggested that the reference point is set only once at the beginning. This may cause a very faraway reference point for those problems with a very large feasible region in the objective space (as illustrated in Fig. 1), as the solution set is gradually converging to the PF.

There is an issue when applying this strategy to some problems with special PF shapes. For example, the inverted-DTLZ1 [19] has an inverted-triangular PF, so many solutions in the final solution set will distribute on the boundary of the PF (Fig. 2a) [13]–[15]. Although it does not affect the distributions of the solution set in triangular PF problems (Fig. 2c comparing with Fig. 2d), it is necessary to use a proper reference point specification during the algorithm run. The reason is explained in detail in [14].

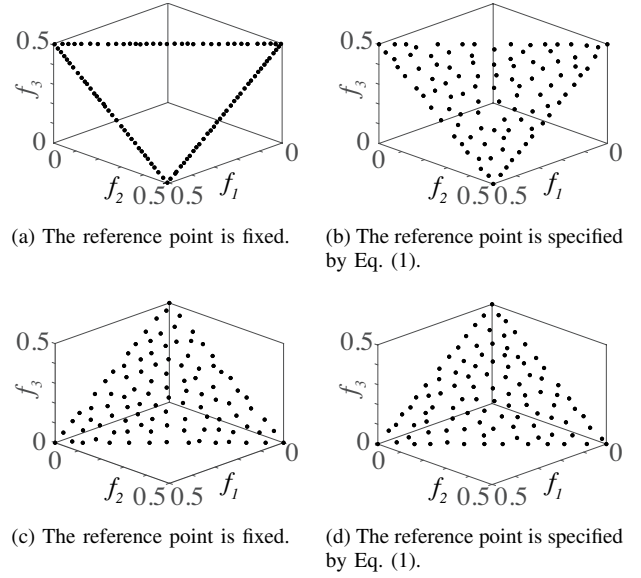


Fig. 2: The final distribution of the solution set on inverted-DTLZ1 (2a and 2b) and DTLZ1 (2c and 2d). The algorithm is SMS-EMOA with population size = 100 and total evaluation number = 20000. 2a and 2c: the reference point is calculated only once at the initial step; 2b and 2d: the reference point is specified by the Eq. (1) with  $r = 1.1$ . All the solutions in the final distribution are on the boundary of the PF in (2a), which shows the bad effect of a faraway reference point on the final distribution of inverted-triangular problems. This bad effect can not be observed on triangular problems in (2c).

### A. Original Reference Point Specification

In the original paper of SMS-EMOA [7], the reference point is specified as the estimated nadir point increased by 1.0 (in two-dimension, a sufficiently large reference point is chosen). And in [15], the reference point is set as  $(r, r, \dots, r)$  in the normalized objective space. However in the current implementation of PlatEMO [20] (a well-known EMO framework), the following mechanism is always used:

$$\mathbf{R} = r \cdot \mathbf{N}, \quad (1)$$

where  $\mathbf{R} \in \mathbb{R}^m$  is the reference point in each generation, and  $\mathbf{N} \in \mathbb{R}^m$  is the estimated nadir point of the last front of the current population.  $r$  is specified as 1.1 in PlatEMO. In the source code of PlatEMO, the values of  $r$  in HV-based algorithms are specified as follow: 1.1 (SMS-EMOA [7]) and 1.2 (HypE [8]). In the process of SMS-EMOA, we use the HV contribution to evaluate each solution, when the solutions in the current population are obtained. And the reference point used to calculate the HV is calculated by the formula above.

However, fixing the  $r$  value to 1.1 is not recommended, especially on problems with the inverted-triangular PF [12]. The research on specifying the value of  $r$  is limited, as the effect of the location of the reference point on the PF is not fatal on some benchmark problems (e.g., triangular PF problems).

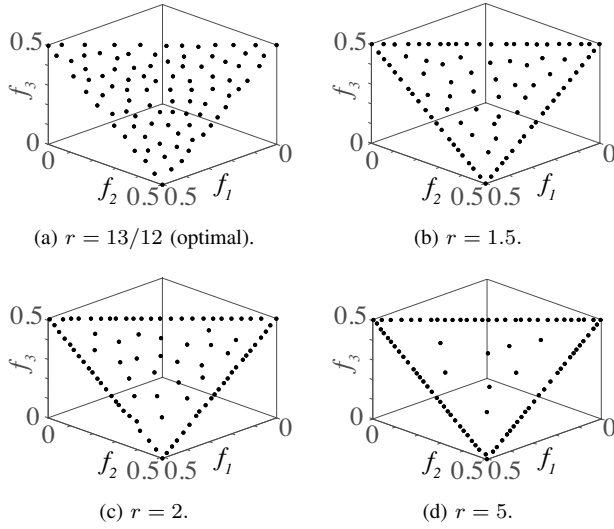


Fig. 3: The final distribution of the solution set on inverted-DTLZ1. The algorithm is SMS-EMOA with population size = 91 ( $H = 12$ ) and total evaluation number = 20000.  $r = 13/12$  is the optimal setting and we observed a uniform distribution in (3a). As the increasing of  $r$ , solutions are more likely to be on the boundary (3b-3d).

#### B. Reference Point Specification Proposed in [12]

In [12]–[14], the suggested value of  $r$  is investigated on linear PF problems. Specifically, on inverted-triangular problems (e.g., inverted-DTLZ1 [14]), the suggested value of  $r$  is:

$$r = 1 + \frac{1}{H}, \quad (2)$$

where  $H$  is a parameter used in MOEA/D [21] for generating uniformly distributed weight vectors [15]. Given the population size  $\mu$  and dimensionality  $m$ , the value of  $H$  can be calculated by the following formula:

$$C_{m-1}^{H+m-1} \leq \mu < C_{m-1}^{H+m}. \quad (3)$$

In Fig. 3a,  $r = 13/12$  ( $H = 12$ ) is the optimal setting for a 91-individual 3-dimensional inverted-DTLZ1 problem and a uniform distribution is observed. In Fig. 3b - 3d, the number of the inner solutions is decreased and many solutions move to the boundary of the PF with the increase of  $r$ .

Basically, the process of EMOAs can be separated into two stages:

1) *Early Stage*: In this stage, all the solutions are far away from the PF. The main task is to converge the solutions to the PF. We also call this stage the convergence stage.

2) *Final Stage*: In this stage, all the solutions are inside or near the PF. So the main task is to make the distribution of solutions more evenly on the PF. We also call this stage the diversity stage.

#### C. Specify the Value of $r$ for Better Searching Behavior

In the early stage, the solution set is not close to the PF, which makes the estimated ideal and nadir points calculated

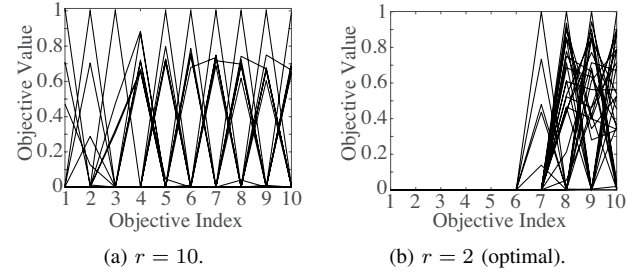


Fig. 4: The final distribution of SMS-EMOA on the 10-objective DTLZ2 with different settings of  $r$ . The population number is 30 ( $H = 1$ ), the total evaluation number is 10000.

by current solutions far away from true ideal and nadir points. And considering the problems with 10 objectives, if the external solutions have the same HV contribution with the inner solutions, the exploration of solutions will be poor. As a result, some dimensionalities of solutions will be missing and the breadth-diversity [22] will be poor. An example is given in Fig. 4. The first 6 dimensionalities are all 0 for the solutions obtained by SMS-EMOA on 10-objective DTLZ2 problems (Fig. 4b). In [15], a larger value of  $r$  than  $1 + 1/H$  is suggested in the early stage.

#### D. Specify the Value of $r$ for Uniform Distribution

When the algorithm reaches the final stage, all solutions are near the PF. To get a uniform solution distribution,  $r$  should be specified as  $1 + 1/H$ , as in Eq. (2). On inverted-triangular problems, the distribution of solutions on PF strongly depends on the value of  $r$  (as shown in Fig. 3).

The sensitivity of the  $r$  value on the solutions distribution is also observed in some real-world problems, for example, distance minimization problems. This observation shows the potential use of the dynamic reference point specification [15].

### III. DYNAMIC REFERENCE POINT SPECIFICATION MECHANISM

For different purposes in the two stages, the  $r$  should be treated differently [15]. Not only the reference point but also the value of  $r$  needs to be adapted in each iteration of the algorithm. This is called dynamic reference point specification. Based on Eq. (1), we define the dynamic reference point specification as:

$$\mathbf{R} = r(t) \cdot \mathbf{N}, t = 0, 1, \dots, T, \quad (4)$$

where  $T$  is the total number of generations, and  $r = r(t)$  is a function of the current generation  $t$ . The value of  $r$  is adapted during the process of the algorithm.

But unfortunately, there is no best mechanism on how to specify the value of  $r$  dynamically that outperforming the others in all problems and all experiment settings. One mechanism may be the best when working on some specific experiment conditions, but may not be good on other conditions.

### A. Linearly Decreasing Mechanism Proposed in [15]

Based on the description above,  $r$  is suggested to be specified dynamically at different stages of the algorithm (at the early stage, a larger  $r$  is specified; at the final stage,  $r = 1 + 1/H$  is specified).

In [15], a linearly decreasing mechanism has been proposed:

$$r(t) = r_{Initial} \frac{(T-t)}{T} + (1 + 1/H) \frac{t}{T}, t = 0, 1, \dots, T, \quad (5)$$

where  $T$  is the total number of generations, and  $r_{Initial}$  is the initial value of  $r$ , which is larger than  $1 + 1/H$ . It is a simple and practical mechanism. In Eq. (5), the value of  $r$  starts from  $r_{Initial}$ , then gradually decreases to the suggested value in a linearly decreasing process.

In the next section, another dynamic mechanism based on weak convergence detection criterion is proposed. We show that it outperforms the simple linearly decreasing mechanism on some specific problems.

### B. A New Dynamic Reference Point Specification Mechanism

In this section, we will introduce a new mechanism that uses a weak convergence detection criterion to decide when to change the value of  $r$  from  $r_{Initial}$  to  $1 + 1/H$ .

As we have explained before, a larger  $r$  is suggested at the early stage of the algorithms. But for good diversity at the final stage, it is needed to set  $r$  to its suggested value ( $1 + 1/H$ ). For this purpose, it is necessary to detect whether the algorithm is converged. If solutions are all close to the PF, we change the value of  $r$  to  $r_{Optimal}$ ; otherwise, we set the value of  $r$  to  $r_{Initial}$ . The mechanism is shown below:

$$r(t) = \begin{cases} r_{Initial}, & t < t_{Converged} \\ 1 + 1/H, & t \geq t_{Converged} \end{cases} \quad t = 0, 1, \dots, T. \quad (6)$$

$r(t)$  equals to  $r_{Initial}$  before reaching the converged generation  $t_{Converged}$ , and changes to  $1 + 1/H$  after  $t_{Converged}$ .  $t_{Converged}$  is determined by a weak convergence detection criterion.

Various indicators including convergence detection indicators which are used to detect the stagnation have been proposed in the literature [23]–[29]. They focus on the accuracy of convergence, which is not the purpose in our approach. After algorithm converged, we still need some generations in order to get a uniform distribution of solution set. We summarize our weak convergence detection criterion as follow:

1) *Inaccuracy*: It is not necessary to have an accurate convergence detection. The convergence can be reported if current solutions are close to the PF. In other words, the estimated ideal and nadir points based on the current solutions are close to the true ideal and nadir points.

2) *Saving time*: We should not spend too much time in convergence detection for the reason that the state-of-the-art HV-based algorithms such as SMS-EMOA and HypE, are time-consuming when the dimension of the objective space is very high.

HV is not a good choice of our weak convergence detection indicator. The reason is that during the process of algorithm,

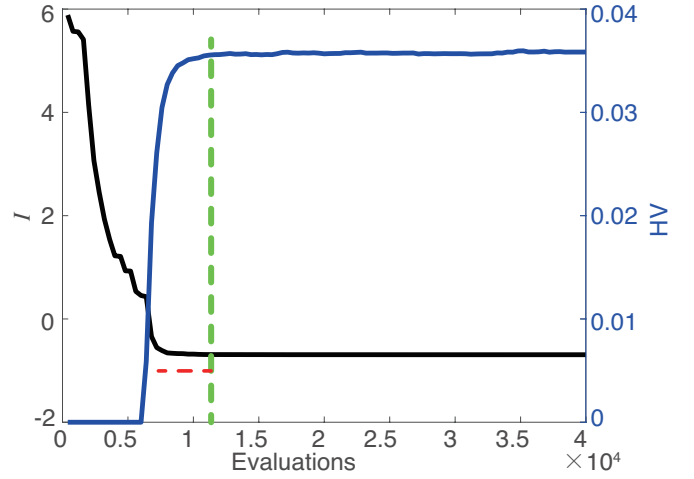


Fig. 5: Example of nadir point and HV value on the 3-objective inverted-DTLZ1 with SMS-EMOA. The blue curve is the change of HV while the black curve is the change of the indicator  $I$ : best logarithmic nadir point so-far. The green dotted line shows the evaluation where the convergence is detected and the red dotted line shows the considered window for evaluated number  $t_{Converged}$ .

the reference point is calculated by Eq. (1), which means that they are different among generations. So, we can not simply compare HV calculated in algorithm among different generations.

We should consider other good indicators satisfying our convergence detection criterions. In the progress of the algorithm, the HV of the current solution set increases while the estimated nadir point of the current solution set is gradually approaching the PF. The estimated nadir point can be a good choice for our purpose. More specifically, for a minimization problem, we consider the indicator  $I$  as follows:

$$\begin{aligned} \mathbf{N}_t &= [f_{t1}, f_{t2}, \dots, f_{tm}]^\top \in \mathbb{R}^m, \\ I_0 &= \frac{1}{m} \sum_{i=1}^m \ln f_{0i}, \\ I_t &= \min(I_{t-1}, \frac{1}{m} \sum_{i=1}^m \ln f_{ti}), t = 1, 2, \dots, T, \end{aligned} \quad (7)$$

where  $T$  is the total number of generations,  $\mathbf{N}_t$  is the estimated nadir point at the  $t^{th}$  generation with  $m$  elements:  $f_{t1}, f_{t2}, \dots, f_{tm}$ .  $I_0$  is the average value of logarithmic nadir point calculated by the initial population. And  $I_t$  is the minimum value of  $I$  before the  $t^{th}$  generation (including the  $t^{th}$  generation). Fig. 5 shows the change of HV and indicator  $I$  of SMS-EMOA on 3-objective inverted-DTLZ1. When current solutions are close to the PF, the estimated nadir point of current solutions is close to the true nadir point.

After choosing the indicator, the next step is to detect the stagnation of the indicator. We use a basic linear regression method called Simple Least Squares [17] with a simple least squares convergence detection strategy introduced in [24]. If

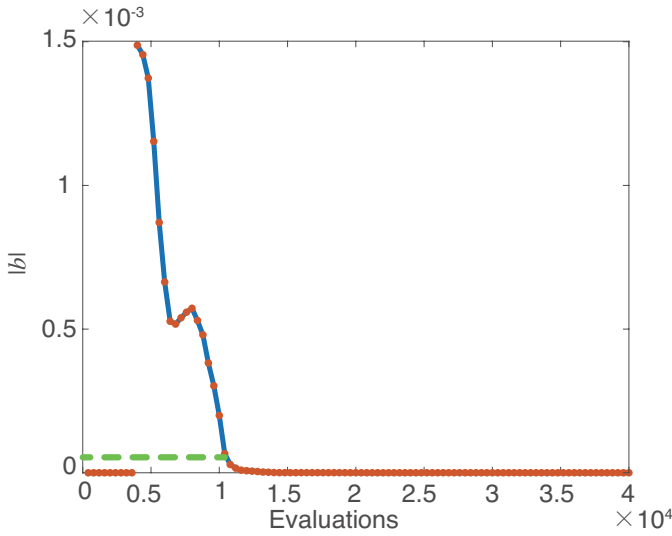


Fig. 6: Example of  $|b|$  on the 3-objective inverted-DTLZ1 (window size  $w_l = 4000$  and population size  $\mu = 100$ ). The green dotted line shows the threshold.

the absolute value of the slope of the linear regression is below a threshold, the convergence is reported. Briefly speaking, considering a simple linear regression model  $I(t) = a + bt$ , the intercept  $a$  and slope  $b$  of the  $t^{th}$  generation can be calculated with the following matrix-based formula:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum t_i^2 & \sum t_i \\ \sum t_i & w_l \end{bmatrix}^{-1} * \begin{bmatrix} \sum t_i * I_{t_i} \\ \sum I_{t_i} \end{bmatrix}, \quad (8)$$

where  $w_l$  is the length of the chosen window (in the example of Fig. 5, the chosen window for  $t_{Converged}$  is represented with the red dotted line), and  $t_i$  is the evaluated number in the chosen window, which holds the following condition:  $t_i \in (t', t], t - t' = w_l$ . The value of slope  $b$  is shown in Fig. 6 (Note that the value in the first  $w_l$  evaluations is 0, and we should not consider the first  $w_l$  evaluations).

With the above Eq. (8), we report the convergence if the following condition holds:

$$|b| < thres. \quad (9)$$

The value of  $t_{Converged}$  equals to  $t$  (i.e., the current evaluation number) when the convergence is first detected. The whole process of weak convergence detection is also clearly described in Algorithm 1. If Algorithm 1 returns True, we report the convergence, vice versa. The choice of the  $thres$  value is trivial as the report ahead or delay is not fatal to the algorithm or the final solution set. We choose  $thres$  value as  $10^{-5}$  after some experimental computation with the window size  $w_l = 4000$ . If we do not change the window size, this threshold can be applied to other problems or other HV-based algorithms because of a weak convergence detection purpose.

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#### Algorithm 1: Weak Convergence Detection

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**Input:**

$w_l$ , // Window size.  
 $S_t$ , // Solution set when evaluation number is  $t$ .  
 $I = \{I_0, I_1, \dots, I_{t-1}\}$ , // Stored indicator values.  
 $thres$ , // The chosen threshold for slope.

**Output:** Return True if converged, otherwise return False.

Calculate nadir point  $N_t$  of  $S_t$ ;  
 Calculate indicator  $I_t$ ; // By Eq. (7).

$I \leftarrow I \cup \{I_t\}$ ;

**if**  $t \geq w_l$  **then**

    Calculate  $b$ ; // By Eq. (8).

**if**  $|b| < thres$  **then**

        Return True; // Converged.

**end if**

**end if**

Return False; // Not converged.

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## IV. COMPUTATIONAL EXPERIMENTS

### A. Experimental Settings

In this section, the two different dynamic reference point specification mechanism (i.e., the linearly decreasing mechanism and the weak convergence detection mechanism) are tested with the state-of-the-art algorithm SMS-EMOA [7]. The DTLZ test suite [30], WFG test suite [31], their minus-versions [32], and Multi-Point Distance Minimization Problem (MPDMP) [18] are used in this experiment. We consider the problems with 10 objectives. All the code in this section is implemented in PlatEMO framework [20] with the following settings:

- Population size: 30 ( $H = 1$ ),
- Total evaluation number: 100,000 solution evaluations,
- Initial value of  $r$  ( $r_{Initial}$ ): 10,
- Crossover: Simulated binary crossover,
- Crossover probability: 1.0,
- Mutation: Polynomial mutation,
- Mutation probability:  $1/D$ ,
- Distribution index of Crossover and Mutation: 20,
- Number of decision variables  $D$ :
  - 14 (DTLZ1 and minus-DTLZ1),
  - 2 (MPDMP),
  - 19 (other problems),
- Number of runs: 20 runs.

### B. Computational Results

In our experiments, four versions of SMS-EMOA algorithm with different reference point specification mechanisms are considered. These algorithms are named as SMS-EMOA-10, SMS-EMOA-Opt, SMS-EMOA-LD, and SMS-EMOA-CD. For SMS-EMOA-10, the value  $r$  is set to 10. The value  $r$  for SMS-EMOA-opt is set to  $r = 1 + 1/H$ . SMS-EMOA with the linearly decreasing mechanism and with the weak convergence detection mechanism are referred to as SMS-EMOA-LD and SMS-EMOA-CD, respectively. The HV values



TABLE I: Mean HV and standard deviation over 20 independent runs for DTLZ and WFG test problems.

Problem	$M$	$D$	SMS-EMOA-10	SMS-EMOA-Opt	SMS-EMOA-LD	SMS-EMOA-CD
DTLZ1	10	14	6.8448e-1 (4.61e-1) $\approx$	2.0261e-1 (2.23e-1) $-$	<b>8.8369e-1 (2.00e-1) <math>\approx</math></b>	6.9062e-1 (3.95e-1)
DTLZ2	10	19	<b>1.0234e+3 (4.72e-1) <math>\approx</math></b>	9.9315e+2 (3.94e+1) $-$	1.0234e+3 (7.28e-1) $\approx$	1.0184e+3 (1.85e+1)
DTLZ3	10	19	0.0000e+0 (0.00e+0) $\approx$	<b>1.9068e+1 (8.53e+1) <math>\approx</math></b>	0.0000e+0 (0.00e+0) $\approx$	0.0000e+0 (0.00e+0)
DTLZ4	10	19	<b>8.0029e+2 (2.21e+2) <math>\approx</math></b>	5.2691e+2 (1.79e+2) $-$	6.6262e+2 (2.17e+2) $\approx$	7.2840e+2 (2.26e+2)
WFG1	10	19	2.2025e+12 (2.55e+11) $\approx$	2.2597e+12 (2.64e+11) $\approx$	<b>2.2005e+12 (1.81e+11) <math>\approx</math></b>	<b>2.2601e+12 (3.42e+11)</b>
WFG2	10	19	<b>3.3604e+12 (2.92e+10) <math>\approx</math></b>	3.3493e+12 (2.39e+10) $+$	3.3487e+12 (7.64e+10) $\approx$	<b>3.3420e+12 (8.46e+10)</b>
WFG3	10	19	4.7217e-3 (1.04e-2) $-$	<b>2.5310e-2 (3.07e-2) <math>\approx</math></b>	1.7230e-2 (2.35e-2) $\approx$	2.3443e-2 (2.84e-2)
WFG4	10	19	3.7667e+12 (1.77e+10) $\approx$	<b>3.5602e+12 (9.48e+10) <math>-</math></b>	3.7585e+12 (3.74e+10) $\approx$	<b>3.7737e+12 (1.63e+10)</b>
WFG5	10	19	3.6535e+12 (7.68e+9) $\approx$	<b>3.5027e+12 (1.16e+11) <math>-</math></b>	<b>3.6555e+12 (8.64e+9) <math>\approx</math></b>	3.6523e+12 (1.93e+10)
WFG6	10	19	3.6082e+12 (7.78e+10) $\approx$	<b>3.5432e+12 (6.24e+10) <math>-</math></b>	3.5829e+12 (5.75e+10) $\approx$	<b>3.6099e+12 (4.49e+10)</b>
WFG7	10	19	<b>3.7967e+12 (7.58e+9) <math>\approx</math></b>	<b>3.7326e+12 (5.73e+10) <math>-</math></b>	3.7959e+12 (1.35e+10) $\approx$	3.7922e+12 (1.27e+10)
WFG8	10	19	3.7512e+12 (1.51e+10) $\approx$	<b>3.7087e+12 (3.64e+10) <math>-</math></b>	<b>3.7613e+12 (1.13e+10) <math>+</math></b>	3.7524e+12 (1.23e+10)
WFG9	10	19	3.5727e+12 (1.85e+11) $\approx$	<b>3.3072e+12 (2.80e+11) <math>-</math></b>	<b>3.6302e+12 (1.57e+11) <math>+</math></b>	3.5146e+12 (2.31e+11)
$+/-/\approx$			0/1/12	1/9/3	2/0/11	

TABLE II: Mean HV and standard deviation over 20 independent runs for minus-DTLZ, minus-WFG and MPDMP.

Problem	$M$	$D$	SMS-EMOA-10	SMS-EMOA-Opt	SMS-EMOA-LD	SMS-EMOA-CD
minus-DTLZ1	10	14	4.3889e+28 (5.56e+26) $\approx$	<b>4.4237e+28 (5.99e+26) <math>\approx</math></b>	4.4120e+28 (4.79e+26) $\approx$	<b>4.3872e+28 (7.72e+26)</b>
minus-DTLZ2	10	19	1.2299e+7 (1.38e+5) $-$	1.4737e+7 (1.55e+5) $-$	<b>1.4891e+7 (1.11e+5) <math>\approx</math></b>	1.4844e+7 (1.45e+5)
minus-DTLZ3	10	19	1.1227e+35 (3.16e+33) $-$	1.3385e+35 (3.41e+33) $\approx$	<b>1.3702e+35 (3.42e+33) <math>\approx</math></b>	1.3582e+35 (3.78e+33)
minus-DTLZ4	10	19	1.1638e+7 (2.11e+5) $-$	<b>1.4899e+7 (1.20e+5) <math>+</math></b>	1.4895e+7 (1.27e+5) $+$	1.3585e+7 (1.57e+6)
minus-WFG1	10	19	4.4808e+10 (5.25e+8) $\approx$	<b>4.4894e+10 (4.81e+8) <math>\approx</math></b>	4.4890e+10 (4.35e+8) $\approx$	<b>4.4705e+10 (5.60e+8)</b>
minus-WFG2	10	19	6.6225e+10 (1.05e+8) $-$	6.7628e+10 (7.51e+8) $\approx$	6.7870e+10 (3.31e+7) $-$	<b>6.7907e+10 (1.60e+7)</b>
minus-WFG3	10	19	6.3620e+10 (5.49e+8) $\approx$	<b>6.4321e+10 (2.29e+8) <math>+</math></b>	6.3642e+10 (4.93e+8) $\approx$	6.3799e+10 (5.87e+8)
minus-WFG4	10	19	1.6478e+11 (3.22e+9) $-$	<b>1.9896e+11 (2.20e+9) <math>\approx</math></b>	1.9752e+11 (2.39e+9) $\approx$	1.9260e+11 (1.34e+10)
minus-WFG5	10	19	1.6534e+11 (2.09e+9) $-$	1.9817e+11 (1.82e+9) $\approx$	<b>1.9897e+11 (1.93e+9) <math>\approx</math></b>	1.9819e+11 (1.40e+9)
minus-WFG6	10	19	1.6532e+11 (2.11e+9) $-$	1.9865e+11 (1.83e+9) $\approx$	<b>1.9989e+11 (2.03e+9) <math>\approx</math></b>	1.9960e+11 (1.94e+9)
minus-WFG7	10	19	1.6534e+11 (2.88e+9) $-$	1.9870e+11 (1.30e+9) $-$	<b>1.9978e+11 (2.21e+9) <math>\approx</math></b>	1.9937e+11 (2.97e+9)
minus-WFG8	10	19	1.6808e+11 (1.46e+9) $-$	1.9932e+11 (1.56e+9) $-$	<b>2.0220e+11 (1.71e+9) <math>+</math></b>	2.0065e+11 (1.77e+9)
minus-WFG9	10	19	1.6415e+11 (3.16e+9) $-$	1.9777e+11 (2.21e+9) $\approx$	1.9557e+11 (2.17e+9) $-$	<b>1.9852e+11 (1.86e+9)</b>
MPDMP	10	2	1.4848e+5 (2.50e+2) $-$	1.5051e+5 (3.10e+2) $-$	1.5017e+5 (1.98e+2) $-$	<b>1.5097e+5 (1.64e+2)</b>
$+/-/\approx$			0/11/3	2/4/8	2/3/9	

after 100,000 evaluations are obtained for each algorithm. TABLE I and TABLE II show the computational results for HV metrics. The best result in each row is highlighted in bold, and the worst result is shaded. The Wilcoxon rank sum test is used to show the statistical significance for SMS-EMOA-10, SMS-EMOA-Opt, SMS-EMOA-LD in comparison to our proposed SMS-EMOA-CD. The three symbols “+”, “-”, “ $\approx$ ” mean significantly better, significantly worse and no significant difference.

In Table I, the results show that SMS-EMOA-Opt performs the worst (9 out of 13 significantly worse than SMS-EMOA-CD) among the algorithms. This result shows that when applying  $r = 1 + 1/H$  all the time, bad searching behavior is obtained. We can not tell the differences between SMS-EMOA-10 and SMS-EMOA-Opt, as the Wilcoxon rank sum tests show almost all the results are “ $\approx$ ”. This is probably because of the small influence of the reference point on the triangular PF problems. As for SMS-EMOA-LD, two better results are obtained when compared to SMS-EMOA-CD. This

indicates that SMS-EMOA-LD is slightly better than SMS-EMOA-CD on triangular PF problems.

Table II shows the results obtained for inverted-triangular PF problems (i.e., minus-DTLZ1-4, minus-WFG1-9, and MPDMP). The results show that SMS-EMOA-10 performs the worst (12 out of 14) on most of the inverted-triangular PF problems, and the Wilcoxon rank sum tests show that almost all the results from SMS-EMOA-10 are significantly worse than SMS-EMOA-CD (11 out of 14 worse and no better results). This result can be explained by the setting of  $r$ . Since the value of  $r$  is set to 10 for the whole process of SMS-EMOA-10, many solutions are on the boundary of PF. Solutions on the boundary of PF As compared to SMS-EMOA-CD, the performances of SMS-EMOA-Opt (2 better but 4 worse results) and SMS-EMOA-LD (2 better but 3 worse results) are slightly worse on inverted-triangular PF problems. The results of MPDMP shows that SMS-EMOA-CD is the best among the four algorithms.

Fig. 7 shows the plot of HV for the four mechanisms on the

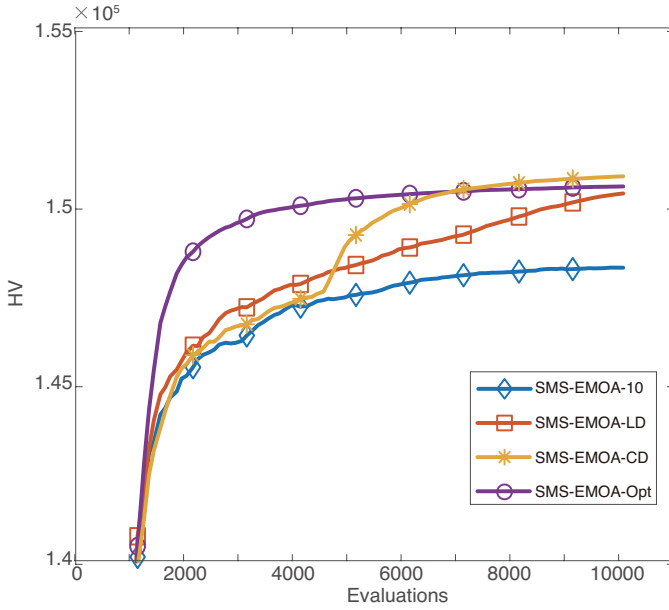


Fig. 7: The comparison of four mechanisms. The value of each point is the average of 20 independent runs in each evaluation and the problem is the 10-objective MPDMP in TABLE II.

10-dimensional MPDMP problem. In Fig. 7, the HV of SMS-EMOA-LD (the red curve) gradually increases and finally reaches the same level as SMS-EMOA-Opt, as the value of  $r$  is gradually decreased to  $1 + 1/H$ . The HV of SMS-EMOA-CD (the yellow curve) firstly reaches a stable level similar to SMS-EMOA-10, for that their values of  $r$  are both 10 before 4,500 evaluations. The convergence detection is reported at about 4,500 evaluations for SMS-EMOA-CD. Then, the values of  $r$  in SMS-EMOA-CD reaches the optimal value of  $1 + 1/H$  and the HV increases. The reason is due to the decrease of the boundary solutions, and on the other hand, the inner solutions increase. Finally, the HV of SMS-EMOA-CD is better than SMS-EMOA-Opt and SMS-EMOA-LD.

## V. CONCLUSIONS

In this paper, we emphasize the importance of reference point specification in SMS-EMOA by a simple example. We have demonstrated that without a good reference point specification mechanism, a poor diversity of the final solutions on inverted-shape PF problems will be obtained. This phenomenon is hardly observed on the triangular PF problems when the reference point is worse than the nadir point. We introduced the dynamic reference point specification mechanism with the illustration by two aspects:

1) *Better Searching Behavior*: In the early stage, the solutions may be far away from the true PF. A larger value of  $r$  can achieve better searching behavior. We give an example of DTLZ2.

2) *Uniform Distribution*: Considering the linear triangular problems, the optimal setting of the reference point is  $r = 1 + 1/H$ .

We summarize the basic idea of dynamic reference point specification mechanism as follows: the value of  $r$  should be specified larger than  $1 + 1/H$  at the initial stage and equal to  $1/(1 + H)$  in the final stage, as in Eqs. (4)-(6).

In this paper, we have proposed a new dynamic reference point specification mechanism. A weak convergence detection mechanism is used in the proposed method. The new dynamic reference point specification mechanism is tested on SMS-EMOA algorithm and compared with other SMS-EMOAs under different settings of  $r$ . The results show that SMS-EMOA with  $r = 1 + 1/H$  performs the worst on the triangular PF problems, and SMS-EMOA with  $r = 10$  performs the worst on the inverted-triangular PF problems. This gives us a hint on the necessary of the dynamic mechanism. We have also compared our proposed mechanism with the linearly decreasing mechanism. The results show that our new mechanism outperforms the linearly decreasing mechanism on some test problems, specifically on the MPDMP problem.

In the future, we plan to investigate the behavior of our new mechanism and further improve it. Besides that, the problems with different PF shapes will be tested and analyzed.

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