Responses to reviewer 13

Corrections to your comments are highlighted in colors ■ and ■. In the revised version, we have made the following changes to the symbols and terminologies used in the manuscript.

Original version	Revised version
Controller,disturbance	Player I, Player II
$a \in \mathcal{A}$, $d \in \mathcal{D}$	$a \in \mathcal{A}$, $b \in \mathcal{B}$
$\mathscr{A}$, $\mathscr{D}$	$\mathscr{A}_0$, $\mathscr{B}_0$
Given time horizon $\bar{T}$	T or $T_1, ..., T_M$
Given admissible cost $\bar{J}$	J or $J_1, ..., J_M$
$\phi_{t_0}^{t_1}(., s_0, a, d)$	$\phi_{s,t}^{a,b}(.)$
$\min\limits_{a(.)} \max\limits_{d(.)} \ t_f$	$\sup\limits_{\beta(.)} \inf\limits_{a(.)} \ t_f$ (Emphasize that Player II plays nonanticipative strategies)
$J_{t_0}^{t_1}(s_0, a, d)$	$\mathcal{J}_{t_0}^{t_1}(s, a, b)$
$\min\limits_{a(.)} \max\limits_{d(.)} \ J_{t_0}^{t_1}(s_0, a, d)$	$\sup\limits_{\beta(.)} \inf\limits_{a(.)} \ \mathcal{J}_{t_0}^{t_1}(s, a, \beta[a])$
Given final time t_f	$\bar{T}$ (The symbol of the free final time t_f remains unchanged)
$\widehat{W}_{t_f}^{c}(.)$	$W_{\bar{T}}^{c}(., 0)$, $\left(W_{\bar{T}}^{c}(s, \tau) = \sup\limits_{\beta(.)} \inf\limits_{a(.)} \ \mathcal{J}_{\tau}^{\bar{T}}(s, a, \beta[a]) \right)$

1. line 3 below (1): the mapping f is not fully defined.

Response: In the revised version, we have fixed this mistake

2. below (5) the dissipation function and the Hamiltonian are mentioned, but they are not defined. Please add a definition for the sake of completeness.

Response: In the level set method, the solution of the HJ PDE is not differentiable everywhere. Therefore, a viscosity solution is introduced. The spatial derivative is replaced by the left and right derivatives [2, 3], take the direction along x axis as an example:

$$\min_{a \in \mathcal{A}} \max_{b \in \mathcal{B}} p_x f_x(s, a, b) = \min_{a \in \mathcal{A}} \max_{b \in \mathcal{B}} \left[\frac{p_x^+ + p_x^-}{2} f_x(s, a, b) \right] - \alpha_x \frac{p_x^+ - p_x^-}{2} \tag{1}$$

where $p_x = \frac{\partial V}{\partial x}$, $f_x(s, a, b)$ is the component of $f(s, a, b)$ along x axis, p_x^- and p_x^+ are the left and right derivatives, respectively, α_x is the dissipation function, and

$$\alpha_x = \max \left| \frac{\partial}{\partial p_x} \left[\min_a \max_b p^{\mathrm{T}} f(s, a, b) \right] \right| \tag{2}$$

From the above equation, to compute α_x, it is necessary to traverse the set $\mathcal{A}$ and $\mathcal{B}$, as well as the entire computational domain, because the choice of a and b is related to p. For the sake of simplicity, consider the following equation:

$$\max_b p^{\mathrm{T}} f(s, b) \tag{3}$$

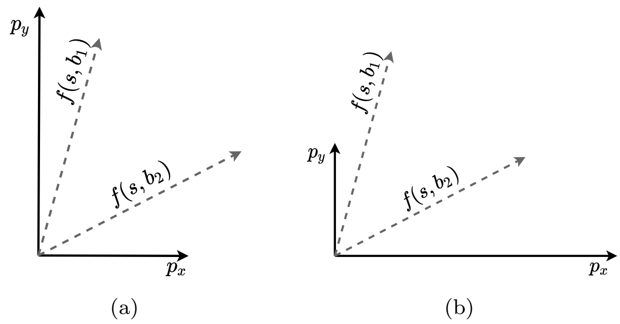

Figure 1:

In the case shown in Fig. 1(a), $\arg\max_{b\in\{b_1,b_2\}} p^{\mathrm{T}} f(s,b) = b_1$, in the case shown in Fig. 1(b), $\arg\max_{b\in\{b_1,b_2\}} p^{\mathrm{T}} f(s,b) = b_2$. A too small α_x will lead to instability of the viscous solution and too large α_x will lead to computational errors. Therefore, overestimation of α_x is the preferable choice. If the system is of affine nonlinear form, i.e.,

$$f_x(s,a,d) = f_x^s(s) + g_1(s)a + g_2(s)b \tag{4}$$

α_x can be overestimated by [2]:

$$\alpha_x \leq |f_x^s(s)| + \max\left[|g_1(s)|\, a_{\max}, |g_1(s)|\, a_{\min}\right] + \max\left[|g_2(s)|\, b_{\max}, |g_2(s)|\, b_{\min}\right] \tag{5}$$

As previously mentioned, the method in this paper does not involve viscosity solutions, and there is no need to compute the dissipation function. Since the reason for not having to compute the dissipation function is that this paper uses a recursion-interpolation-based approach, and the use of recursion and interpolation cannot be considered as an advantage of the proposed method, the description of the dissipation function has been removed in the revised version. In the revised version, we put more emphasis on the properties of the value function itself, rather than on the method of computing the value function.

3. In (6) add a space in the objective function between max and t_f

Response: This problem has been fixed in the revised version.

4. The concept of cost-limited backward reachable sets is well-explained. However, I believe it could also be realized using the standard techniques in Section 2.1. To this end, introduce a new state z, say, and differential equation $\dot{z} = c(s,a), z(t_0) = 0$. Then, $z(t_1)$ is the cost accumulated until t_1. With $K = \{(s,z) \in \mathbb{R}^{n+1} | l(s) \leq 0, z \leq \bar{J}\}$ this would fit into the theory presented in Section 2.1 (with the initial state of z fixed to 0, so the state dimension in the value function will not increase). What is the actual benefit of the approach in Section 2.2 and Section 3? Please comment!

Response: The practice of adding a new state z is theoretically valid. However, solving the HJ PDE requires iteration over all grid points in the computational domain (although we are interested in only a fraction of these grid points, i.e., those that satisfy $z = 0$). Because the values of the function V at each grid point interact with each other, it is necessary to estimate the spatial derivative using the difference between them. One of the developers of the Level Set Toolbox has published a paper [1] in recent years. This paper studied the feasible region for forced aircraft landings. After simplification, the state space contains four states: Position of the aircraft (x,y,z), where z is the altitude, and heading angle of the aircraft Ψ. The authors are interested in the feasible region of the landing, which is a subset of the region satisfying $z = 0$. However, the authors still compute the reachable tube in the entire four-dimensional state space, and then find the intersection of this reachable tube and the $z = 0$ region. Therefore, adding a dimension to the state space significantly increases the number of grids and thus the amount of computation.

5. Is t_f free in (10)? Then it should be added as an optimization variable.

Response: Yes, t_f is free. In the revised version, we have emphasized this point.

6. What is γ in Theorem 1?

Response: We assume $\min_{s\in\mathbb{R}^n, a\in\mathcal{A}} c(s,a) = \gamma$, and γ is a positive number. See Assumption 1 in the manuscript.

7. line 2 below (24): D = [-1,1]

Response: This problem has been fixed in the revised version.

8. line 1 below (25): analyse

Response: This problem has been fixed in the revised version.

9. The numerical results in Table 1 suggest that the numerical scheme is not appropriate as no convergence can be observed (neither in time nor in space, see last row). This behaviour should be investigated carefully and more thoroughly. Given the data in Table 1, the results have to be considered not very reliable.

Response: In the revision, we replaced the time-discrete form from forward Euler method to fourth-order Runge-Kutta method. The results show that the error decreases as the grid size decreases, and the convergence rate is comparable to that of the level set method. see Fig. 4 in the revision.

10. The game aspect is not very prominent throughout the paper. I suggest to modify the title to, e.g. "Cost-limited reachability: A new problem in reachability analysis".

Response: In the modified version, we have added an elaboration of the information pattern in the differential game. It also highlights the information pattern we adopted, and the reasons for adopting it.

References

[1] Anayo K Akametalu, Claire J Tomlin, and Mo Chen. Reachability-based forced landing system. *Journal of Guidance, Control, and Dynamics*, 41(12):2529–2542, 2018.

[2] Ian M Mitchell. A toolbox of level set methods. *UBC Department of Computer Science Technical Report TR-2007-11*, 2007.

[3] Stanley Osher and Ronald Fedkiw. *Level set methods and dynamic implicit surfaces*, volume 153. Springer Science & Business Media, 2006.