# Responses to reviewer 20

Corrections to your comments are highlighted in colors ■ and ■. In the revised version, we have made the following changes to the symbols and terminologies used in the manuscript.

| Original version | Revised version |
|---|---|
| Controller,disturbance | Player I, Player II |
| $a \in \mathcal{A}$, $d \in \mathcal{D}$ | $a \in \mathcal{A}$, $b \in \mathcal{B}$ |
| $\mathscr{A}$, $\mathscr{D}$ | $\mathscr{A}_0$, $\mathscr{B}_0$ |
| Given time horizon $\bar{T}$ | $T$ or $T_1, ..., T_M$ |
| Given admissible cost $\bar{J}$ | $J$ or $J_1, ..., J_M$ |
| $\phi_{t_0}^{t_1}(., s_0, a, d)$ | $\phi_{s,t}^{a,b}(.)$ |
| $\min\limits_{a(.)} \max\limits_{d(.)} \ t_f$ | $\sup\limits_{\beta(.)} \inf\limits_{a(.)} \ t_f$ (Emphasize that Player II plays nonanticipative strategies) |
| $J_{t_0}^{t_1}(s_0, a, d)$ | $\mathcal{J}_{t_0}^{t_1}(s, a, b)$ |
| $\min\limits_{a(.)} \max\limits_{d(.)} \ J_{t_0}^{t_1}(s_0, a, d)$ | $\sup\limits_{\beta(.)} \inf\limits_{a(.)} \ \mathcal{J}_{t_0}^{t_1}(s, a, \beta[a])$ |
| Given final time $t_f$ | $\bar{T}$ (The symbol of the free final time $t_f$ remains unchanged) |
| $\widehat{W}_{t_f}^{c}(.)$ | $W_{\bar{T}}^{c}(., 0)$, $\left(W_{\bar{T}}^{c}(s, \tau) = \sup\limits_{\beta(.)} \inf\limits_{a(.)} \ \mathcal{J}_{\tau}^{\bar{T}}(s, a, \beta[a])\right)$ |

1. mathematical definitions for reachability and dynamic games are wrong. For example, eq (2) introduces the BRT. This equation says, there exists $t$, for all $d(.)$, there exists $a(.)$ such that $\phi_0^{\bar{T}}(t, s_0, a, d) \in K$ This definition implies that the disturbance signal $d(.)$ for all time first plays and the control signal $a(.)$ second plays. In other words, $a(.)$ chooses its action with knowing the information of $d(.)$ for all time. This is not what the authors want. In the dynamic game, at each time $t$, if $a(t)$ first plays, then $d(t)$ second plays. To have a better mathematical expression, the concept of the non-anticipative strategy has been developed. The below paper first provided rigorous proof of HJI PDE with the non-anticipative strategy. The reviewer thinks that the authors should use the non-anticipative strategy. Note that, if the authors stick to use the definition in eq (2) and (3), (5) is wrong.
In addition, the order of the arguments in eq (2) is wrong. The correct definition is the BRT is the set of states such that for all disturbance's non-anticipative strategy, there exists a control signal $a(.)$ such that there exists a time $t$: the state trajectory at $t$ is in the set $K$. The authors need to be very careful with ordering arguments. This mistake also made in eq (9).

**Response:** In the revision, we have added a description of the information pattern and emphasized that the information pattern in this paper is that Player II plays the non-anticipative strategy. We also describe the reasons for choosing this information pattern from a safety perspective. After introducing the information pattern, we fixed these wrong equations.

2. Mathematical definition of W in eq (6) is wrong. The correct definition is $W(s_0) = \min\limits_{a(.)} \max\limits_{d(.)} t_f$ subject to the five following constraints without having the condition: 'if for all $d(.)$,.....'. The definition (6) means that, for $s$ such that for all $d(.)$ there exists $a(.)$ such that the state trajectory gets in $K$ at some time, find another minimax solution pair $(a(.), d(.))$ with respect to the cost $t$). Also, the 'otherwise' case with the infinity output is not necessary since if, for all $d(.)$, there is no $a(.)$ such that the state trajectory gets in $K$, $W(s_0)$ is obviously infinity. This mistake is also made in eq (10).

**Response:** In the revision, we removed the conditions "if for all $d(.)$,....." and "otherwise...", and introduced the information pattern. However, we emphasize in the context of these equations the case where these value function take infinity.

3. The proof for Theorem 1 is not rigorous. In eq (18), the last equality is not obvious to skip. The reviewer checked that the last equality is only true if $\tilde{J} < \gamma t_f$. Otherwise, this equality is not true.

**Response:** Yes, here we make a clerical error. The "Therefore, $\widehat{W}_{t_f}^{c}(s_0) < \tilde{J} \leq \gamma t_f$..." should be

"Therefore, $\widehat{W}^c_{t_f}(s_0) \leq \tilde{J} < \gamma t_f ...$". It is important to mention here that, in the original manuscript, the meanings of "$t_f$" in Eq. (6) and Eq. (10) are different from those in Eq. (15), (16), (17) and (18). The former are the free, not given, while the later are not free, but given. To avoid ambiguity, in the revision, the given $t_f$ is replaced with $\bar{T}$, and $\widehat{W}^c_{t_f}(.)$ is replaced with $W^c_{\bar{T}}(.,0)$. Take Example 1 in the manuscript as an example, the following animation visually displays this theorem (opening this pdf file with software "Adobe Acrobat Pro DC" will play this animation). It can be seen that as $\bar{T}$ increases, the range that makes function $W^c_{\bar{T}}(.,0)$ and function $W^c(.)$ equal becomes larger and larger,

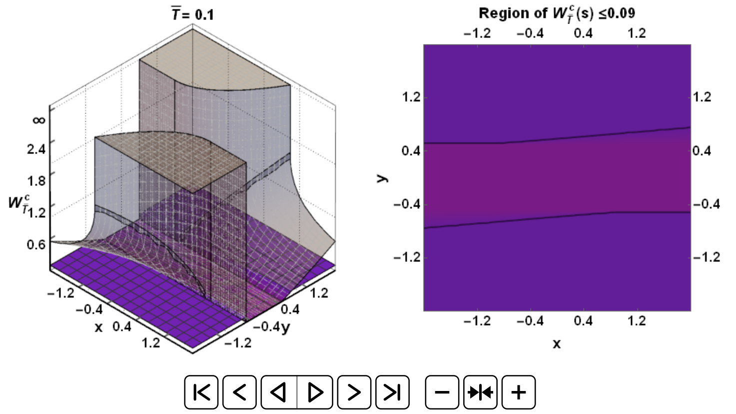



Figure 1: The variation of function $W^c_{\bar{T}}(.,0)$ with $\bar{T}$.

In the animation, the gray surface represents the analytic expression of function $W^c(.)$, i.e.:

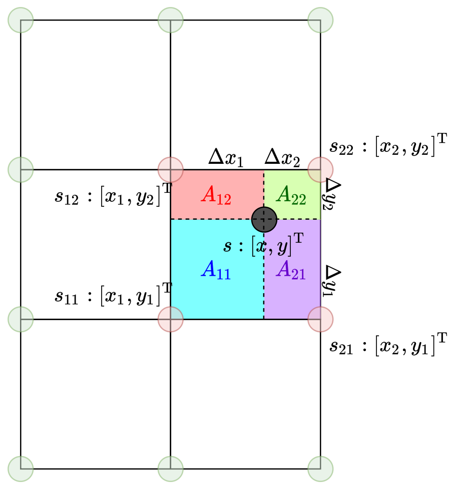
$$W^c(s) = \begin{cases} -\sqrt{x^2+2x-2y+2}+x+1 & x^2+2x-2y+2 \geq 0 \wedge y > 0.5 \wedge x \geq -1 \\ -\sqrt{x^2-2x+2y+2}-x+1 & x^2-2x+2y+2 \geq 0 \wedge y < -0.5 \wedge x \leq 1 \\ 0 & |y| \leq 0.5 \\ \infty & \left(x^2+2x-2y+2 < 0 \vee x < -1\right) \wedge y \geq 0.5 \\ \infty & \left(x^2-2x+2y+2 < 0 \vee x > 1\right) \wedge y \leq -0.5 \end{cases} \tag{1}$$

4. The reviewer understands what eq (20) means, but this can be understood by experts in this domain. This mathematical description is not clear. The reviewer suggests checking above Evans's paper that has a very similar equation with clear mathematical explanations.

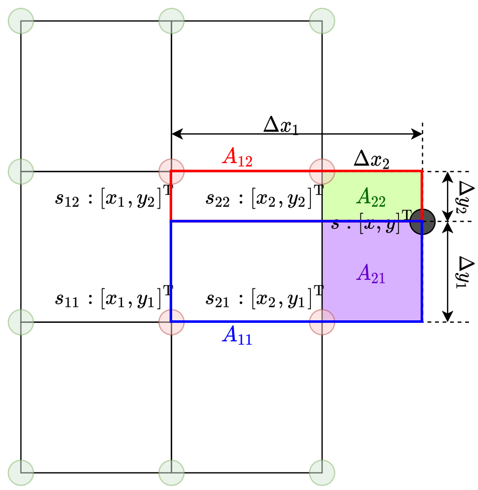
**Response:** In the revision, we refer to Theorem 3.1 of Evans's paper and improve the derivation process.

5. The numerical algorithm does not explain how to handle the state out of the grid window. For example, $\widehat{F}(s_0, a, d)$ can be placed out of the grid window. Do the authors use any extrapolation? Handling this issue is also very carefully considered in the level set methods. The reviewer suggests checking the following book.

**Response:** This is handled with reference to the Level Set Toolbox. The linear extrapolation is applied. Take a two dimensional problem as an example:




(a) $s$ lies inside the computational domain.




(b) $s$ lies outside the computational domain.

Figure 2: Linear extrapolation

As shown in the figure, the value of the function $V(.)$ at point $s$ is to be estimated. In our method, the cell closest to $s$ (marked by the red points) is used to construct a bilinear interpolation function to estimate $V(s)$, i.e.

$$V(s) = \frac{V(s_{11})A_{22} + V(s_{21})A_{12} + V(s_{12})A_{21} + V(s_{22})A_{11}}{A} \tag{2}$$

where

$$A = \Delta x \Delta y, A_{11} = \Delta x_1 \Delta y_1, A_{21} = \Delta x_2 \Delta y_1, A_{12} = \Delta x_1 \Delta y_2, A_{22} = \Delta x_2 \Delta y_2 \tag{3}$$

and

$$\begin{aligned} \Delta x = x_2 - x_1, \Delta x_1 = x - x_1, \Delta x_2 = x_2 - x \\ \Delta y = y_2 - y_1, \Delta y_1 = y - y_1, \Delta y_2 = y_2 - y \end{aligned} \tag{4}$$

Regardless of whether $s$ lies in the computational domain or not, the estimation is done as shown in Eq. (2). Only when $s$ lies inside the computational domain, $\Delta x_1, \Delta x_2, \Delta y_1, \Delta y_2$ are all positive, and when $s$ lies outside the computational domain, some of these are negative. For example, in the case shown in Figure 2(b), $\Delta x_2$ is negative.

6. Choosing the time-step can be done by following CFL condition analysis in PDE. This analysis can be also found in the above book.

**Response:** CFL is a necessary condition for the numerical stability of the finite difference method. However, our method does not involve difference operations. see the following figure.

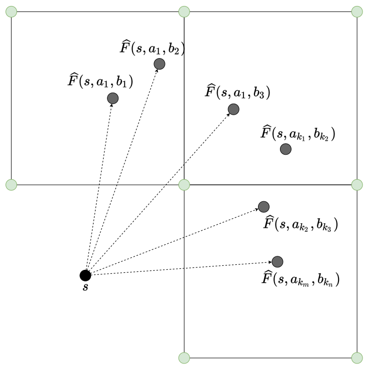
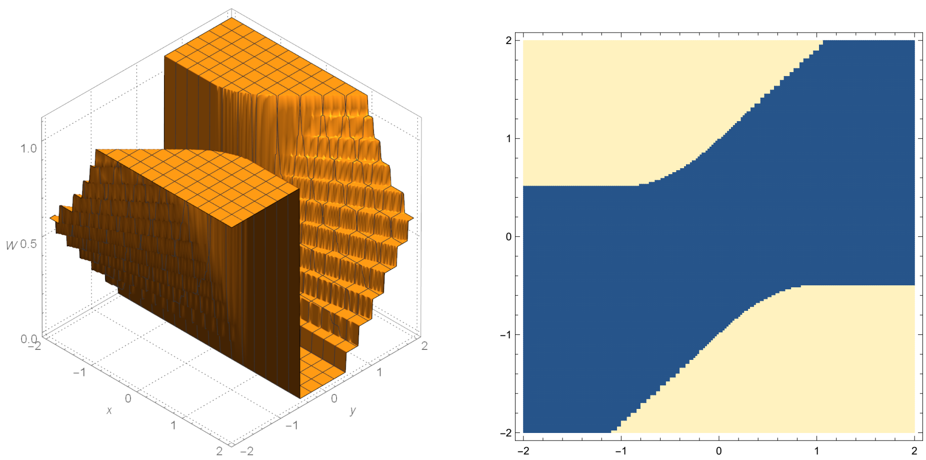


Figure 3:

There are two parts to be computed, one is the cost consumed during the transition $\widehat{C}(s_0, a, b)$ and the other is the value of the function at the transferred state $W_{\frac{c}{T}}(\widehat{F}(s_0, a, b), k\Delta t)$. $W_{\frac{c}{T}}(\widehat{F}(s_0, a, b), k\Delta t)$ can be obtained by the bilinear interpolation of the four nearest grid points to $\widehat{F}(s_0, a, b)$. Therefore, both parts do not involve the spatial derivatives or the numerical stability problems common in the process of numerically solving PDEs. In fact, we tested in the first example a time step of length 0.1 (the grids number is $201 \times 201$), which is much larger than the time step determined by the CFL condition, numerical instability does not occur. The results are shown in the following figure:



(a) Value function (b) $R_K(1)$

Figure 4:

The surface representing the value function is basically consistent with the analytic solution except for some folds, and the computed reachable set also roughly matches the analytic solution except for some "jagged" edges.

In the revision, we replaced the time-discrete form from forward Euler method to fourth-order Runge-Kutta method, then we found that there is no significant effect on the computational accuracy when the time step is within a certain range, and the variation of computational accuracy with the number of grids is comparable to that of the level set method, see Fig. 4 of the revision. We have the following speculations about the reasons for the no significant effect: In our method, the computational error consists of two parts, one is the error in computing $\widehat{F}(s_0, a, b)$ and $\widehat{C}(s_0, a, b)$, which increases with the time step size, and the other is the cumulative error of repeated interpolation $W_{\frac{c}{T}}(\widehat{F}(s_0, a, b), k\Delta t)$, which increases with the number of time steps (i.e., it decreases with the time step size, a larger time step means fewer recursions are needed). It should also be noted that the method used to compute the Jaccard index in this paper is also a numerical method (Numerical integration in Mathematica 12), therefore, there is some error in the computation results.

7. In eq (3), $V(s0, \bar{T}) = l(s0)$ is not necessarily written since this is not an additional condition but simply derived by the definition.

**Response:** This condition has been removed in the revision.

8. The dissipation function in the level set toolboxes is determined by the system dynamics.

**Response:** The general form of the dissipation function is:

$$\alpha_x = \max \left| \frac{\partial}{\partial p_x} \left[ \min_a \max_b p^{\mathrm{T}} f(s,a,b) \right] \right| \tag{5}$$

where $p = \frac{\partial V}{\partial s}$. If the system is of affine nonlinear form, i.e.,

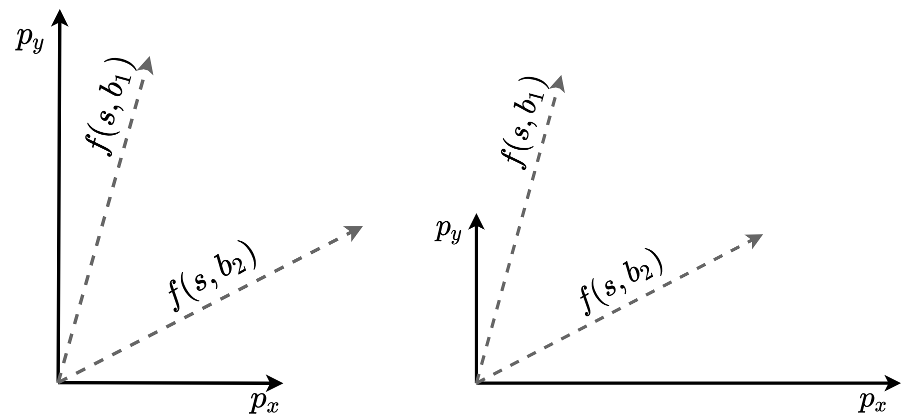
$$f(s,a,b) = f^s(s) + g_1(s)a + g_2(s)b \tag{6}$$

$\alpha_x$ can be overestimated by [1]:

$$\alpha_x \leq |f_x^s(s)| + \max\left[|g_1(s)|\, a_{\max}, |g_1(s)|\, a_{\min}\right] + \max\left[|g_2(s)|\, b_{\max}, |g_2(s)|\, b_{\min}\right] \tag{7}$$

Eq. (7) is determined by the system dynamics. However, the general form of the dissipation function is dependent on $p$, because the choice of $a$ and $b$ is related to $p$. For the sake of simplicity, consider the following equation:

$$\max_b p^{\mathrm{T}} f(s,b) \tag{8}$$




(a) (b)

Figure 5:

In the case shown in Fig. 5(a), $\arg\max_{b\in\{b_1,b_2\}} p^{\mathrm{T}} f(s,b) = b_1$, in the case shown in Fig. 5(b), $\arg\max_{b\in\{b_1,b_2\}} p^{\mathrm{T}} f(s,b) = b_2$.

Our method does not need to compute the dissipation function because the method to compute the value function is recursion and interpolation rather than the viscosity solution of the PDE. It is not so much an advantage as it is a choice of last resort. This is because the value functions constructed by our method are often discontinuous. We removed the description of the dissipation function in the revised version. But such a value function brings some benefits. For example, when saving multiple BRTs with different time horizons (or CBRTs with different admissible costs), our method consumes less storage space compared to the level set method. In the first example of the revised version, we visually demonstrate this advantage.

9. The wording 'the Lagrangian methods' is rarely used. The word is too ambiguous to understand even though this word is used in one of your references.

**Response:** In the modified version, ellipsoidal method and polyhedral method are no longer collectively referred to as Lagrangian method, but are described separately.

10. In the third paragraph in the introduction, what is 'RBS'? Is it 'BRS'? The definition for this is necessary. 'the BRS can then be expressed as the sub-zero level set of solution of the HJ PDE' is more correct?

**Response:** We are very sorry, these are all slip-ups, our method cannot compute backward reachable set (BRS). These mistakes are fixed in the revised version.

11. In the second paragraph in the introduction, it is not clear the meaning of 'verifying each state one by one'. Does it mean that spatial-discretization requires an exponential computational complexity in state dimension? If this means just verification for a single initial state, the computation can be really fast.

**Response:** Since the simulation-based approach is not a hot research topic in recent years and the sentence is prone to ambiguity, it has been removed in the revised version.

# References


[1] Ian M Mitchell. A toolbox of level set methods. *UBC Department of Computer Science Technical Report TR-2007-11*, 2007.