Zienkiewicz. $Z_K = \mathbb{P}_2(K) + \mathrm{span}\{\lambda_i^2\lambda_j - \lambda_i\lambda_j^2,\ 1\le \lambda_i < \lambda_j \le 3\}$.

$\Sigma_K = \{v(a_i),\ \partial_x v(a_i),\ \partial_y v(a_i):\ i=1,2,3\}$

记 $\xi_i = x_j - x_k$, $\eta_i = y_j - y_k$ $w_i = x_j y_k - x_k y_j$.

$\lambda_i = \frac{1}{J}(\eta_i x - \xi_i y + w_i)$ $J = 2|T| = w_1 + w_2 + w_3$.

可得 $\xi_j\eta_k - \xi_k\eta_j = J$. $\partial_n\lambda_i|_{e_j} = -\frac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|}$.

syms λ_1 λ_2 λ_3 x_1 x_2 x_3 y_1 y_2 y_3.

$\xi_i = x_j - x_k$ $\eta_i = y_j - y_k$. $J = \det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$ $\partial_x\lambda_i = \frac{\eta_i}{J}$. $\partial_y\lambda_i = -\frac{\xi_i}{J}$.

$\varphi = [\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2, \lambda_2^2\lambda_3 - \lambda_2\lambda_3^2, \lambda_3^2\lambda_1 - \lambda_1\lambda_3^2]$

计算形函数自由度的值. $AC = I_9$. $\Rightarrow C = A\backslash I_9$.

latex(C) =

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & -\xi_3/2 & \xi_3/2 & 0 & -\eta_3/2 & \eta_3/2 & 0 \\
0 & 1 & 1 & 0 & -\xi_1/2 & \xi_1/2 & 0 & -\eta_1/2 & \eta_1/2 \\
1 & 0 & 1 & \xi_2/2 & 0 & -\xi_2/2 & \eta_2/2 & 0 & -\eta_2/2 \\
1 & -1 & 0 & -\xi_3/2 & -\xi_3/2 & 0 & -\eta_3/2 & -\eta_3/2 & 0 \\
0 & 1 & -1 & 0 & -\xi_1/2 & -\xi_1/2 & 0 & -\eta_1/2 & -\eta_1/2 \\
-1 & 0 & 1 & -\xi_2/2 & 0 & -\xi_2/2 & -\eta_2/2 & 0 & -\eta_2/2
\end{bmatrix}$$

$\xi_i = \lambda_i^2(3 - 2\lambda_i) + 2b_K$

$\theta_i = \lambda_i^2(\xi_j\lambda_k - \xi_k\lambda_j) + \frac{\xi_j - \xi_k}{2} b_K$

$w_i = \lambda_i^2(\eta_j\lambda_k - \eta_k\lambda_j) + \frac{\eta_j - \eta_k}{2} b_K$

$(b_K = \lambda_1\lambda_2\lambda_3)$

校正空间 $b_K\mathbb{P}_1$ 在 $\{v(a_i),\ \partial_x v(a_i),\ \partial_y v(a_i)\}$ 上值为 0

记 $\{\varphi_i\}_{i=1}^{3}$ 是以 $\{f_{e_i}\ p_2\ \partial_n v\}$ 为自由度的基函数.

令 $\varphi_i = \lambda_i\lambda_j\lambda_k(c_i\lambda_i + c_j\lambda_j + c_k\lambda_k)$.

$\partial_n \varphi_i|_{e_j} = -|\nabla\lambda_j|\lambda_i\lambda_k(c_i\lambda_i + c_k\lambda_k)$.

$f_{e_j} p_2 \partial_n \varphi_i = -\dfrac{|\nabla\lambda_j|}{360}(c_i + c_k)$.

$$\begin{cases} -\dfrac{|\nabla\lambda_i|}{360}(c_j + c_k) = 1 \\ c_i + c_k = 0 \\ c_i + c_j = 0 \end{cases}$$

$c_i = \dfrac{180}{|\nabla\lambda_i|}$ $\quad c_j = c_k = -\dfrac{180}{|\nabla\lambda_i|}$.

$\varphi_i = \dfrac{180}{|\nabla\lambda_i|}\lambda_1\lambda_2\lambda_3(2\lambda_i - 1)$.

$\tilde{\varphi}_i = b_K(2\lambda_i - 1)$

$\zeta_i = \lambda_i^2(3 - 2\lambda_i) + 2\lambda_1\lambda_2\lambda_3$.

$\partial_n \zeta_i|_{e_i} = -2|\nabla\lambda_i|\lambda_j\lambda_k$ $\quad f_{e_i} p_2 \partial_n \zeta_i = -\dfrac{|\nabla\lambda_i|}{90}$

$\partial_n \zeta_i|_{e_j} = 6\dfrac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|}(\lambda_i^2 - \lambda_i) - 2|\nabla\lambda_j|\lambda_i\lambda_k$. $\quad f_{e_j} p_2 \partial_n \zeta_i = -\dfrac{\nabla\lambda_i\cdot\nabla\lambda_j}{30|\nabla\lambda_j|} - \dfrac{|\nabla\lambda_j|}{90}$

$$\tilde{\zeta}_i = \zeta_i + 2\sum_{i=1}^{3} b_K(2\lambda_i - 1) + 6\sum_{j\neq i}\frac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|^2}\tilde{\varphi}_j.$$

$$= \lambda_i^2(3 - 2\lambda_i) + 6\sum_{j\neq i}\frac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|^2}\tilde{\varphi}_j.$$

$$\theta_i = \lambda_i^2(\xi_j\lambda_k - \xi_k\lambda_j) + \frac{\xi_j - \xi_k}{2}\lambda_1\lambda_2\lambda_3$$

$$\partial_n\theta_i|_{e_i} = -\frac{\xi_j - \xi_k}{2}|\nabla\lambda_i|\lambda_j\lambda_k \qquad f_{e_i} P_2\partial_n\theta_i = -\frac{\xi_j-\xi_k}{360}|\nabla\lambda_i|$$

$$\partial_n\theta_i|_{e_j} = -2\xi_j\frac{\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|}\lambda_i\lambda_k - \left(\xi_j\frac{\nabla\lambda_k\cdot\nabla\lambda_j}{|\nabla\lambda_j|} - \xi_k|\nabla\lambda_j|\right)\lambda_i^2$$
$$-\frac{\xi_j - \xi_k}{2}|\nabla\lambda_j|\lambda_i\lambda_k.$$

$$f_{e_j} P_2\partial_n\theta_i = -\frac{\xi_j\nabla\lambda_i\cdot\nabla\lambda_j}{90|\nabla\lambda_j|} + \frac{1}{180}\left(\xi_j\frac{\nabla\lambda_k\cdot\nabla\lambda_j}{|\nabla\lambda_j|} - \xi_k|\nabla\lambda_j|\right) - \frac{\xi_j-\xi_k}{360}|\nabla\lambda_j|.$$
$$= -\frac{\xi_j\nabla\lambda_i\cdot\nabla\lambda_j}{60|\nabla\lambda_j|} - \frac{\xi_j+\xi_k}{180}|\nabla\lambda_j| - \frac{\xi_j-\xi_k}{360}|\nabla\lambda_j|.$$

$$\partial_n\theta_i|_{e_k} = 2\xi_k\frac{\nabla\lambda_i\cdot\nabla\lambda_k}{|\nabla\lambda_k|}\lambda_i\lambda_j - \left(\xi_j|\nabla\lambda_k| - \xi_k\frac{\nabla\lambda_j\cdot\nabla\lambda_k}{|\nabla\lambda_k|}\right)\lambda_i^2$$
$$-\frac{\xi_j-\xi_k}{2}|\nabla\lambda_k|\lambda_i\lambda_j$$

$$f_{e_k} P_2\partial_n\theta_i = \frac{\xi_k\nabla\lambda_i\cdot\nabla\lambda_k}{90|\nabla\lambda_k|} + \frac{1}{180}\left(\xi_j|\nabla\lambda_k| - \xi_k\frac{\nabla\lambda_j\cdot\nabla_k}{|\nabla\lambda_k|}\right) - \frac{\xi_j-\xi_k}{360}|\nabla\lambda_k|.$$
$$= \frac{\xi_k\nabla\lambda_i\cdot\nabla\lambda_k}{60|\nabla\lambda_k|} + \frac{\xi_j+\xi_k}{180}|\nabla\lambda_k| - \frac{\xi_j-\xi_k}{360}|\nabla\lambda_k|$$

$$\tilde{\theta}_i = \lambda_i^2(\xi_j\lambda_k - \xi_k\lambda_j) + 2(\xi_j+\xi_k)b_K(\lambda_j-\lambda_k)$$
$$+\frac{3\xi_j\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|^2}\tilde{\varphi}_j - \frac{3\xi_k\nabla\lambda_i\cdot\nabla\lambda_k}{|\nabla\lambda_k|^2}\tilde{\varphi}_k$$

here $\tilde{\varphi}_i = b_K(2\lambda_i - 1)$.

$$\tilde{\omega}_i = \lambda_i^2(\eta_j\lambda_k - \eta_k\lambda_j) + 2(\eta_j+\eta_k)b_K(\lambda_j-\lambda_k)$$
$$+\frac{3\eta_j\nabla\lambda_i\cdot\nabla\lambda_j}{|\nabla\lambda_j|^2}\tilde{\varphi}_j - \frac{3\eta_k\nabla\lambda_i\cdot\nabla\lambda_k}{|\nabla\lambda_k|^2}\tilde{\varphi}_k$$