Operations Research Letters 46 (2018) 103–108

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

A closed-form solution of the Black–Litterman model with conditional value at risk

Tao Pang [a,*], Cagatay Karan [b]

[a] Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA
[b] Operations Research, North Carolina State University, Raleigh, NC 27695-7913, USA

ARTICLE INFO

Article history:
Received 7 October 2016
Received in revised form 18 October 2017
Accepted 24 November 2017
Available online 14 December 2017

Keywords:
Black–Litterman model
Portfolio optimization
CVaR
Elliptical distribution

ABSTRACT

We consider a portfolio optimization problem of the Black–Litterman type, in which we use the conditional value-at-risk (CVaR) as the risk measure and we use the multi-variate elliptical distributions, instead of the multi-variate normal distribution, to model the financial asset returns. We propose an approximation algorithm and establish the convergence results. Based on the approximation algorithm, we derive a closed-form solution of the portfolio optimization problems of the Black–Litterman type with CVaR.

1. Introduction

In the classical Markowitz portfolio optimization model, the historical mean vector and covariance matrix of the risky assets are used to obtain the optimal portfolio allocation while normal distributions are assumed (Markowitz [16]). Mean–variance optimization considers only the first two moments of the return distribution. This restriction is consistent with reality only if asset returns follow normal distributions (Fabozzi et al. [7] and Meucci [17]). But we know that the asset market returns' co-skewness and co-kurtosis values differ from normality (see more at Harvey et al. [10] and Jondeau and Rockinger [15]). Additionally, historical returns are not good estimates of the future returns, which are very difficult to estimate. Furthermore, Markowitz's optimal allocation vectors are often lack of diversification and (or) they could be corner solutions.

Aiming to solve the above issues with the classical Markowitz portfolio optimization model, Black and Litterman [6] propose a portfolio optimization technique in which the investor's view can be integrated with the historical performance to obtain the optimal portfolio. The Black–Litterman Model (BLM) combines the views of the investor on the selected assets with their historical information to update their mean vector and covariance matrix using Bayesian framework. The BLM assumes that the expected returns are random variables themselves which are normally distributed and centered at the capital asset price model (CAPM) equilibrium returns with historical covariance matrix. There are two different types of BLMs in the literature: the original (canonical) model (see He and Litterman [11], Satchell and Scowcroft [22], Idzorek [14] and Bertsimas et al. [5]) and the alternate model proposed later (see Walters [25], Meucci [17,18], and Xiao and Valdez [26]). Here we consider the alternate model.

Most of current literatures on the BLM use normal distributions with variance as the risk measure. However, normal distributions are not good models for financial market returns. In this paper, we consider the BLM with elliptical distributions. Many widely used distributions, such as normal, student-t, etc. are elliptical distributions. In addition, the CAPM holds as long as return distributions are elliptical (see Meucci [17] and references therein). Derivation of the posterior distribution for this case is given by Xiao and Valdez [26] (see Proposition 2).

On the other hand, as conditional value-at-risk (CVaR) has become more and more popular as a coherent risk measure in the financial industry, we use CVaR, instead of variance, as the risk measure. Xiao and Valdez [26] also consider the BLM with VaR and CVaR as risk measures. In particular, they consider the unconstrained trade-off model, and no explicit formula is derived. In this paper, we consider the constrained model in which the goal is to maximize the expected return with constraints on the maximal accepted risk in terms of CVaR. Direct derivation of the optimal solution analytically for the BLM with CVaR is extremely difficult. Here we propose an efficient approximation algorithm, and based on the approximation algorithm, we derive the closed-form solution for the BLM with elliptical distributions and CVaR. In addition, our results can be used to solve the unconstrained case, too. More details can be found in Section 5. To our best

* Corresponding author.
E-mail addresses: tpang@ncsu.edu (T. Pang), ckaran@ncsu.edu (C. Karan).

https://doi.org/10.1016/j.orl.2017.11.014

knowledge, no closed-form solution for BLM with CVaR has been derived before.

The remainder of the paper is organized as follows. Definitions of the portfolio allocation problems (PAPs) are given in Section 2. The BLM with CVaR is reviewed in Section 3. We propose an efficient approximation algorithm for optimization problems with CVaR in Section 4. The closed-form optimal solution and some discussions are given in Section 5. We conclude the paper in Section 6.

2. Portfolio allocation problems (PAP)

We consider a market with n risky assets and one riskless asset. Risky asset returns are denoted by a random vector $\mathbf{r} \in \mathbb{R}^n$ which is defined on the probability space $(\Omega, \mathcal{F}, P)$. Vectors are defined as column vectors unless otherwise stated. The mean vector and the covariance matrix of risky asset returns are denoted by $\boldsymbol{\mu} = \mathbb{E}[\mathbf{r}] \in \mathbb{R}^n$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$, respectively. We assume $\boldsymbol{\mu}$ is finite, and $\boldsymbol{\Sigma}$ is finite and positive semi-definite. The risk free rate of return is denoted as $r_f \in \mathbb{R}_+ \cup 0$. Moreover, $\mathbf{x} \in \mathbb{R}^n$ is the portfolio weight vector of risky assets and $(1 - \mathbf{e}'\mathbf{x})$ is the allocation on the risk-free asset, where $\mathbf{e} = (1, 1, \ldots, 1)'$ is a vector of ones in $\mathbb{R}^n$.

First we define the space of portfolio returns for a given number of available risky assets and a risk-free asset.

Definition 1 (*Space of Portfolio Returns*). The space of portfolio returns is defined by

$$\mathcal{V} = \{\tilde{v} \in \mathbb{R} : \exists (r_f, \mathbf{x}) \text{ s.t. } \tilde{v} = \mathbf{r}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f\}. \tag{1}$$

From Definition 1, it is easy to get that each portfolio return can be represented as a combination of return with certainty ($\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f$) and return with uncertainty ($(\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}$).

Next we define the constrained Markowitz's portfolio allocation problem (PAP) (Markowitz [16]):

Definition 2 (*Markowitz's PAP*).

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \le L\}, \tag{2}$$

where $L \in \mathbb{R}_+$ is the investor's risk tolerance level.

In Markowitz's PAP, the variance is used as the risk measure. There are other widely used risk measures, such as value at risk (VaR) and conditional value at risk (CVaR).

Definition 3 (*VaR and CVaR*). Given $\alpha \in (0, 1)$ and a return random variable (r.v.) $\tilde{v}$, the VaR of the r.v. $\tilde{v}$ with a confidence level α is

$$VaR_\alpha(\tilde{v}) = \inf\{y \in \mathbb{R} : Pr(y + \tilde{v} \le 0) \le 1 - \alpha\}.$$

The CVaR of $\tilde{v}$ with confidence level α is

$$CVaR_\alpha(\tilde{v}) = -\mathbb{E}[\tilde{v} | \tilde{v} \le -VaR_\alpha(\tilde{v})].$$

VaR_α is not a coherent risk measure (see Artzner et al. [1]). In particular, diversification benefits may not present under VaR. On the other hand, CVaR is coherent. (The properties of CVaR can be found in Rockafellar and Uryasev [20,21].) It is one of the main reasons that CVaR is now a very popular risk measure.

Here we consider the PAP with CVaR:

Definition 4 (*PAP with CVaR*).

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : CVaR_\alpha(\mathbf{r}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f) \le L\}.$$

We want to solve the PAP given in Definition 4 for elliptical distributions. Elliptical distributions can be used to model the leptokurtic behavior of asset returns. Moreover, if we assume elliptical distributions, then we can get the closed form representation of CVaR.

Proposition 1 (*CVaR for Elliptical Distributions*). *Assume that the asset returns* $\mathbf{r}$ *follows a multivariate elliptical distribution,* $\mathbf{r} \sim ED_n(\boldsymbol{\mu}, \mathbf{D}, g_n)$*, and* $\tilde{v} \in \mathcal{V}$ *where* $\mathcal{V}$ *is given by (1). Assume that* $\boldsymbol{\Sigma}$ *is the covariance matrix of* $\mathbf{r}$*. Then we have*

$$CVaR_\alpha(\tilde{v}) = -(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f) + \beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}, \tag{3}$$

where β_α *is constant that depends on the CVaR confidence level* α *and the return distributions.*

Using Lemma 4.1 in Xiao and Valdez [26] and the fact that $\boldsymbol{\Sigma} = c\mathbf{D}$ where c is a constant depending on the distribution, we can obtain Proposition 1 very easily.

Next we give the value of β_α for some widely used elliptical distributions.

(i) If $\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $\beta_\alpha = f(z_{1-\alpha})/(1-\alpha)$, where $f(\cdot)$ is the standard normal density function and $z_{1-\alpha}$ is the z-score of the standard normal distribution.

(ii) If $\mathbf{r}$ follows a multivariate Student-t distribution, i.e. $\mathbf{r} \sim t(\boldsymbol{\mu}, \boldsymbol{\Sigma}, m)$ where m stands for the degree of freedom, then

$$\beta_\alpha = \frac{c_2 m}{(1-\alpha)(m-1)}\left(1 + \frac{y_{1-\alpha}^2}{m}\right)^{\frac{1-m}{2}},$$

where $c_2 = \frac{(\pi m)^{-1/2}\Gamma((m+1)/2)}{\Gamma(m/2)}$ and $y_{1-\alpha}$ satisfies $\int_{-\infty}^{y_{1-\alpha}} g_1(u^2)\, du = 1 - \alpha$ where $g_n(u^2) = c_2\left(1 + \frac{u^2}{m}\right)^{\frac{-(1+m)}{2}}$ is the density generator function of the multivariate student-t distribution.

These results are pretty standard and can be derived using results in Xiao and Valdez [26] or Landsman and Valdez [13]. So we omit the derivation details here.

On the other hand, instead of solving the PAP given in Definition 4 for elliptical distributions one can use robust optimization and elliptical uncertainty sets to get the explicit CVaR formula (see Natarajan et al. [19], Bertsimas and Brown [4], and Ben Tal and Nemirovski [2,3]). Moreover, BLM in robust optimization setting is given by Bertsimas et al. [5].

3. BLM with CVaR for elliptical distributions

We continue with the BLM under elliptical distributions ($ED_n(\cdot)$) given in Xiao and Valdez [26]. Let $\mathbf{r} \sim ED_n(\boldsymbol{\mu}, \mathbf{D}, g_n)$ be an n dimensional vector, where $\boldsymbol{\mu}, \mathbf{D}$ and g_n are the location parameter, dispersion matrix and the density generator function, respectively (more details on elliptical distributions can be found in Fang et al. [8]). Here we assume that $\boldsymbol{\mu} = \boldsymbol{\Pi}$ where $\boldsymbol{\Pi}$ is the CAPM equilibrium expected excess return and can be found via inverse optimization (for details see Meucci [17]). The conditional random view vector is $\mathbf{v}|\mathbf{r} \sim ED_k(\mathbf{P}\boldsymbol{\Pi}, \Omega, g_k(\cdot; p(\mathbf{r})))$ where $p(\mathbf{r}) = (\mathbf{r} - \boldsymbol{\Pi})'\mathbf{D}^{-1}(\mathbf{r} - \boldsymbol{\Pi})$. The posterior distribution is given by the following proposition.

Proposition 2 (*Xiao and Valdez [26]*). *The posterior distribution is*

$$\mathbf{r}|\mathbf{v} \sim ED_k(\boldsymbol{\mu}_{BL}, \mathbf{D}_{BL}, g_n(\cdot; q(\mathbf{v}))),$$

where

$$\boldsymbol{\mu}_{BL} = \boldsymbol{\mu} + \mathbf{D}\mathbf{P}'(\Omega + \mathbf{P}\mathbf{D}\mathbf{P}')^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\mu}),$$
$$\mathbf{D}_{BL} = \mathbf{D} - \mathbf{D}\mathbf{P}'(\Omega + \mathbf{P}\mathbf{D}\mathbf{P}')^{-1}\mathbf{P}\mathbf{D},$$
$$q(\mathbf{v}) = (\mathbf{v} - \mathbf{P}\boldsymbol{\mu})'(\Omega + \mathbf{P}\mathbf{D}\mathbf{P}')^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\mu}),$$
$$\boldsymbol{\Sigma}_{BL} = \mathbf{D}_{BL}C_k(q(\mathbf{v})/2),$$

and C_k *is a distribution specific function from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$*.*

For more details about this proposition, please refer to Xiao and Valdez [26] (Proposition 3.1 and the discussion after it), and we omit the proof here.

The key assumption of BLM is that every player in the market solves Markowitz's problem. In other words, BLM takes CAPM equilibrium as prior for the excess return distribution. However, in our case, investors have views under the CVaR risk measure.

There are some other generalizations for the BLM with CAPM equilibrium. For example, Silva et al. [23] give the BLM under active management, and Giacometti et al. [9] propose a model where asset returns follow stable distributions with different types of risk measures. Unlike models in those papers, here we consider elliptical distributions for asset returns and solve the constrained model.

Consider the unconstrained PAP with CVaR given by Definition 4. Using Proposition 1, we can represent the objective function as follows:

$$\begin{aligned}\mathcal{G}(\mathbf{x}) &= \boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f - \lambda CVaR_\alpha(\tilde{v}) \\ &= (1+\lambda)[\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f] - \lambda\beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}.\end{aligned}$$

We now take the partial derivative with respect to $\mathbf{x}$ to get the first order necessary condition and we can use inverse optimization to find the expected excess return vector:

$$\boldsymbol{\Pi} = \left(\frac{\lambda\beta_\alpha}{1+\lambda}(\mathbf{x}'_{mkt}\boldsymbol{\Sigma}\mathbf{x}_{mkt})^{-1/2}\right)\boldsymbol{\Sigma}\mathbf{x}_{mkt}, \tag{4}$$

where we take $\mathbf{x} = \mathbf{x}_{mkt}$ as the market weights.

We get the return distribution for the updated excess return vector by using Proposition 2. Once an elliptical distribution for the asset returns is chosen, the parameter β_α can be determined by the CVaR confidence level α. We also need to determine the investor risk aversion parameter λ, which is related with the risk reward trade off. Then we can solve the PAP with CVaR given by Definition 4 with the new (updated) excess return vector and dispersion matrix.

4. An approximation algorithm

Consider the constrained PAP with CVaR given by Definition 4. By Proposition 1, we can get the Lagrangian:

$$\begin{aligned}\mathcal{L}(\mathbf{x},\delta) &= \boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f \\ &\quad - \delta\left[CVaR_\alpha(\mathbf{r}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f) - L\right] \\ &= (1-\delta)[(\boldsymbol{\mu}-\mathbf{e}r_f)'\mathbf{x}] - \delta[\beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} - L - r_f].\end{aligned}$$

Now, take the partial derivative with respect to $\mathbf{x}$ to get the first order necessary condition.

$$(1+\delta)\left(\boldsymbol{\mu}-\mathbf{e}r_f\right) - \delta\left(\beta_\alpha(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})^{-1/2}\boldsymbol{\Sigma}\mathbf{x}\right) = 0$$

Landsman [12] derives the closed-form solution of the problem of minimizing the root of a quadratic functional subject to some affine constraints. In addition to that, we can find the numerical solution using convex (or semidefinite) programming. But there is no explicit formulas of the solutions for constrained PAP with CVaR yet. Hence, we propose an algorithm to find the closed-form formula of the optimal solution.

The asset returns are assumed to be elliptically distributed. Furthermore, historical mean vector and covariance matrix are taken as the mean of the asset returns and covariance matrix, respectively.

Define

$$\tilde{L}(\mathbf{x}) \equiv \frac{\left(L + \boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f\right)}{\beta_\alpha}. \tag{5}$$

Then, by virtue of Proposition 1, we can verify that

$$CVaR_\alpha(\mathbf{r}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f) \le L \iff \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \le \tilde{L}(\mathbf{x}). \tag{6}$$

To solve the PAP with CVaR, we propose an approximation algorithm. First, we define the feasible solution space $\mathcal{P}$ as

$$\mathcal{P} \equiv \{\mathbf{x} \in \mathbb{R}^n : CVaR_\alpha(\mathbf{r}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f) \le L\}. \tag{7}$$

Next, we define a sequence of vectors $\{\mathbf{x}_j\}_{j\ge 0}$ as follows:

$$\mathbf{x}_0 \in \mathcal{P}, \tag{8}$$

$$\mathbf{x}_{j+1} = \arg\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \le \tilde{L}(\mathbf{x}_j)\}, \quad j \ge 0. \tag{9}$$

In other words, $\mathbf{x}_{j+1}$ is the solution of the following PAP

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \le \tilde{L}(\mathbf{x}_j)\}. \tag{10}$$

We want to point out that the initial vector $\mathbf{x}_0$ can be any vector in $\mathcal{P}$. For example, we can choose $\mathbf{x}_0 = \mathbf{0}$. More details will be given in Proposition 3.

We have the following results.

Lemma 1. *Let $\{\mathbf{x}_j\}_{j\ge 0}$ be given by (8)–(9). Then $\mathbf{x}_j \in \mathcal{P}$, $\forall j \ge 0$ and $\{\tilde{L}(\mathbf{x}_j)\}_{j\ge 0}$ is a non-decreasing sequence.*

Proof. We prove the result by induction. First, from (8), we see that $\mathbf{x}_0 \in \mathcal{P}$. Further, by virtue of (7) and (6), it is easy to check that $\sqrt{\mathbf{x}_0'\boldsymbol{\Sigma}\mathbf{x}_0} \le \tilde{L}(\mathbf{x}_0)$. By virtue of the definition of $\mathbf{x}_1$ (see (9)), we can get that

$$\boldsymbol{\mu}'\mathbf{x}_0 + (1-\mathbf{e}'\mathbf{x}_0)r_f \le \boldsymbol{\mu}'\mathbf{x}_1 + (1-\mathbf{e}'\mathbf{x}_1)r_f.$$

Then, by the definition of $\tilde{L}(\mathbf{x})$ (5), we have

$$\tilde{L}(\mathbf{x}_0) \le \tilde{L}(\mathbf{x}_1).$$

Using the definition of $\mathbf{x}_1$ (see (9)) and the above inequality, we have that

$$\sqrt{\mathbf{x}_1'\boldsymbol{\Sigma}\mathbf{x}_1} \le \tilde{L}(\mathbf{x}_0) \le \tilde{L}(\mathbf{x}_1).$$

Therefore, $\mathbf{x}_1 \in \mathcal{P}$. Now we can assume that $\mathbf{x}_j \in \mathcal{P}$ and $\tilde{L}(\mathbf{x}_{j-1}) \le \tilde{L}(\mathbf{x}_j)$. Then using the same argument as we used for $\mathbf{x}_0$ and $\mathbf{x}_1$, we can show that

$$\mathbf{x}_{j+1} \in \mathcal{P}, \quad \tilde{L}(\mathbf{x}_j) \le \tilde{L}(\mathbf{x}_{j+1}).$$

This completes the proof. □

Unlike the classical mean–variance optimization problem which always have a solution, the PAP with CVaR may not have a bounded solution. For example, if $n = 1$, the CVaR constraint becomes $-(\mu - r_f)x - r_f + \beta_\alpha\sigma x \le L$, which might become redundant when $\mu - r_f > \sigma\beta_\alpha$, and the optimal solution is $x = \infty$. Therefore, some extra conditions are needed for the PAP with CVaR.

Define the risk-adjusted return vector $\tilde{\boldsymbol{\mu}}$ and a constant d as

$$\tilde{\boldsymbol{\mu}} \equiv \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\mu}-\mathbf{e}r_f), \tag{11}$$

$$d \equiv \sqrt{(\boldsymbol{\mu}-\mathbf{e}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{e}r_f)}. \tag{12}$$

It is easy to see that $d = \left\|\tilde{\boldsymbol{\mu}}\right\|$. We assume the following condition holds:

$$\beta_\alpha - d > 0. \tag{13}$$

This condition is not a strong condition. For example, for $n = 1$ with normal distributions, $\beta_\alpha = 2.0627$ for $\alpha = 95\%$ and d is nothing but the Sharp ratio of the risky asset. Assume that $\mu = 20\%$, $r_f = 0\%$ and $\sigma = 16\%$, we can get $d = (\mu - r_f)/\sigma = 1.25$. Note that the risk-reward trade off is usually taken as $\delta = 1.25$ (see He and Litterman [11] and Bertsimas et al. [5]). So condition (13) is easily satisfied.

Proposition 3 (*Conv. of* $\tilde{L}(\mathbf{x}_j)$). *Assume that* (13) *holds and define*

$$\tilde{L}^* \equiv \frac{L + r_f}{\beta_\alpha - d}. \tag{14}$$

Let $\{\mathbf{x}_j\}_{j\geq 0}$ *be given by* (8)–(9). *Then we have*

$$\tilde{L}(\mathbf{x}_j) \leq \tilde{L}^*, \quad \forall j \geq 0, \tag{15}$$

$$\text{and} \quad \lim_{j\to\infty} \tilde{L}(\mathbf{x}_j) = \tilde{L}^*, \quad \forall \mathbf{x}_0 \in \mathcal{P}. \tag{16}$$

Further, we have

$$\tilde{L}(\mathbf{x}) \leq \tilde{L}^*, \quad \forall \mathbf{x} \in \mathcal{P}. \tag{17}$$

Proof. By virtue of the definition of $\mathbf{x}_{j+1}$ (see (9)), we know that $\mathbf{x}_{j+1}$ is the optimal solution of (10). The closed-form solution of this problem is well-known:

$$\mathbf{x}_{j+1} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f)}{2\delta}, \tag{18}$$

where δ is the Lagrange multiplier given by $\delta = \frac{d}{2\tilde{L}(\mathbf{x}_j)}$ and d is defined by (12). So we can get

$$\mathbf{x}_{j+1} = \frac{\tilde{L}(\mathbf{x}_j)}{d}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{e}). \tag{19}$$

Using the definition of $\tilde{L}(\cdot)$, we have

$$\tilde{L}(\mathbf{x}_{j+1}) = \frac{L + r_f}{\beta_\alpha} + \frac{(\boldsymbol{\mu} - r_f\mathbf{e})'\mathbf{x}_{j+1}}{\beta_\alpha}. \tag{20}$$

It is easy to verify that (19) and (20) imply

$$\tilde{L}(\mathbf{x}_{j+1}) = \frac{L + r_f}{\beta_\alpha} + \frac{d\tilde{L}(\mathbf{x}_j)}{\beta_\alpha}. \tag{21}$$

By virtue of Lemma 1, we have that $\tilde{L}(\mathbf{x}_{j+1}) \geq \tilde{L}(\mathbf{x}_j)$. So

$$\frac{L + r_f}{\beta_\alpha} + \frac{d\tilde{L}(\mathbf{x}_j)}{\beta_\alpha} \geq \tilde{L}(\mathbf{x}_j),$$

which is equivalent to

$$\tilde{L}(\mathbf{x}_j) \leq \frac{L + r_f}{\beta_\alpha - d}.$$

This holds for any $j \geq 0$. Therefore, (15) holds.

Further, using the recursive formula (21), we can get

$$\tilde{L}(\mathbf{x}_j) = \frac{L + r_f}{\beta_\alpha}\sum_{i=0}^{j-1}\left(\frac{d}{\beta_\alpha}\right)^i + \left(\frac{d}{\beta_\alpha}\right)^j\tilde{L}(\mathbf{x}_0). \tag{22}$$

By virtue of (13), we know that $\frac{d}{\beta_\alpha} < 1$. Therefore,

$$\lim_{j\to\infty}\tilde{L}(\mathbf{x}_j) = \frac{L + r_f}{\beta_\alpha}\frac{1}{1 - \frac{d}{\beta_\alpha}} = \frac{L + r_f}{\beta_\alpha - d} = \tilde{L}^*,$$

and it is true for any $\mathbf{x}_0 \in \mathcal{P}$. Therefore, (16) holds.

Finally, for any $\mathbf{x} \in \mathcal{P}$, we can take $\mathbf{x}_0 = \mathbf{x}$ in (8). Then by (15), we can get (17). This completes the proof. □

Now we can present the main result of this section.

Theorem 1. *Assume that* (13) *holds. Then* $\mathbf{x}^*$ *is an optimal solution to the PAP with CVaR (Definition 4) if and only if* $\tilde{L}(\mathbf{x}^*) = \tilde{L}^*$ *and* $\mathbf{x}^* \in \mathcal{P}$, *where* $\tilde{L}(\cdot)$, $\tilde{L}^*$ *and* $\mathcal{P}$ *are defined by* (5), (14) *and* (7), *respectively.*

Proof. Define $\hat{\mathbf{x}}$ as a solution of

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq \tilde{L}^*\}. \tag{23}$$

Let $\{\mathbf{x}_j\}_{j\geq 0}$ be given by (8)–(9). By virtue of (15), we know that $\tilde{L}(\mathbf{x}_j) \leq \tilde{L}^*$. Then, by virtue of (23) and (9), we can get that

$$\boldsymbol{\mu}'\mathbf{x}_{j+1} + (1 - \mathbf{e}'\mathbf{x}_{j+1})r_f \leq \boldsymbol{\mu}'\hat{\mathbf{x}} + (1 - \mathbf{e}'\hat{\mathbf{x}})r_f.$$

By the definition of $\tilde{L}(\cdot)$ (see (5)), we can get $\tilde{L}(\mathbf{x}_{j+1}) \leq \tilde{L}(\hat{\mathbf{x}})$, $\forall j \geq 0$, which implies that

$$\tilde{L}^* = \lim_{j\to\infty}\tilde{L}(\mathbf{x}_j) \leq \tilde{L}(\hat{\mathbf{x}}). \tag{24}$$

Since $\hat{\mathbf{x}}$ is a solution of (23), the above equation implies that

$$\sqrt{\hat{\mathbf{x}}'\boldsymbol{\Sigma}\hat{\mathbf{x}}} \leq \tilde{L}(\hat{\mathbf{x}}). \tag{25}$$

Then, using (6), we can get that $\hat{\mathbf{x}} \in \mathcal{P}$. Now by virtue of (17), we know that

$$\tilde{L}(\hat{\mathbf{x}}) \leq \tilde{L}^*.$$

Together with (24), the above inequality implies that

$$\tilde{L}(\hat{\mathbf{x}}) = \tilde{L}^*. \tag{26}$$

Let $\mathbf{x}^*$ be an optimal solution of the PAP with CVaR:

$$\mathbf{x}^* \in \operatorname{argmax}\{(\boldsymbol{\mu} - \mathbf{e})'\mathbf{x} + r_f : CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \leq L\}.$$

Then we have $\mathbf{x}^* \in \mathcal{P}$. Based on (25), we know that $\hat{\mathbf{x}} \in \mathcal{P}$. Since $\mathbf{x}^*$ is the optimal solution, we must have

$$(\boldsymbol{\mu} - \mathbf{e}r_f)'\hat{\mathbf{x}} + r_f \leq (\boldsymbol{\mu} - \mathbf{e}r_f)'\mathbf{x}^* + r_f \tag{27}$$

which implies $\tilde{L}(\hat{\mathbf{x}}) \leq \tilde{L}(\mathbf{x}^*)$. Now, taking $\mathbf{x}_0$ as $\mathbf{x}^*$ and using Lemma 1, we can get that

$$\tilde{L}(\mathbf{x}^*) \leq \tilde{L}^* = \tilde{L}(\hat{\mathbf{x}}).$$

Therefore, we must have $\tilde{L}(\mathbf{x}^*) = \tilde{L}(\hat{\mathbf{x}}) = \tilde{L}^*$.

On the other hand, if there is an $\mathbf{x}^* \in \mathcal{P}$ such that $\tilde{L}(\mathbf{x}^*) = \tilde{L}^*$, then we can use (17) to derive that $\mathbf{x}^*$ maximizes $\tilde{L}(\cdot)$ over $\mathcal{P}$. By the definition of $\tilde{L}(\mathbf{x})$ (see (5)), we can see that maximizing $\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}r_f)\mathbf{x}$ is equivalent to maximizing $\tilde{L}(\mathbf{x})$. Therefore, $\mathbf{x}^*$ maximizes $\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f$ over $\mathcal{P}$ and it is an optimal solution of the PAP with CVaR given by Definition 4. This completes the proof. □

5. Closed-form solutions of BLM with CVaR

Now, we are ready to give the closed-form solution for the PAP with CVaR under elliptical distributions for the BLM. Under the BLM with CVaR, we have an updated mean vector $\boldsymbol{\mu}_{BL}$ and covariance matrix $\boldsymbol{\Sigma}_{BL}$ which are given by Proposition 3. Define

$$d_{BL} \equiv \sqrt{(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}, \tag{28}$$

$$\tilde{L}_{BL}^* \equiv \frac{L + r_f}{\beta_\alpha - d_{BL}}. \tag{29}$$

Theorem 2. *Let* d_{BL}, $\tilde{L}_{BL}^*$ *be given by* (28), (29), *respectively. Assume that* $\beta_\alpha > d_{BL}$. *Define*

$$\hat{\mathbf{x}} \equiv \frac{\tilde{L}_{BL}^*}{d_{BL}}\left(\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\right). \tag{30}$$

Then $\hat{\mathbf{x}}$ *is an optimal solution of the PAP with CVaR.*

Proof. By virtue of Theorem 1, it is sufficient to show that $\hat{\mathbf{x}} \in \mathcal{P}$ and $\tilde{L}(\hat{\mathbf{x}}) = \tilde{L}_{BL}^*$. First, by virtue of (28) and (30), we can get that

$$\sqrt{\hat{\mathbf{x}}'\boldsymbol{\Sigma}_{BL}\hat{\mathbf{x}}}$$
$$= \frac{\tilde{L}_{BL}^*}{d_{BL}}\sqrt{(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\boldsymbol{\Sigma}_{BL}^{-1}\boldsymbol{\Sigma}_{BL}\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}$$

$$= \frac{\tilde{L}^*_{BL}}{d_{BL}} d_{BL} = \tilde{L}^*_{BL},$$

and

$$(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\hat{\mathbf{x}} = \frac{\tilde{L}^*_{BL}}{d_{BL}} (\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\boldsymbol{\Sigma}^{-1}_{BL}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f) = \frac{\tilde{L}^*_{BL}}{d_{BL}} d^2_{BL} = \tilde{L}^*_{BL} d.$$

Then, by virtue of (29), we have

$$\begin{aligned} &CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\hat{\mathbf{x}} + r_f) \\ &= -(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\hat{\mathbf{x}} - r_f + \beta_\alpha\sqrt{\hat{\mathbf{x}}'\boldsymbol{\Sigma}_{BL}\hat{\mathbf{x}}} \\ &= -\tilde{L}^*_{BL}d_{BL} - r_f + \beta_\alpha \tilde{L}^*_{BL} = (\beta_\alpha - d)\tilde{L}^*_{BL} + r_f = L. \end{aligned}$$

Therefore, $\hat{\mathbf{x}} \in \mathcal{P}$. Second, using (5) and (28), we have

$$\tilde{L}(\hat{\mathbf{x}}) = \frac{L + r_f}{\beta_\alpha} + \frac{1}{\beta_\alpha}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\hat{\mathbf{x}} = \frac{\tilde{L}^*_{BL}(\beta_\alpha - d_{BL})}{\beta_\alpha} + \frac{1}{\beta_\alpha}\tilde{L}^*_{BL}d_{BL} = \tilde{L}^*_{BL}.$$

This completes the proof. □

Although the explicit formula (30) is derived for the constrained problem, we can actually use it to get the optimal solution for the unconstrained trade-off case. Consider the unconstrained model of PAP with variance:

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \frac{\lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}{2} \right\}, \tag{31}$$

where λ denotes the investor's risk (variance in this case) reward trade-off. The optimal solution of this problem is

$$\mathbf{x}^* = \lambda^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f). \tag{32}$$

Next, for the constrained PAP as given by Definition 2, we can get the optimal solution (refer to (18), (19))

$$\mathbf{x}^* = \left(\frac{d}{L}\right)^{-1} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f), \tag{33}$$

where d is defined by (12).

From (32) and (33), we can see that when the investor has an upper limit for her portfolio risk, then she may change her risk-reward coefficient (i.e., setting $\lambda = \frac{d}{L}$), as if she is solving an unconstrained model with an updated risk-reward trade-off coefficient (see also Steinbach [24]), to obtain the optimal solution.

Further, by Theorem 2, we can get the optimal solution for the BLM with CVaR:

$$\mathbf{x}^* = \left(\frac{d_{BL}}{\tilde{L}^*_{BL}}\right)^{-1} \boldsymbol{\Sigma}^{-1}_{BL}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f), \tag{34}$$

where $\tilde{L}^*_{BL}$ is given by (29). Here, we can see that when the investor uses CVaR in an constrained setting then her risk-reward coefficient is updated with $\lambda = \frac{d_{BL}}{\tilde{L}^*_{BL}}$ instead of $\lambda = \frac{d}{L}$ in the variance case. This is interesting and one can perturb the new risk-reward trade-off in order to understand the behavior of the optimal portfolio vector under different problem settings.

From the above discussion, we can see that the optimal solution of the unconstrained case the constrained case will actually be the same, if we choose the corresponding parameter values by using $\lambda = \frac{d_{BL}}{\tilde{L}^*_{BL}}$ (for CVaR) or $\lambda = \frac{d}{L}$ (for variance). Therefore, our results can be used to solve the unconstrained PAP, too.

In addition, our model is a generalization of the multivariate normal case with variance as the risk measure, in which the CAPM equilibrium is centered at $\boldsymbol{\Pi} = 2\delta\boldsymbol{\Sigma}\mathbf{x}_{mkt}$ with $\delta = 1.25$ fixed. If we assume multivariate normal distribution for asset returns and take λ in (4) such that

$$\frac{f(z_{1-\alpha})}{(1-\alpha)}\frac{\lambda}{(1+\lambda)}(\mathbf{x}'_{mkt}\boldsymbol{\Sigma}\mathbf{x}_{mkt})^{-1/2} = 2.5,$$

then we will get the same $\boldsymbol{\Pi}$ value. In addition, if we take the risk level L as $L(\beta_\alpha - d) + r_f$, then we have the same setting as the multivariate normal models (see, for example, He and Litterman [11]).

6. Conclusion

Black and Litterman [6] propose the BLM in order to overcome the Markowitz Model's drawbacks. Their model uses the Bayesian framework to combine the intuitions and/or the views about the selected assets or market parameters with the historical information of the market to update the mean vector and covariance matrix. In our work, we use CVaR as a risk measure, instead of the variance risk measure proposed in the original model. In addition, elliptical distributions are used to capture the non-normal behavior of the asset returns. For constrained problems, we propose an efficient approximation algorithm for the BLM with CVaR and establish the convergence results. Based on the approximation, we derived a closed-form solution of the BLM with CVaR.

Acknowledgments

We sincerely appreciate the anonymous reviewers to early versions for their invaluable comments that result in a much better version of this paper.

References

[1] P. Artzner, F. Delbaen, J.M. Eber, D. Heath, Coherent measures of risk, Math. Finan. 9 (3) (1999) 203–228.

[2] A. Ben-Tal, A. Nemirovski, Robust solutions of uncertain linear programs, Oper. Res. Lett. 25 (1) (1999) 1–13.

[3] A. Ben-Tal, A. Nemirovski, in: Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications, vol. 2, SIAM, 2001.

[4] D. Bertsimas, D.B. Brown, Constructing uncertainty sets for robust linear optimization, Oper. Res. 57 (6) (2009) 1483–1495.

[5] D. Bertsimas, V. Gupta, I.C. Paschalidis, Inverse optimization: A new perspective on the Black-Litterman model, Oper. Res. 60 (6) (2012) 1389–1403.

[6] F. Black, R. Litterman, Global portfolio optimization, Financ. Anal. J. 48 (5) (1992) 28–43.

[7] F.J. Fabozzi, P.N. Kolm, D.A. Pachamanova, S.M. Focardi, Robust Portfolio Optimization and Management, John Wiley & Sons, 2007.

[8] K.T. Fang, S. Kotz, K.W. Ng, Symmetric Multivariate and Related Distributions, Chapman and Hall, 1990.

[9] R. Giacometti, M. Bertocchi, S.T. Rachev, F.J. Fabozzi, Stable distributions in the Black-Litterman approach to asset allocation, Quant. Finance 7 (4) (2007) 423–433.

[10] C.R. Harvey, J.C. Liechty, M.W. Liechty, P. Muller, Portfolio selection with higher moments, Quant. Finance 10 (5) (2010) 469–485.

[11] G. He, R. Litterman., The intuition behind Black-Litterman model portfolios, in: Investment Management Research, Goldman Sachs & Company, New York, 1999.

[12] Z. Landsman, Minimization of the root of a quadratic functional under a system of affine equality constraints with application to portfolio management, J. Comput. Appl. Math. 220 (1) (2008) 739–748.

[13] Z.M. Landsman, E.A. Valdez, Tail conditional expectations for elliptical distributions, N. Am. Actuar. J. 7 (4) (2003) 55–71.

[14] T.M. Idzorek, A step-by-step guide to the Black-Litterman model, in: Forecasting Expected Returns in the Financial Markets, 2002, p. 17.

[15] E. Jondeau, M. Rockinger, Optimal portfolio allocation under higher moments, Eur. Financ. Manag. 12 (1) (2006) 29–55.

[16] H. Markowitz, Portfolio selection, J. Finance 7 (1) (1952) 77–91.

[17] A. Meucci, Risk and Asset Allocation, Springer Science & Business Media, 2005.

[18] A. Meucci, The Black Litterman Approach: Original Model and Extensions, in: The Encyclopedia of Quantitative Finance, Wiley, 2010 Available at http://ssrn.com/abstract=1117574.

[19] K. Natarajan, D. Pachamanova, M. Sim, Constructing risk measures from uncertainty sets, Oper. Res. 57 (5) (2009) 1129–1141.
[20] R.T. Rockafellar, S. Uryasev, Optimization of conditional value-at-risk, J. Risk 2 (2000) 21–42.
[21] R.T. Rockafellar, S. Uryasev, Conditional value-at-risk for general loss distributions, J. Bank. Finance 26 (7) (2002) 1443–1471.
[22] S. Satchell, A. Scowcroft, A demystification of the Black-Litterman model: Managing quantitative and traditional portfolio construction, J. Asset Manag. 1 (2) (2000) 138–150.
[23] A.S. Da Silva, W. Lee, B. Pornrojnangkool, The Black-Litterman model for active portfolio management, J. Portfolio Manag. 35 (2) (2009) 61.
[24] M.C. Steinbach, Markowitz revisited: Mean–variance models in financial portfolio analysis, SIAM Rev. 43 (1) (2001) 31–85.
[25] J. Walters, The Black-Litterman model: A detailed exploration, 2014. http://www.blacklitterman.org. (Accessed 15 June 2014).
[26] Y. Xiao, E.A. Valdez, A Black-Litterman asset allocation model under Elliptical distributions, Quant. Finance 15 (3) (2015) 509–519.