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# Kolmogorov Approximation

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## 1 Introduction

Many different approaches to approximation of probability distributions are studied in the literature [12, 15, 16]. The papers vary in the types random variables involved, how they are represented, and in the criteria used for evaluation of the quality of the approximations. This paper is on approximating discrete distributions represented as explicit probability mass functions with ones that are simpler to store and to manipulate. This is needed, for example, when a discrete distribution is given as a large data-set, obtained, e.g., by sampling, and we want to represent it approximately with a small table.

The main contribution of this paper is an efficient algorithm for computing the best possible approximation of a given random variable with a random variable whose complexity is not above a prescribed threshold, where the measures of the quality of the approximation and the complexity of the random variable are as specified in the following two paragraphs.

We measure the quality of an approximation by the distance between the original variable and the approximate one. Specifically, we use the Kolmogorov distance which is one of the most used in statistical practice and literature. Given two random variables  $X$  and  $X'$  whose cumulative distribution functions (cdfs) are  $F_X$  and  $F_{X'}$ , respectively, the Kolmogorov distance between  $X$  and  $X'$  is  $d_K(X, X') = \sup_t |F_X(t) - F_{X'}(t)|$  (see, e.g., [9]). We say that  $X'$  is a good approximation of  $X$  if  $d_K(X, X')$  is small.

The complexity of a random variable is measured by the size of its support, the number of values that it can take,  $|\text{support}(X)| = |\{x: \Pr(X = x) \neq 0\}|$ . When distributions are maintained as explicit tables, as done in many implementations of statistical software, the size of the support of a variable is proportional to the amount of memory needed to store it and to the complexity of the computations around it. In summary, the exact notion of optimality of the approximation targeted in this paper is:

**Definition 1.** A random variable  $X'$  is an optimal  $m$ -approximation of a random variable  $X$  if  $|\text{support}(X')| \leq m$  and there is no random variable  $X''$  such that  $|\text{support}(X'')| \leq m$  and  $d_K(X, X'') < d_K(X, X')$ .

The main contribution of the paper is an efficient algorithm that takes  $X$  and  $m$  as parameters and constructs an optimal  $m$ -approximation of  $X$ .

The rest of the paper is organized as follows. In Section 2 we describe how our work relates to other algorithms and problems studied in the literature. In Section 3 we detail the proposed algorithm,

analyze its properties, and prove the main theorem. In Section 4 we demonstrate how the proposed approach performs on the problem of estimating the probability of hitting deadlines is plans and compare it to alternatives approximation approaches from the literature. We also demonstrate the performance of our approximation algorithm on some randomly generated random variables. The paper is concluded with a discussion in Section 5.

## 2 Related Work

The problem studied in this paper is related to the theory of Sparse Approximation (aka Sparse Representation) that deals with sparse solutions for systems of linear equations, as follows.

Given a matrix  $D \in \mathbb{R}^{n \times p}$  and a vector  $x \in \mathbb{R}^n$ , the most studied sparse representation problem is finding the sparsest possible representation  $\alpha \in \mathbb{R}^p$  satisfying  $x = D\alpha$ :

$$\min_{\alpha \in \mathbb{R}^p} \|\alpha\|_0 \text{ subject to } x = D\alpha.$$

where  $\|\alpha\|_0 = |\{i : \alpha_i \neq 0, i = 1, \dots, p\}|$  is the  $\ell_0$  pseudo-norm, counting the number of non-zero coordinates of  $\alpha$ . This problem is known to be NP-Hard with a reduction to NP-complete subset selection problems.

In these terms, using also the  $\ell_\infty$  norm that represents the maximal coordinate and the  $\ell_1$  norm that represents the sum of the coordinates, our problem can be phrased as:

$$\min_{\alpha \in [0, \infty)^p} \|x - D\alpha\|_\infty \text{ subject to } \|\alpha\|_0 = m \text{ and } \|\alpha\|_1 = 1.$$

where  $D$  is the all-ones triangular matrix (the entry at row  $i$  and column  $j$  is one if  $i \leq j$  and zero otherwise),  $x$  is related to  $X$  such that the  $i$ th coordinate of  $x$  is  $F_X(x_i)$  where  $\text{support}(X) = \{x_1 < x_2 < \dots < x_n\}$  and  $\alpha$  is related to  $X'$  such that the  $i$ th coordinate of  $\alpha$  is  $f_{X'}(x_i)$ . The functions  $F_X$  and  $f_{X'}$  represent, respectively, the cumulative distribution function of  $X$  and the mass distribution function of  $X'$ . This, of course, means that the coordinates of  $x$  are assumed to be positive and monotonically increasing and that the last coordinate of  $x$  is assumed to be one. We demonstrate an application for this specific sparse representation problem and show that it can be solve in  $O(n^2m)$  time and  $O(m^2)$  memory.

Another related research is dealing with credit scoring [21] where the main objective is to separate good applicants from bad applicants. The Kolmogorov–Smirnov statistic KS is a standard measure of model strength or model performance in credit scoring. currently there are three computational methods of KS in terms of score binning: (1) the method with equal-width binning [11], (2) the method with equal-size binning [17], and (3) the method without binning [2, 19], this method uses a kind of binning in which each score is treated as a bin, or equal-width binning with a width of 0. Essentially, this work aims at presenting a comparison study of the three methods in 3 aspects: Values, Rank Ordering of Scores and Geometrical Way. We would like to suggest our approximation method, the KolmogorovApprox algorithm, described in this work as another comparison method which provides optimality guarantees that are not given in the binning technique.

The present study is also a continuation of the work of Pavlikov and Uryasev [15], where a procedure to produce a random variable  $X'$  that optimally approximates a random variable  $X$  is presented. Their approximation scheme, achieved using convex and linear programming, is designed for a different notion of distance (called CVaR). The new contribution of the present work in this context is that our method is direct, not using linear or convex programming, thus allowing tighter analysis of time and memory complexity.

### 66 3 An Algorithm for Optimal Approximation

67 In the scope of this section, let  $X$  be a given random variable with a finite support of size  $n$ , and let  
 68  $0 < m \leq n$  be a given complexity bound. We first develop notations and collect facts towards an  
 69 algorithm for finding an optimal  $m$ -approximation of  $X$ .

70 The first useful fact is that it is enough to limit our search to approximations  $X'$ 's such that  
 71  $\text{support}(X') \subseteq \text{support}(X)$ :

72 **Lemma 2.** *There is an optimal  $m$ -approximation  $X'$  of  $X$  such that  $\text{support}(X') \subseteq \text{support}(X)$ .*

73 *Proof.* [[DF: will continue tomorrow, please don't touch]] Let  $X''$  be any random variable whose  
 74 support is of size smaller or equal to  $m$ , we find a random variable  $X'$  with  $\text{support}(X') \subseteq$   
 75  $\text{support}(X)$  and  $d_K(X, X') = d_K(X, X'')$ . Let  $\{x_1, \dots, x_n\} = \text{support}(X)$ , and let  $x_0 =$   
 76  $-\infty, x_{n+1} = \infty$ . Consider the the random variable  $X'$  whose probability mass function is  $f_{X'}(x_i) =$   
 77  $P(x_{i-1} < X'' < x_i)$  for  $i = 1, \dots, n-1$ ,  $f_{X'}(x_n) = P(x_n - 1 < X'' < x_{n+1})$  and  $F_{X'}(x) = 0$   
 78 if  $x \notin \text{support}(X)$ . First see that for every  $x_i < x < x_{i+1}$  for  $i = 0, \dots, n-1$  we have  
 79  $|F_X(x) - F_{X'}(x)| \leq |F_X(x_i) - F_{X'}(x)|$

80 By definition  $d_K(X, X') = d_K(X, X'')$ . Since  $\text{support}(X') \subseteq \text{support}(X)$  and  $|\text{support}(X')| <$   
 81  $|\text{support}(X'')| \leq m$ , we get that  $d_K(X, X'') \geq \varepsilon$ . Thus,  $\varepsilon$  is a global minimum and any  
 82  $X' \in \text{argmin}\{d_K(X, X') : \text{support}(X') \subseteq \text{support}(X), |\text{support}(X')| \leq m\}$  is an  $m$ -optimal  
 83 approximation of  $X$ .  $\square$

84 Next, note that every random variable  $X''$  with support of size at most  $m$  that is contained in  
 85  $\text{support}(X)$  can be described by first setting the (at most  $m$ ) elements of the support of  $X''$ ; then for  
 86 every such option, determine  $X''$  by setting probability values for the elements in the chosen support  
 87 of  $X'$ , and setting 0 for rest of the elements.

88 Denote the set of random variables with support  $S \subseteq \text{support}(X)$  by  $\mathbb{X}_S$ . In Step 1 below, we find a  
 89 random variable in  $\mathbb{X}_S$  that minimizes the Kolmogorov distance from  $X$ , and denote this distance  
 90 by  $\varepsilon(X, S)$ . Next, in Step 2, that we will describe later, we will show how to efficiently find  $S$   
 91 that minimizes  $\varepsilon(X, S)$  among all the sets that satisfy  $S \subset \text{support}(X)$  and  $|S| \leq m$ . Then the  
 92 minimized random variable  $\mathbb{X}_S$  from the minimal  $S$ , is the  $m$ -optimal approximation to  $X$ .

#### 93 3.1 Step 1: Finding an $X'$ in $\mathbb{X}_S$ that minimizes $d_K(X, X')$

94 We first fix a set  $S \subseteq \text{support}(X)$  of size at most  $m$ , and among all the random variables in  
 95  $\mathbb{X}_S$  find one with a minimal distance from  $X$ . Denote the elements of  $S$  in increasing order by  
 96  $S = \{x_1 < \dots < x_m\}$  and let  $x_0 = -\infty$ , and  $x_{m+1} = \infty$ . For every  $1 < i \leq m$  let  $\hat{x}_i$  be the  
 97 maximal element of  $\text{support}(X)$  that is smaller than  $x_i$ .

98 Next, as the elements of  $S$  are also elements of  $\text{support}(X)$ , we can define the following weight  
 99 function:

100 **Definition 3.** *For  $0 \leq i \leq m$  let*

$$w(x_i, x_{i+1}) = \begin{cases} P(x_i < X < x_{i+1}) & \text{if } i = 0 \text{ or } i = m; \\ P(x_i < X < x_{i+1})/2 & \text{otherwise.} \end{cases}$$

101 Note that  $x_i = -\infty$  for  $i = 0$  and  $x_i = \infty$  for  $i = m + 1$ . Also note that  $P(x_i < X < x_{i+1}) =$   
 102  $F_X(\hat{x}_{i+1}) - F_X(x_i)$ , a fact that we will use throughout this section.

103 **Definition 4.** Let  $\varepsilon(X, S) = \max_{i=0, \dots, m} w(x_i, x_{i+1})$ .

104 We first show that  $\varepsilon(X, S)$  is a lower bound. That is, every random variable in  $\mathbb{X}_S$  has a distance at  
 105 least  $\varepsilon(X, S)$ . Then, we present a random variable  $X' \in \mathbb{X}_S$  with distance  $\varepsilon(X, S)$ . It then follows  
 106 that such  $X'$  is an optimal  $m$ -approximation random variable among all random variables in  $\mathbb{X}_S$ .

107 The intuition behind choosing these specific weights and  $\varepsilon(X, S)$  being a lower bound is as follows.  
 108 Since for every  $X' \in \mathbb{X}_S$  the probability values of  $X'$  for the elements not in  $S$  are set to 0, we have  
 109 that  $F_{X'}(\hat{x}_{i+1}) = F_{X'}(x_i)$ . Therefore the distance between  $X'$  and  $X$  at points  $x_i$  and  $\hat{x}_{i+1}$  that we  
 110 have to take into additional account is increased by  $F_X(\hat{x}_{i+1}) - F_X(x_i) = P(x_i < X < x_{i+1})$ .

111 Formally we have the following.

112 **Proposition 5.** If  $X' \in \mathbb{X}_S$  then  $d_K(X, X') \geq \varepsilon(X, S)$ .

113 *Proof.* By definition, for every  $0 \leq i \leq m$ ,  $d_K(X, X') \geq \max\{|F_X(\hat{x}_{i+1}) - F_{X'}(\hat{x}_{i+1})|, |F_X(x_i) -$   
 114  $F_{X'}(x_i)|\}$ . Note that  $F_{X'}(\hat{x}_{i+1}) = F_{X'}(x_i)$  since the probability values for all the elements not in  $S$   
 115 are set to 0.

116 If  $i = 0$ , that is  $x_i = -\infty$ , we have that  $F_X(x_i) = F_{X'}(x_i) = F_{X'}(\hat{x}_{i+1}) = 0$  and therefore  
 117  $d_K(X, X') \geq |F_X(\hat{x}_{i+1})| = |F_X(\hat{x}_{i+1}) - F_X(x_i)| = P(x_i < X < x_{i+1}) = w(x_i, x_{i+1})$ .

118 If  $i = m$ , that is  $x_{i+1} = \infty$ , we have that  $F_X(\hat{x}_{i+1}) = F_{X'}(\hat{x}_{i+1}) = F_{X'}(x_i) = 1$ . and therefore  
 119  $d_K(X, X') \geq |1 - F_X(\hat{x}_i)| = |F_X(\hat{x}_{i+1}) - F_X(x_i)| = P(x_i < X < x_{i+1}) = w(x_i, x_{i+1})$ .

120 Otherwise for every  $1 \leq i < m$ , we use the fact that  $\max\{|a|, |b|\} \geq |a - b|/2$  for every  $a, b \in$   
 121  $\mathbb{R}$ , to have  $d_K(X, X') \geq 1/2|F_X(\hat{x}_{i+1}) - F_X(x_i) + F_{X'}(x_i) - F_{X'}(\hat{x}_{i+1})|$ . So  $d_K(X, X') \geq$   
 122  $1/2|F_X(\hat{x}_{i+1}) - F_X(x_i)| = P(x_i < X < x_{i+1})/2 = w(x_i, x_{i+1})$ .

123 Therefore since  $d_K(X, X') \geq w(x_i, x_{i+1})$  for every  $0 \leq i \leq m$ , by definition of  $\varepsilon(X, S)$  proof  
 124 follows.  $\square$

125 Next we show a random variable  $X' \in \mathbb{X}_S$  with a distance of  $\varepsilon(X, S)$  from  $X$ . Thus  $X'$  is an optimal  
 126  $m$ -approximation among the set  $\mathbb{X}_S$ . We define  $X'$  as follows:

127 **Definition 6.** Let  $f_{X'}(x_i) = w(x_{i-1}, x_i) + w(x_i, x_{i+1}) + f_X(x_i)$  for  $i = 1, \dots, m$  and  $f_{X'}(x) = 0$   
 128 for  $x \notin S$ .

129 We first show that  $X'$  is a properly defined random variable:

130 **Lemma 7.**  $f_{X'}$  is a probability mass function.

131 *Proof.* From definition  $f_{X'}(x_i) \geq 0$  for every  $i$ . To see that  $\sum_i f_{X'}(x_i) = 1$ , we have  $\sum_i f_{X'}(x_i) =$   
 132  $\sum_i (w(x_{i-1}, x_i) + w(x_i, x_{i+1}) + f_X(x_i)) = \sum_{x_i \in S} f_X(x_i) + w(x_0, x_1) + \sum_{0 < i < m} 2w(x_i, x_{i+1}) +$   
 133  $w(x_m, x_{m+1}) = \sum_{x_i \in S} P(X = x_i) + P(x_0 < X < x_1) + \sum_{0 < i < m} P(x_i < X < x_{i+1}) +$   
 134  $P(x_m < X < x_{m+1}) = 1$  since this sum is the entire cpt of  $X$ .  $\square$

135 Note that  $X'$  can be constructed in linear time to the size of the cdf of  $X$ . Intuitively the setting of  
 136  $X'$  allows to take an "advantage" of distance from  $X$  at the elements of  $\text{support}(X')$ , to avoid the  
 137 overall increased distance of  $X$  from  $X'$  at the elements that are not at  $\text{support}(X)$  and in which  
 138  $f_{X'}$  is set to 0. Formally we have the following.

139 **Lemma 8.** Let  $x \in \text{support}(X)$  and  $0 \leq i \leq m$  be such that  $x_i \leq x \leq x_{i+1}$  then  $-w(x_i, x_{i+1}) \leq$   
 140  $F_X(x) - F_{X'}(x) \leq w(x_i, x_{i+1})$ .

141 *Proof.* We prove by induction on  $0 \leq i < m$ .

142 First see that  $F_{X'}(j) = 0$  for every  $x_0 < j < x_1$  and therefore  $F_X(j) - F_{X'}(j) = F_X(j) - 0 \leq$   
 143  $F_X(\hat{x}_1) = F_X(\hat{x}_1) - F_X(x_0) = w(x_0, x_1)$ . For  $j = x_1$  we have  $F_X(x_1) - F_{X'}(x_1) = F_X(\hat{x}_1) +$   
 144  $f_X(x_1) - (w(x_0, x_1) + w(x_1, x_2) + f_X(x_1)) = w(x_0, x_1) + f_X(x_1) - (w(x_0, x_1) + w(x_1, x_2) +$   
 145  $f_X(x_1)) = -w(x_1, x_2)$ .

146 Next assume that  $F_X(\hat{x}_i) - F_{X'}(\hat{x}_i) = w(x_{i-1}, x_i)$ . Then  $F_X(x_i) - F_{X'}(x_i) = F_X(\hat{x}_i) + f_X(x_i) -$   
 147  $(w(x_{i-1}, x_i) + w(x_i, x_{i+1}) + f_X(x_i)) = w(x_{i-1}, x_i) + f_X(x_i) - (w(x_{i-1}, x_i) + w(x_i, x_{i+1}) +$   
 148  $f_X(x_i)) = -w(x_i, x_{i+1})$ .

149 As before we have that for all  $x_i < j < x_{i+1}$ , we have  $F_X(j) - F_{X'}(j) = F_X(j) - F_{X'}(\hat{x}_{i+1}) \leq$   
 150  $F_X(\hat{x}_{i+1}) - F_{X'}(\hat{x}_{i+1})$ . Then  $F_X(\hat{x}_{i+1}) - F_{X'}(\hat{x}_{i+1}) = (F_X(x_i) + P(x_i < x < x_{i+1})) -$   
 151  $F_{X'}(x_i) = -w(x_i, x_{i+1}) + 2w(x_i, x_{i+1}) = w(x_i, x_{i+1})$ .

152 Finally for  $x_m \leq j \leq x_{m+1}$  we have that  $F_{X'}(x_m) = 1$  therefore  $F_X(x_m) - F_{X'}(x_m) = (1 -$   
 153  $P(x_m < X < x_{m+1})) - 1 = P(x_m < X < x_{m+1}) = w(x_m, x_{m+1})$ , and for every  $x_m < j <$   
 154  $x_{m+1}$  we have  $F_X(j) - F_{X'}(j) < (1 - P(x_m < X < x_{m+1})) - 1 < -P(x_m < X < x_{m+1}) =$   
 155  $-w(x_m, x_{m+1})$  as required.  $\square$

156 From Lemma 8, by the definition of  $\varepsilon(X, S)$ , we then have:

157 **Corrolary 9.**  $d_K(X, X') = \varepsilon(X, S)$ .

### 158 **3.2 Step 2: Finding an $S$ that minimizes $\varepsilon(X, S)$**

159 Chakravarty, Orlin, and Rothblum [3] proposed a polynomial-time method that, given a certain  
 160 objective functions (additive), finds an optimal consecutive partition. Their method involves the  
 161 construction of a graph such that the (consecutive) set partitioning problem is reduced to the problem  
 162 of finding the shortest path in that graph.

163 The KolmogorovApprox algorithm (Algorithm 1) starts by constructing a directed weighted graph  
 164  $G$  similar to the method of Chakravarty, Orlin, and Rothblum [3]. The nodes  $V$  consist of the  
 165 support of  $X$  together with an extra two nodes,  $-\infty$  and  $\infty$  for technical reasons, whereas the  
 166 edges  $E$  connect every pair of nodes in one direction (lines 1-2). The weight  $w$  of each edge  
 167  $e = (x, y) \in E$  is determined by one of two cases as in Definition 3. The values taken are  
 168 non inclusive, since we are interested only in the error value. The source node of the shortest  
 169 path problem at hand corresponds to the  $-\infty$  node added to  $G$  in the construction phase, and  
 170 the target node is the extra node  $\infty$ . The set of all solution paths in  $G$ , i.e., those starting at  
 171  $-\infty$  and ending in  $\infty$  with at most  $m$  edges, is called  $paths(G, -\infty, \infty)$ . The goal is to find the  
 172 path  $l$  in  $paths(G, -\infty, \infty)$  with the lightest bottleneck (line 3). This can be achieved by using  
 173 the *Bellman – Ford* algorithm with two tweaks. The first is to iterate the graph  $G$  in order to  
 174 find only paths with length of at most  $m$  edges. The second is to find the lightest bottleneck as  
 175 opposed to the traditional objective of finding the shortest path. This is performed by modifying the  
 176 manner of “relaxation” to  $bottleneck(x) = \min[\max(bottleneck(v), w(e))]$ , done also in [10, 18].  
 177 Consequently, we find the lightest maximal edge in a path of length  $\leq m$ , which represents the  
 178 minimal error,  $\varepsilon(X, S)$ , defined in Definition 4 where the nodes in path  $l$  represent the elements in set  
 179  $S$ . The approximated random variable  $X'$  is then derived from the resulting path  $l$  (lines 4-5). Every  
 180 node  $x \in l$  represent a value in the new calculated random variable  $X'$ , we than iterate the path  $l$  to  
 181 find the probability of the event  $f_{X'}(x)$  as described in Definition 6.

182 **Theorem 10.**  $KolmogorovApprox(X, m)$  is an  $m$ -optimal-approximation of  $X$ .

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**Algorithm 1:** KolmogorovApprox( $X, m$ )

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```
1  $S = \text{support}(X) \cup \{\infty, -\infty\}$ 
2  $G = (V, E) = (S, \{(x, y) : x < y\})$ 
3  $(x_0, \dots, x_{m+1}) = l \in \text{argmin}_{l \in \text{paths}(G, -\infty, \infty), |l| \leq m} \max\{w(e) : e \in l\}$ 
4 for  $0 < i < m + 1$  do
5    $f_{X'}(x_i) = w(x_{i-1}, x_i) + w(x_i, x_{i+1}) + f_X(x_i)$ 
6 return  $X'$ 
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183 *Proof.* If we consider the vertexes  $S = l \setminus \{-\infty, \infty\}$  for a path  $l \in \text{paths}(G, -\infty, \infty)$  we have  
184 that  $\max\{w(e) : e \in l\} = \varepsilon(X, S)$ . Therefore, line 3 of the algorithm essentially computes a set  
185  $S \in \text{argmin}_{S \subseteq \text{support}(X), |S| \leq m} \varepsilon(X, S)$ . By Corollary 9, the variable  $X'$  constructed in lines 4 and  
186 5 satisfies  $d_K(X, X') = \varepsilon(X, S)$  and by the minimality of  $S$  and by Proposition 5, it is an optimal  
187 approximation.  $\square$

188 **Theorem 11.** The KolmogorovApprox( $X, m$ ) algorithm runs in time  $O(mn^2)$ , using  $O(n^2)$  mem-  
189 ory where  $n = |\text{support}(X)|$ .

190 *Proof.* Constructing the graph  $G$  takes  $O(n^2)$ . The number of edges is  $O(E) \approx O(n^2)$  and for every  
191 edge the weight is at most the sum of all probabilities between the source node  $-\infty$  and the target  
192 node  $\infty$ , which can be done efficiently by aggregating the weights of already calculated edges. The  
193 construction is also the only stage that requires memory allocation, specifically  $O(E + V) = O(n^2)$ .  
194 Finding the shortest path takes  $O(m(E + V)) \approx O(mn^2)$ .

195 [[GW: put a reference to the work of the fellows from the Technion to avoid some of this?]]

196 Since  $G$  is DAG (directed acyclic graph) finding a shortest path takes  $O(E + V)$ . We only need  
197 to find paths of length  $\leq m$ , which takes  $O(m(E + V))$ . Deriving the new random variable  $X'$   
198 from the computed path  $l$  takes  $O(m)$ . For every node  $x_i$  in  $l$  (at most  $m$  nodes), use the already  
199 calculated weights to find the probability mass function  $f_{X'}(x_i)$ . To conclude, the worst case run-time  
200 complexity is  $O(n^2 + mn^2 + m) = O(mn^2)$  and memory complexity is  $O(E + V) = O(n^2)$ .  $\square$

## 201 4 A case study and experimental results

202 The case study examined in our experiments is the problem of task trees with deadlines [5, 4].  
203 Hierarchical planning is a well-established field in AI [6, 7, 8], and is still relevant nowadays [1, 20].  
204 A hierarchical plan is a method for representing problems of automated planning in which the  
205 dependency among tasks can be given in the form of networks, here we focus on hierarchical plans  
206 represented by task trees. The leaves in a task tree are *primitive* actions (or tasks), and the internal  
207 nodes are either *sequence* or *parallel* actions. The plans we deal with are of stochastic nature, and  
208 the task duration is described as probability distributions in the leaf nodes. We assume that the  
209 distributions are independent but *not* necessarily identically distributed and that the random variables  
210 are discrete and have a finite support.

211 A sequence node denotes a series of tasks that should be performed consecutively, whereas a parallel  
212 node denotes a set of tasks that begin at the same time. A *valid* plan is one that is fulfilled before some  
213 given *deadline*, i.e., its *makespan* is less than or equal to the deadline. The objective in this context is  
214 to compute the probability that a given plan is valid, or more formally computing  $P(X < T)$ , where  
215  $X$  is a random variable representing the makespan of the plan and  $T$  is the deadline. The problem of  
216 finding the probability that a task tree satisfies a deadline is known to be NP-hard. In fact, even the

problem of summing a set of random variables is NP-hard [13]. This is an example of an explicitly given random variable that we need to estimate deadline meeting probabilities for.

The first experiment we focus on is the problem of task trees with deadlines, and consider three types of task trees. The first type includes logistic problems of transporting packages by trucks and airplanes (from IPC2 <http://ipc.icaps-conference.org/>). Hierarchical plans of those logistic problems were generated by the JSHOP2 planner [14], one parallel node with all descendant task nodes being in sequence. The second type consists of task trees used as execution plans for the ROBIL team entry in the DARPA robotics challenge (DRC simulation phase), and the third type is of linear plans (sequential task trees). The primitive tasks in all the trees are modeled as discrete random variables with support of size  $M$  obtained by discretization of uniform distributions over various intervals. The number of tasks in a tree is denoted by  $N$ .

We implemented the approximation algorithm for solving the deadline problem with four different methods of approximation. The first two are for achieving a one-sided Kolmogorov approximation – the OptTrim [4] and the Trim [5] operators, and the third is a simple sampling scheme. We used those methods as a comparison to the Kolmogorov approximation with the suggested KolmogorovApprox algorithm. The parameter  $m$  of OptTrim and KolmogorovApprox corresponds to the inverse of  $\varepsilon$  given to the Trim operator. Note that in order to obtain some error  $\varepsilon$ , one must take into consideration the size of the task tree  $N$ , therefore,  $m/N = 1/(\varepsilon \cdot N)$ . We ran also an exact computation as a reference to the approximated one in order to calculate the error. The experiments conducted with the following operators and their parameters: KolmogorovApprox operator with  $m = 10 \cdot N$ , the OptTrim operator with  $m = 10 \cdot N$ , the Trim as operator with  $\varepsilon = 0.1/N$ , and two simple simulations, with a different samples number  $s = 10^4$  and  $s = 10^6$ .

Task Tree	$M$	KolmogorovApprox	OptTrim	Trim	Sampling	
		$m/N=10$	$m/N=10$	$\varepsilon \cdot N=0.1$	$s=10^4$	$s=10^6$
Logistics ( $N=34$ )	2	0	0	0.0019	0.007	0.0009
	4	0.0024	0.0046	0.0068	0.0057	0.0005
Logistics ( $N=45$ )	2	0.0002	0.0005	0.002	0.015	0.001
	4	0	0.003	0.004	0.008	0.0006
DRC-Drive ( $N=47$ )	2	0.0014	0.004	0.009	0.0072	0.0009
	4	0.001	0.008	0.019	0.0075	0.0011
Sequential ( $N=10$ )	2	0.0093	0.015	0.024	0	0.0009
	4	0.008	0.024	0.04	0.008	0.0016
	10	0	0.028	0.06	0.0117	0.001

Table 1: Comparison of estimated errors with respect to the reference exact computation on various task trees.

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Table 1 shows the results of the case study experiment. The quality of the solutions provided by using the KolmogorovApprox operator are better than those provided by the Trim and OptTrim operators, following the optimality guarantees, but is interesting to see that the quality gaps happen in practice in each of the examined task trees. However, in some of the task trees the sampling method produced better results than the approximation algorithm with KolmogorovApprox. Nevertheless, the approximation algorithm comes with an inherent advantage of providing an exact quality guarantees, as opposed to the probabilistic guarantees provided by sampling.

In order to better understand the quality gaps in practice between KolmogorovApprox, OptTrim, and Trim, we investigate their relative errors when applied on single random variables with support size  $n = 100$ , and different support sizes of the resulting random variable approximation ( $m$ ). In each

instance of this experiment, a random variable is randomly generated by choosing the probabilities of each element in the support uniformly and then normalizing these probabilities so that they sum to 1.

Figure 1 present the error produced by the above methods. The depicted results are averages over several instances (50 instances) of random variables. The curves in the figure show the average error of OptTrim and Trim operators with comparison to the average error of the optimal approximation provided by KolmogorovApprox as a function of  $m$ . According to the depicted results it is evident that increasing the support size of the approximation  $m$  reduces the error, as expected, in all three methods. However, errors produced by the KolmogorovApprox are significantly smaller, a half of the error produced by OptTrim and Trim.

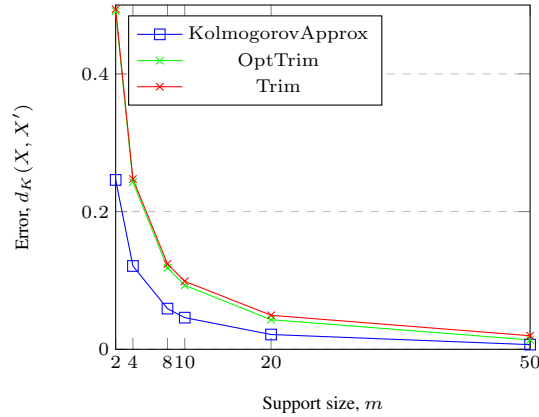


Figure 1: Error comparison between KolmogorovApprox, OptTrim, and Trim, on randomly generated random variables as function of  $m$ .

We also examined how our algorithm compares to linear programming as described and discussed, for example, in [15]. We ran an experiment to compare the run-time between the KolmogorovApprox algorithm with the run-time of a state-of-art implementation of linear programming. We used the “Minimize” function of Wolfram Mathematica and fed it with the equations  $\min_{\alpha \in \mathbb{R}^n} \|x - \alpha\|_\infty$  subject to  $\|\alpha\|_0 \leq m$  and  $\|\alpha\|_1 = 1$ . The run-time comparison results were clear and persuasive, for a random variable with support size  $n = 10$  and  $m = 5$ , the LP algorithm run-time was 850 seconds, where the KolmogorovApprox algorithm run-time was less than a tenth of a second. For  $n = 100$  and  $m = 5$ , the KolmogorovApprox algorithm run-time was 0.14 seconds and the LP algorithm took more than a day. Due to these timing results of the LP algorithm we did not proceed to examine it any further. Since it is not trivial to formally analyze the run-time of the LP algorithm, we conclude by the reported experiment that in this case the LP algorithm might not be as efficient as KolmogorovApprox algorithm whose complexity is proven to be polynomial in Theorem 11.

## 5 Discussion

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