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# Kolmogorov Approximation

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## 1 Introduction

Many different approaches to approximation of probability distributions are studied in the literature [9, 12, 13]. The papers vary in the types random variables involved, how they are represented, and in the criteria used for evaluation of the quality of the approximations. This paper is on approximating discrete distributions represented as explicit probability mass functions with ones that are simpler to store and to manipulate. This is needed, for example, when a discrete distribution is given as a large data-set, obtained, e.g., by sampling, and we want to represent it approximately with a small table.

The main contribution of this paper is an efficient algorithm for computing the best possible approximation of a given random variable with a random variable whose complexity is not above a prescribed threshold, where the measures of the quality of the approximation and the complexity of the random variable are as specified in the following two paragraphs.

We measure the quality of an approximation by the distance between the original variable and the approximate one. Specifically, we use the Kolmogorov distance which is one of the most used in statistical practice and literature. Given two random variables  $X$  and  $X'$  whose cumulative distribution functions (cdfs) are  $F_X$  and  $F_{X'}$ , respectively, the Kolmogorov distance between  $X$  and  $X'$  is  $d_K(X, X') = \sup_t |F_X(t) - F_{X'}(t)|$  (see, e.g., [8]). We say that  $X'$  is a good approximation of  $X$  if  $d_K(X, X')$  is small.

The complexity of a random variable is measured by the size of its support, the number of values that it can take,  $|\text{support}(X)| = |\{x: \Pr(X = x) \neq 0\}|$ . When distributions are maintained as explicit tables, as done in many implementations of statistical software, the size of the support of a variable is proportional to the amount of memory needed to store it and to the complexity of the computations around it.

In summary, the exact notion of optimality of the approximation targeted in this paper is:

**Definition 1.** A random variable  $X'$  is an optimal  $m$ -approximation of a random variable  $X$  if  $|\text{support}(X')| \leq m$  and there is no random variable  $X''$  such that  $|\text{support}(X'')| \leq m$  and  $d_K(X, X'') < d_K(X, X')$ .

The main contribution of the paper is an efficient algorithm that takes  $X$  and  $m$  as parameters and constructs an optimal  $m$ -approximation of  $X$ .

The rest of the paper is organized as follows. In Section 2 we describe how our work relates to other algorithms and problems studied in the literature. In Section 3 we detail the proposed algorithm,

31 analyze its properties, and prove Theorem ???. In Section 4 we demonstrate how the proposed  
 32 approach performs on the problem of estimating the probability of hitting deadlines is plans and  
 33 compare it to alternatives approximation approaches from the literature. We also demonstrate the  
 34 performance of our approximation algorithm on some randomly generated random variables. The  
 35 paper is concluded with a discussion in Section 5.

## 36 2 Related Work

37 The problem studied in this paper is related to the theory of Sparse Approximation (aka Sparse  
 38 Representation) that deals with sparse solutions for systems of linear equations, as follows.

Given a matrix  $D \in \mathbb{R}^{n \times p}$  and a vector  $x \in \mathbb{R}^n$ , the most studied sparse representation problem is finding the sparsest possible representation  $\alpha \in \mathbb{R}^p$  satisfying  $x = D\alpha$ :

$$\min_{\alpha \in \mathbb{R}^p} \|\alpha\|_0 \text{ subject to } x = D\alpha.$$

39 where  $\|\alpha\|_0 = |\{i : \alpha_i \neq 0, i = 1, \dots, p\}|$  is the  $\ell_0$  pseudo-norm, counting the number of non-zero  
 40 coordinates of  $\alpha$ . This problem is known to be NP-Hard with a reduction to NP-complete subset  
 41 selection problems.

In these terms, using also the  $\ell_\infty$  norm that represents the maximal coordinate and the  $\ell_1$  norm that represents the sum of the coordinates, our problem can be phrased as:

$$\min_{\alpha \in [0, \infty)^p} \|x - D\alpha\|_\infty \text{ subject to } \|\alpha\|_0 = m \text{ and } \|\alpha\|_1 = 1.$$

42 where  $D$  is the all-ones triangular matrix (the entry at row  $i$  and column  $j$  is one if  $i \leq j$  and zero  
 43 otherwise),  $x$  is related to  $X$  such that the  $i$ th coordinate of  $x$  is  $F_X(x_i)$  where  $\text{support}(X) = \{x_1 <$   
 44  $x_2 < \dots < x_n\}$  and  $\alpha$  is related to  $X'$  such that the  $i$ th coordinate of  $\alpha$  is  $f_{X'}(x_i)$ . The functions  $F_X$   
 45 and  $f_{X'}$  represent, respectively, the cumulative distribution function of  $X$  and the mass distribution  
 46 function of  $X'$ . This, of course, means that the coordinates of  $x$  are assumed to be positive and  
 47 monotonically increasing and that the last coordinate of  $x$  is assumed to be one. We demonstrate an  
 48 application for this specific sparse representation problem and show that it can be solve in  $O(n^2m)$   
 49 time and  $O(m^2)$  memory.

## 50 3 An Algorithm for Optimal Approximation

51 Let, in the scope of this section,  $X$  be a given random variable with a finite support of size  $n$ , and let  
 52  $0 < m \leq n$  be a given complexity bound. We will now develop notations and collect facts towards  
 53 an algorithm for finding an optimal  $m$ -approximation of  $X$ .

54 The first useful fact is that it is enough to limit our search to approximations  $X'$ 's such that  
 55  $\text{support}(X') \subseteq \text{support}(X)$ :

56 **Lemma 2.** *There is an optimal  $m$ -approximation  $X'$  of  $X$  such that  $\text{support}(X') \subseteq \text{support}(X)$ .*

57 *Proof.* Since minimizing  $d_K(X, X')$  subject to the constraints that  $\text{support}(X') \subseteq \text{support}(X)$   
 58 and  $|\text{support}(X')| \leq m$  is a convex optimization problem, it has a solution  $\varepsilon$ . Let  $X''$  be any  
 59 random variable whose support is of size smaller or equal to  $m$ , let  $\{x_1, \dots, x_n\} = \text{support}(X)$ ,  
 60 and let  $x_{n+1} = \infty$ . Consider the the random variable  $X'$  whose probability mass function is  
 61  $f_{X'}(x_i) = P(x_i \leq X'' < x_{i+1})$  for  $i = 1, \dots, n$  and  $F_{X'}(x) = 0$  if  $x \notin \text{support}(X)$ . By def-  
 62 inition  $d_K(X, X') = d_K(X, X'')$ . Since  $\text{support}(X') \subseteq \text{support}(X)$  and  $|\text{support}(X')| <$

63  $|\text{support}(X'')| \leq m$ , we get that  $d_K(X, X'') \geq \varepsilon$ . Thus,  $\varepsilon$  is a global minimum and any  
64  $X' \in \text{argmin}\{d_K(X, X') : \text{support}(X') \subseteq \text{support}(X), |\text{support}(X')| \leq m\}$  is an  $m$ -optimal  
65 approximation of  $X$ .  $\square$

66 Next, note that every random variable  $X''$  with support of size at most  $m$  that is contained in  
67  $\text{support}(X)$  can be described by first setting the (at most  $m$ ) elements of the support of  $X''$ ; then for  
68 every such option, determine  $X''$  by setting probability values for the elements in the chosen support  
69 of  $X'$ , and setting 0 for rest of the elements.

70 Denote the set of random variables with support  $S \subseteq \text{support}(X)$  by  $\mathbb{X}_S$ . In Step 1 below, we find a  
71 random variable in  $\mathbb{X}_S$  that minimizes the Kolmogorov distance from  $X$ , and denote this distance  
72 by  $\varepsilon(X, S)$ . Next, in Step 2, that we will describe later, we will show how to efficiently find  $S$  that  
73 minimizes  $\varepsilon(X, S)$  among all the sets that satisfy  $S \subseteq \text{support}(X)$  and  $|S| \leq m$ .

### 74 **3.1 Step 1: Finding a random variable $X'$ in $\mathbb{X}_S$ that minimizes $d_K(X, X')$**

75 We first fix a set  $S \subseteq \text{support}(X)$  of size at most  $m$ , and among all the random variables in  $\mathbb{X}_S$   
76 find one with a minimal distance from  $X$ . To that, set  $S = \{x_1 < \dots < x_m\} \subseteq \text{support}(X)$ . To  
77 simplify the proofs set  $x_0 = -\infty$ , and  $x_{m+1} = \infty$ . Then  $x_0 < x_1$  and  $x_m < x_{m+1}$ . In addition  
78 recall that for every random variable  $X''$ ,  $F_{X''}(-\infty) = 0$  and  $F_{X''}(\infty) = 1$ . Finally, for every  
79  $1 \leq i \leq m$  let  $\hat{x}_i$  be the maximal element of  $\text{support}(X)$  that is smaller than  $x_i$ . For the rest of this  
80 section we assume  $S$  is fixed and therefore is not necessarily included in the notation.

81 Next, as the elements of  $S$  are also elements of  $\text{support}(X)$ , we can define the following weight  
82 function that we use to find the  $m$ -optimal distance  $\varepsilon(X, S)$ .

83 **Definition 3.** For  $0 \leq i \leq m$  let

$$w(x_i, x_{i+1}) = \begin{cases} P(x_i < X < x_{i+1}) & \text{if } i = 0 \text{ or } i = m; \\ P(x_i < X < x_{i+1})/2 & \text{otherwise.} \end{cases}$$

84 Note that  $x_i = -\infty$  for  $i = 0$  and  $x_i = \infty$  for  $i = m + 1$ . Also note that  $P(x_i < X < x_{i+1}) =$   
85  $F_X(\hat{x}_{i+1}) - F_X(x_i)$ , a fact that we will use throughout this section.

86 **Definition 4.** Let  $\varepsilon(X, S) = \max_{i=0, \dots, m} w(x_i, x_{i+1})$ .

87 We first show that  $\varepsilon(X, S)$  is a lower bound. That is, every random variable in  $\mathbb{X}_S$  has a distance at  
88 least  $\varepsilon(X, S)$ . Then, we present a random variable  $X' \in \mathbb{X}_S$  with distance  $\varepsilon(X, S)$ . It then follows  
89 that such  $X'$  is an optimal  $m$ -approximation random variable among all random variables in  $\mathbb{X}_S$ .

90 The intuition behind choosing these specific weights and  $\varepsilon(X, S)$  being a lower bound is as follows.  
91 Since for every  $X' \in \mathbb{X}_S$  the probability values of  $X'$  for the elements not in  $S$  are set to 0, we have  
92 that  $F_{X'}(\hat{x}_{i+1}) = F_{X'}(x_i)$ . Therefore the distance between  $X'$  and  $X$  at points  $x_i$  and  $\hat{x}_{i+1}$  that we  
93 have to take into additional account is increased by  $F_X(\hat{x}_{i+1}) - F_X(x_i) = P(x_i < X < x_{i+1})$ .

94 Formally we have the following.

95 **Proposition 5.** If  $X' \in \mathbb{X}_S$  then  $d_k(X, X') \geq \varepsilon(X, S)$ .

96 *Proof.* By definition, for every  $0 \leq i \leq m$ ,  $d_k(X, X') \geq \max\{|F_X(\hat{x}_{i+1}) - F_{X'}(\hat{x}_{i+1})|, |F_X(x_i) -$   
97  $F_{X'}(x_i)|\}$ . Note that  $F_{X'}(\hat{x}_{i+1}) = F_{X'}(x_i)$  since the probability values for all the elements not in  $S$   
98 are set to 0.

99 If  $i = 0$ , that is  $x_i = -\infty$ , we have that  $F_X(x_i) = F_{X'}(x_i) = F_{X'}(\hat{x}_{i+1}) = 0$  and therefore  
 100  $d_k(X, X') \geq |F_X(\hat{x}_{i+1})| = |F_X(\hat{x}_{i+1}) - F_X(x_i)| = P(x_i < X < x_{i+1}) = w(x_i, x_{i+1})$ .

101 If  $i = m$ , that is  $x_{i+1} = \infty$ , we have that  $F_X(\hat{x}_{i+1}) = F_{X'}(\hat{x}_{i+1}) = F_{X'}(x_i) = 1$ . and therefore  
 102  $d_k(X, X') \geq |1 - F_X(\hat{x}_i)| = |F_X(\hat{x}_{i+1}) - F_X(x_i)| = P(x_i < X < x_{i+1}) = w(x_i, x_{i+1})$ .

103 Otherwise for every  $1 \leq i < m$ , we use the fact that  $\max\{|a|, |b|\} \geq |a - b|/2$  for every  $a, b \in$   
 104  $\mathbb{R}$ , to have  $d_k(X, X') \geq 1/2|F_X(\hat{x}_{i+1}) - F_X(x_i) + F_{X'}(x_i) - F_{X'}(\hat{x}_{i+1})|$ . So  $d_k(X, X') \geq$   
 105  $1/2|F_X(\hat{x}_{i+1}) - F_X(x_i)| = P(x_1 < X < x_2)/2 = w(x_i, x_{i+1})$ .

106 Therefore since  $d_k(X, X') \geq w(x_i, x_{i+1})$  for every  $0 \leq i \leq m$ , by definition of  $\varepsilon(X, S)$  proof  
 107 follows.  $\square$

108 Next we show a random variable  $X' \in \mathbb{X}_S$  with a distance of  $\varepsilon(X, S)$  from  $X$ . Thus  $X'$  is an optimal  
 109  $m$ -approximation among the set  $\mathbb{X}_S$ . We define  $X'$  as follows:

110 **Definition 6.** Let  $f_{X'}(x_i) = w(x_{i-1}, x_i) + w(x_i, x_{i+1}) + f_X(x_i)$  for  $i = 1, \dots, m$  and  $f_{X'}(x) = 0$   
 111 for  $x \notin S$ .

112 We first show that  $X'$  is a properly defined random variable:

113 **Lemma 7.**  $f_{X'}$  is a probability mass function.

114 *Proof.* From definition  $f_{X'}(x_i) \geq 0$  for every  $i$ . To see that  $\sum_i f_{X'}(x_i) = 1$ , we have  $\sum_i f_{X'}(x_i) =$   
 115  $\sum_i (w(x_{i-1}, x_i) + w(x_i, x_{i+1}) + f_X(x_i)) = \sum_{x_i \in S} f_X(x_i) + w(x_0, x_1) + \sum_{0 < i < m} 2w(x_i, x_{i+1}) +$   
 116  $w(x_m, x_{m+1}) = \sum_{x_i \in S} P(X = x_i) + P(x_0 < X < x_1) + \sum_{0 < i < m} P(x_i < X < x_{i+1}) +$   
 117  $P(x_m < X < x_{m+1}) = 1$  since this sum is the entire cpt of  $X$ .  $\square$

118 Note that  $X'$  can be constructed in linear time to the size of the cdf of  $X$ . Intuitively the setting of  
 119  $X'$  allows to take an "advantage" of distance from  $X$  at the elements of  $\text{support}(X')$ , to avoid the  
 120 overall increased distance of  $X$  from  $X'$  at the elements that are not at  $\text{support}(X)$  and in which  
 121  $f_{X'}$  is set to 0. Formally we have the following.

122 **Lemma 8.** Let  $x \in \text{support}(X)$  and  $0 \leq i \leq m$  be such that  $x_i \leq x \leq x_{i+1}$  then  $-w(x_i, x_{i+1}) \leq$   
 123  $F_X(x) - F_{X'}(x) \leq w(x_i, x_{i+1})$ .

124 *Proof.* We prove by induction on  $0 \leq i < m$ .

125 First see that  $F_{X'}(j) = 0$  for every  $x_0 < j < x_1$  and therefore  $F_X(j) - F_{X'}(j) = F_X(j) - 0 \leq$   
 126  $F_X(\hat{x}_1) = F_X(\hat{x}_1) - F_X(x_0) = w(x_0, x_1)$ . For  $j = x_1$  we have  $F_X(x_1) - F_{X'}(x_1) = F_X(\hat{x}_1) +$   
 127  $f_X(x_1) - (w(x_0, x_1) + w(x_1, x_2) + f_X(x_1)) = w(x_0, x_1) + f_X(x_1) - (w(x_0, x_1) + w(x_1, x_2) +$   
 128  $f_X(x_1)) = -w(x_1, x_2)$ .

129 Next assume that  $F_X(\hat{x}_i) - F_{X'}(\hat{x}_i) = w(x_{i-1}, x_i)$ . Then  $F_X(x_i) - F_{X'}(x_i) = F_X(\hat{x}_i) + f_X(x_i) -$   
 130  $(w(x_{i-1}, x_i) + w(x_i, x_{i+1}) + f_X(x_i)) = w(x_{i-1}, x_i) + f_X(x_i) - (w(x_{i-1}, x_i) + w(x_i, x_{i+1}) +$   
 131  $f_X(x_i)) = -w(x_i, x_{i+1})$ .

132 As before we have that for all  $x_i < j < x_{i+1}$ , we have  $F_X(j) - F_{X'}(j) = F_X(j) - F_{X'}(\hat{x}_{i+1}) \leq$   
 133  $F_X(\hat{x}_{i+1}) - F_{X'}(\hat{x}_{i+1})$ . Then  $F_X(\hat{x}_{i+1}) - F_{X'}(\hat{x}_{i+1}) = (F_X(x_i) + P(x_i < x < x_{i+1})) -$   
 134  $F_{X'}(x_i) = -w(x_i, x_{i+1}) + 2w(x_i, x_{i+1}) = w(x_i, x_{i+1})$ .

135 Finally for  $x_m \leq j \leq x_{m+1}$  we have that  $F_{X'}(x_m) = 1$  therefore  $F_X(x_m) - F_{X'}(x_m) = (1 -$   
 136  $P(x_m < X < x_{m+1})) - 1 = P(x_m < X < x_{m+1}) = w(x_m, x_{m+1})$ , and for every  $x_m < j <$   
 137  $x_{m+1}$  we have  $F_X(j) - F_{X'}(j) < (1 - P(x_m < X < x_{m+1})) - 1 < -P(x_m < X < x_{m+1}) =$   
 138  $-w(x_m, x_{m+1})$  as required.  $\square$

139 From Lemma 8, by the definition of  $\varepsilon(X, S)$ , we then have:

140 **Corrolary 9.**  $d_k(X, X') = \varepsilon(X, S)$ .

### 141 3.2 Step 2: Finding an $S$ that minimizes $\varepsilon(X, S)$

142 Chakravarty, Orlin, and Rothblum [2] proposed a polynomial-time method that, given a certain  
143 objective functions (additive), finds an optimal consecutive partition. Their method involves the  
144 construction of a graph such that the (consecutive) set partitioning problem is reduced to the problem  
145 of finding the shortest path in that graph.

146 The KolmogorovApprox algorithm (Algorithm 1) starts by constructing a directed weighted graph  
147  $G$  similar to the method of Chakravarty, Orlin, and Rothblum [2]. The nodes  $V$  consist of the  
148 support of  $X$  together with an extra two nodes,  $-\infty$  and  $\infty$  for technical reasons, whereas the  
149 edges  $E$  connect every pair of nodes in one direction (lines 1-2). The weight  $w$  of each edge  
150  $e = (x, y) \in E$  is determined by one of two cases as in Definition 3. The first is where nodes  
151  $x$  or  $y$  are the source or target nodes respectively. In this case, the weight is the probability of  $X$   
152 to get a value between  $x$  and  $y$ , non inclusive, i.e.,  $w(e) = \Pr(x < X < y)$ . The second case  
153 is where  $x$  and  $y$  are not a source or target nodes, here the weight is the probability of  $X$  to get a  
154 value between  $x$  and  $y$ , non inclusive, divided by two i.e.,  $w(e) = \Pr(x < X < y)/2$ . The values  
155 taken are non inclusive, since we are interested only in the error value. The source node of the  
156 shortest path problem at hand corresponds to the  $-\infty$  node added to  $G$  in the construction phase,  
157 and the target node is the extra node  $\infty$ . The set of all solution paths in  $G$ , i.e., those starting at  
158  $-\infty$  and ending in  $\infty$  with at most  $m$  edges, is called  $paths(G, -\infty, \infty)$ . The goal is to find the  
159 path  $l$  in  $paths(G, -\infty, \infty)$  with the lightest bottleneck (line 3). This can be achieved by using the  
160 *Bellman – Ford* algorithm with two tweaks. The first is to iterate the graph  $G$  in order to find only  
161 paths with length of at most  $m$  edges. The second is to find the lightest bottleneck as opposed to  
162 the traditional objective of finding the shortest path. This is performed by modifying the manner of  
163 “relaxation” to  $bottleneck(x) = \min[\max(bottleneck(v), w(e))]$ , done also in [14]. Consequently,  
164 we find the lightest maximal edge in a path of length  $\leq m$ , which represents the minimal error,  
165  $\varepsilon(X, S)$ , defined in Definition ?? where the nodes in path  $l$  represent the elements in set  $S$ . The  
166 approximated random variable  $X'$  is then derived from the resulting path  $l$  (lines 4-5). Every node  
167  $x \in l$  represent a value in the new calculated random variable  $X'$ , we than iterate the path  $l$  to find  
168 the probability of the event  $f_{X'}(x)$  as described in Definition 6. For every edge  $(x_i, x_j) \in l$  we  
169 determine: if  $(x_i, x_j)$  is the first edge in the path  $l$  (i.e.  $x_i = -\infty$ ), then node  $x_j$  gets the full weight  
170  $w(x_i, x_j)$  and it's own weight in  $X$  such that  $f_{X'}(x_j) = f_X(x_j) + w(x_i, x_j)$ . If  $(x_i, x_j)$  in not the  
171 first nor the last edge in path  $l$  then we divide it's weight between nodes  $x_i$  and  $x_j$  in addition to their  
172 own original weight in  $X$  and the probability that already accumulated. If  $(x_i, x_j)$  is the last edge  
173 in the path  $l$  (i.e.  $x_j = \infty$ ) then node  $x_i$  gets the full weight  $w(x_i, x_j)$  in addition to what was already  
174 accumulated such that  $f_{X'}(x_j) = f_{X'}(x_i) + w(x_i, x_j)$ .

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#### Algorithm 1: KolmogorovApprox( $X, m$ )

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1  $S = \text{support}(X) \cup \{\infty, -\infty\}$ 
2  $G = (V, E) = (S, \{(x, y) : x < y\})$ 
3  $(x_0, \dots, x_{m+1}) = l \in \text{argmin}_{l \in paths(G, -\infty, \infty), |l| \leq m} \max\{w(e) : e \in l\}$ 
4 for  $0 < i < m + 1$  do
5    $\lfloor f_{X'}(x_i) = w(x_{i-1}, x_i) + w(x_i, x_{i+1}) + f_X(x_i)$ 
6 return  $X'$ 

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---

175 **Theorem 10.** KolmogorovApprox( $X, m$ ) is an  $m$ -optimal-approximation of  $X$ .

176 *Proof.* If we consider the vertexes  $S = l \setminus \{-\infty, \infty\}$  for a path  $l \in \text{paths}(G, -\infty, \infty)$  we have  
 177 that  $\max\{w(e) : e \in l\} = \varepsilon(X, S)$ . Therefore, line 3 of the algorithm essentially computes a set  
 178  $S \in \operatorname{argmin}_{S \subseteq \operatorname{support}(X), |S| \leq m} \varepsilon(X, S)$ . By Corollary 9, the variable  $X'$  constructed in lines 4 and  
 179 5 satisfies  $d_K(X, X') = \varepsilon(X, S)$  and by the minimality of  $S$  and by Proposition 5, it is an optimal  
 180 approximation.  $\square$

181 **Theorem 11.** *The KolmogorovApprox( $X, m$ ) algorithm runs in time  $O(mn^2)$ , using  $O(n^2)$  mem-*  
 182 *ory where  $n = |\operatorname{support}(X)|$ .*

183 *Proof.* Constructing the graph  $G$  takes  $O(n^2)$ . The number of edges is  $O(E) \approx O(n^2)$  and for every  
 184 edge the weight is at most the sum of all probabilities between the source node  $-\infty$  and the target  
 185 node  $\infty$ , which can be done efficiently by aggregating the weights of already calculated edges. The  
 186 construction is also the only stage that requires memory allocation, specifically  $O(E + V) = O(n^2)$ .  
 187 Finding the shortest path takes  $O(m(E + V)) \approx O(mn^2)$ .

188 [[GW: put a reference to the work of the fellows from the Technion to avoid some of this?]]

189 Since  $G$  is DAG (directed acyclic graph) finding shortest path takes  $O(E + V)$ . We only need  
 190 to find paths of length  $\leq m$ , which takes  $O(m(E + V))$ . Deriving the new random variable  $X'$   
 191 from the computed path  $l$  takes  $O(mn)$ . For every node in  $l$  (at most  $m$  nodes), calculating the  
 192 probability  $P(s < X < \infty)$  takes at most  $n$  steps. To conclude, the worst case run-time complexity  
 193 is  $O(n^2 + mn^2 + mn) = O(mn^2)$  and memory complexity is  $O(E + V) = O(n^2)$ .  $\square$

## 194 4 A case study and experimental results

195 The case study examined in our experiments is the problem of task trees with deadlines [4, 3].  
 196 Hierarchical planning is a well-established field in AI [5, 6, 7], and is still relevant nowadays [1, 15].  
 197 A hierarchical plan is a method for representing problems of automated planning in which the  
 198 dependency among tasks can be given in the form of networks, here we focus on hierarchical plans  
 199 represented by task trees. The leaves in a task tree are *primitive* actions (or tasks), and the internal  
 200 nodes are either *sequence* or *parallel* actions. The plans we deal with are of stochastic nature, where  
 201 the duration of a primitive action is given by a random variable.

202 A sequence node denotes a series of tasks that should be performed consecutively, whereas a parallel  
 203 node denotes a set of tasks that begin at the same time. A *valid* plan is one that is fulfilled before some  
 204 given *deadline*, i.e., its *makespan* is less than or equal to the deadline. The objective in this context  
 205 is to compute the probability that a given plan is valid, or more formally computing  $P(X < T)$ ,  
 206 where  $X$  is a random variable representing the makespan of the plan and  $T$  is the deadline. As said  
 207 above, resource consumption (task duration) is uncertain, and described as probability distributions  
 208 in the leaf nodes. We assume that the distributions are independent but *not* necessarily identically  
 209 distributed and that the random variables are discrete and have a finite support.

210 The problem of finding the probability that a task tree satisfies a deadline is known to be NP-hard. In  
 211 fact, even the problem of summing a set of random variables is NP-hard [10]. This is an example of  
 212 an explicitly given random variable that we need to estimate deadline meeting probabilities for.

213 In the first experiment we focus on is the problem of task trees with deadlines, and consider three  
 214 types of task trees. The first type includes logistic problems of transporting packages by trucks and  
 215 airplanes (from IPC2 <http://ipc.icaps-conference.org/>). Hierarchical plans of those logistic problems  
 216 were generated by the JSHOP2 planner [11] (see example problem, Figure 1, one parallel node with  
 217 all descendant task nodes being in sequence). The second type consists of task trees used as execution

plans for the ROBIL team entry in the DARPA robotics challenge (DRC simulation phase), and the third type is of linear plans (sequential task trees). The primitive tasks in all the trees are modeled as discrete random variables with support of size  $M$  obtained by discretization of uniform distributions over various intervals. The number of tasks in a tree is denoted by  $N$ .

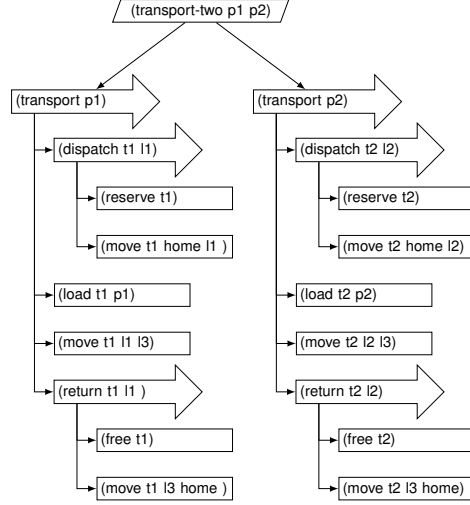


Figure 1: A plan generated by the JSHOP2 algorithm. Arrow shapes represent sequence nodes, parallelograms represent parallel nodes, and rectangles represent primitive nodes.

We implemented the approximation algorithm for solving the deadline problem with four different methods of approximation. The first two are for achieving a one-sided Kolmogorov approximation – the OptTrim [3] and the Trim [4] operators, and the third is a simple sampling scheme. We used those methods as a comparison to the Kolmogorov approximation with the suggested KolmogorovApprox algorithm. The parameter  $m$  of OptTrim and KolmogorovApprox corresponds to the inverse of  $\varepsilon$  given to the Trim operator. Note that in order to obtain some error  $\varepsilon$ , one must take into consideration the size of the task tree  $N$ , therefore,  $m/N = 1/(\varepsilon \cdot N)$ . We ran also an exact computation as a reference to the approximated one in order to calculate the error. The experiments conducted with the following operators and their parameters: KolmogorovApprox operator with  $m = 10 \cdot N$ , the OptTrim operator with  $m = 10 \cdot N$ , the Trim as operator with  $\varepsilon = 0.1/N$ , and two simple simulations, with a different samples number  $s = 10^4$  and  $s = 10^6$ .

| Task Tree                | $M$ | KolmogorovApprox | OptTrim  | Trim                      | Sampling |          |
|--------------------------|-----|------------------|----------|---------------------------|----------|----------|
|                          |     | $m/N=10$         | $m/N=10$ | $\varepsilon \cdot N=0.1$ | $s=10^4$ | $s=10^6$ |
| Logistics<br>( $N=34$ )  | 2   | 0                | 0        | 0.0019                    | 0.007    | 0.0009   |
|                          | 4   | 0.0024           | 0.0046   | 0.0068                    | 0.0057   | 0.0005   |
| Logistics<br>( $N=45$ )  | 2   | 0.0002           | 0.0005   | 0.002                     | 0.015    | 0.001    |
|                          | 4   | 0                | 0.003    | 0.004                     | 0.008    | 0.0006   |
| DRC-Drive<br>( $N=47$ )  | 2   | 0                | 0.004    | 0.009                     | 0.0072   | 0.0009   |
|                          | 4   | 0.001            | 0.008    | 0.019                     | 0.0075   | 0.0011   |
| Sequential<br>( $N=10$ ) | 2   | 0.0093           | 0.015    | 0.024                     | 0        | 0        |
|                          | 4   | 0                | 0.024    | 0.04                      | 0.008    | 0.0016   |
|                          | 10  | 0                | 0.028    | 0.06                      | 0.0117   | 0.001    |

Table 1: Comparison of estimated errors with respect to the reference exact computation on various task trees.

Table 1 shows the results of the main experiment. The quality of the solutions provided by using the OptTrim operator are better (lower errors) than those provided by the Trim operator, following the

optimality guarantees, but is interesting to see that the quality gaps happen in practice in each of the examined task trees. However, in some of the task trees the sampling method produced better results than the approximation algorithm with OptTrim. Nevertheless, the approximation algorithm comes with an inherent advantage of providing an exact quality guarantees, as opposed to the probabilistic guarantees provided by sampling.

In order to better understand the quality gaps in practice between KolmogorovApprox, OptTrim, and Trim, we investigate their relative errors when applied on single random variables with support size  $n = 100$ , and different support sizes of the resulting random variable approximation ( $m$ ). In each instance of this experiment, a random variable is randomly generated by choosing the probabilities of each element in the support from a uniform distribution and then normalizing these probabilities so that they sum to one.

Figure 2 present the error produced by the above methods. The depicted results are averages over several instances (50 instances) of random variables. The curves in the figure show the average error of OptTrim and Trim operators with comparison to the average error of the optimal approximation provided by KolmogorovApprox as a function of  $m$ .

According to the depicted results it is evident that increasing the support size of the approximation  $m$  reduces the error, as expected, in all three methods. However, errors produced by the KolmogorovApprox are significantly smaller, safe to say, a half of the error produced by OptTrim and Trim, it is clear both in the table (the relative error is mostly above 1) and in the graph.

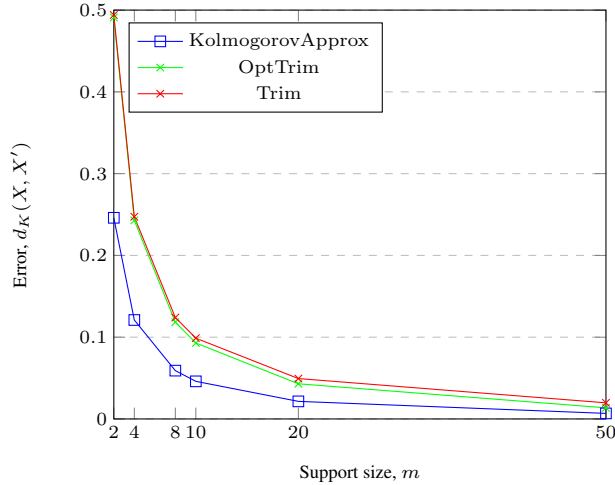


Figure 2: Error comparison between KolmogorovApprox, OptTrim, and Trim, on randomly generated random variables as function of  $m$ .

We also examined how our algorithm compares to linear programming as described and discussed, for example, in [12]. We ran an experiment to compare the run-time between the KolmogorovApprox algorithm with the run-time of a state-of-art implementation of linear programming. We used the “Minimize” function of Wolfram Mathematica and fed it with the equations  $\min_{\alpha \in \mathbb{R}^n} \|x - \alpha\|_\infty$  subject to  $\|\alpha\|_0 \leq m$  and  $\|\alpha\|_1 = 1$ . The run-time comparison results were clear and persuasive, for a random variable with support size  $n = 10$  and  $m = 5$ , the LP algorithm run-time was 850 seconds, where the KolmogorovApprox algorithm run-time was less than a tenth of a second. For  $n = 100$  and  $m = 5$ , the KolmogorovApprox algorithm run-time was 0.14 seconds and the LP algorithm took more than a day. Due to these timing results of the LP algorithm we did not proceed to examine it any further. Since it is not trivial to formally analyze the run-time of the LP algorithm,



we conclude by the reported experiment that in this case the LP algorithm might not be as efficient as KolmogorovApprox algorithm whose complexity is proven to be polynomial in Theorem 11.

## 5 Discussion

## References

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