**SC1007 Data Structures and Algorithms** **2022/23 Semester 2**

# 7: Analysis of Algorithms

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## 7.1 Asymptotic Algorithm Analysis

In algorithm analysis, we are interested to understand how the resources (e.g. **time** and **memory space**) used by an algorithm when the input size is increased. **Asymptotic Analysis**, a.k.a. **Asymptotics** is the study of describing the limiting behavior of algorithms. It usually describes as a function of a parameter, $N$, as $N$ becomes larger and larger without bound. In this lecture, we will only discuss problems of the single-parameter asymptotics. For example, $f(n) = n^3 + 2n$. As $n$ becomes very large, the term $2n$ can be negligible. In this case, $f(n)$ is asymptotically equivalent to $n^3$. $n$ denotes the input size of an algorithm. In the graph problems, you may see some two-parameter problems, where you need to use functions of two parameters, the number of vertices and the number of edges.

## 7.2 Time and Space Complexity

There are two kinds of efficiency of algorithms: **time efficiency** and **space efficiency**.
**Time efficiency**, a.k.a time complexity, indicates the amount of time used by an algorithm. We need to **count the number of primitive operations** in the algorithm and express it in terms of problem size (e.g. a function of $n$). We also need to understand how the parameters affect the performance of the algorithm. It is related to algorithmic aspects.
**Space efficiency**, a.k.a space complexity indicates the amount of memory units used by an algorithm. We need to understand how the data (inputs and some intermediate results) are storaged. It is related to data structures in the algorithm.

- Primitive operations: declaration (e.g. int x), assignment (e.g. x=1), arithmetic (+, -, *, /, %) and logic (==, ! =, <, >, &&, ||) operations
  These basic steps are usually performed in **constant time**

- Repetition Structure: for-loop, while-loop

- Selection Structure: if/else statement, switch-case statement

### 7.2.1 Example 1: while loop

**Algorithm 1** ‘while’ Loop

```
1: $j \leftarrow 0$                                   ▷ Constant time: $c_0$
2: while $j <= n$ do                                  ▷ Run $n$ iterations
3:     $factorial \leftarrow factorial * j$           ▷ $c_1$
4:     $j \leftarrow j + 1$                           ▷ $c_2$
                                   ▷ Time complexity $= c_1 n + c_2 n$
```

In this example, $c_0$, $c_1$ and $c_2$ are constant time for respective primitive operations. The time complexity of this ‘while’ algorithm is a function of $n$. The function increases **linearly** with $n$.

### 7.2.2 Example 2: Nested ‘for’ loop

**Algorithm 2** Nested ‘for’ Loop

```
1: for $j \leftarrow 1, m$ do                         ▷ Run $m$ iterations
2:     for $k \leftarrow 1, n$ do                     ▷ Run $n$ iterations
3:         $sum \leftarrow sum + M[j][k]$             ▷ $c_1$
                              ▷ $c_2$ additional cost of outer loop
                              ▷ Time complexity $= m(c_2 + c_1 n)$
```

The time complexity of this nested loop example is $m(c_2 + c_1 n)$. There are many additional costs in a loop. Not only $j{+}{+}$, branch back to the first line of the loop and check the conditional statement require to take time. These operations are usually constant time. When m=n, the time complexity can be simplified to a funciton of $n^2$. The number of operations increases **quadratically** with $n$.

### 7.2.3 Example 3: Nested loop with cubic time complexity

```
    for (i=1; i<=n; i++)
        M[i] = 0;
        for (j=i; j>0; j--)
            for (k=i; k>0; k--)
                M[i] += A[j]*B[k];
```

In this example, the inner nested loops run $i^2$ iterations where $i$ is from 1 to $n$. The total number of iterations is

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{7.1}$$

## 7.3 Different Cases of Complexity Analysis

When the algorithm involves selection structure, not all operations will be executed every time. Given different inputs, different operation blocks will be selected. Hence, the number of operations will be different. In the algorithm analysis, we need to further consider the following three cases:

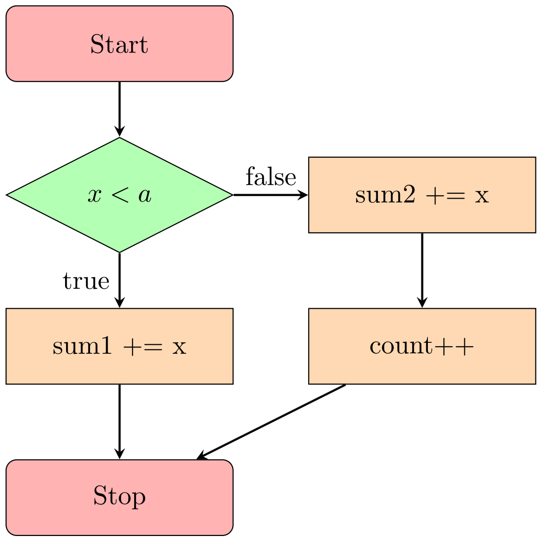
- Best-case analysis: The minimum number of primitive operations performed by the algorithm on any input of size $n$. It is known as the **best-case time complexity**

- Worst-case analysis: The maximum number of primitive operations performed by the algorithm on any input of size $n$. It is known as the **worst-case time complexity**

- Average-case analysis: The average number of primitive operations performed by the algorithm on all inputs of size $n$. It is known as the **average time complexity**

### 7.3.1 Example 4: 'if-else' selection structure

```
if(x < a)
        sum1 += x;
else{
        sum2+=x;
        count++;
        }
```



Let $c_1$ be the cost of $x < a$ branch, $c_2$ be the cost of its *else* case and $\mathbf{p}(<)$ be the probability that $x < a$ is true

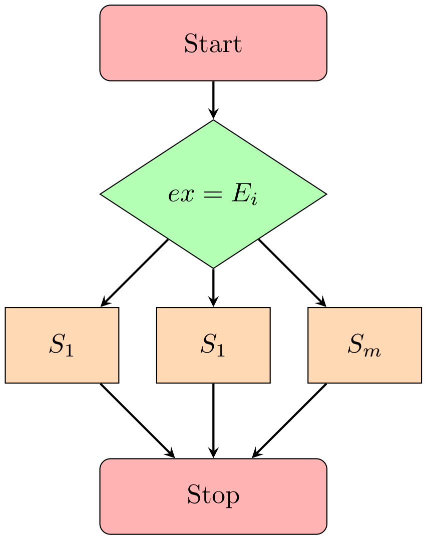
- Best-case analysis: $c_1$

- Worst-case analysis: $c_2$

- Average-case analysis: $\mathbf{p}(<) * c_1 + (1 - \mathbf{p}(<)) * c_2$

### 7.3.2 Example 5: ‘switch-case’ multiple selection structure

switch(ex)
    case $E_1$: $\mathbf{S_1}$;
    case $E_2$: $\mathbf{S_2}$;
    ...
    case $E_m$: $\mathbf{S_m}$;}



Let **C** be the time complexity of the switch-case multiple selection structure, $\mathbf{T_i}$ be the time complexity of each block case. The three complexity analyses of the algorithm are:

- Best-case analysis: $C + T_{min}$
- Worst-case analysis: $C + T_{max}$
- Average-case analysis: $C + \sum_{i=1}^{m} \mathbf{p}(i) T_i$

```
switch (choice){
    case 1: computer the sum; break;    // 5n instructions
    case 2: search BST; break;          // 6lgn instructions
    case 3: print BST; break;           // 3n instructions
    case 4: search for the minimum; break; //4lgn instructions
}
```

Listing 1: switch-case Statement: The cost above are for examples only

- Best-case analysis: $C + T_{min} = C + 4 \log_2 n$
- Worst-case analysis: $C + T_{max} = C + 5n$
- Average-case analysis: $C + \sum_{i=1}^{4} \mathbf{p}(i) T_i$
  Let’s assume that the probabilities of cases are 0.1, 0.4, 0.3 and 0.2 respectively. Then we obtain $C + 1.4n + 3.2 \log_2 n$

### 7.3.3 Example 6: Time complexity of Sequential Search

```c
pt=head;
while (pt->key != a){
   pt = pt->next;
   if(pt == NULL) break;
}
```

Listing 2: Searching an item in a linked list

Let $c_1$ be the cost of checking the first node and $c_2$ be the cost of each iteration.
When the search key **a**, is always in the list, we have:

- Best-case analysis: $c_1$ when **a** is the first item in the list
- Worst-case analysis: $c_1 + c_2(n-1)$ when **a** is the last item in the list
- Average-case analysis: assuming the probability to search for any item is equal, $\frac{1}{n}$

$$\begin{aligned}\frac{1}{n}\sum_{n}^{i=1}(c_1 + c_2(i-1)) &= \frac{1}{n}[nc_1 + c_2\sum_{i=1}^{n-1}(i-1)]\\ &= c_1 + \frac{c_2}{n}\frac{(n-1)(1+(n-1))}{2}\\ &= c_1 + \frac{c_2(n-1)}{2}\end{aligned}$$

When the item **a**, is not in the list, we have:

- When the search key **a** is not in the array, the number of comparisons, $c_1 + nc_2$.
- Combine success cases and failure cases with their probability, $P(succ) + P(fail) = 1$:

$$P(succ)(c_1 + \frac{c_2(n-1)}{2}) + (1 - P(succ))(c_1 + nc_2) = c_1 + nc_2 - P(succ)(\frac{c_2(n+1)}{2})$$

- the time complexity is a linear function

Since the data is stored in unordered. To search a key, every element is required to read and compare. This brute-force approach properly is the only way if we do not do any preprocessing on the data. Its time complexity is in linear. The constant term can be ignored when the $n$ is large.

## 7.4 Time Complexity of Recursive Functions

To obtain the time complexity of a recursive function, we need to determine:

- number of primitive operations for each recursive call
- number of recursive calls

The following example is the recursive version of Algorithm 1.

```c
int factorial (int n)
{
    if(n==1) return 1;
    else return n*factorial(n-1);
}
```

Let the cost of each recursive call when $n > 1$ be $c_1$ and when $n = 1$ be $c_2$.
The total number of recursive calls is $n - 1$ (factorial(n-1), factorial(n-2), . . . , factorial(2), factorial(1)).
Time complexity is

$$c_1(n - 1) + c_2$$

.

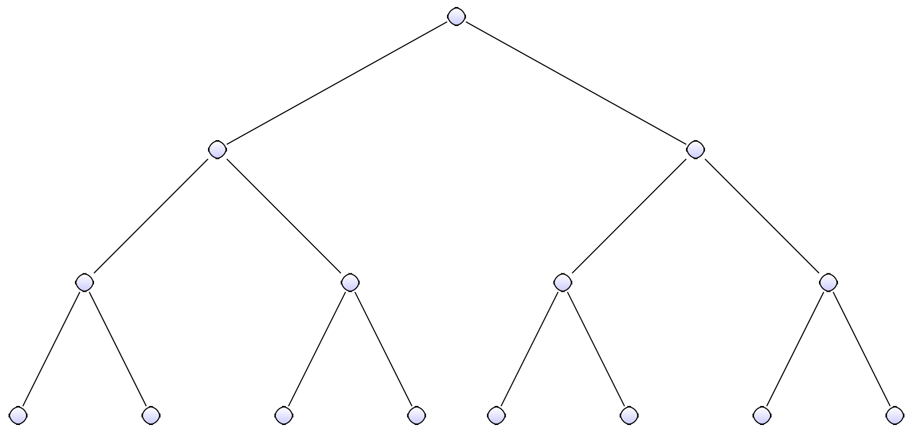
In the next example, the algorithm is counting the number of item, **a** in the array.

```c
int count (int array[],int n, int a)
{
    if(n==1)
          if(array[0]==a)
             return 1;
          else return 0;
    if(array[0]==a)
         return 1+ count(&array[1], n-1, a);
    else
         return count (&array[1], n-1, a);
}
```

Here we only determine the number of comparisons (array[0]==a).

$$\begin{aligned}
W_1 &= 1\\
W_n &= 1 + W_{n-1}\\
&= 1 + 1 + W_{n-2}\\
&= 1 + 1 + 1 + W_{n-3}\\
\ldots &= \ldots\\
&= 1 + 1 + \ldots + 1 + W_1\\
&= n
\end{aligned}$$

The total number of recursive call is $n-1$. We have done $n-1$ comparison. When $n = 1$, we do 1 comparison.
Total number of comparison is $n$.
This is a method of backward substitutions.
When there are multiple recursive calls, the analysis becomes a bit complex. See the following binary tree example:



```
preorder (simple_t* tree)
{
   if(tree != NULL){
      tree->item *= 10;
      preorder (tree->left);
      preorder (tree->right);
   }
}
```

When the tree is empty (NULL), there is no recursive call and the function will simply return back to the caller. Let us assume that it is a complete binary tree. The number of nodes is $2^k - 1$ where $k$ is the depth of the binary tree.

$$\begin{aligned}
W_0 &= 0\\
W_1 &= 1\\
W_2 &= 1 + W_1 + W_1 = 1 + 2 = 3\\
W_3 &= 1 + W_2 + W_2 = 1 + 2(1+2) = 1 + 2 + 4 = 7\\
\ldots &= \ldots\\
W_{k-1} &= 1 + 2 * W_{k-2} = 1 + 2 + 4 + 8 + \ldots + 2^{k-2}\\
W_k &= 2^k - 1
\end{aligned}$$

We can easily observe that it is a geometric series. Since the function above is visiting every node of the binary tree, the number of recursive calls is the number of nodes, $2^k - 1$.
This is a method of forward substitutions.

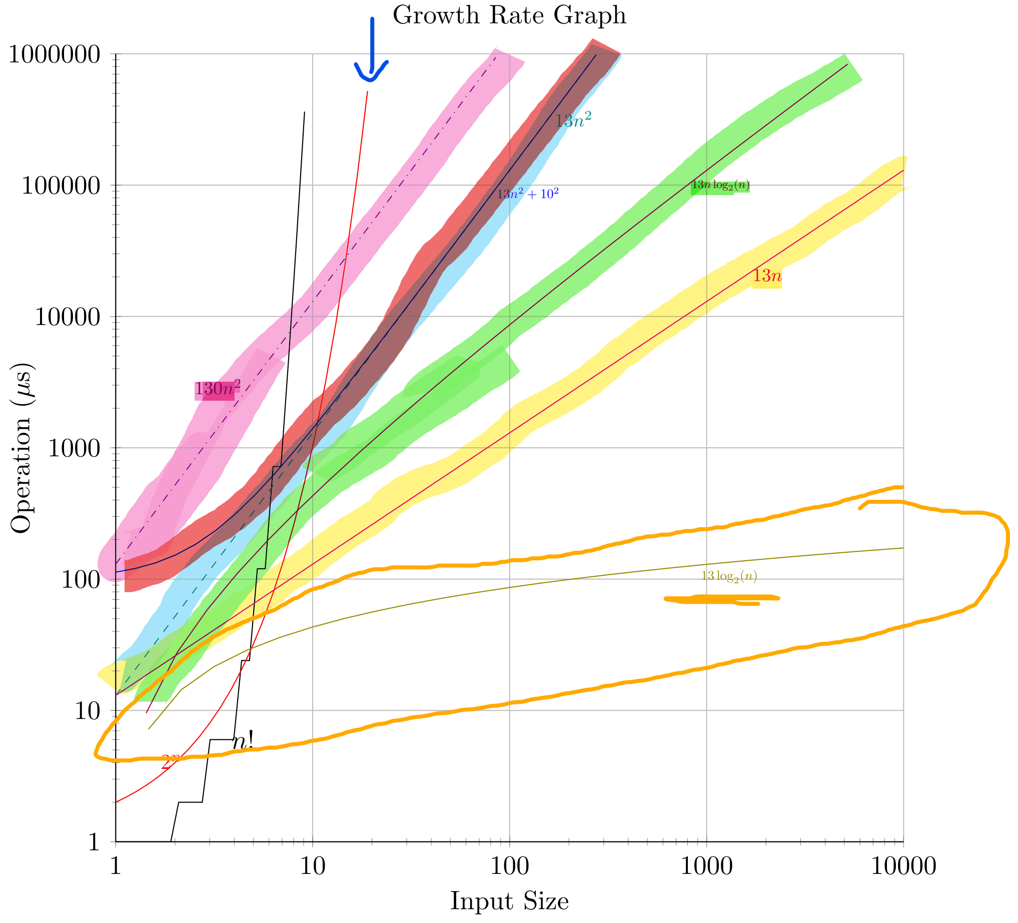
When we analyse the time complexity of an algorithm, we concern on its complexity when the problem size is large. When the problem size is large, some terms in the complexity may not be important. Thus, we are not really interested at its exact time complexity. We just would like to know its order of growth in practice.

## 7.5 Order of Growth

We are interested at growth rate of time complexity. To analysis the efficiency of an algorithm, we would like to know its running time for large input sizes. Thus, the constants are not significant and the multipliers are not important for relative growth rate.

| **Algorithm** | **linear** | **linearithmic** | **quadratic 1** | **quadratic 2** | **quadratic 3** | **exponential** |
|---|---|---|---|---|---|---|
| **Input / Operation(s)** | $13n$ | $13n\log_2 n$ | $13n^2$ | $130n^2$ | $13n^2+10^2$ | $2^n$ |
| 10 | 0.00013 | 0.00043 | 0.0013 | 0.013 | 0.0014 | 0.001024 |
| 100 | 0.0013 | 0.0086 | 0.13 | 1.3 | 0.1301 | $4 \times 10^{16}$ years |
| $10^4$ | 0.13 | 1.73 | 22mins | 3.61hrs | 22mins | |
| $10^6$ | 13 | 259 | 150days | 1505days | 150days | |

Growth Rate Graph



From the growth rate functions above, we can observe the following characteristic of functions

1. The factorial of n ($n!$) is the fastest growth when $n > 10$.
2. When n is large enough, the growth rate is in the following order

$$\text{constant} < \log_{10}(n) < \log_2(n) < n < nlog_2(n) < n^2 < n^3 < 10n^3 < 2^n < n! < n^n$$

3. When n is large enough, the constant, $10^2$, can be ignored
4. $13n^2$ and $130n^2$ are almost parallel when n is large. It implies that both have similar growth rate but $130n^2$ is slightly faster.

### 7.5.1 Faster Computer Versus Faster Algorithm

Can we simply use faster computer to resolve the 'difficult' computation problems? The answer is NO. To illustrate this problem, let us compare an old computer with a $10\times$ faster new computer. The old computer executes 10k basic operations per hour and the new computer can execute 100k operations per hour. The following table shows the problem size can be solved by the old computer ($n$) and the new computer ($n'$) in an hour.

| $f(n)$ | $n$ | $n'$ | Change | $\frac{n'}{n}$ |
|---|---|---|---|---|
| $10n$ | 1000 | 10k | $n' = 10n$ | 10 |
| $20n$ | 500 | 5k | $n' = 10n$ | 10 |
| $5n \log n$ | 250 | 1842 | $3.16n < n' < 10n$ | 7.37 |
| $2n^2$ | 70 | 223 | $n' = 3.16n$ | 3.16 |
| $2^n$ | 13 | 16 | $n' = n + 3$ | 1.23 |

For linear complexity problem, the improvement is $10\times$. For harder problems with faster-growing function, the improvement is poorer than the linear problems. The exponential complexity problem is hardly improved by using $10\times$ faster computer. We only manage to increase the number of data size from 13 to 16.

Compare $n^2$ algorithm with $n \log n$ algorithm When data size, $n = 1024$,

- $n^2$ algorithm takes $1024 \times 1024 = 1,048,576$ primitive steps
- $n \log n$ algorithm takes $1024 \times \log 1024 = 10,240$ primitive steps

The improvement from the $n^2$ to the $n \log_2 n$ is a factor of **100**.
When data size increases to $n = 2048$,

- $n^2$ algorithm takes $2048 \times 2048 = 4,194,304$ primitive steps
- $n \log n$ algorithm takes $2048 \times \log 2048 = 22,528$ primitive steps

The improvement from the $n^2$ to the $n \log_2 n$ is a factor of **200**.

Moore's Law asserts that the number of transistors on a microchip doubles every two years, though the cost of computers is halved. In other words, we can expect that the speed and capability of our computers will increase every couple of years, and we will pay less for them. However, we can observe that the computers is getting closer to the physical limits of Moore's Law. It is hardly to make the clock speed beyond 5GHz. We are trying to use parallel process to improve the performance. Moreover, It is always good that you have a more efficient algorithm.

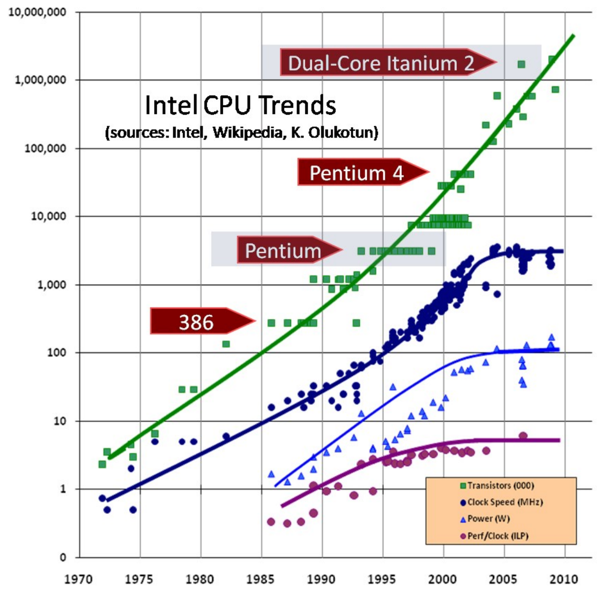


Figure 7.1: Moore's Law: https://cs.stackexchange.com/questions/27875/moores-law-and-clock-speed

## 7.6 Asymptotic Notation

When we consider the order of growth and efficiency of an algorithm, three asymptotic notations: $\Omega$ (big-Omega), $\Theta$ (big-Theta), $\mathcal{O}$ (big-Oh) are used.

### 7.6.1 Big-Oh Notation: $\mathcal{O}$

**Definition 7.1** *$\mathcal{O}$-notation: Let $f$ and $g$ be two functions such that $f(n) : \mathbb{N} \to \mathbb{R}^+$ and $g(n) : \mathbb{N} \to \mathbb{R}^+$, $f(n)$ is said to be in $\mathcal{O}(g(n))$, denoted $f(n) \in \mathcal{O}(g(n))$, if $f(n)$ is **bounded above** by some constant multiple of $g(n)$ for all large $n$, i.e., the set of functions can be defined as*

$$\mathcal{O}(g(n)) = \{f(n) : \exists \text{positive constants, } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \quad \forall n \geq n_0\}$$

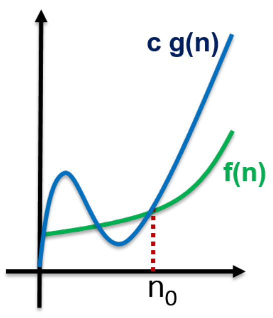


Figure 7.2: Big-Oh Notation

**Example 1 (a):**
Given that $f(n) = 4n + 3$ and $g(n) = n$,
If let $c = 5$ and $n_0 = 3$, then based on $\mathcal{O}$-notation definition, we have

$$\begin{aligned} f(n) &\leq cg(n) \quad \forall n \geq n_0 \\ f(n) &\leq 5g(n) \quad \forall n \geq 3 \end{aligned}$$

$\therefore f(n) = \mathcal{O}(g(n))$ or $4n + 3 \in \mathcal{O}(n)$

**Example 2 (a):**
Given that $f(n) = 4n + 3$ and $g(n) = n^3$,
If let $c = 1$ and $n_0 = 3$, then based on $\mathcal{O}$-notation definition, we have

$$f(n) \leq g(n) \quad \forall n \geq 3$$

$\therefore f(n) = \mathcal{O}(g(n))$ or $4n + 3 \in \mathcal{O}(n^3)$

The following alternative definition of $\mathcal{O}$-notation can help you to find the complexity class of the given function easily via their limit

**Definition 7.2** *$\mathcal{O}$-notation: Let $f$ and $g$ be two functions such that $f(n) : \mathbb{N} \to \mathbb{R}^+$ and $g(n) : \mathbb{N} \to \mathbb{R}^+$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, then $f(n) \in \mathcal{O}(g(n))$ or $f(n) = \mathcal{O}(g(n))$.*

**Example 1 (b) :**
Given that $f(n) = 4n + 3$ and $g(n) = n$,

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{4n+3}{n} \\ &= 4 < \infty \end{aligned}$$

$\therefore f(n) = \mathcal{O}(g(n))$ or $4n + 3 \in \mathcal{O}(n)$

**Example 2 (b):**
Given that $f(n) = 4n + 3$ and $g(n) = n^3$,

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{4n+3}{n^3} \\ &= 0 < \infty \end{aligned}$$

$\therefore f(n) = \mathcal{O}(g(n))$ or $4n + 3 \in \mathcal{O}(n^3)$

In certain cases, we may need to use L'Hôpital's Rule **Example 3 :**
Given that $f(n) = 4n + 3$ and $g(n) = e^n$,

Apply L'Hôpital's Rule to find $\lim_{n\to\infty} \frac{4n+3}{e^n}$

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \lim_{n\to\infty}\frac{4n+3}{e^n}\\ &= \lim_{n\to\infty}\frac{4}{e^n}\\ &= 0 < \infty\end{aligned}$$

$\therefore f(n) = \mathcal{O}(g(n))$ or $4n+3 \in \mathcal{O}(e^n)$

### 7.6.2 Big-Omega Notation: $\Omega$

**Definition 7.3** *$\Omega$-notation: Let $f$ and $g$ be two functions such that $f(n) : \mathbb{N} \to \mathbb{R}^+$ and $g(n) : \mathbb{N} \to \mathbb{R}^+$, $f(n)$ is said to be in $\Omega(g(n))$, denoted $f(n) \in \Omega(g(n))$, if $f(n)$ is* ***bounded below*** *by some constant multiple of $g(n)$ for all large $n$, i.e., the set of functions can be defined as*

$$\Omega(g(n)) = \{f(n) : \exists \text{positive constants, } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \quad \forall n \ge n_0\}$$

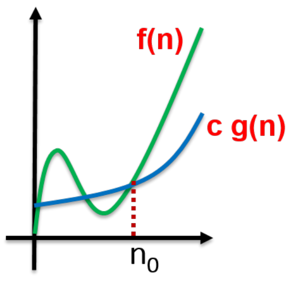


Figure 7.3: Big-Omega Notation

The following alternative Definition of $\Omega$-notation can help you to find the complexity class of the given function easily via their limit

**Definition 7.4** *$\Omega$-notation: Let $f$ and $g$ be two functions such that $f(n) : \mathbb{N} \to \mathbb{R}^+$ and $g(n) : \mathbb{N} \to \mathbb{R}^+$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$, then $f(n) \in \Omega(g(n))$ or $f(n) = \Omega(g(n))$.*

**Example 1 (a):**
Given that $f(n) = 4n+3$ and $g(n) = 5n$,
Let $c = \frac{1}{5}$, $n_0 = 0$
Then

$$\begin{gathered} f(n) \ge \frac{1}{5}g(n)\\ 4n+3 \ge n \quad \forall n \ge 0\end{gathered}$$

$\therefore f(n) = \Omega(g(n))$ or $4n+3 \in \Omega(n)$

**Example 1 (b):**
Given that $f(n) = 4n + 3$ and $g(n) = 5n$,

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{4n+3}{5n} \\ &= \frac{4}{5} > 0 \end{aligned}$$

$\therefore f(n) = \Omega(g(n))$ or $4n + 3 \in \Omega(n)$

**Example 2:**
Given that $f(n) = n^3 + 2n$ and $g(n) = 5n$,

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{n^3+2n}{5n} \\ &= \infty > 0 \end{aligned}$$

$\therefore f(n) = \Omega(g(n))$ or $4n + 3 \in \Omega(5n)$

### 7.6.3 Big-Theta Notation: $\Theta$

**Definition 7.5** $\Theta$-*notation: Let $f$ and $g$ be two functions such that $f(n) : \mathbb{N} \to \mathbb{R}^+$ and $g(n) : \mathbb{N} \to \mathbb{R}^+$, $f(n)$ is said to be in $\Theta(g(n))$, denoted $f(n) \in \Theta(g(n))$, if $f(n)$ is* ***bounded both above and below*** *by some constant multiples of $g(n)$ for all large $n$, i.e., the set of functions can be defined as*

$$\Theta(g(n)) = \{f(n) : \exists \text{positive constants, } c_1,\ c_2 \text{ and } n_0 \text{ such that } c_1 g(n) \le f(n) \le c_2 g(n) \quad \forall n \ge n_0\}$$

The following alternative Definition of $\Omega$-notation can help you to find the complexity class of the given function easily via their limit

**Definition 7.6** $\Theta$-*notation: Let $f$ and $g$ be two functions such that $f(n) : \mathbb{N} \to \mathbb{R}^+$ and $g(n) : \mathbb{N} \to \mathbb{R}^+$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ where $0 < c < \infty$, then $f(n) \in \Theta(g(n))$ or $f(n) = \Theta(g(n))$.*

**Example 1 (a):**
Given that $f(n) = 2n^2 + 7$ and $g(n) = 7n^2 + n$,
Using alternative definition above,

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{2n^2+7}{7n^2+n} \\ &= \frac{2}{7} \end{aligned}$$

$\therefore f(n) = \Theta(g(n))$ or $4n + 3 \in \Theta(n)$

### 7.6.4 Summary of Asymptotic Notation

| $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ | $f(n) \in \mathcal{O}(g(n))$ | $f(n) \in \Omega(g(n))$ | $f(n) \in \Theta(g(n))$ |
|---|---|---|---|
| $0$ | ✓ | | |
| $0 < c < \infty$ | ✓ | ✓ | ✓ |
| $\infty$ | | ✓ | |

The $\mathcal{O}$, $\Omega$ and $\Theta$ notations are used in studying the asymptotic efficiency of an algorithm.

- If $f(n) = \mathcal{O}(g(n))$, it implies that $g(n)$ is asymptotic upper bound of $f(n)$
- If $f(n) = \Omega(g(n))$, it implies that $g(n)$ is asymptotic lower bound of $f(n)$
- If $f(n) = \Theta(g(n))$, it implies that $g(n)$ is asymptotic tight bound of $f(n)$

In practice, $\mathcal{O}$-notation is the most useful notation.

When time complexity of algorithm A **grows faster** than algorithm B for the same problem, we say A is **inferior** to B.

### 7.6.5 How to determine the big-Oh notation from an algorithm?

1. Count primitive operations to derive complexity function $f(n)$ (in terms of problem size)
2. Discard constant terms and multipliers in $f(n)$
3. Determine dominant term in $f(n)$
4. Dominant term = big-Oh notation for $f(n)$ (= big-Oh notation for algorithm)

The dominant term can refer 7.6.7.

### 7.6.6 Asymptotic Notation in Equations and Its Simplification

When an asymptotic notation appears in an equation, we interpret it as standing for some anonymous function that we do not care to name.
**Examples**:

- $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$
- $T(n) = T(\frac{n}{2} + \Theta(n)$
- $2n^2 + 3n + 1 = 2n^2 + \Theta(n) = \Theta(n^2)$

Some simplication rules for asymptotic analysis:

1. If $f(n) = \mathcal{O}(cg(n))$ for any constant $c > 0$,
   then $f(n) = \mathcal{O}(g(n))$
2. If $f(n) = \mathcal{O}(g(n))$ and $g(n) = \mathcal{O}(h(n))$,
   then $f(n) = \mathcal{O}(h(n))$.
   e.g. $f(n) = 2n$, $g(n) = n^2$, $h(n) = n^3$
   $\Rightarrow f(n) = \mathcal{O}(g(n))$ and $g(n) = \mathcal{O}(h(n))$
   $\therefore 2n = \mathcal{O}(n^3)$
3. If $f_1(n) = \mathcal{O}(g_1(n))$ and $f_2(n) = \mathcal{O}(g_2(n))$,
   then $f_1(n) + f_2(n) = \mathcal{O}(\max(g_1(n), g_2(n)))$
   e.g. $5n + 3\lg n = \mathcal{O}(n)$
4. If $f_1(n) = \mathcal{O}(g_1(n))$ and $f_2(n) = \mathcal{O}(g_2(n))$,
   then $f_1(n)f_2(n) = \mathcal{O}(g_1(n)g_2(n))$
   e.g. $f_1(n) = 3n^2$, $f_2(n) = \lg n$, $f_1(n) = \mathcal{O}(n^2)$, $f_2(n) = \mathcal{O}(\lg n)$,
   then $3n^2 \lg n = \mathcal{O}(n^2 \lg n)$

   Some properties of $\mathcal{O}$, $\Omega$ and $\Theta$:

   - $\mathcal{O}$, $\Omega$ and $\Theta$ are **Reflexive**:
     - $f(n) = \mathcal{O}(f(n))$
     - $f(n) = \Omega(f(n))$
     - $f(n) = \Theta(f(n))$
   - $\mathcal{O}$, $\Omega$ and $\Theta$ are**Transitive**:
     - If $f(n) = \mathcal{O}(g(n))$ and $g(n) = \mathcal{O}(h(n))$, then $f(n) = \mathcal{O}(h(n))$
     - If $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$, then $f(n) = \Omega(h(n))$
     - If $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, then $f(n) = \Theta(h(n))$
   - $\Theta$ is**Symmetric**:
     - If $f(n) = \Theta(g(n))$, then $g(n) = \Theta(f(n))$ because it is tight bounded.

### 7.6.7 Common Complexity Classes

| Order of Growth | Class | Example |
|---|---|---|
| $1$ | constant | Finding midpoint of an array |
| $\log n$ | logarithmic | Binary search |
| $n$ | linear | Linear Search |
| $n \log_2 n$ | linearithmic | Merge Sort |
| $n^2$ | quadratic | Insertion Sort |
| $n^3$ | cubic | Matrix Inversion (Gauss-Jordan elimination) |
| $2^n$ | exponential | The Tower of Hanoi Problem |
| $n!$ | factorial | Travelling Salesman Problem |

1. Constant order: the running time is independent to the problem size, $n$. It denotes as $f(n) \in \mathcal{O}(1)$
   **Example:** $sum = \frac{n}{2}(n+1)$;
   We can count in the statement, 1 addition, 1 multiplication, 1 division and 1 assignment. There are 4 operations, which is independent of n. We have $f(n) = 4$, which means $f(n)$ is big Oh of 4, i.e. $\mathcal{O}(4)$. Formally, if you wish to verify that 4 is in $\mathcal{O}(1)$, you can pick $c = 4$ and any $n_o = 0$, such that $\forall n \geq 0$, $f(n)$, which is $\leq 4 \times 1$. As such, we can deduce that $f(n) \in \mathcal{O}(1)$.

2. Logarithmic order: $f(n) \in \mathcal{O}(\log n)$. $\log n$ grows slower than $n$ which means the running time of $f(n)$ increases slower than its problem size n.
   **Example:**

```
    for (i=n; i>=1; i/=2)
        sum++;
```

   It is noted that this example let $i = n$ and $i$ is reduced to $\frac{n}{2}$, then $\frac{n}{4}$ until it reaches 1. Details will be discussed in the tutorial but you can see that it will take $\lfloor \log_2 n \rfloor + 1$ iteration.
   $\therefore f(n) \in \mathcal{O}(\log n)$
   **Note 1**: Prove that Growth rate of $\log n$ is slower than $n^\varepsilon$ for all $\varepsilon > 0$

$$\begin{aligned}\lim_{n\to\infty} \frac{\log n}{n^\varepsilon} &= \lim_{n\to\infty} \frac{\frac{1}{\ln 10} \cdot \frac{1}{n}}{\varepsilon n^{\varepsilon - 1}} \\ &= \lim_{n\to\infty} \frac{c}{\varepsilon n^\varepsilon} \\ &= 0\end{aligned}$$

   **Note 2**: Base of log is convertible with a constant multiplier. Thus it is not important. $\log_b n = \frac{\log_c n}{\log_c b}$ where $\log_c b$ is a constant

3. Linear Order: $f(n) \in \mathcal{O}(n)$

```
    for (i=1; i<=n; j++)
        sum++;
```

   The number of iterations is $n$. $\therefore f(n) \in \mathcal{O}(n)$

4. Linearithmic Order:$f(n) \in \mathcal{O}(n \log n)$. It is commonly seen in algorithms that break a problem into sub-problems, solve them independently and combine the solutions. e.g. merge sort.
   For example, consider this set of recurrent equation, that represents the time complexity function of a recursive algorithm.
   $W(2) = 1$
   $W(n) = 2W(\frac{n}{2}) + n - 1$
   After you solve this recurrent equation, you will obtain a complexity class of $n \log n$. Please try to derive it out.

5. Polynomial Order:$f(n) \in \mathcal{O}(n^p)$ for $\exists p \in \mathbb{N}$
   **Example:**

```
    for (i=1; i<=n; i++)
        for (j=1; j<=n; j++)
            for (k=1; k<=n; k++)
                M[i][j] = A[i][k]*B[k][j];
```

   Consider this piece of codes above, consisting of a triple nested for loop, the number of primitive operations is proportional to $n^3$, where $n$ is the problem size.
   $\therefore f(n) \in \mathcal{O}(n^3)$

6. Exponential Order:$f(n) \in \mathcal{O}(a^n)$ for $\exists a \in \mathbb{N}$ usually it is not practical for normal use especially when the problem size is large. **Example:**Print all subsets of a set of n elements
   $\therefore f(n) \in \mathcal{O}(2^n)$

## 7.7 Space Complexity

For space complexity, we count the number of basic storage units in an algorithm. We first determine the number entities in problem (or problem size, $n$). Instead of count the number of primitive operations, we concern about the storage usage of the algorithm. The storage units can be integer (**int**), floating point number (**float**), or character (**char**).
**Example**:

1. The space complexity of an array of $n$ integers is $\mathbf{\Theta(n)}$.
2. A matrix used for storing edge information of a graph, i.e. $G[x][y] = 1$ if there exists an edge from $x$ to $y$. The space complexity of a graph with $n$ vertices is $\mathbf{\Theta(n^2)}$.

**Spaece/ Time Tradeoff Principle** It is important to note that there is typically a tradeoff between space complexity and time complexity. In other words, the reduction in time complexity can be achieved by sacrificing space complexity and vice versa. We shall see some examples of this in the later part of this course.

Arithmetic Series

$$A_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a_0+a_{n-1}]$$

Geometric Series

$$G_n = \frac{a(1-r^n)}{1-r}$$

Arithmetico-geometric Series

$$\sum_{t=1}^{k} t2^{t-1} = 2^k(k-1)+1$$

Faulhaber's Formula for the sum of the p-th powers of the first n positive integers

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$