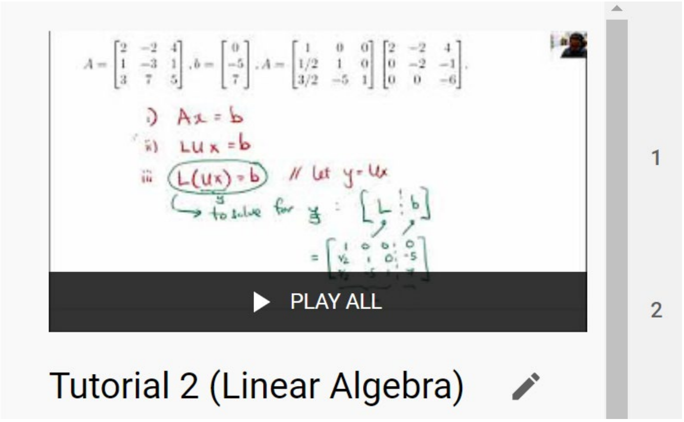
Maths/LA/Tut2

Chng Eng Siong

9 Sep 2020

Tutorial 2 Help links



Youtube link: playlist

https://www.youtube.com/channel/UCBzG5jg3huxiPkCt-Serrjw/playlists

https://www.youtube.com/watch?v=pq8AsdXJbP4&list=PLki3aFwg-9ey6evKRpWoCdtokuUWBuyjY

PDF

https://www.dropbox.com/s/zdthw24mv7ranuh/Tut2_Q1_8_ces.pdf?dl=0

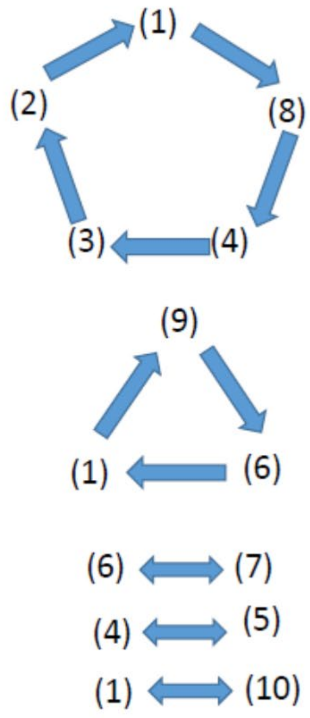
Good References:

Q1-6

- Invertible Matrix Theorem

Theorem 2.4. *The Invertible Matrix Theorem*
Let A be a square $n \times n$ matrix. Then the following statements are equivalent, i.e., for a given A, the statements are either all true or all false.

1. *A is an invertible matrix.*
2. *A is row equivalent to I_n.*
3. *A has n pivot positions.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *$A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The columns of A span $\mathbb{R}^n$.*
8. *There is an $n \times n$ matrix C such that CA=I.*
9. *There is an $n \times n$ matrix D such that AD=I.*
10. *A^T is an invertible matrix.*



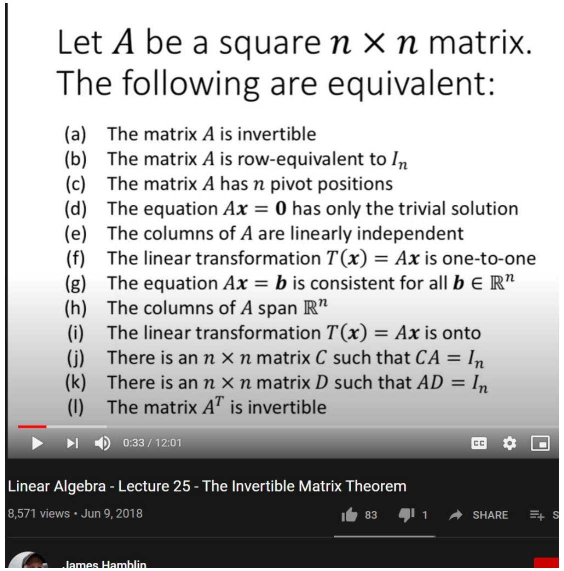
Invertibility Matrix Theorem

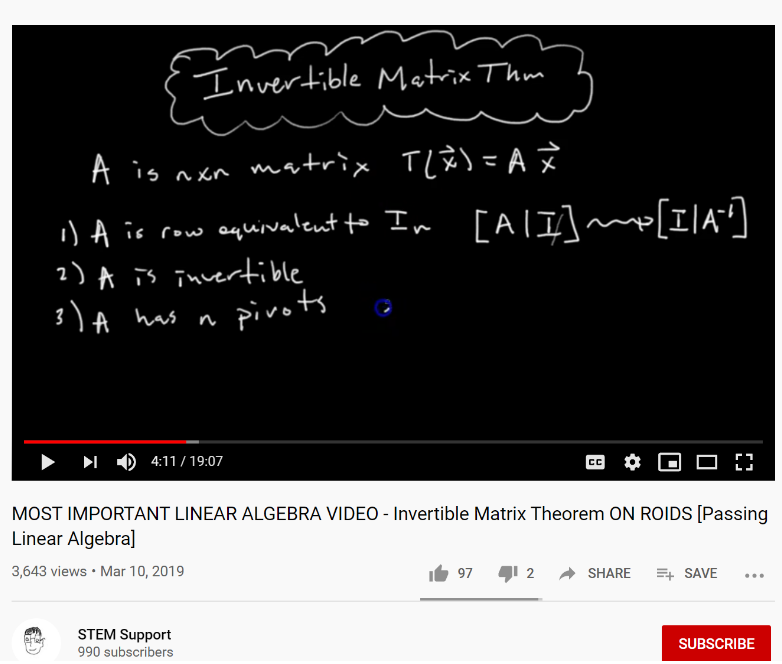
- [James Hamblin: https://www.youtube.com/watch?v=4eZA0GTbZN0](https://www.youtube.com/watch?v=4eZA0GTbZN0)

- [STEM Support: https://youtu.be/P74M9suLIEU](https://youtu.be/P74M9suLIEU)

Jason Gregersen

- https://www.youtube.com/watch?v=mTryd7gPHOQ





Tut2, Q2,3) Invertible matrixes properties

https://www.quora.com/If-the-product-of-two-square-matrices-AxB-is-invertible-does-it-mean-that-each-matrix-is-invertible?ch=10&share=a8d6d515&srid=aF7K

https://yutsumura.com/the-product-of-two-nonsingular-matrices-is-nonsingular/

- https://yutsumura.com/properties-of-nonsingular-and-singular-matrices/
- https://yutsumura.com/a-matrix-is-invertible-if-and-only-if-it-is-nonsingular/
- https://yutsumura.com/two-matrices-are-nonsingular-if-and-only-if-the-product-is-nonsingular/#more-4875

https://yutsumura.com/two-matrices-are-nonsingular-if-and-only-if-the-product-is-nonsingular/#more-4875)

Problem 562

An $n \times n$ matrix A is called **nonsingular** if the only vector $\mathbf{x} \in \mathbb{R}^n$ satisfying the equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
Using the definition of a nonsingular matrix, prove the following statements.

(a) If A and B are $n \times n$ nonsingular matrix, then the product AB is also nonsingular.

(b) Let A and B be $n \times n$ matrices and suppose that the product AB is nonsingular. Then:

1. The matrix B is nonsingular.
2. The matrix A is nonsingular. (You may use the fact that a nonsingular matrix is invertible.)

Proof.

(a) If A and B are $n \times n$ nonsingular matrix, then the product AB is also nonsingular.

Let $\mathbf{v} \in \mathbb{R}^n$ and suppose that $(AB)\mathbf{x} = \mathbf{0}$.
Our goal is to prove that $\mathbf{x} = \mathbf{0}$.
Let $\mathbf{y} := B\mathbf{x} \in \mathbb{R}^n$. Then we have

$$A\mathbf{y} = A(B\mathbf{x}) = (AB)\mathbf{x} = \mathbf{0}.$$

Since A is nonsingular, this implies that the vector $\mathbf{y} = \mathbf{0}$.
Hence we have $\mathbf{y} = B\mathbf{x} = \mathbf{0}$.
Since B is nonsingular, this further implies that $\mathbf{x} = \mathbf{0}$.

It follows that if $(AB)\mathbf{x} = \mathbf{0}$, then we must have $\mathbf{x} = \mathbf{0}$.
By definition, this means that the matrix AB is nonsingular.

(b)-1. If AB is nonsingular, then B is nonsingular.

Suppose that $B\mathbf{x} = \mathbf{0}$. We prove that $\mathbf{x} = \mathbf{0}$.
Since $B\mathbf{x} = \mathbf{0}$, it yields that

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

As the matrix AB is nonsingular, it follows from $(AB)\mathbf{x} = \mathbf{0}$ that $\mathbf{x} = \mathbf{0}$.
This proves that the matrix B is nonsingular.

(b)-2. If AB is nonsingular, then A is nonsingular.

By part (1), we know that B is nonsingular, hence it is invertible.
The inverse matrix B^{-1} and the matrix AB are both nonsingular.
Hence it follows from part (a) that the product of AB and B^{-1} is also nonsingular.
Thus,

$$A = (AB)B^{-1}$$

is a nonsingular matrix.

Nonsingular if and only if Invertible

For the proof of the fact we used in the proof of (b)-2 that a matrix is nonsingular if and only if it is invertible, see the post↴
A Matrix is Invertible If and Only If It is Nonsingular

Q6: Ax = b

TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

These two questions will appear throughout the text, in many different guises. This section and the next will show how to answer these questions via row operations on the augmented matrix.

EXAMPLE 2 Determine if the following system is consistent:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

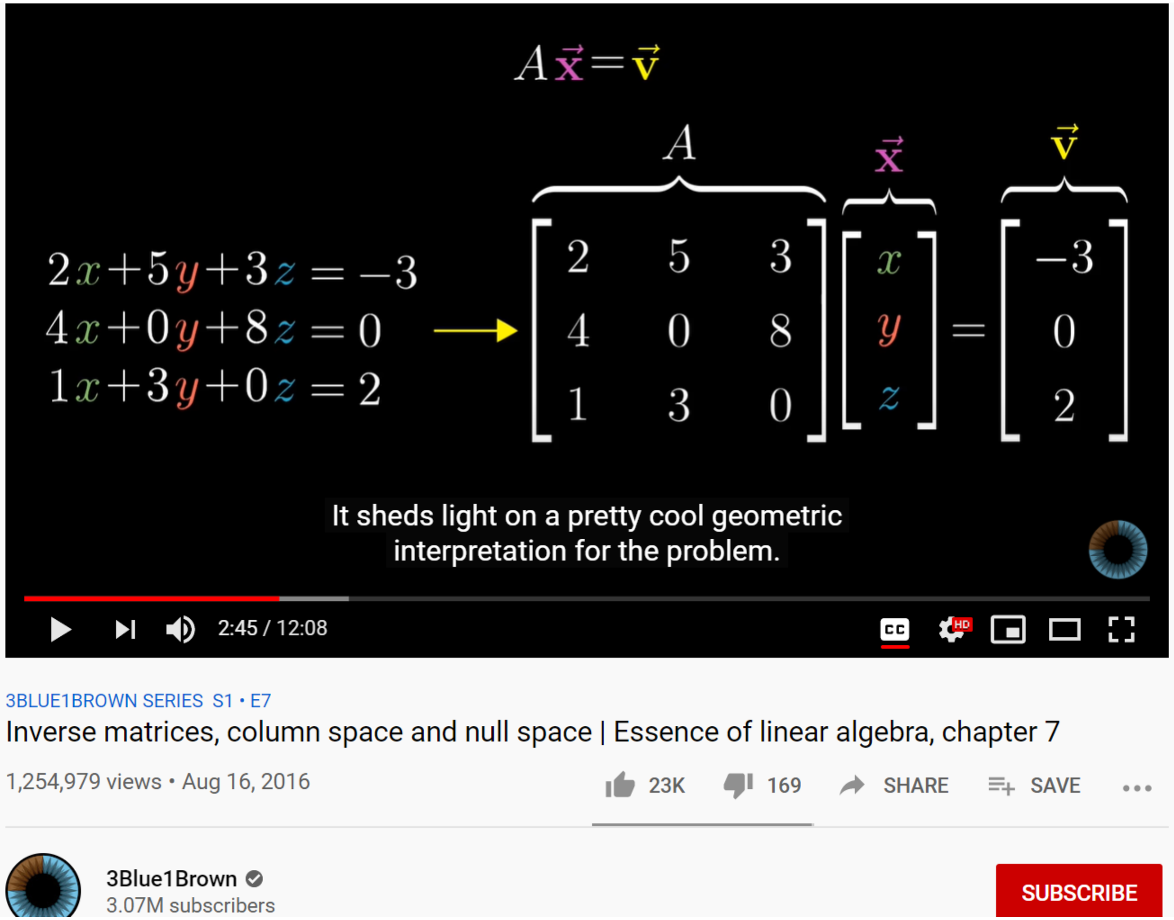
SOLUTION This is the system from Example 1. Suppose that we have performed the row operations necessary to obtain the triangular form

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Q6)

3Blue1Brown:
https://www.youtube.com/watch?v=uQhTuRlWMxw

Explaning Ax=b from column space



Q6)

- https://math.stackexchange.com/questions/2438825/linear-system-ax-0-with-invertible-a-has-unique-solution-x-0

Linear system $Ax = 0$ with invertible A has unique solution $x = 0$?

Ask Question

Asked 2 years, 11 months ago Active 2 years, 11 months ago Viewed 579 times

7

If $Ax = 0$ and A is an invertible matrix, then $x = 0$ is the unique solution?

I think it is a very basic question but I forget some knowledge in linear algebra. For an invertible A, the inverse A^{-1} is well-defined, thus we can left-multiply A^{-1} to each side of the equation to get $A^{-1}Ax = A^{-1}0$, which leads to $Ix = 0$ and thus $x = 0$. I think this calculation says $x = 0$ must be a solution but dose not guarantee the uniqueness. It is not a proof.

linear-algebra matrices

1 Answer

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3

I think you're thinking wrongly about your proof, but that doesn't show up too much in the proof itself.

The proof is fine. Except the two last sentences of course: it shows not that $x = 0$ is a solution, but that no other solutions exists. And the last sentece is also wrong as it is a proof.

To be a bit elaborate we have that if u is a solution, that is $Au = 0$. Left multiplication by A^{-1} **implies** that $Iu = A^{-1}Au = A^{-1}0 = 0$. And this in turn implies(*) that $u = 0$.

Note that this hasn't shown that $u = 0$ is a solution, but rather that if u is a solution it must be 0. However hopefully you already know that's an solution (and you also have that each step in the above proof is an equivalence, but that's not at obvious as they are implications).

(*) That $Iu = 0$ implies that $u = 0$ is for example shown by RAA by in addition to $Iu = 0$ assuming that $u \neq 0$ nand then using the property of the identity that is $Iu = u =\neq 0$ which contradicts the assumtion $Iu = 0$.

share cite follow

answered Sep 21 '17 at 12:55

skyking

15.5k 1 12 33

Q6) Unique

- https://yutsumura.com/the-inverse-matrix-is-unique/

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Hint.

That the inverse matrix of A is unique means that there is only one inverse matrix of A.
(That's why we say "the" inverse matrix of A and denote it by A^{-1}.)
So to prove the uniqueness, suppose that you have two inverse matrices B and C and show that in fact $B=C$.

Recall that B is the inverse matrix if it satisfies

$$AB=BA=I,$$

where I is the identity matrix.

Proof.

Suppose that there are two inverse matrices B and C of the matrix A. Then they satisfy

$$AB=BA=I \tag{*}$$

and

$$AC=CA=I. \tag{**}$$

To show that the uniqueness of the inverse matrix, we show that $B=C$ as follows. Let I be the $n\times n$ identity matrix. We have

$$\begin{aligned} B &= BI \\ &= B(AC) && \text{by (**)} \\ &= (BA)C && \text{by the associativity} \\ &= IC && \text{by (*)} \\ &= C. \end{aligned}$$

Thus, we must have $B=C$, and there is only one inverse matrix of A.

Q7) How to use LU to solve Ax=b

- https://andreask.cs.illinois.edu/cs357-s15/public/notes/03-lu.pdf

- Advantages:
- https://www.cl.cam.ac.uk/teaching/1314/NumMethods/supporting/mcmaster-kiruba-ludecomp.pdf

Does an LU factorization help us solve Ax = b?

Certainly: Just plug the factorization in.

$Ax=b \qquad A=LU$

$\Downarrow$

$L\underbrace{Ux}_{\text{"}y\text{"}}=b$ ← a new unknown that we just invented

$\Downarrow$

$Ly=b$ ← solvable by forward substitution

$\Downarrow$ → now know y

$Ux=y$ ← solvable by backward substitution

→ now know x

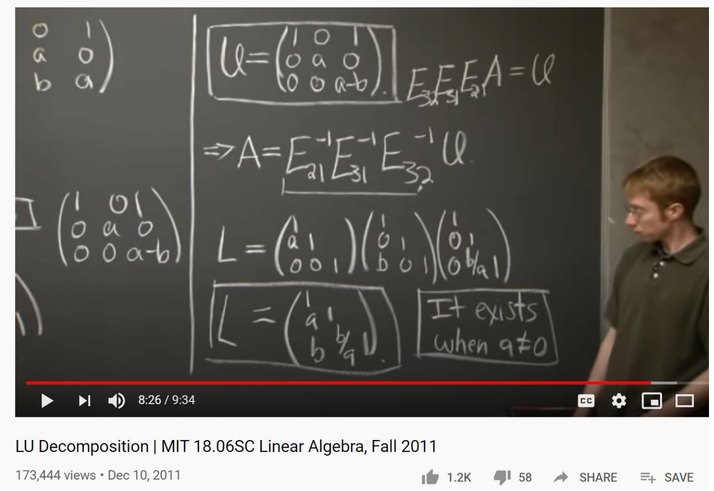
$\Downarrow$

$Ax=b$ solved.

LU Decomposition

LU decomposition is a better way to implement Gauss elimination, especially for repeated solving a number of equations with the same left-hand side. That is, for solving the equation $Ax = b$ with different values of b for the same A.

Q7) Good notes for LU



MIT tutorial (Ben Harris) showing how to perform LU

- https://www.youtube.com/watch?v=rhNKncraJMk

Why we can collect permutation matrix together

- https://math.stackexchange.com/questions/2043317/lu-decomposition-do-permutation-matrices-commute

- What is pivoting

http://buzzard.ups.edu/courses/2014spring/420projects/math420-UPS-spring-2014-reid-LU-pivoting.pdf

Can non-square matrix be LU

- https://math.stackexchange.com/questions/186972/how-can-lu-factorization-be-used-in-non-square-matrix

UIUC notes

- https://andreask.cs.illinois.edu/cs357-s15/public/notes/03-lu.pdf
- https://andreask.cs.illinois.edu/cs357-s15/public/notes/04-lu-applications.pdf
- https://andreask.cs.illinois.edu/cs357-s15/public/notes/

Q8) Matrix Decomposition

General Discussions:

- https://en.wikipedia.org/wiki/Matrix_decomposition
- https://www.quora.com/Linear-Algebra-What-are-the-most-important-matrix-factorizations

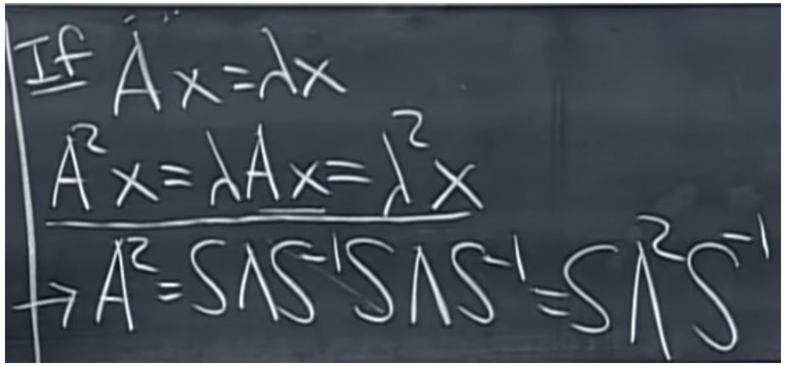
Python Examples:

- https://people.duke.edu/~ccc14/sta-663/LinearAlgebraMatrixDecompWithSolutions.html
- https://machinelearningmastery.com/introduction-to-matrix-decompositions-for-machine-learning/

Q8) Why is $A = S\Lambda S^{-1}$ interesting?
E.g calculating $(A)^K$ - power of A

L22,
Strang

(9:16)



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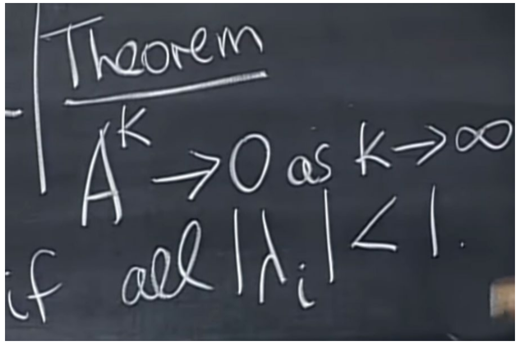

Assumption: S is full rank , so its inverse exist!
Link:18.06, Lect 22
https://www.youtube.com/watch?v=13r9QY6cmjc

Overview

- https://www.samartigliere.com/math/linear-algebra/

Linear Algebra

- Introduction
- Vectors
  - Linear combinations of vectors
  - Vector lengths and dot products
  - Linear equations

WELCOME

My name is Sam Artigliere. I am interested in theoretical physics, mathematics, neuroscience and computer science. I am particularly fascinated by the relationship between science and religion. My goal on this site is to share my interests and views on these subjects