Tutorial 2

Matrix Algebra

1. Consider an $m \times n$ matrix A and an $n \times p$ matrix B. Show that if the columns of B are linearly dependent, then so are the columns of AB.

2. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.

3. Suppose A, B and X are $n \times n$ matrices with A, X and $A - AX$ invertible. Also, suppose $(A - AX)^{-1} = X^{-1}B$. Solve for X. If you need to invert a matrix, explain why that matrix is invertible.

4. a. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system with n linear equations and n unknowns. If $A\mathbf{x} = 0$ has only the trivial solution, show that for any positive integer k, the system $A^k\mathbf{x} = 0$ also has only the trivial solution.

   b. Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations and let $\mathbf{x}_1$ be a fixed solution. Show that every solution to the system can be written in the from $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$ where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$.

5. a. If the columns of an $n \times n$ matrix A are linearly independent, show that the columns of A^2 span $\mathbb{R}^n$.

   b. Show that if AB is invertible, so is A.

6. Using the notion of pivots and free variables only, answer the following questions:

   a. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why the equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.

   b. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

7. Solve the equation $A\mathbf{x} = \mathbf{b}$ using the LU factorization given for A:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}.$$

8. (*Spectral Factorization*) Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the

diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. Find A^2 and A^3 and hence, a simple formula for A^k (where k is a positive integer). This factorization is useful when computing high powers of A.

Answers

7. $\mathbf{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

8. $A^k = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^k & 0 \\ 0 & 0 & 1/3^k \end{bmatrix} P^{-1}$

End