MH1812 Tutorial
Chapter 3: Predicate Logic

Q1: Consider the predicates $M(x,y)$ = "x has sent an email to y", and $T(x,y)$ = "x has called y". The predicate variables x, y take values in the domain $D = \{\text{students in the class}\}$. Express these statements using symbolic logic.

1. There are at least two students in the class such that one student has sent the other an email, and the second student has called the first student.

   Solution: We need two predicate variables since at least 2 students are involved, say x and y. There are at least two students in the class becomes

   $$x \in D, y \in D.$$

   Then x sent an email to y, that is $M(x,y)$ and y has called x, that is $T(y,x)$, thus

   $$M(x,y) \wedge T(y,x).$$

   Furthermore, we need to take into account the fact that there are at least "two" students, so x and y have to be distinct! Thus the final answer is

   $$\exists x \in D, \exists y \in D, x \neq y \wedge M(x,y) \wedge T(y,x).$$

   $\square$

2. There are some students in the class who have emailed everyone.

   Solution: There are students becomes

   $$\exists x \in D,$$

   then x has emailed everyone, that is

   $$\exists x \in D, (\forall y \in D, M(x,y)).$$

   Note that the order of the quantifiers is important. $\square$

Q2: Consider the predicate $P(x,y)$="x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain D=$\{\text{courses}\}$. Express each statement by an English sentence.

1. $\exists x \in S, P(x, \mathrm{MH1812})$.

   Solution: There exists a student such that this student is enrolled in the class MH1812, that is some student enrolled in the class MH1812. $\square$

2. $\exists y \in D, P(\mathrm{Carol}, y)$.

   Solution: There exists a course such that Carol is enrolled in this course, that is, Carol is enrolled in some course, or Carol is enrolled in at least one course. $\square$

3. $\exists x \in S, (P(x, \mathrm{MH1812}) \wedge P(x, \mathrm{CZ2002}))$.

   Solution: There exists a student, such that this student is enrolled in MH1812 and in CZ2002, that is some student is enrolled in both MH1812 and CZ2002. $\square$

4. $\exists x \in S, \exists x' \in S, \forall y \in D, ((x \neq x') \wedge (P(x, y) \leftrightarrow P(x', y)))$.

   Solution: There exist two distinct students x and x', such that for all courses, x is enrolled in the course if and only if x' is enrolled in the course. In other words, there exist two students which are enrolled in exactly the same courses. $\square$

Q3: Consider the predicate $P(x, y, z) =$ "$xyz = 1$", for $x, y, z \in \mathbb{R}, x, y, z > 0$. What are the truth values of the following statements? Justify your answer.

1. $\forall x, \forall y, \forall z, P(x, y, z)$.

   Solution: $\forall x, \forall y, \forall z, P(x, y, z)$ is false: take $x = 1$ and $y = 1$, then whenever $z \neq 1$, $xyz = z \neq 1$. $\square$

2. $\exists x, \exists y, \exists z, P(x, y, z)$.

   Solution: $\exists x, \exists y, \exists z, P(x, y, z)$ is true: take $x = y = z = 1$. $\square$

3. $\forall x, \forall y, \exists z, P(x, y, z)$.

   Solution: $\forall x, \forall y, \exists z, P(x, y, z)$ is true: choose any x and any y, then there exists a z, namely $z = \frac{1}{xy}$ such that $xyz = 1$. $\square$

4. $\exists x, \forall y, \forall z, P(x, y, z)$.

   Solution: $\exists x, \forall y, \forall z, P(x, y, z)$ is false: one cannot find a single x such that $xyz = 1$ no matter what are y and z. Assume such x exists, then for any $y_1, z_1 \neq 0$ and y_1+1, z_1, $xy_1z_1 = 1$ and $x(y_1+1)z_1 = 1$ result in valid solution, hence contradiction. $\square$

Q4: 1. Express

$$\neg(\forall x, \forall y, P(x, y))$$

in terms of existential quantification.

Solution: We see that $\neg(\forall x, \forall y, P(x, y))$ is a negation of two universal quantifications. Denote $Q(x) =$ "$\forall y, P(x, y)''$, then $\neg(\forall x, Q(x))$ is $(\exists x, \neg Q(x))$, thus

$$\neg(\forall x, \forall y, P(x, y)) \equiv \exists x, \neg(\forall y, P(x, y))$$

and now we iterate the same rule on the next negation, to get

$$\neg(\forall x, \forall y, P(x, y)) \equiv \exists x, \exists y, \neg P(x, y).$$

$\square$

2. Express

$$\neg(\exists x, \exists y, P(x, y))$$

in terms of universal quantification.

Solution: We repeat the same procedure with the negation of two existential quantifications, by setting this time $Q(x) =$ "$\exists y, P(x, y)''$:

$$\begin{aligned}\neg(\exists x, \exists y, P(x, y)) &\equiv \neg(\exists x, Q(x)) \\ &\equiv \forall x, \neg Q(x) \\ &\equiv \forall x, \neg(\exists y, P(x, y)) \\ &\equiv \forall x, \forall y, \neg P(x, y).\end{aligned}$$

$\square$

Q5: Consider the predicate $P(x, y) =$"x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $C = \{\text{courses}\}$. Form the negation of these statements:

1. $\exists x, (P(x, \text{MH1812}) \wedge P(x, \text{CZ2002}))$.

   Solution: We have

$$\begin{aligned}&\neg(\exists x, (P(x, \text{MH1812}) \wedge P(x, \text{CZ2002})) \\ \equiv&\forall x, \neg(P(x, \text{MH1812}) \wedge P(x, \text{CZ2002})) \\ \equiv&\forall x, \neg P(x, \text{MH1812}) \vee \neg P(x, \text{CZ2002})\end{aligned}$$

   where the first equivalence is the negation of quantification, and the second equivalence De Morganś law. $\square$

2. $\exists x, \exists y, \forall z, ((x \neq y) \wedge (P(x, z) \leftrightarrow P(y, z)))$.

   Solution:

$$\begin{aligned}&\neg(\exists x, \exists y, \forall z, ((x \neq y) \wedge (P(x, z) \leftrightarrow P(y, z)))) \\ \equiv&\forall x, \neg(\exists y, \forall z, ((x \neq y) \wedge (P(x, z) \leftrightarrow P(y, z)))) \\ \equiv&\forall x, \forall y, \neg(\forall z, ((x \neq y) \wedge (P(x, z) \leftrightarrow P(y, z)))) \\ \equiv&\forall x, \forall y, \exists z, \neg((x \neq y) \wedge (P(x, z) \leftrightarrow P(y, z))) \\ \equiv&\forall x, \forall y, \exists z, \neg(x \neq y) \vee \neg(P(x, z) \leftrightarrow P(y, z))\end{aligned}$$

   using three times the negation of quantification, and lastly the De Morgan's law. Next $\neg(x \neq y) \equiv (x = y)$ and using that

$$P(x, z) \leftrightarrow P(y, z) \equiv (P(x, z) \rightarrow P(y, z)) \wedge (P(y, z) \rightarrow P(x, z))$$

we get

$$\neg(P(x,z) \leftrightarrow P(y,z)) \equiv \neg(P(x,z) \rightarrow P(y,z)) \vee \neg(P(y,z) \rightarrow P(x,z))$$

so that, using the Conversion theorem to get $\neg(\neg P(x,z) \vee P(y,z)) = P(x,z) \wedge \neg P(y,z)$ OR $\neg(\neg P(y,z) \vee P(x,z)) = P(y,z) \wedge \neg P(x,z)$

$$\begin{aligned}
&\neg(\exists x, \exists y, \forall z, ((x \neq y) \wedge (P(x,z) \leftrightarrow P(y,z)))) \\
\equiv &\forall x, \forall y, \exists z, (x = y) \vee [(P(x,z) \vee P(y,z)) \wedge (\neg P(x,z) \vee \neg P(y,z))] \\
&\textbf{Note}\text{: } (P(x,z) \wedge \neg P(y,z)) \vee (P(y,z) \wedge \neg P(x,z)) \\
&\equiv (P(x,z) \vee P(y,z)) \wedge (\neg P(x,z) \vee \neg P(y,z)).
\end{aligned}$$

When many steps are involved, it is often a good idea to check the sanity of the answer. If we look at $\neg(P(x,z) \leftrightarrow P(y,z))$, it is false exactly when $P(x,z)$ and $P(y,z)$ are taking the same truth value (either both true or both false). Now we look at $(P(x,z) \vee P(y,z)) \wedge (\neg P(x,z) \vee \neg P(y,z))$: when $P(x,z)$ and $P(y,z)$ are taking the same value, we get false, and true otherwise. This makes sense! □

Q6: Show that $\forall x \in D, P(x) \rightarrow Q(x)$ is equivalent to its contrapositive.

Solution: For every instantiation of x, $(\forall x \in D, P(x) \rightarrow Q(x))$ is a proposition, thus we can use the conversion theorem:

$$\begin{aligned}
&(\forall x \in D, P(x) \rightarrow Q(x)) \\
\equiv &(\forall x \in D, \neg P(x) \vee Q(x)) \\
\equiv &(\forall x \in D, Q(x) \vee \neg P(x)) \\
\equiv &(\forall x \in D, \neg\neg Q(x) \vee \neg P(x)) \\
\equiv &(\forall x \in D, \neg Q(x) \rightarrow \neg P(x)).
\end{aligned}$$

□

Q7: Show that

$$\neg(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x, P(x) \wedge \neg Q(x).$$

Solution:

$$\begin{aligned}
&\neg(\forall x, P(x) \rightarrow Q(x)) \\
\equiv &\exists x, \neg(P(x) \rightarrow Q(x)) \\
\equiv &\exists x, \neg(\neg P(x) \vee Q(x)) \\
\equiv &\exists x, \neg\neg P(x) \wedge \neg Q(x) \\
\equiv &\exists x, P(x) \wedge \neg Q(x)
\end{aligned}$$

□