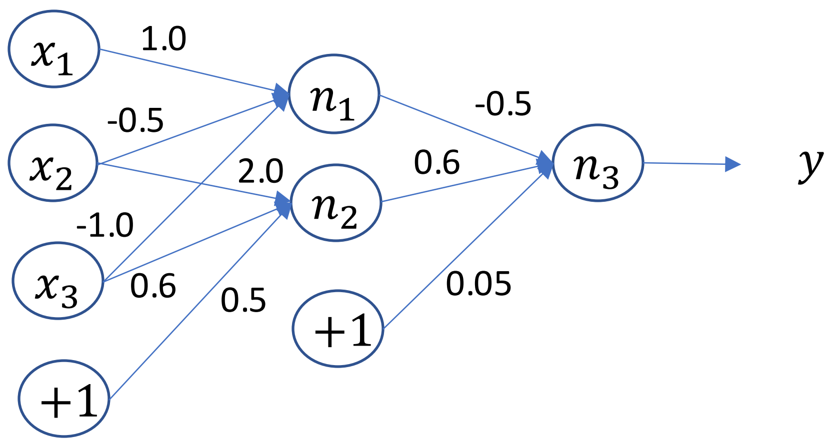
Fundamentals of Neural Networks

SC4001 – Tutorial 1



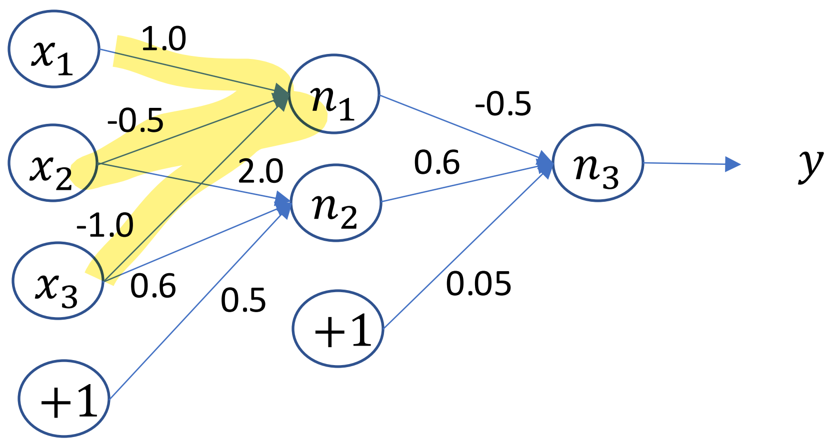
1. Figure 1 shows a two-layer feedforward neural network receiving 3-dimensional inputs $(x_1, x_2, x_3) \in \boldsymbol{R}^3$. The connection weights and biases of the neurons $n_1, n_2,$ and n_3 are as indicated in the figure. The hidden-layer neurons have activation functions given by $g(u) = \frac{1.0}{1+e^{-0.5u}}$ where u denotes the synaptic input to the neuron. The activation function $f(u)$ of the output neuron is a ReLU function: $f(u) = \max\{0, u\}$.

    a. Write weight vectors and biases connected to individual neurons, and the weight matrix and bias vector connected to the hidden layer.

    b. Find the synaptic inputs and activations of the neurons for the following input signals:

    (i) (1.0, –0.5, 1.0) (ii) (-1.0, 0.0, –2.0) (iii) (2.0, 0.5, –1.0).

Q1



Weights:

n_1 neuron: $\boldsymbol{w}_1 = \begin{pmatrix} 1.0 \\ -0.5 \\ -1.0 \end{pmatrix}$; n_2 neuron: $\boldsymbol{w}_2 = \begin{pmatrix} 0.0 \\ 2.0 \\ 0.6 \end{pmatrix}$; and n_3 neuron: $\boldsymbol{w}_3 = \begin{pmatrix} -0.5 \\ 0.6 \end{pmatrix}$.

Biases:

n_1 neuron: b_1= 0.0; n_2 neuron: b_2= 0.5; and n_3 neuron: b_3= 0.05

To the hidden-layer

Weight matrix $\boldsymbol{W} = \begin{array}{cc} n_1 & n_2 \\ \multicolumn{2}{c}{\begin{pmatrix} 1.0 & 0.0 \\ -0.5 & 2.0 \\ -1.0 & 0.6 \end{pmatrix}} \end{array}$ and bias $\boldsymbol{b} = \begin{pmatrix} 0.0 \\ 0.5 \end{pmatrix}$

Q1

Activation functions:

hidden-layer neurons: $\mathrm{g}(u) = \frac{1.0}{1+e^{-0.5u}}$

Output neuron $f(u) = \max\{0.0, \quad u\}$

Apply $\boldsymbol{x}_1 = (1.0 \quad -0.5 \quad 1.0)^T$:

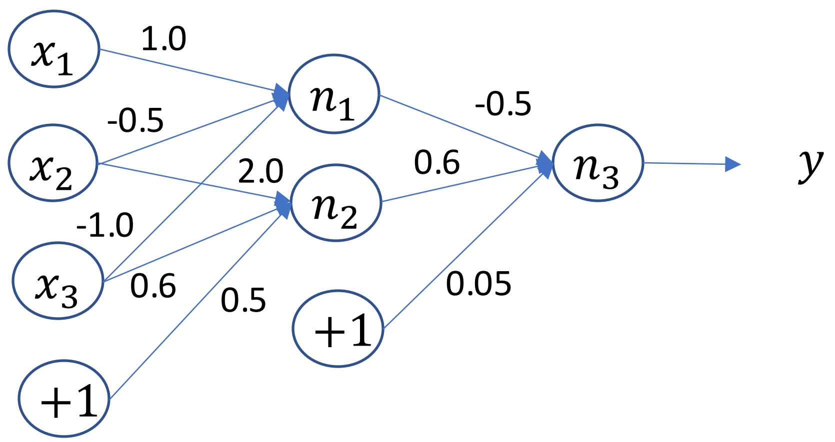
Synaptic input to neuron n_1, $u_1 = \boldsymbol{x}_1{}^T \boldsymbol{w}_1 + b_1 = (1.0 \quad -0.5 \quad 1.0)\begin{pmatrix} 1.0 \\ -0.5 \\ -1.0 \end{pmatrix} + 0.0 = 0.25$

Output of neuron n_1, $y_1 = \mathrm{g}(u_1) = \frac{1.0}{1+e^{-0.5u_1}} = 0.531$

Synaptic input to neuron n_2, $u_2 = \boldsymbol{x}_1{}^T \boldsymbol{w}_2 + b_2 = (1.0 \quad -0.5 \quad 1.0)\begin{pmatrix} 0.0 \\ 2.0 \\ 0.6 \end{pmatrix} + 0.5 = 0.1$

Output of neuron n_2, $y_2 = \mathrm{g}(u_2) = \frac{1.0}{1+e^{-0.5u_2}} = 0.512$

Q1



Output of the hidden layer $\boldsymbol{z} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.531 \\ 0.512 \end{pmatrix}$

Synaptic input to neuron n_3, $u_3 = \boldsymbol{z}^T \boldsymbol{w}_3 + b_3 = (0.531 \quad 0.512) \begin{pmatrix} -0.5 \\ 0.6 \end{pmatrix} + 0.05 = 0.092$

Output of neuron n_3, $y_3 = f(u_3) = \max\{0.0, u_3\} = 0.092$

Q1

Similarly, for other two inputs:

x	u_1	y_1	u_2	y_2	z	u_3	y_3
$\begin{pmatrix} -1.0 \\ 0.0 \\ -2.0 \end{pmatrix}$	1.00	0.622	-0.700	0.413	$\begin{pmatrix} 0.622 \\ 0.413 \end{pmatrix}$	-0.013	0.0
$\begin{pmatrix} 2.0 \\ 0.5 \\ -1.0 \end{pmatrix}$	2.75	0.798	0.900	0.611	$\begin{pmatrix} 0.798 \\ 0.611 \end{pmatrix}$	0.017	0.017

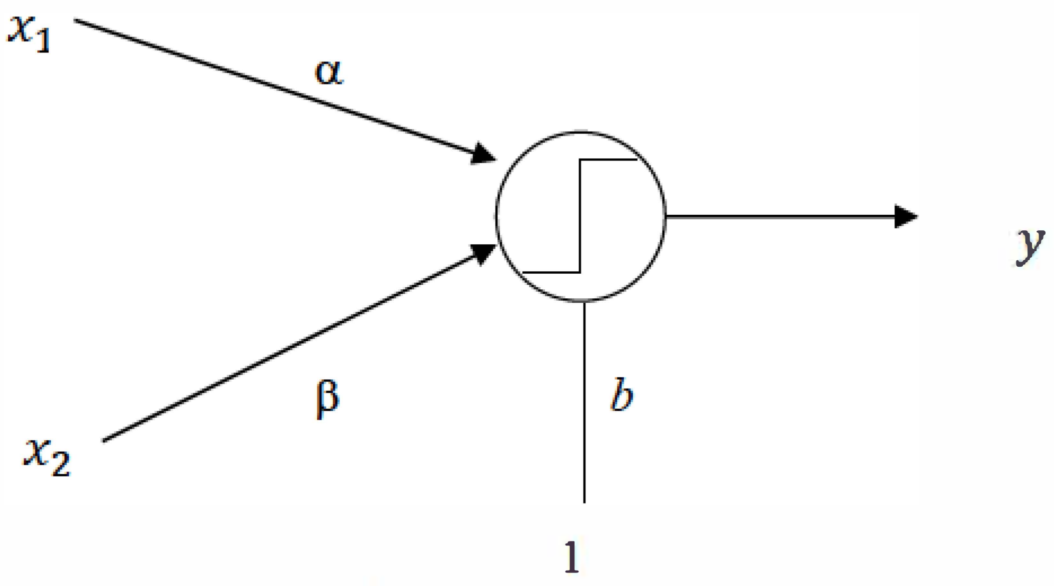


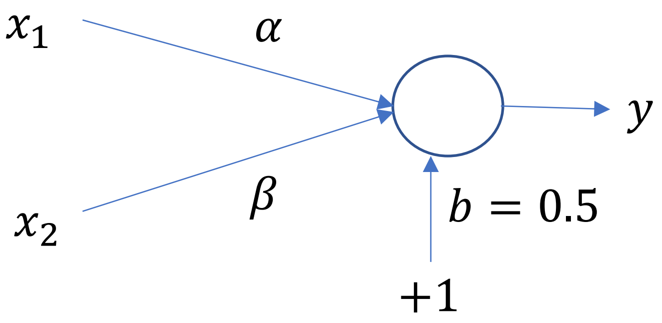
Figure 2

2. Two input binary neuron shown in figure 2 has a unit step activation function with bias $b = 0.5$ and receives two-dimensional input $(x_1, x_2) \in \boldsymbol{R}^2$.

(a) Find the space of possible values of weights (α, β) if the neuron is
  (i) ON for input (1.0, 1.0)
  (ii) ON for input (0.5, -1.0)
  (iii) OFF for input (2.0, -0.5).

(b) Indicate the weight space in 2-D α-β plot and show that (-0.2, 0.2) is in this space.

Q2



Weight $\boldsymbol{w} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and bias $b = 0.5$.

$$y = f(u) = 1(u > 0)$$

Input $\boldsymbol{x} = (1.0 \quad 1.0)^T$: ON

Synaptic input $u = \boldsymbol{x}^T\boldsymbol{w} + b$

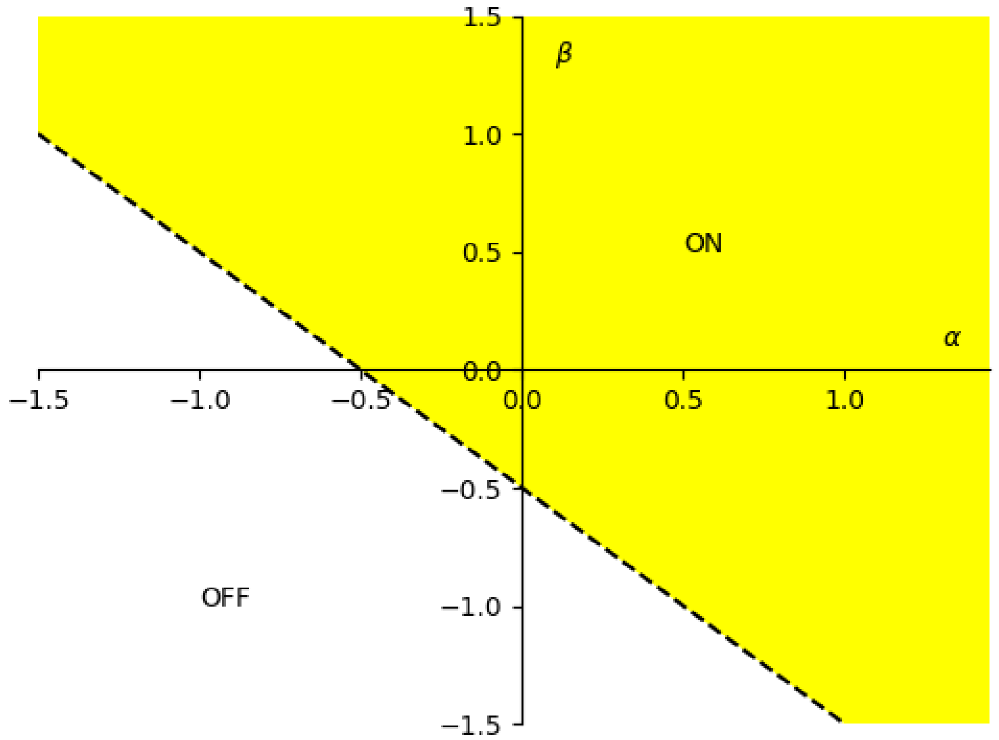
$$= (1.0 \quad 1.0)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} + 0.5$$

$$= \alpha + \beta + 0.5$$

Neuron is ON;

So, $u = \alpha + \beta + 0.5 > 0.0$

$$\beta > -\alpha - 0.5 \qquad (1)$$



Boundary: $\beta = -\alpha - 0.5$

Q2

Input $\boldsymbol{x} = (0.5 \quad -1.0)^T$: ON

Synaptic input $u = \boldsymbol{x}^T \boldsymbol{w} + b$

$$= (0.5 \quad -1.0)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} + 0.5$$

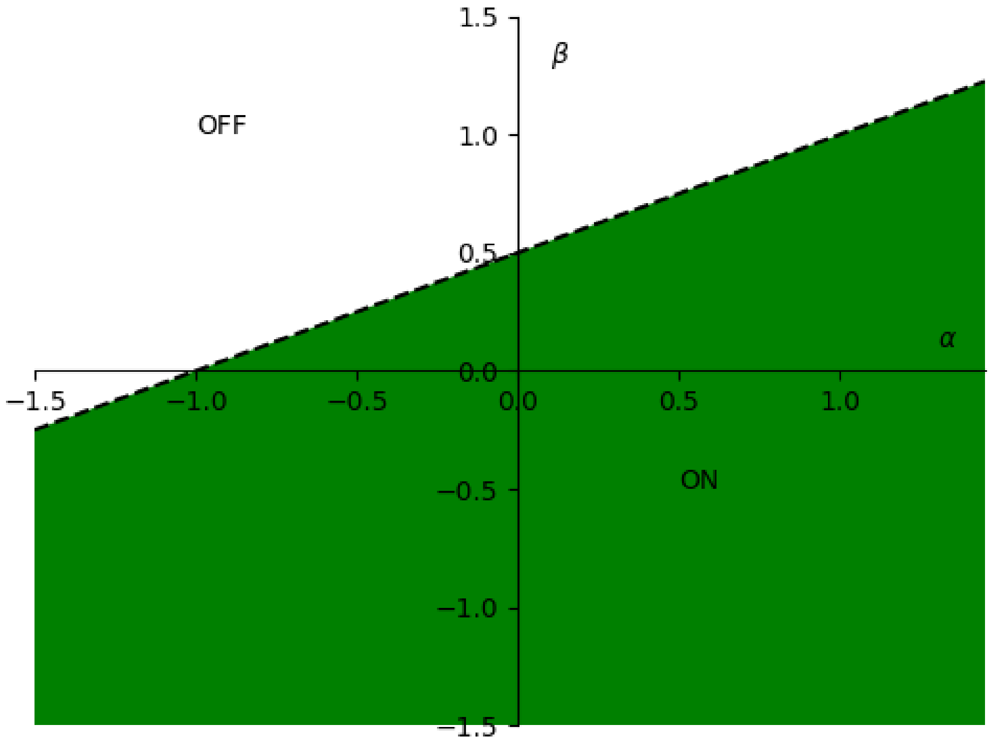
$$= 0.5\alpha - \beta + 0.5$$

Neuron is ON;

So, $u = 0.5\alpha - \beta + 0.5 > 0$

$$\beta < 0.5\alpha + 0.5 \qquad (2)$$

Boundary: $\beta = 0.5\alpha + 0.5$



Q2

Input $\boldsymbol{x} = (2.0 \quad -0.5)^T$: OFF

Synaptic input $u = \boldsymbol{x}^T \boldsymbol{w} + b$

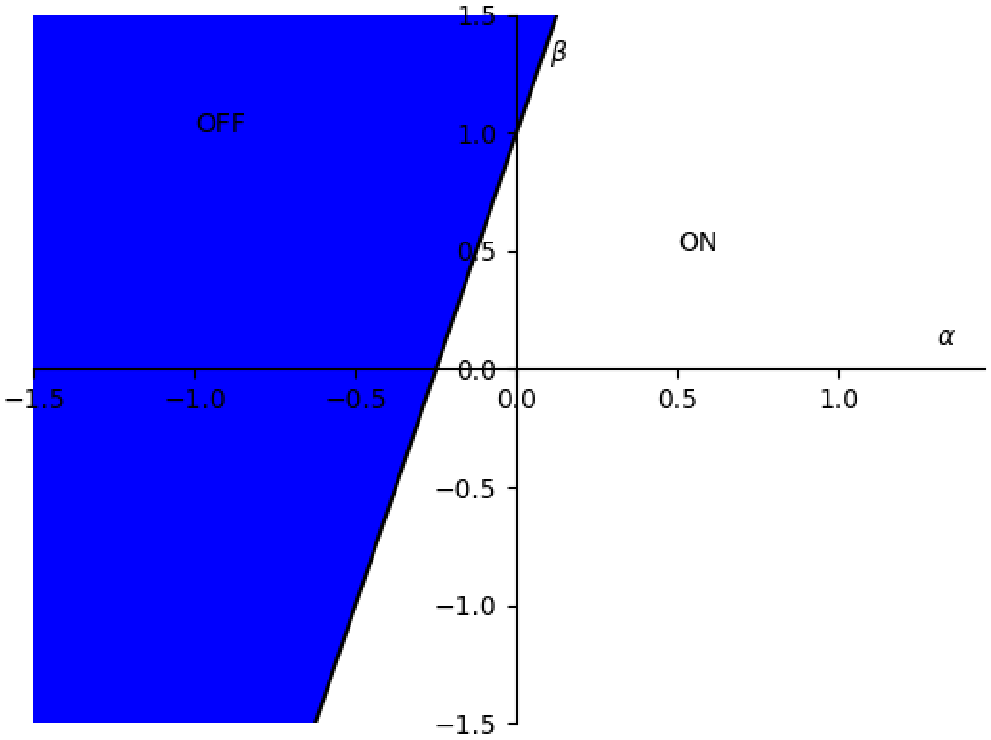
$$= (2.0 \quad -0.5)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} + 0.5$$

$$= 2\alpha - 0.5\beta + 0.5$$

Neuron is OFF;

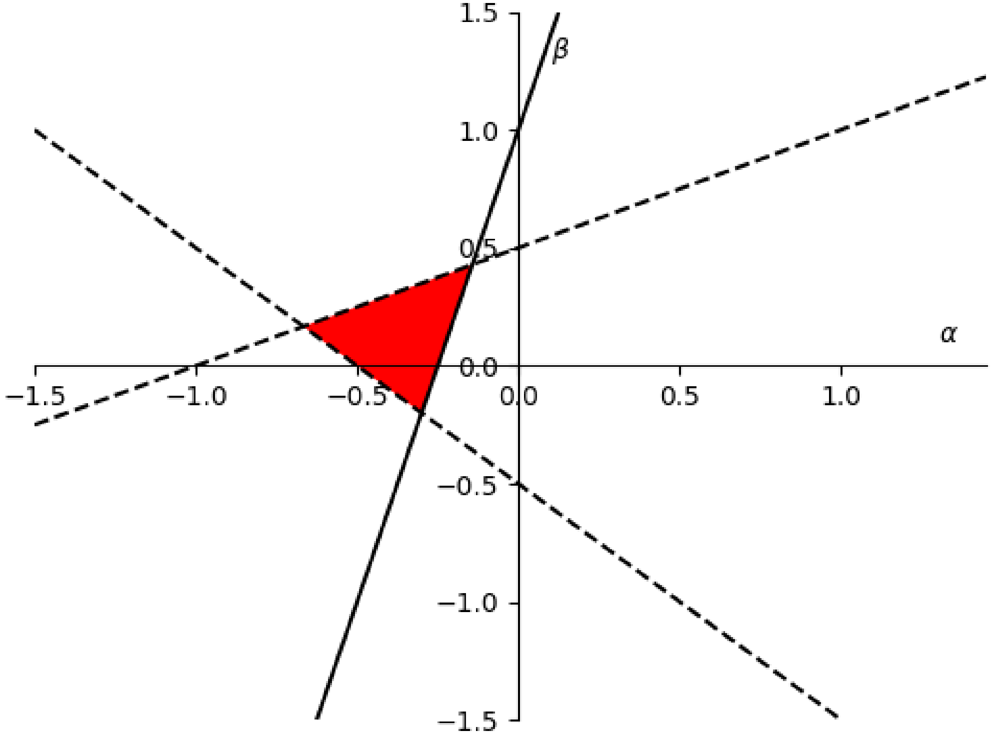
So, $u = 2\alpha - 0.5\beta + 0.5 \leq 0$

$$\beta \geq 4\alpha + 1.0 \qquad (3)$$



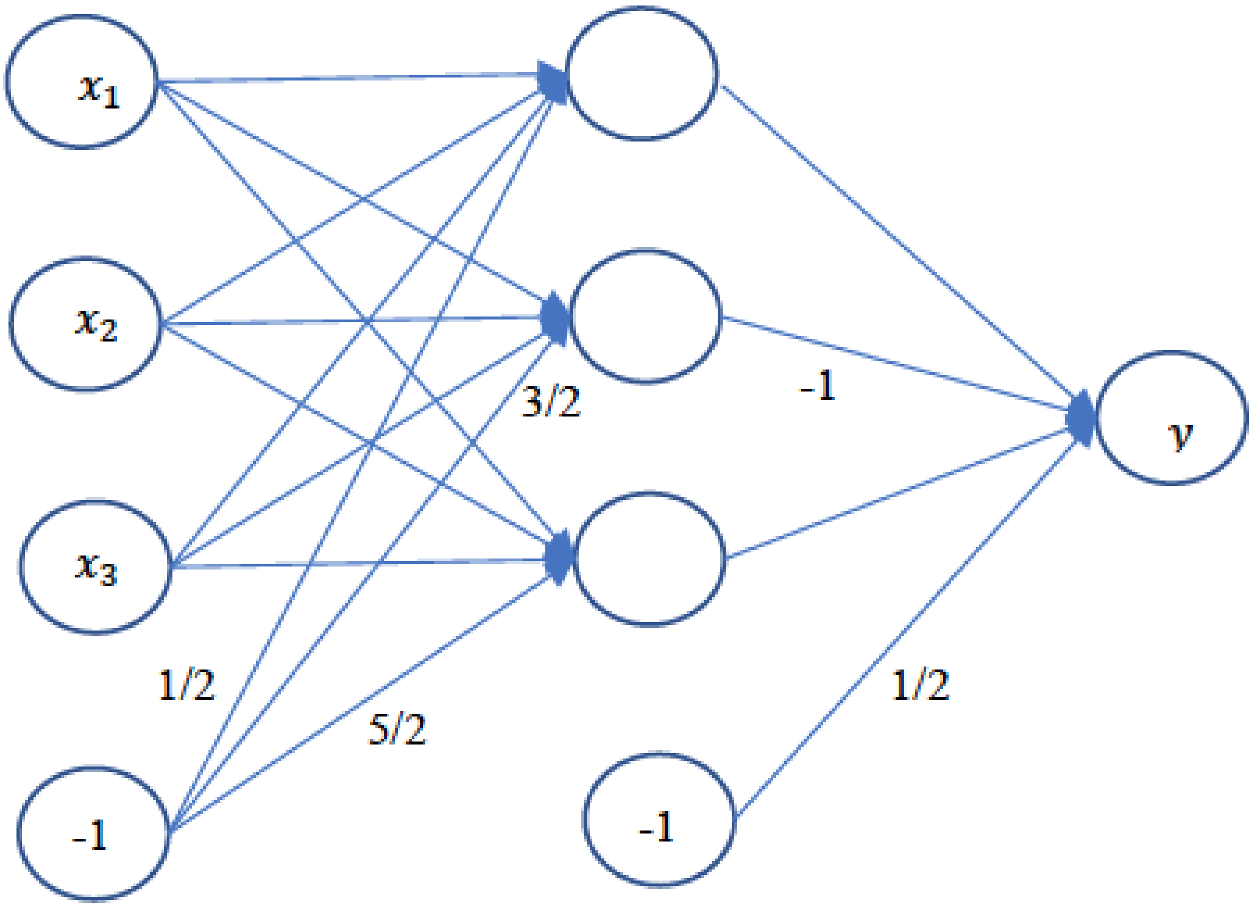
Q2

The neuron should satisfy conditions (1), (2) and (3).
Intersection of the colored regions gives the space of (α, β).

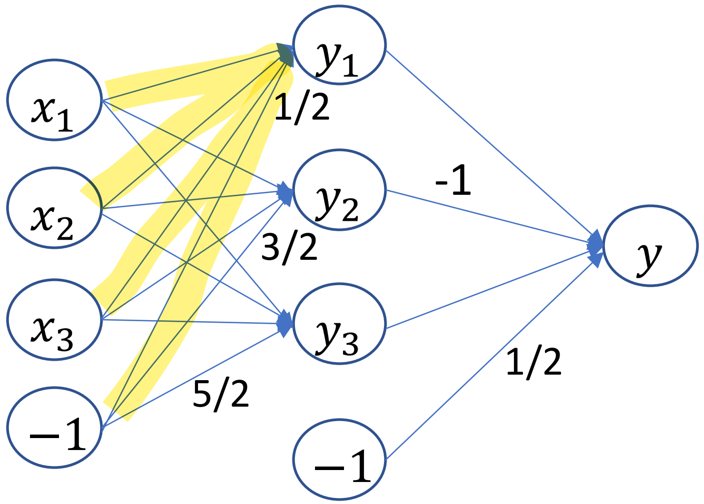


From the figure, (-0.2, 0.2) is within the space of (α, β).

Q3 The network shown in figure 3 has neurons having threshold activation functions and receives three-bit binary patterns $(x_1, x_2, x_3) \in \{0,1\}^3$. By analyzing for the outputs for all possible three-bit input patterns, determine the logic function that the network implements. All unlabeled weights shown in figure 3 are of unity weight.



Q3



Input

$$\boldsymbol{x} = (x_1, x_2, x_3) \in \{0, 1\}^3$$

Synaptic inputs to hidden neurons:

$$u_1 = x_1 + x_2 + x_3 - 0.5$$
$$u_2 = x_1 + x_2 + x_3 - 1.5$$
$$u_3 = x_1 + x_2 + x_3 - 2.5$$

Outputs of hidden neurons:

$$y_1 = 1(u_1 > 0.0)$$
$$y_2 = 1(u_2 > 0.0)$$
$$y_3 = 1(u_3 > 0.0)$$

Synaptic input to output neuron:

$$u = y_1 - y_2 + y_3 - 0.5$$

Output of the network:

$$y = 1(u > 0.0)$$

Q3

$$u_1 = x_1 + x_2 + x_3 - 0.5$$
$$u_2 = x_1 + x_2 + x_3 - 1.5$$
$$u_3 = x_1 + x_2 + x_3 - 2.5$$

$$y = f(u) = 1(u > 0.0)$$

$$u = y_1 - y_2 + y_3 - 0.5$$

Note that $\boldsymbol{x} = (x_1, x_2, x_3) \in \{0, 1\}^3$

x1	x2	x3	u1	y1	u2	y2	u3	y3	u	y
0	0	0	-1/2	0	-3/2	0	-5/2	0	-1/2	0
0	0	1	1/2	1	-1/2	0	-3/2	0	1/2	1
0	1	0	1/2	1	-1/2	0	-3/2	0	1/2	1
0	1	1	3/2	1	1/2	1	-1/2	0	-1/2	0
1	0	0	1/2	1	-1/2	0	-3/2	0	1/2	1
1	0	1	3/2	1	1/2	1	-1/2	0	-1/2	0
1	1	0	3/2	1	1/2	1	-1/2	0	-1/2	0
1	1	1	5/2	1	3/2	1	1/2	1	1/2	1

$$y = \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3 + x_1x_2x_3$$

Q3 (optional)

$$
\begin{aligned}
y &= \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3 + x_1x_2x_3 \\
&= \bar{x}_1\left(\bar{x}_2x_3 + x_2\bar{x}_3\right) + x_1(\bar{x}_2\bar{x}_3 + x_2x_3) \\
&= \bar{x}_1\left(x_2 \oplus x_3\right) + x_1\left(\overline{x_2 \oplus x_3}\right) \\
&= x_1 \oplus \left(x_2 \oplus x_3\right) \\
&= x_1 \oplus x_2 \oplus x_3
\end{aligned}
$$

$\oplus \rightarrow$ Exclusive OR

The function implemented by the network is ‘Three input Exclusive OR’