MH1812 Tutorial
Chapter 2: Propositional Logic

Q1: Decide whether the following statements are propositions. Justify your answer.

1. $2 + 2 = 5$.
   Solution: Yes, because this statement always takes the truth value "false". □
2. $2 + 2 = 4$.
   Solution: Yes, because this statement always takes the truth value "true". □
3. $x = 3$.
   Solution: No, because this statement can be "true" when x is 3 and "false" when x is not 3. □
4. Every week has a Sunday.
   Solution: Yes, because this statement always takes the truth value "true". □
5. Have you read "Catch 22"?
   Solution: No, because the truth value depends on who is answering the question. □

Q2: Show the second law of de Morgan:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

Solution: We show the equivalence using truth tables:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Since both truth tables are the same, the two logical expressions are equivalent. □

Q3: Show that second absorption law $p \wedge (p \vee q) \equiv p$ holds.

Solution: We show the equivalence using a truth table:

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Since the columns of p and $p \wedge (p \vee q)$ are identical, so these two logical expressions are equivalent. $\square$

Q4: These two laws are called distributivity laws. Show that they hold:

1. Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$.
   Solution:

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$p \vee r$	$q \vee r$	$(p \vee r) \wedge (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	F	T	T	T	T
F	T	F	F	F	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

$\square$

2. Show that $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$.
   Solution:

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$p \wedge r$	$q \wedge r$	$(p \wedge r) \vee (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

$\square$

Q5: Verify $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$ by

- constructing a truth table,
  Solution:

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg p \wedge \neg q$	$\neg(p \vee \neg q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	F	T	F	T
F	F	T	T	T	F	T	T

□

- developing a series of logical equivalences.

  Solution:

$$\begin{aligned}
\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) &\equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q) \quad \text{(de Morgan)} \\
&\equiv \neg p \wedge (q \vee \neg q) \quad \text{(distributivity)} \\
&\equiv \neg p \wedge T \quad (\text{since } (q \vee \neg q) \equiv T) \\
&\equiv \neg p.
\end{aligned}$$

□

Q6: Using a truth table, show that:

$$\neg q \to \neg p \equiv p \to q.$$

Solution:

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

□

Q7: Show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$.

Solution:

$$\begin{aligned}
p \vee q \to r &\equiv (p \vee q) \to r \quad \text{(precedence)} \\
&\equiv \neg(p \vee q) \vee r \quad \text{(conversion theorem)} \\
&\equiv (\neg p \wedge \neg q) \vee r \quad \text{(de Morgan)} \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) \quad \text{(distributivity)} \\
&\equiv (p \to r) \wedge (q \to r) \quad \text{(conversion theorem)}
\end{aligned}$$

□

Q8: Are $(p \to q) \vee (q \to r)$ and $p \to r$ equivalent statements?

Solution: They are not equivalent. Here is a proof using truth table:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \vee (q \to r)$	$p \to r$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

We can see that the second row are giving different truth values, for example. This can be done using equivalences as well:

$$\begin{aligned}(p \to q) \vee (q \to r) &\equiv (\neg p \vee q) \vee (\neg q \vee r) \quad \text{(conversion theorem)} \\ &\equiv \neg p \vee r \vee T \quad \text{(since } \neg q \vee q \equiv T\text{)} \\ &\equiv T\end{aligned}$$

Since $p \to r$ is not equivalent to T, both statements cannot be equivalent. □

Q9: Show that this argument is valid:

$$\begin{aligned}&\neg p \to F; \\ \therefore\ &p.\end{aligned}$$

Solution: The premise is $\neg p \to F \equiv p \vee F$, which is true only when p is true. □

Q10: Show that this argument is valid, where C denotes a contradiction.

$$\begin{aligned}&\neg p \to C; \\ \therefore\ &p.\end{aligned}$$

Solution: The premise is $\neg p \to C \equiv p \vee C$, which is true only when p is true. □

Q11: Determine whether the following argument is valid:

$$\begin{aligned}&\neg p \to r \wedge \neg s; \\ &t \to s; \\ &u \to \neg p; \\ &\neg w; \\ &u \vee w; \\ \therefore\ &t \to w.\end{aligned}$$

Solution: Reasoning in the succinct form:

Step	Formula	Reason
(1)	$u \vee w$	Premise
(2)	$\neg w$	Premise
(3)	u	(1) + (2), disjunctive syllogism
(4)	$u \to \neg p$	Premise
(5)	$\neg p$	(3) + (4), modus ponens
(6)	$\neg p \to r \wedge \neg s$	Premise
(7)	$r \wedge \neg s$	(5) + (6), modus ponens
(8)	$\neg s$	(7), conjunction simplification
(9)	$t \to s$	Premise
(10)	$\neg t$	(8) + (9), modus tollens
(11)	$\neg t \vee w$	(10), disjunctive addition
(12)	$t \to w$	equivalent form of (11)

Alternatively, one can write the above reasoning in a verbose form as follows.

We start by noticing that we have by disjunctive syllogism that

$$u \vee w;\ \neg w;\ \therefore u.$$

Next, by modus ponens,

$$u \to \neg p;\ u;\ \therefore \neg p.$$

Again by modus ponens,

$$\neg p \to r \wedge \neg s;\ \neg p;\ \therefore r \wedge \neg s,$$

Then by conjunction simplification,

$$r \wedge \neg s;\ \therefore \neg s.$$

Finally, by modus tollens,

$$t \to s;\ \neg s;\ \therefore \neg t$$

It follows by disjunctive addition that

$$\neg t;\ \therefore \neg t \vee w$$

or, equivalently,

$$\neg t \vee w \equiv t \to w$$

using the Conversion theorem, which shows that the argument is valid. □

Q12: Determine whether the following argument is valid:

$$\begin{aligned} & p; \\ & p \vee q; \\ & q \to (r \to s); \\ & t \to r; \\ \therefore\ & \neg s \to \neg t. \end{aligned}$$

Solution: For this question, there is no obvious way to combine the known statements with inference rules. The only 2 related statements are p and $p \vee q$, and assuming that both are true, all can be deduced is that q is either true or false (this gives no information about q at all). Now if q is false, $q \to (r \to s)$ is always true, while if q is true, $q \to (r \to s)$ is true only if $(r \to s)$ is true, which excludes the possibility $r = T$ and $s = F$. Now we look at the last premise $t \to r$. For it to be true, we need t false, or t true and r true. If s is true, then $\neg s$ is always false, and the conclusion is always true. We thus focus on s is false, and $\neg t$ is false, that is t is true. So we have a counter-example (which makes all premises true and conclusion false):

$$q = F,\ r = T,\ s = F,\ t = T.$$

One can also draw the truth table and find a counterexample from the critical rows. $\square$