MATHS/ LA/ P2/ T6/ Q1

Q1 Support

Q1) Discuss dot product, norm, normalizing a vector, scaled version of a vector.

Dot Product

Ref 6.1.3, pg 2

$v, u \in R^n$

$u \cdot v = \|u\| \, \|v\| \cos\theta$. — (1)

norm of u = length of u

$\measuredangle$ 0..π between u and v

// Ref 6.1.3, pg 5 derived from Law of cosine

$= u_1 v_1 + u_2 v_2 + \dots u_n v_n$

↳ component form of dot product,

$= \sum_{i=1}^{n} u_i v_i$

// Ref 6.1.3 pg 8 row, col^n vector form

$= u^T v = v^T u$

↳ matrix, vector form of dot product

// Ref 6.1.3 pg 2

$\therefore \cos\theta = \dfrac{u \cdot v}{\|u\| \|v\|}$

Norm of vector

Ref 6.1.2 pg 2

Length / magnitude of vector

$\|v\| = \sqrt{\sum_{i=1}^{n} (v_i)^2}$; $v \in R^n$

Normalizing a Vector

Ref 6.1.2 pg 4

Making a vector to be unit length.

$u = \dfrac{v}{\|v\|}$

// u is a vector in same direction with v and has norm = 1.

Scaled Version of a vector

Ref 6.2.2 pg 3

$\omega_1 = \mathrm{Proj}_a u$

$= \left(\dfrac{u \cdot a}{\|a\|^2} \right) a$

$= \left(\dfrac{u \cdot a}{a \cdot a} \right) a$

scaler

along direction of a.

MATHS/LA/TUT 6/Q1

T6
1

//Ref 6.1.3 pg 2

1) $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

Find $u \cdot u$, $v \cdot u$, $\frac{v \cdot u}{u \cdot u}$

a) $u \cdot u = [-1 \ \ 2]\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \sum_{i=1}^{2} (u_i)^2$

$= 1 + 4 = 5 = (-1)^2 + (2)^2$ ▣

b) $v \cdot u = \|u\| \|v\| \cos\theta$. [Definition of dot product.]

$= \sum_{i=1}^{2} v_i u_i$

$= [4 \ \ 6]\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 12 = 8$ ▣

c) $\frac{v \cdot u}{u \cdot u} = \frac{8}{5}$ ▣

4) $\frac{u}{u \cdot u} = \frac{1}{u \cdot u}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$

To normalize, we should

$\frac{u}{\|u\|} = \frac{1}{\sqrt{u \cdot u}} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \Rightarrow$

$\left\| \frac{u}{\|u\|} \right\| = \left(-\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{1}{5} + \frac{4}{5} = 1$

5) $\left(\frac{u \cdot v}{v \cdot v}\right) v = \left(\frac{8}{16+36}\right)\begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$= \left(\frac{2}{13}\right)\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$ ▣

This is the formula of $\text{Proj}_v u$ [project u onto v].

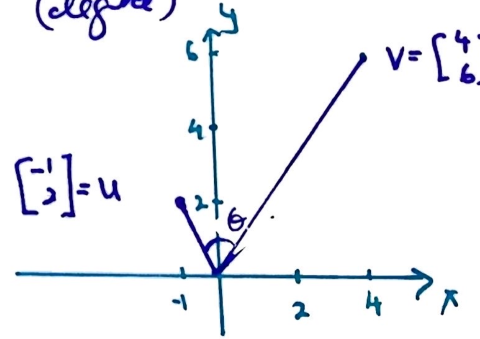
- The closest point along v to u.
- The orthogonal component $e = u - \text{Proj}_v u$ is $\perp$ to v

(b extra) what is θ? ∡ between u and v.

$\cos\theta = \frac{v \cdot u}{\|u\| \|v\|} = \frac{8}{\sqrt{5}\sqrt{52}} = 0.4961$ (rad). 0.n.

$\theta = 1.0517$ rad.

$\therefore \theta = 60.25°$ (degree)



2a) $v \cdot v = \|v\|^2$ = [True]

Definition of $\|\cdot\|$ = length of vector $v \in \mathbb{R}^n$

(Sec 6.1.2 pg 2)

does the same thing

$$= \sqrt{v_1^2 + v_2^2 + \dots v_n^2}$$

$$= \sqrt{\sum_{i=1..n} v_i^2} = \sqrt{v^T v}$$

$$= \sqrt{v \cdot v}$$

//Ref 6.1.3 pg 7

2b) for any scaler c, $\underset{\sim}{u} \cdot (c\underset{\sim}{v}) = c(\underset{\sim}{u} \cdot \underset{\sim}{v})$ [True]

// Note to differentiate scaler and vector, we put $\underset{\sim}{u}$ here.
If it is clear from context, then we do not.
In text book, vectors are in bold.

Theorem 1c (pg 331 4th edition)

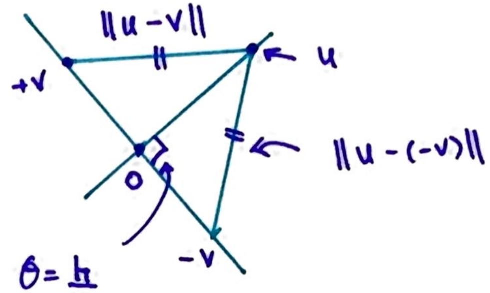
$$(c\underset{\sim}{u}) \cdot \underset{\sim}{v} = c(\underset{\sim}{u} \cdot \underset{\sim}{v}) = \underset{\sim}{u} \cdot (c\underset{\sim}{v})$$

and $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$

2c) if distance of u to v == u to $(-v)$ then u and v are orthogonal.

Ans: [True]

see pg 333, Lay 4th edition, fig 5.



$\theta = \frac{\pi}{2}$
right ∡

$$(\text{dist}(u, -v))^2$$
$$= \|u - (-v)\|^2$$
$$= \|u + v\|^2$$
$$= (u+v) \cdot (u+v)$$
$$= u \cdot u + v \cdot u + u \cdot v + v \cdot v$$
$$= \|u\|^2 + \|v\|^2 + 2u \cdot v$$

$$(\text{dist}(u, v))^2$$
$$= \|u - v\|^2$$
$$= (u - v) \cdot (u - v)$$
$$= u^2 - vu - uv + v^2$$
$$= \|u\|^2 + \|v\|^2 - 2vu$$
$$= \|u\|^2 + \|v\|^2 - 2u \cdot v$$

LHS == RHS
only when $u \cdot v = 0$
which is when $\theta = \frac{\pi}{2}$

16 Q2 d,e

2d) For a square matrix A, vectors in $Col(A)$ are orthogonal to $Null(A)$.

Ans : $\boxed{\text{False}}$.

(A) Counter Example : $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

$u = col(A) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$V = NuLL(A) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u \cdot v = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \Rightarrow u \cdot v$ is NOT $\perp$.

(B)

More explanation :

If A is $m \times n$ matrix, where $(m \neq n)$.

then $col(A) \in R^m$

$NuLL(A) \in R^n$

$\therefore$ cannot even take dot product as vectors are not same size.

(C) More explanation :

Row Space $\perp$ NULL Space.
dimension r dimension $(n-r)$

Col Space $\perp$ Left Null Space $N(A^T)$.
dimension r dimension $(m-r)$.

Ref : lecture 14 MIT : Orthogonal vectors and subspace MIT 18.06SC Linear Algebra. Fall 2011

2e) If vector $\{v_1, \dots v_p\}$ span a subspace W, if x is $\perp$ to each v_j, $j=1..p$, then $x \in W^\perp$; $v_j \in R^n$

Ans: True

see proof in (Q3).

also $\boxed{W^\perp \oplus W = R^n}$ } direct sum of sub space.

direct sum = span $(W^\perp \cup W)$

// see Maths stack exchange

"Proof vector space is a direct sum of subspace and its orthogonal complement".

Q3) Let $W \subset R^n$,
$W^{\perp}$ be set of vectors $\perp$ to W.
Show that $W^{\perp}$ is a subspace of R^n.

Ans:

Again we need to prove the 3 properties that a subspace must have.

a) $\underline{0} \in W^{\perp}$
c) closed under addition
b) closed under scaler multiplication.

a) $\underline{0}$ is $\perp$ to W
since $u \in W$, $u \cdot \underline{0} = 0$.
$\therefore$ actually $\underline{0} \in W$ for W to be a subspace.
$\Rightarrow W \cap W^{\perp} = \{\underline{0}\}$
contains the zero vector.

Note: This question is related to row space $C(A^T)$ vs $N(A)$ that each is a subspace of R^n.

b) if $z \in W^{\perp}$, $u \in W$, $c \in R$ (scaler)

then $(cz) \cdot u = c(z \cdot u) = c\,0 = 0$
since u is any element in W,
$\therefore cz \in W^{\perp}$

c) let $z_1, z_2 \in W^{\perp}$
$u \in W$,

$$(z_1 + z_2) \cdot u = z_1 \cdot u + z_2 \cdot u$$
$$= 0 + 0$$
$$= 0$$

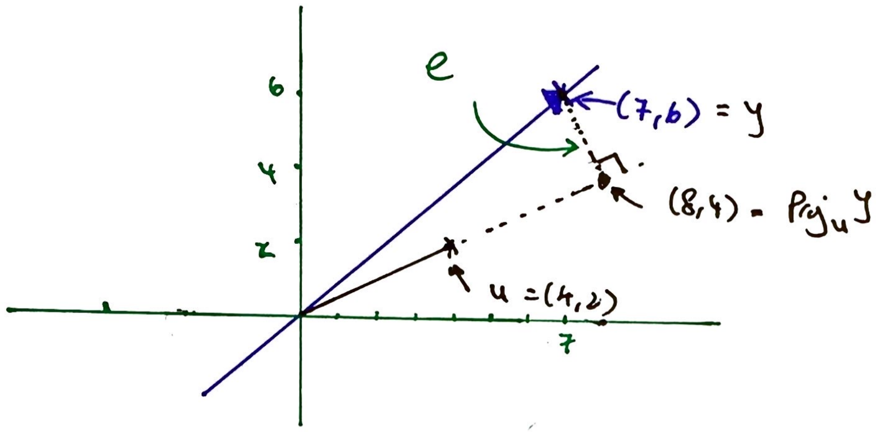
$\therefore (z_1 + z_2) \in W^{\perp}$

$\therefore$ closed under addition.

$\therefore W^{\perp}$ is a subspace.

Q Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, find orthogonal proj of y onto u. Write y as sum of 2 orthogonal vector, one in span $\{u\}$ and one orthogonal to u.

Ans:



orthogonal projection of y onto u = $\text{Proj}_u y$

$$\text{Proj}_u y = \hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{[7\ 6]\begin{bmatrix} 4 \\ 2 \end{bmatrix}}{16+4} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \left(\frac{40}{20}\right)\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 4 \end{bmatrix} \boxtimes$$

$$e \perp u = y - \text{Proj}_u y$$

$$= \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \boxtimes$$

Sanity check $e \perp \text{Proj}_u y$

$$= [-1\ 2]\begin{bmatrix} 8 \\ 4 \end{bmatrix} = -8 + 8 = 0.$$

$$\therefore y = \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \hat{y} + e$$

$$= \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Key idea: $\hat{y} = \text{Proj}_u y$ is closest point along u to y. and can be proofed by geometry, the error $e = y - \hat{y}$ is orthogonal to span $\{u\}$.

Ref: see 6.2.2 pg 4

5a) Determine if the follow set of vectors is orthonormal, if a set is only orthogonal, normalize the vectors to produce an orthonormal set.

Ans: orthogonal means $u \cdot v = 0$.
orthonormal means $u \cdot v = 0$
and $\|u\|, \|v\| = 1$.

$$u = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \quad v = \begin{bmatrix} -1/2 \\ 0 \\ +1/2 \end{bmatrix}.$$

$$\therefore \quad u \cdot v = [\,1/3 \;\; 1/3 \;\; 1/3\,]\begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = -1/6 + 0 + 1/6 = 0$$

$\Rightarrow$ orthogonal.

$$\text{But } \|u\| = \sqrt{u \cdot u} = \sqrt{1/3^2 + 1/3^2 + 1/3^2}$$
$$= \sqrt{1/3}$$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{(-1/2)^2 + 0^2 + (1/2)^2} = \sqrt{1/2}$$

To produce a vector with unit length.

$$\hat{u} = \frac{u}{\|u\|} = \frac{1}{\sqrt{1/3}}\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \quad ; \quad \frac{1}{\sqrt{1/3}} = \sqrt{3}$$

$$= \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}.$$

Sanity check $\|\hat{u}\|^2 = [\,\sqrt{3}/3 \;\; \sqrt{3}/3 \;\; \sqrt{3}/3\,]\begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$

$$= \frac{3}{9} + \frac{3}{9} + \frac{3}{9} = 1 \quad \blacksquare$$

To produce $\hat{v}$, normalize v,

$$\hat{v} = \frac{v}{\|v\|} = \frac{1}{\sqrt{1/2}}\begin{bmatrix} -1/2 \\ 0 \\ +1/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \quad \blacksquare$$

MATH8/LA/TUT/Q 5b/6a/6b

b) Is $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}$ orthogonal, orthonormal?

u v

$\|u\| = 1$, $\|v\| = 1$.

But $u \cdot v = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\-1\\0\end{bmatrix} = -1$

$\neq 0$ $\therefore$ these two vectors are Not orthogonal.

$\{u, v\}$ is NOT an orthogonal set.

Q6) a) Not every linearly independent set in R^n is an orthogonal set.

Ans: TRUE

example $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ is independent for R^2 BUT not orthogonal.

Q5b, Q23, 6a, b

6b) If y is a linear combination of non-zero vectors from an orthogonal set, then the weights in the linear combination can be computed w/o row operations on a matrix.

Ans: TRUE.

let $y = Ax$ where A has orthogonal col

$= \begin{bmatrix}\uparrow & \uparrow & & \uparrow\\ u_1 & u_2 & \dots & u_n\\ \downarrow & \downarrow & & \downarrow\end{bmatrix}\begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix}$; $A \in R^{m \times n}$; $x \in R^n$; $y \in R^m$; col A forms an orthogonal set

Ans: TRUE

//Ref: 6.2.5 pg2 Theorem 8

Theorem 5 of Lay (pg 339) 4th edit

$\{u_1, u_2, \dots u_n\}$ is an orthogonal set.

$y = u_1 x_1 + u_2 x_2 + \dots u_n x_n$, to find x_i, dot product the eq by x_i on both sides.

since $u_i \cdot u_j = 0$ when $i \neq j$;

$\therefore$ each weight can be found by :

$$x_i = \frac{u_i \cdot y}{u_i \cdot u_i} \quad (i = 1 \dots n)$$

6c) If the vectors in an orthogonal set is normalized, some of the new vectors may not be orthogonal.

Ans: False: if $\{v_1, v_2, \dots v_n\}$ are orthogonal, but each col vector is not unit length, then

$$\{\hat{v}_1, \hat{v}_2, \dots \hat{v}_n\} = \left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots \frac{v_n}{\|v_n\|}\right\}$$

normalized.

and $$\frac{v_i}{\|v_i\|} \cdot \frac{v_j}{\|v_j\|} = \left(\frac{1}{\|v_i\|}\frac{1}{\|v_j\|}\right)(v_i \cdot v_j)$$

$$= 0 \text{ if } i \neq j$$

Scaling each vector by its norm to form unit norm vector does not change it direction.

6d) A matrix with orthonormal col is an orthogonal matrix.

Ans: False.

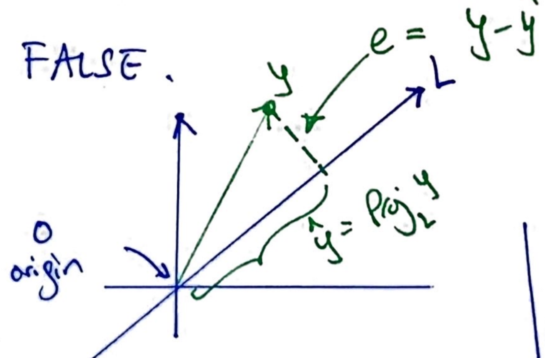
Reason: orthogonal matrix MUST BE SQUARE.

A matrix with orthonormal col need NOT be square.

6e) If L is a line through $\underline{0}$ and if $\hat{y}$ is the orthogonal projection of y onto L, then $\|\hat{y}\|$ gives the distance from y to L.

Ans: FALSE.

see 6.2.5/Pg 4



distance y to L
$= \|e\|$
$= \|y - \hat{y}\|$.

e = vector $\perp$ to L.

$\hat{y}$ = Nearest pt on L to y.

$\|\hat{y}\|$ = length of approximating y using vector along L.

MATHS/LA/TUT/Q6 (2nd q).

Q6
24a,b

24a) Not every orthogonal set in R^n is independent.

Ans: TRUE. = "bcos it can contain $\underline{0}$ vector".

But every orthogonal set of "Non-zero" vectors is linearly independent.

Note: Once a set contains the Null vector, (is orthogonal to every vector,) the set of vectors is linearly dependent.

Why? If $\{v_1, v_2, \dots v_n\}$ are linearly independent,

$$c_1 v_1 + c_2 v_2 + \cdots c_N v_n = 0$$

iff $c_1, c_2, \dots c_N = 0$.

$\therefore$ if one of the vector, e.g $v_1 = \underline{0}$

then $c_1 \neq 0$, $v_1 = 0$

$c_2 \cdots c_N = 0$ $\therefore$ rhs $= 0$

24b) If a set $S = \{u_1, u_2, \dots u_p\}$ has the property $u_i \cdot u_j = 0$ whenever $i \neq j$ then S is an orthonormal set:

Ans: False. It is an orthogonal set.

To be orthonormal, each $\|u_i\| = 1$

Note:

An orthogonal matrix is a square matrix, and its col norm = 1 (orthonormal) and is orthogonal to other col.

see 6.2.4 pg 3+4

MATHS/LA/TUT/Q6 (2nd q)

24c) If the col of $m \times n$ matrix are orthonormal, then the linear mapping $x \to Ax$ preserve lengths.

Ans: TRUE.

see Theorem 7 of Lay pg 343 — see 6.2.4 pg 41 Theorem 7

Theorem 7a) $\|Ux\| = \|x\|$.

Note $(Ux)\cdot(Uy) = [Ux]^T[Uy]$

row vector — col vector

$= x^T U^T U y$

$= x^T y.$

$= x \cdot y.$

Hence if $x = y$ $\therefore (Ux)\cdot(Ux) = x \cdot x$ ☒

Q6 24c,d e

24d) The orthogonal projection of y onto v is the same as orthogonal projection of y onto cv whenever $c \neq 0$.

Ans: TRUE.

Proof: suppose $\text{Proj}_v y = \frac{y\cdot v}{v\cdot v} v$

replace v by cv $\Rightarrow \frac{y\cdot cv}{cv\cdot cv} cv$

$= \frac{c(y\cdot v)}{c^2(v\cdot v)} cv$

$= \frac{y\cdot v}{v\cdot v} v$ ☒

$\therefore$ $\text{Proj}_v y$ does not depend on c, i.e only need non-zero v.

24e) An orthogonal matrix is invertible. — see sec 6.2.4/pg 3

Ans: TRUE, since $Q^TQ = I = QQ^T$

Q = orthogonal matrix $Q^{-1} = Q^T$