MH1812 Tutorial
Chapter 4: Proof Techniques

Q1: Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Solution: We will prove by contrapositive: "if $\sqrt{q}$ is NOT irrational, then q is NOT irrational", which is equivalent to "if $\sqrt{q}$ is rational, then q is rational". If $\sqrt{q}$ is rational, it can be written as $\sqrt{q} = a/b$ with $a, b \in \mathbb{Z}$, then $q = (\sqrt{q})^2 = a^2/b^2$, which is also rational by definition since both a^2 and b^2 are integers. $\square$

Q2: Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Solution: First of all, the first n odd positive integers refer to the set $1, 3, 5, \ldots, 2n-1$, so the sum is $S_n = \sum_{i=1}^{n}(2i-1)$.
Prove by mathematical induction:
Base Step: when $n = 1$, $S_1 = \sum_{i=1}^{1}(2i-1) = 2 \times 1 - 1 = 1 = n^2$.
Inductive Step: assume for $n = k$, the equation is true, i.e.,

$$S_k = \sum_{i=1}^{k}(2i-1) = k^2.$$

Then for $n = k+1$,

$$S_{k+1} = \sum_{i=1}^{k+1}(2i-1) = S_k + (2(k+1)-1) = k^2 + (2k+1) = (k+1)^2,$$

which implies that the equation is true for $n = k+1$.

Hence, by mathematical induction, the equation holds for all positive integers n. $\square$

Q3: Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer. Can you modify your argument to show a stronger result that $n^3 - n$ is always divisible by 6?

Solution: Base Step: the smallest positive integer is 1, so we check the basis case $n = 1$. When $n = 1$, $n^3 - n = 1^3 - 1 = 0$, which is multiple of 3 (actually 0 is multiple of any non-zero integer).
Inductive Step: assume for $n = k$, $n^3 - n = k^3 - k = 3r$ for some $r \in \mathbb{Z}$. Then, for

$n = k+1$, $n^3 - n = (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = k^3 + 3k^2 + 2k = 3r + k + 3k^2 + 2k = 3(r + k^2 + k)$, which is also a multiple of 3.
Hence, the proposition is proved by mathematical induction.

To strengthen the result to $6|(n^3 - n)$, we modify the inductive step as follows. For $n = k$, suppose that $n^3 - n = k^3 - k = 6r$ for some integer r. Then for $n = k+1$, $n^3 - n = k^3 + 3k^2 + 3k + 1 = 6r + 3k(k+1)$. Note that one of k and $k+1$ must be even, hence $k(k+1) = 2s$ for some integer s, and $n^3 - n = 6(r+s)$. This completes the proof of the inductive step. □

Q4: Prove by mathematical induction that

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Solution: Here n is a positive integer. The base case is $n = 1$.
Base Step: for $n = 1$, LHS $= 1^2 = 1$, and RHS $= \frac{1}{6} \times 1 \times (1+1) \times (2 \cdot 1 + 1) = 1$, verified.
Inductive Step: assume for $n = k$, the statement is true, i.e.,

$$1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1).$$

For $n = k+1$,

$$\begin{aligned}
1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= \frac{1}{6}(k+1)(k(2k+1) + 6(k+1)) \\
&= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1).
\end{aligned}$$

Thus the equation holds when $n = k+1$.

Hence, by mathematical induction, the equation holds for all $n \geq 1$. □

Q5: Prove using mathematical induction that for every integer $n \geq 1$ and real number $x \geq -1$,

$$(1+x)^n \geq 1 + nx.$$

Solution: Base step: When $n = 1$, both the left- and the right-hand sides are $1 + x$, and the inequality holds trivially.

Inductive step: Suppose the inequality holds for some $n = k$. For $n = k+1$, we have

$$\begin{aligned}
(1+x)^n = (1+x)^{k+1} &= (1+x)^k \cdot (1+x) \\
&\geq (1+kx)(1+x) \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x,
\end{aligned}$$

where, in the first inequality, we used the inductive hypothesis and the fact that $1+x \geq 0$. Hence the inequality holds for $n = k + 1$.

It follows that the inequality holds for all $n \geq 1$ by mathematical induction. $\square$

Q6: Prove using mathematical induction that

$$2^n > n^2 + 6, \quad n \geq 5.$$

Solution: Base step: When $n = 5$, we have $2^n = 32$ and $n^2+6 = 31$, thus the inequality holds.

Inductive step: Suppose the inequality holds for some $n = k$. For $n = k + 1$, we have by induction hypothesis that

$$\begin{aligned} 2^{k+1} - ((k+1)^2 + 6) = 2 \cdot 2^k - (k^2 + 2k + 7) &> 2(k^2 + 6) - (k^2 + 2k + 7) \\ &= k^2 - 2k + 5 \\ &= (k-1)^2 + 4 > 0. \end{aligned}$$

Hence the inequality holds for $n = k + 1$.

It follows that the inequality holds for all $n \geq 5$ by mathematical induction. $\square$