**Maths/LA/Tutorial/Ch6 (Rev 24 July 2021)**

Sep 2020, Rev A

Q1) Lay5e/ch6.1/pg 338/Ex1

Compute the quantities in Exercises 1–8 using the vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

1. $\mathbf{u} \cdot \mathbf{u}$, $\mathbf{v} \cdot \mathbf{u}$, and $\dfrac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$
2. $\mathbf{w} \cdot \mathbf{w}$, $\mathbf{x} \cdot \mathbf{w}$, and $\dfrac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$
3. $\dfrac{1}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}$
4. $\dfrac{1}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$
5. $\left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$
6. $\left(\dfrac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{x} \cdot \mathbf{x}}\right)\mathbf{x}$
7. $\|\mathbf{w}\|$
8. $\|\mathbf{x}\|$

Q2) Lay5e/ch6.1/ pg 338/ Ex19

In Exercises 19 and 20, all vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

19. a. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

 b. For any scalar $c$, $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.

 c. If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

 d. For a square matrix $A$, vectors in Col $A$ are orthogonal to vectors in Nul $A$.

 e. If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace $W$ and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\mathbf{x}$ is in $W^{\perp}$.

Ans: T,T,T,F,T

Q3) Lay5e /ch6.1/ pg 339/Ex30)

**30.** Let $W$ be a subspace of $\mathbb{R}^n$, and let $W^\perp$ be the set of all vectors orthogonal to $W$. Show that $W^\perp$ is a subspace of $\mathbb{R}^n$ using the following steps.

a. Take $\mathbf{z}$ in $W^\perp$, and let $\mathbf{u}$ represent any element of $W$. Then $\mathbf{z} \cdot \mathbf{u} = 0$. Take any scalar $c$ and show that $c\mathbf{z}$ is orthogonal to $\mathbf{u}$. (Since $\mathbf{u}$ was an arbitrary element of $W$, this will show that $c\mathbf{z}$ is in $W^\perp$.)

b. Take $\mathbf{z}_1$ and $\mathbf{z}_2$ in $W^\perp$, and let $\mathbf{u}$ be any element of $W$. Show that $\mathbf{z}_1 + \mathbf{z}_2$ is orthogonal to $\mathbf{u}$. What can you conclude about $\mathbf{z}_1 + \mathbf{z}_2$? Why?

c. Finish the proof that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Q4) Lay5e/ Example 3, pg 342,

**EXAMPLE 3** Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$. Then write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span $\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

Q5) Lay5e/ch6.2/pg 347/Ex17,18

In Exercises 17–22, determine which sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

**17.** $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

**18.** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

Q6) Lay5e/ch6.2/pg 347/Ex23+24

In Exercises 23 and 24, all vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

23. a. Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

b. If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

c. If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d. A matrix with orthonormal columns is an orthogonal matrix.

e. If $L$ is a line through $\mathbf{0}$ and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto $L$, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to $L$.

24. a. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b. If a set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then $S$ is an orthonormal set.

c. If the columns of an $m \times n$ matrix $A$ are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.

d. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

e. An orthogonal matrix is invertible.

Q7) Lay5e/ch6.2/pg 347/Ex31

31. Show that the orthogonal projection of a vector $\mathbf{y}$ onto a line $L$ through the origin in $\mathbb{R}^2$ does not depend on the choice of the nonzero $\mathbf{u}$ in $L$ used in the formula for $\hat{\mathbf{y}}$. To do this, suppose $\mathbf{y}$ and $\mathbf{u}$ are given and $\hat{\mathbf{y}}$ has been computed by formula (2) in this section. Replace $\mathbf{u}$ in that formula by $c\mathbf{u}$, where $c$ is an unspecified nonzero scalar. Show that the new formula gives the same $\hat{\mathbf{y}}$.

Q8) Lay5e/ch6.3/pg351,pg353/Example2+3+4

**EXAMPLE 2** Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in $W$ and a vector orthogonal to $W$.

**EXAMPLE 3** If $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, as in Example 2, then the closest point in $W$ to $\mathbf{y}$ is

**EXAMPLE 4** The distance from a point $\mathbf{y}$ in $\mathbb{R}^n$ to a subspace $W$ is defined as the distance from $\mathbf{y}$ to the nearest point in $W$. Find the distance from $\mathbf{y}$ to $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Q9) Proof Theorem 10 of Lay5e/ch6.3/pg 353

**THEOREM 10**

If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace $W$ of $\mathbb{R}^n$, then

$$\text{proj}_W\, \mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p \tag{4}$$

If $U = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_p\,]$, then

$$\text{proj}_W\, \mathbf{y} = UU^T\mathbf{y} \quad \text{for all } \mathbf{y} \text{ in } \mathbb{R}^n \tag{5}$$

Q10) Lay5e/ch6.3/pg355/Ex21

In Exercises 21 and 22, all vectors and subspaces are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

**21.** a. If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and to $\mathbf{u}_2$ and if $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z}$ must be in $W^{\perp}$.

b. For each $\mathbf{y}$ and each subspace $W$, the vector $\mathbf{y} - \text{proj}_W\,\mathbf{y}$ is orthogonal to $W$.

c. The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace $W$ can sometimes depend on the orthogonal basis for $W$ used to compute $\hat{\mathbf{y}}$.

d. If $\mathbf{y}$ is in a subspace $W$, then the orthogonal projection of $\mathbf{y}$ onto $W$ is $\mathbf{y}$ itself.

e. If the columns of an $n \times p$ matrix $U$ are orthonormal, then $UU^T\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of $U$.

Q11) Lay/ch6.4/pg 359/Ex19+20

**19.** Suppose $A = QR$, where $Q$ is $m \times n$ and $R$ is $n \times n$. Show that if the columns of $A$ are linearly independent, then $R$ must be invertible. [*Hint:* Study the equation $R\mathbf{x} = \mathbf{0}$ and use the fact that $A = QR$.]

**20.** Suppose $A = QR$, where $R$ is an invertible matrix. Show that $A$ and $Q$ have the same column space. [*Hint:* Given $\mathbf{y}$ in Col $A$, show that $\mathbf{y} = Q\mathbf{x}$ for some $\mathbf{x}$. Also, given $\mathbf{y}$ in Col $Q$, show that $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$.]

Q12) Lay5e/ch6.4/pg359/Example4

**EXAMPLE 4** Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Q13) Lay5e/ch6.4/pg 360/

**2.** Suppose $A = QR$, where $Q$ is an $m \times n$ matrix with orthogonal columns and $R$ is an $n \times n$ matrix. Show that if the columns of $A$ are linearly dependent, then $R$ cannot be invertible.

============== End of Tut Ch 6 =================