

SC2001/CX2101

Algorithm Design and Analysis

Tutorial 3

Analysis Techniques

(Week 9)

This tutorial helps you develop skills in the learning outcome of the course: “Able to conduct complexity analysis of recursive algorithms: solve recurrences using the substitution method, the iteration method, the master theorem, the characteristic equation.”

Question 1 (1)

Solve the following recurrence by the iteration method

$$T(1) = 1, \text{ and for } n \geq 2, T(n) = 3T(n-1) + 2$$

Question 1 (1)

Solve the following recurrence by the iteration method

$$T(1) = 1, \text{ and for } n \geq 2, T(n) = 3T(n-1) + 2$$

$$\begin{aligned} T(n) &= 3T(n-1) + 2 \\ &= 3(3T(n-2) + 2) + 2 \\ &= 3^2T(n-2) + 3*2 + 2 \\ &= 3^2(3T(n-3) + 2) + 3*2 + 2 \\ &= 3^3T(n-3) + 3^2*2 + 3*2 + 2 \\ &= 3^{n-1}T(1) + 3^{n-2}*2 + \dots + 3^2*2 + 3*2 + 2 \end{aligned}$$

Question 1 (1)

$$= 3^{n-1} + 2(3^{n-2} + \dots + 3^2 + 3 + 1)$$

$$= 3^{n-1} + 2(3^{n-2} + \dots + 3^2 + 3 + 1)$$

$$= 3^{n-1} + 2(3^{n-1} - 1)/(3-1)$$

$$= 3^{n-1} + 3^{n-1} - 1$$

$$\leq 2 * 3^{n-1}$$

$$\leq 3^n$$

$$\text{So } T(n) = O(3^n)$$

Question 1 (2)

Solve the following recurrence by the iteration method

$$T(1) = 1, \text{ and for } n \geq 2, \text{ a power of 2, } T(n) = 2T(n/2) + 6n$$

Question 1 (2)

Solve the following recurrence by the iteration method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

$$\begin{aligned}T(n) &= 2T\left(\frac{n}{2}\right) + 6n \\&= 2\left(2T\left(\frac{n}{2^2}\right) + 6\left(\frac{n}{2}\right)\right) + 6n \\&= 2^2T\left(\frac{n}{2^2}\right) + 6n + 6n \\&= 2^2\left(2T\left(\frac{n}{2^3}\right) + 6\left(\frac{n}{2^2}\right)\right) + 6n + 6n \\&= 2^3T\left(\frac{n}{2^3}\right) + 6n + 6n + 6n \\&= 2^kT(1) + 6kn \quad \text{assume } n = 2^k \quad \text{so } k = \lg n \\&= n + 6n \lg n \\&= O(n \lg n)\end{aligned}$$

Question 2 (1)

Solve the following recurrence by the substitution method

$$T(1) = 1, \text{ and for } n \geq 2, T(n) = 3T(n-1) + 2$$

Question 2 (1)

Solve the following recurrence by the substitution method

$$T(1) = 1, \text{ and for } n \geq 2, T(n) = 3T(n-1) + 2$$

Guess that $T(n) = O(3^n)$

Proof: We will prove $T(n) \leq 3^n - 2$ for $n \geq 1$

(a) Base case: $T(1) = 1 \leq 3^1 - 2$.

(b) Inductive step: assume that $T(k) \leq 3^k - 2$, prove that $T(k+1) \leq 3^{k+1} - 2$.

$$T(k+1) = 3T(k) + 2$$

$$\leq 3(3^k - 2) + 2$$

$$\leq 3^{k+1} - 6 + 2$$

$$\leq 3^{k+1} - 2 \quad \text{Thus } T(n) = O(3^n - 2) = O(3^n)$$

Question 2 (2)

Solve the following recurrence by the substitution method

$$T(1) = 1, \text{ and for } n \geq 2, \text{ a power of } 2, T(n) = 2T(n/2) + 6n$$

Question 2 (2)

Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

Guess $T(n) = O(n \lg n)$

Proof: we show that $T(n) \leq 8 n \lg n$ for any $n \geq 2$

a) Base case:

$$T(2) = 2 \cdot 1 + 6 \cdot 2 = 14, 8n \lg n = 16,$$

so $T(n) \leq 8 n \lg n$ for $n=2$

b) Inductive step: Assume that $T(2^k) \leq 8 k 2^k$

Prove $T(2^{k+1}) \leq 8 (k+1) 2^{k+1}$

Question 2 (2)

$$\begin{aligned}T(2^{k+1}) &= 2 T(2^k) + 6 * 2^{k+1} \\&\leq 2 * 8 k 2^k + 6 * 2^{k+1} \\&= (8 k + 6) * 2^{k+1} \\&\leq 8 (k + 1) * 2^{k+1}\end{aligned}$$

$$T(n) = O(n \lg n)$$

Question 3(1)

Solve the following recurrence by the master method.

$$W(n) = W(n/3) + 5$$

Question 3(1)

Solve the following recurrence by the master method.

$$W(n) = W(n/3) + 5$$

$$n^{\log_b a} = n^{\log_3 1} = n^0, \quad f(n) = 5 = \theta(1) = \theta(n^0)$$

So,

$$W(n) = \theta(n^0 \lg n)$$

Question 3(2)

Solve the following recurrence by the master method.

$$T(n) = 2T(n/2) + n/4$$

Question 3(2)

Solve the following recurrence by the master method.

$$T(n) = 2T(n/2) + n/4$$

$$n^{\log_b a} = n^{\log_2 2} = n^1, \quad f(n) = n/4 = \theta(n) = \theta(n^1)$$

So,

$$W(n) = \theta(n \lg n)$$

Question 3(3)

Solve the following recurrence by the master method.

$$W(n) = 2W(n/4) + \sqrt{n}^3$$

Question 3(3)

Solve the following recurrence by the master method.

$$W(n) = 2W(n/4) + \sqrt{n^3}$$

$$n^{\log_b a} = n^{\log_4 2} = n^{0.5}, \quad f(n) = \sqrt{n^3} = n^{3/2} = \Omega(n^{0.5 + 0.1})$$

$$\begin{aligned} \text{And } a \cdot f(n/b) &= 2f(n/4) = 2 \cdot (n/4)^{3/2} = 2 \cdot n^{3/2} (1/4)^{3/2} \\ &= (1/4) \cdot n^{3/2} \leq (1/4) \cdot n^{3/2} = c \cdot f(n) \quad c = 1/4 \end{aligned}$$

So,

$$W(n) = \theta(n^{1.5})$$

Question 4

Determine which of the following are linear homogeneous recurrence relations with constant coefficients. Also find the degree of those that are.

1) $a_n = 4a_{n-2} + 5a_{n-3}$

Degree 3

2) $a_n = 2na_{n-1} + a_{n-2}$

Not constant coefficient

3) $a_n = a_{n-1} + a_{n-4}$

Degree 4

4) $a_n = a_{n-1}^2 + a_{n-2}$

Not linear

5) $a_n = a_{n-2} + n$

Not homogeneous

Question 5 (no need to cover all parts if running out of time)

(1) Solve the following recurrence relation together with the initial conditions given.

$$a_n = 7a_{n-1} - 10a_{n-2} \text{ for } n \geq 2, a_0 = 1, a_1 = 0$$

Question 5(1)

Solve the following recurrence relation together with the initial conditions given.

$$a_n = 7a_{n-1} - 10a_{n-2} \text{ for } n \geq 2, a_0 = 1, a_1 = 0$$

The characteristic equation:

$$t^2 - 7t + 10 = 0 \Rightarrow (t - 2)(t - 5) = 0$$

Solution: two distinct roots: $t_1 = 2, t_2 = 5$

Thus $a_n = 2^n C + 5^n D$

Question 5(1)

Substitute the initial conditions into $a_n = 2^n C + 5^n D$ to find C and D:

$$a_0 = 1 = C + D, \quad \Rightarrow \quad 2 = 2C + 2D$$

$$a_1 = 0 = 2C + 5D$$

Thus $2 = -3D$, i.e. $D = -2/3$ then $C = 5/3$

So we have $a_n = (5/3) * 2^n - (2/3) * 5^n$

Question 5(2)

Solve the following recurrence relation together with the initial conditions given.

$$a_n = 4a_{n-2} \text{ for } n \geq 2, a_0 = 6, a_1 = 8$$

Question 5(2)

Solve the following recurrence relation together with the initial conditions given.

$$a_n = 4a_{n-2} \text{ for } n \geq 2, a_0 = 6, a_1 = 8$$

The characteristic equation:

$$t^2 - 4 = 0 \Rightarrow (t)^2 = 4$$

Solution: two distinct roots: $t_1 = 2, t_2 = -2$

Thus $a_n = 2^n C + (-2)^n D$

Question 5(2)

Substitute the initial conditions into $a_n = 2^n C + (-2)^n D$ to find C and D:

$$a_0 = 6 = C + D, \quad \Rightarrow \quad 12 = 2C + 2D$$

$$a_1 = 8 = 2C - 2D$$

Thus $20 = 4C$, i.e. $C = 5$ then $D = 1$

So we have $a_n = 5 \cdot 2^n + (-2)^n$

Question 5(3)

Solve the following recurrence relations together with the initial conditions given.

$$a_n = 2a_{n-1} - a_{n-2} \text{ for all } n \geq 2, \quad a_0 = 1, \quad a_1 = 3$$

Question 5(3)

Solve the following recurrence relations together with the initial conditions given.

$$a_n = 2a_{n-1} - a_{n-2} \text{ for all } n \geq 2, \quad a_0 = 1, \quad a_1 = 3$$

The characteristic equation:

$$t^2 - 2t + 1 = 0 \Rightarrow (t-1)^2 = 0$$

Solution: single root: $t = 1$,

Thus $a_n = C + nD$, we have

$$1 = C, \quad D = 2$$

$$a_n = 1 + 2n$$