Maths LA/ Tut 6/ Q12
Part 2/

Q12) Find a QR factorization of

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Note: we will do an economy factorization,
i.e $Q \in \mathbb{R}^{4\times 3}$.

See: Lecture Notes 6.3.2 (pg 4). / Lay 5e, pg 359 (Ex 4)
Lay 5e, pg 356 (Ex 2)

Gram Schmidt process

$$v_1 = x_1 \quad ; \quad v_1 \in \mathbb{R}^4$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \quad ;$$

the set of V spans col^n space of A

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \quad ;$$

$$Q = \left[\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right] \in \mathbb{R}^{4\times 3}$$

Remember: col of Q are orthonormal col^n spanning the space of A!

step 1: $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_1$

step 2: $v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$

$= x_2 - \text{Proj}_{v_1} x_2$ // to get residual of x_2 being approximated by v_1

$$= \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{x_2} - \underbrace{\frac{3}{4}}_{\left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right)} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{v_1}$$

$$v_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \quad (\text{scaled} \times 4) = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

same direction

step 3: $v_3 = x_3 - \text{Proj}_{W_2} x_3$

$$W_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \right\} = \{ v_1, v_2 \}$$

step 3a:

$$\text{Proj}_{W_2} X_3 = \underbrace{\frac{X_3\cdot V_1}{V_1\cdot V_1}V_1 + \frac{X_3\cdot V_2}{V_2\cdot V_2}V_2}$$

This can be used because V_1, V_2 are orthogonal.

→ see Theorem 8 (pg 350 Lay 5e)

→ see lecture Notes 6.3.1 pg 5

$$\text{Proj}_{W_2} X_3 = \underset{\frac{X_3\cdot V_1}{V_1\cdot V_1}}{\frac{2}{4}} \underset{V_1}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} + \underset{\frac{X_3\cdot V_2}{V_2\cdot V_2}}{\frac{2}{12}} \underset{V_2}{\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}}$$

V_2 (V_2 scaled by $\times 4$ to remove $\frac{1}{4}$)!

$$= \begin{bmatrix}0\\2/3\\2/3\\2/3\end{bmatrix}$$

$$\therefore V_3 = X_3 - \text{Proj}_{W_2} X_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\2/3\\2/3\\2/3\end{bmatrix}$$

$$= \begin{bmatrix}0\\-2/3\\1/3\\1/3\end{bmatrix} \quad ☒$$

$$\therefore Q = \left[\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|}\right]$$

$$= \begin{bmatrix}1/2 & -3/\sqrt{12} & 0\\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6}\\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6}\\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6}\end{bmatrix};$$

$$\frac{V_1}{\|V_1\|} \qquad \frac{V_2}{\|V_2\|} \qquad \frac{V_3}{\|V_3\|}$$

Note: $\frac{V_1}{\|V_1\|}$ and $\frac{-V_1}{\|V_1\|}$ all pt to same direction! so both can be used!

$\frac{V_i}{\|V_i\|}$ is not unique!

To get R, remember

$$\underset{(3\times4)}{Q^T}\underset{(4\times3)}{Q} = I_{(3\times3)}.$$

then $Q^T A = Q^T(\underbrace{QR}_{A})$.

$$Q^T A = IR = R \quad ☒$$

Q
Q3

$$\therefore \quad R = \underset{(3\times4)}{\overset{Q^T}{\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}}} \underset{(4\times3)}{\overset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}}$$

$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix} \quad ☒$$

Notes :

1) We performed economy decomposition.

$Q \in \mathbb{R}^{4\times3}$

The col^n space Q == col space A.

To verify :

method 1 : $\left[Q \;\vdots\; a_i \right]$ $a_i = 1..3$ col of A.

Gaussian elimination we can always find sol^n.

method 2 : create Projection matrix

$$P = \underset{(4\times3)}{Q}\,\underset{(3\times4)}{Q^T} = 4\times4.$$

$Pa_i = a_i \;;\; a_i = 1..3.$

showing that P spans same col space

$= PA = A$ ☒

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Maths LA/ Part 2/ Tut 6 4/ Q 13

Suppose $A = QR$, (should be orthonormal)

$Q = m \times n$ matrix with orthogonal col^n

$R = n \times n$ matrix

If A has linearly dependent col^n then R is singular (cannot be inverted).

Ans: since A has dependent col, then

when $A\underline{x} = \underline{0}$,

there is non-trivial (non-zero) $\underline{x}$

s.t $A\underline{x} = 0$.

Method 1: sub $A = QR$.

pre-multiply by Q^T

$(QR)x = \underline{0}$; and $x \neq 0$. since A has dependent col.

$Q^T Q R x = Q^T \underline{0}$; $Q^T Q = I$

$Rx = \underline{0}$; (since $\underline{x} \neq 0$, but rhs $= 0$ ⇒ R must have dependent col) invertible matrix theorem

⇒ R is singular

⇒ R is not invertible.

Method 2: (notes: Lecture Notes 6.2.4 pg 4)

apply Theorem 7 of (pg 345 Lay 5e)

→ if U is $m \times n$ orthonormal col^n

then $\|Ux\| = \|x\|$ ⇒ pre-multiply with U on x does not change length of x.

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Pg 2

$Ax = \underline{0}$; There is $\underline{X} \neq 0$, s.t

$= \|Ax\| = \|\underline{0}\|$ → norm of zero vector $= 0$ → scalar.

$= \|QRx\| = 0$

→ since Q has orthonormal col by Theorem 7.

$= \|Rx\| = 0$

$\Rightarrow \|u\| = 0$ ← $\underline{0}$ vector

By Theorem 1 (Pg 333 Lay 5e)
$\|u\| = 0$ if $u = \underline{0}$
$u \cdot u \geq 0$ and $u \cdot u = 0$ if $u = \underline{0}$

$\Rightarrow$ to get $\underline{u} = \underline{0}$, since $\underline{X} \neq 0$, then

→ R is singular ☒
→ has dependent col
→ by invertible matrix Theorem ☒

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

e.g $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, then $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$QR = qr(A, 0)$$

$$= \underbrace{\begin{bmatrix} -0.5774 & 0.8165 \\ -0.5774 & -0.4082 \\ -0.5774 & -0.4082 \end{bmatrix}}_{q_1 \quad q_2} \begin{bmatrix} -1.7321 & -3.4641 \\ 0 & 0 \end{bmatrix}$$

q_1 q_2 are orthonormal col.

R is singular!

$$Rx = \begin{bmatrix} -1.7321 & -3.4641 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

any N(A) will get $\alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}$; $\alpha \in R$