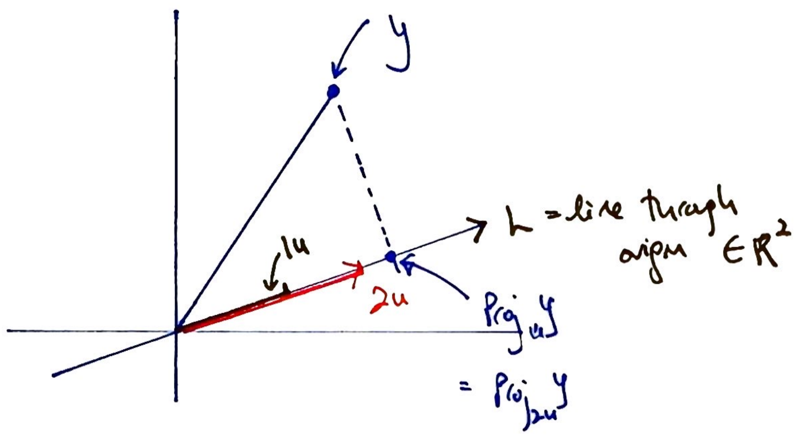
Q7) Show that the orthogonal projection of y onto a line L through origin in $\mathbb{R}^2$ does not depend on the choice of $u \in L$.



Ans: The question ask if

$$\text{Proj } y \text{ onto } u \overset{?}{==} \text{Proj } y \text{ onto } 2u$$

or onto cu ← scaler c.

Yes it is the same.

Proof: $\text{Proj}_{u} y = \left(\frac{y \cdot u}{u \cdot u}\right) u$

$$\text{Proj}_{cu} y = \left(\frac{y \cdot (cu)}{(cu) \cdot (cu)}\right) cu$$

$$= \frac{c\,(y \cdot u)}{c^2\,(u \cdot u)}\, c\, u$$

$$= \left(\frac{c^2\,(y \cdot u)}{c^2\,(u \cdot u)}\right) u$$

$$= \left(\frac{y \cdot u}{u \cdot u}\right) u \quad \boxtimes$$

This shows that y projecting onto u or non-zero cu is the same.

Intuition: we are finding the closest point on L to u, it should be the same pt regardless if u or $2u$ is chosen as the vector along L.

8) $W = \text{span}\{u_1, u_2\}$; $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\text{Proj}_W y$.

Ans : W is a subspace of R^3

- span by 2 orthogonal vector.
- If u_1, u_2 were not orthogonal BUT only independent,

then we can first use Gram-Schmidt to find orthogonal basis.

// since $u_1 \perp u_2$) proof by dot product

$$[2 \; 5 \; -1] \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = -4 + 5 - 1 = 0$$

u_1 u_2

$$= u_1^T u_2$$

orthogonal decomposition theorem

Theorem 8 sec 6.2.5 pg 2

Then to find $\text{Proj}_W y$, given W is a subspace, and $\{u_1, u_2 \cdots u_p\}$ is an orthogonal basis,

$$\text{Proj}_W y = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \cdots + \left(\frac{y \cdot u_p}{u_p \cdot u_p}\right) u_p \rightarrow \text{Theorem 10 eq!}$$

$$\therefore \text{Proj}_W y = \frac{[1 \; 2 \; 3] \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}}{[2 \; 5 \; -1] \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{[1 \; 2 \; 3] \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}{[-2 \; 1 \; 1] \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 \\ 2 \\ +1/5 \end{bmatrix}$$ ☒

Impt

see 6.2.5 / pg 2

Theorem 8 eqn

can only be used for $\{u_1, u_2 \cdots u_p\}$ being a set of ORTHOGONAL VECTORS spanning the space W.

- If $\{u_1, u_2, \cdots u_p\}$ are NOT orthogonal, you must first find an orthogonal basis (e.g using Gram Schmidt) for the space !!!

MATHS/LA/TUT 6/Q8/ex4

8 ex4) Find the (nearest) distance y to subspace W span by $\{u_1, u_2\}$

$$y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$u_1 \cdot u_2 = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 5 - 4 - 1 = 0.$$

$\Rightarrow u_1 \perp u_2 \Rightarrow W$ is span by an orthogonal basis.

$\therefore$ we can use Theorem 8 / sec 6.2.5 pg 2

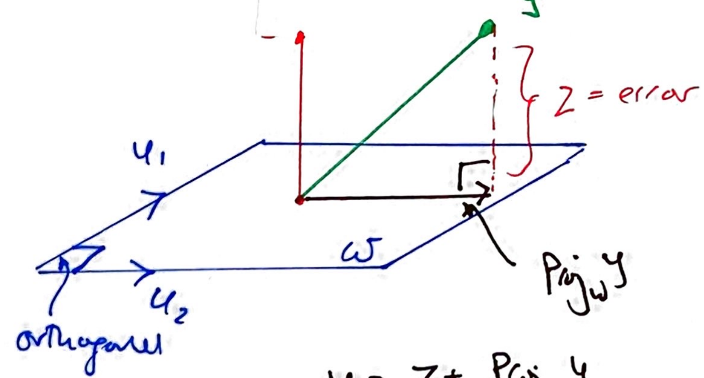
Theorem 8 Orthogonal Decomposition Algorithm : sec 6.2.5 / pg 2

$$y = \hat{y} + z$$
$$= \text{Proj}_W y + z$$

$W = \text{span}\{u_1, u_2, \dots u_p\}$

where $\{u_1, u_2, \dots u_p\}$ are orthogonal basis of W

then z is $\perp W$.



$$y = Z + \text{Proj}_W y.$$

$$\therefore \text{Proj}_W y = \left(\frac{u_1 \cdot y_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{u_2 \cdot y_2}{u_2 \cdot u_2}\right) u_2$$

$$= \frac{15}{30} u_1 + \frac{(-21)}{6} u_2 = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

$$\text{error} = Z = y - \text{Proj}_W y$$

$$= \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

Z is a vector !

The distance of y to $\text{Proj}_W y$ is a scal[e]

it is the $\|Z\|$ (norm),

$$= \sqrt{Z^T Z} = \sqrt{9+36} = \sqrt{45} \quad \blacksquare$$

MATHS/LA/TUT 6/Q9

Proof Theorem 10 (Sec 6.2.5 / pg 7)

- If $\{u_1, u_2, \ldots, u_p\}$ are ORTHONORMAL BASIS of $W \in \mathbb{R}^N$

then $\text{Proj}_W y = (y \cdot u_1) u_1 + \cdots (y \cdot u_p) u_p$

and $= UU^T y \quad \forall y \in \mathbb{R}^N$

Ans: $\underbrace{\text{Theorem 10}}_{\text{need orthonormal basis}}$ is different to $\underbrace{\text{theorem 8}}_{\text{need ONLY orthogonal basis}}$

Note: orthonormal basis means

$u_i \perp u_j \quad i \neq j$

and $\|u_i\| = 1 \quad \forall i = 1 \cdots p$.

In theorem 8:

$$\text{Proj}_W y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

if $\|u_i\| = 1 \Rightarrow u_1 \cdot u_1 = 1$

$\therefore$ Theorem 8 → 10 simplifies the eq^1 to

$$\text{Proj}_W y = \left(\frac{y \cdot u_1}{1}\right) u_1 + \cdots \left(\frac{y \cdot u_p}{1}\right) u_p$$

$$= (y \cdot u_1) u_1 + \cdots (y \cdot u_p) u_p.$$

a) How to get $\begin{bmatrix} y \cdot u_1 \\ y \cdot u_2 \\ \vdots \\ y \cdot u_p \end{bmatrix}$?

Ans: $U^T y$

since
$$\left.\begin{aligned} y \cdot u_1 &= u_1^T y \\ y \cdot u_2 &= u_2^T y \\ &\vdots \\ y \cdot u_p &= u_p^T y \end{aligned}\right\} \begin{bmatrix} \leftarrow u_1^T \rightarrow \\ \leftarrow u_2^T \rightarrow \\ \vdots \\ \leftarrow u_p^T \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ y \\ \downarrow \end{bmatrix}.$$

b) How to get $(y \cdot u_1) u_1 + \cdots + (y \cdot u_p) u_p$.
matrix vector multiplication to realise linear combination.

$$\text{Proj}_W y = \underbrace{\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \cdots & u_p \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} y \cdot u_1 \\ y \cdot u_2 \\ \vdots \\ y \cdot u_p \end{bmatrix}}_{U^T y} = UU^T y$$

10) Answer T/F. Assume all vectors and subspace $\in \mathbb{R}^n$
Justify each answer.

a) If $z \perp u_1$ and $z \perp u_2$
If $W = \text{Span}\{u_1, u_2\}$, then z must be in $W^\perp$.

Ans: TRUE

see Tut 6/Q8 ex 4.

$z = y - \text{Proj}_W y$

is $\perp W$.

explanation see 6.2.5 pg 2 (right side) pg
// Theorem 8

To prove $z \perp W$.

i) show $z \perp u_1$ by

$$z \cdot u_1 = \underbrace{(y - \text{Proj}_W y)}_{z} \cdot u_1$$

$$= y \cdot u_1 - \underbrace{(\text{Proj}_W y)}_{\text{Theorem } 8} \cdot u_1$$

$$\hookrightarrow \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

// since $u_i \cdot u_j = 0 \quad i \neq j$

$$= y \cdot u_1 - \left(\frac{y \cdot u_1}{u_1 \cdot u_1} (u_1 \cdot u_1) + 0 + \cdots 0 \right)$$

$$= y \cdot u_1 - y \cdot u_1$$

$$z \cdot u_1 = 0.$$

showing z is $\perp u_1$.

similarly z is $\perp u_2, \ldots u_p$.

$\therefore z \in W^\perp$.

$\boxed{e}$

10b) TRUE. see (10a), see 6.2.5 pg 2. Theorem 8

c) The orthogonal projection of y on W can depend on orthogonal basis W.

Ans: FALSE

see sec 6.2.5, Theorem 8, pg 2 slide RHS, last paragraph.

To proof:

then $y = \hat{y} + z$

$= \text{Proj}_W y + z$; — (1)

$\Rightarrow y \in W$ and $z \in W^{\perp}$

and $\hat{y}$ is unique.

suppose y can be written as

(2) $y = \hat{y}_1 + z_1$; $\hat{y}_1 \in W$

; $z_1 \in W^{\perp}$

then eq 1 = eq 2

$\Rightarrow \hat{y} + z = \hat{y}_1 + z_1$

$\therefore \hat{y} - y_1 = z_1 - z$; remember that LHS $\in W$, RHS $\in W^{\perp}$.

for them to equal, the only vector is $\underline{0} \in R^n$

This proofs that $y = y_1$ is unique regardless of basis of W. ■

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d) If $y \in$ subspace W, then the orthogonal projection of y onto W is y.

Ans: TRUE

see sec 6.2.5, Theorem 9, pg 4 Left slide, "Properties of orthogonal projection".

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e) If col^s of $n \times p$ matrix U are orthonormal, then UU^Ty is the orthogonal projection of y onto col(U).

Ans: TRUE // see Tut 6/Q9 + Theorem 10.

MATHS/LA/TUT6/Q11

Q11 (19) suppose $A = QR$, where $Q = m \times n$, $R = n \times n$

show that if col^n of A are linearly independent, then R must be invertible.

Ans: for col^n of A to be independent,

i) then $Ax = 0 \Rightarrow x = 0$.

ii) suppose $Rx = 0$. { Note - we have not decided that x need to be zero! yet.

premultiply both side by Q

$\therefore \underbrace{QRx}_{\downarrow} = Q0 = Ax = 0$

$Q0 = 0$

$\therefore$ x must be zero. since A has linearly independent col.

$\Rightarrow$ R must also have linearly independent

By Invertible Matrix Theorem } col. $\Rightarrow$ since R is square, it is invertible.

Q11
19) 20

Q11) (20) Suppose $A = QR$, R is invertible. show that A and Q has same col^n space.

Ans: This question shows us how to find an orthogonal basis of Col (A)!

let $y \in \text{Col}(A)$. then $y = Ax$ for some x.

Then $y = Ax = QRx$

$= Q(\underline{Rx})$

a) Rx = col^n vector = weights multiply by col Q to form y.

$\Rightarrow$ y is a linear combination of col of Q

$\Rightarrow$ y is in col space (Q) } 1^st direction. y in Q col space

b) suppose $y = Qx$. //

since R^{-1} exist, $A = QR$

$\Rightarrow AR^{-1} = Q$

so $y = (AR^{-1})x = A(R^{-1}x)$ } 2^nd direction y in col(A)

= y is linear combination of A. ☒