MH1812 Tutorial
Chapter 5: Combinatorics

Q1: A set menu proposes 2 choices of starters, 3 choices of main dishes, and 2 choices of desserts. How many possible set menus are available?

Solution: $2 \times 3 \times 2 = 12$. $\square$

Q2: (a) In a race with 30 runners where 8 trophies will be given to the top 8 runners (the trophies are distinct, there is a specific trophy for each place), in how many ways can this be done?

(b) In how many ways can you solve the above problem if a certain person, say Jackson, must be one of the top 3 winners?

Solution: Since the trophy for each place is distinct and fixed, we need to place the runners into the top 8 one by one:

(a) $P(30, 8) = 30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23$.

(b) $3 \times P(29, 7) = 3 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23$. $\square$

Q3: A shelf contains 5 Western books and 4 Romances. In how many ways can they be arranged

(a) without any restrictions?

(b) if all the Western are together and all the Romances are together?

(c) if all the Western are together?

(d) if no two Western are together?

Solution:

(a) $9! = 362880$.

(b) $2 \cdot 5! \cdot 4! = 5760$

(c) $5! \cdot 4! \cdot 5 = 14400$, where the last factor of 5 means that there are five possible starting positions to place the Western books.

(d) $5! \cdot 4! = 2880$. Note that the Western books must be exactly one position away from each other. $\square$

Q4: At a party there are 15 men and 20 women.

(a) How many ways are there to form 15 couples consisting of one man and one woman?
(b) How many ways are there to form 10 couples consisting of one man and one woman?

Solution:

1. We choose a woman for each man in sequence. For the first man, there are 20 choices for a woman; for the second man, there are 19 choices, and so on. This results in $20 \times 19 \times \cdots \times 6 = 20!/5!$ ways.
2. We choose 10 from the 15 men and 10 from the 20 women to pair. There are $\binom{15}{10} \cdot \binom{20}{10}$ ways for this step. Next, with the chosen 10 men and 10 women, there are 10! ways to pair them (by a similar argument to the previous part of the question). Hence the total number of ways is

$$\binom{15}{10}\binom{20}{10}10!. \qquad \square$$

Q5: How many ternary strings of length 4 have zero ones?

Solution: 2^4. Since for each ternary bit, there are 2 choices after the exclusion of "1". $\square$

Q6: How many distinguishable arrangements of the 10 letters of the word STATISTICS are possible?

Solution: There are distinct characters with number of occurrences: S(3), T(3), A(1), I(2), C(1). Thus the number of permutations is

$$\frac{10!}{3!\,3!\,1!\,2!\,1!} = 50400. \qquad \square$$

Q7: (Continuation of Q6) Three letters are selected from these ten and the number of distinguishable ways of arranging them is calculated. What is the total number of distinguishable arrangements for all possible such selections?

Solution: We break down the three letters into different cases.

1. Three distinct letters are selected: There are $\binom{5}{3} = 10$ possible cases. The number of distinguishable arrangements for each case is $3! = 6$.
2. Exactly two of the letters are identical: There are 12 cases as below.

   (S,S,A), (S,S,C), (T,T,A), (T,T,C), (S,S,T), (T,T,S), (S,S,I), (T,T,I), (I,I,S), (I,I,T), (I,I,A), (I,I,C)

   The number of distinguishable arrangements for each case is $3!/(2!\,1!) = 3$.
3. Three identical letters: Two possible ways to make such selections (choosing all 3 S's or all 3 Ts). The number of distinguishable arrangements in each case is 1.

Therefore, the total number of distinguishable arrangements is $10{\cdot}6+12{\cdot}3+2{\cdot}1 = 98$. $\square$