Tutorial 1

Systems of Linear Equations

1. Find the values of k for which the equations

$$\begin{array}{ccccccc} x & + & 5y & + & 3z & = & 0 \\ 5x & + & y & - & kz & = & 0 \\ x & + & 2y & + & kz & = & 0 \end{array}$$

have a non-trivial solution.

2. Use Gaussian elimination and back substitution to solve the following system of linear equations:

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned}$$

3. Determine the values of a for which the following linear system has (i) no solutions, (ii) infinite solutions, (iii) exactly one solution:

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

4. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\ \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Is $\mathbf{b}$ in W? How many vectors are in W?

5. Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppose the vectors $\mathbf{u}$ and $\mathbf{v}$ are in Span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v}$ is also in Span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

6. Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

   a. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$?

   b. Do the columns of B span $\mathbb{R}^4$? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each $\mathbf{y}$ in $\mathbb{R}^4$?

c. Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$?

d. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B? Do the columns of B span $\mathbb{R}^3$?

7. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.

8. Suppose A is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does *not* have a solution. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Why?

9. Find the value of h for which the vectors $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$ are linearly *dependent*.

10. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

11. Find the standard matrix of the linear transformation

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$, which first rotates points through $-3\pi/4$ radian (clockwise) and then reflects points through the horizontal $x-$axis.

b. $T : \mathbb{R}^2 \to \mathbb{R}^2$, which first reflects points through the horizontal $x-$axis and then reflects points through the line $y = x$. Show that the transformation is merely a rotation about the origin. What is the angle of rotation?

Answers

1. $k = 1$

2. $x = -1/2, y = 0, z = 1/2$

3. (i) $a = -4$ (ii) $a = +4$ (iii) $a \neq \pm 4$

4. Yes, Infinite

5.

6. a. 3, No b. No, No c. No, No d.No, No

7. One possibility for $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$. For $\mathbf{b}$, take any vector that is not a linear combination of the columns of A.

8. No

9. All values of h.

10. $\begin{bmatrix} 13 \\ 7, \end{bmatrix}$, $\begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$

11. a. $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, b. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\pi/2$ radians

End