worksheet_19

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1 Worksheet 19

Name: **Bowen Li**
UID: **U79057147**

1.0.1 Topics

- Linear Model Evaluation

1.1 Linear Model Evaluation

Notice that R^2 only increases with the number of explanatory variables used. Hence the need for an adjusted R^2 that penalizes for insignificant explanatory variables.

[1]:
```
import numpy as np
from sklearn.linear_model import LinearRegression
from sklearn.preprocessing import PolynomialFeatures

SAMPLE_SIZE = 100
beta = [1, 5]
X = -10.0 + 10.0 * np.random.random(SAMPLE_SIZE)
Y = beta[0] + beta[1] * X + np.random.randn(SAMPLE_SIZE)

for i in range(1, 15):
    X_transform = PolynomialFeatures(degree=i, include_bias=False).
 ↪fit_transform(X.reshape(-1, 1))
    model = LinearRegression()
    model.fit(X_transform, Y)
    print(model.score(X_transform, Y))
```

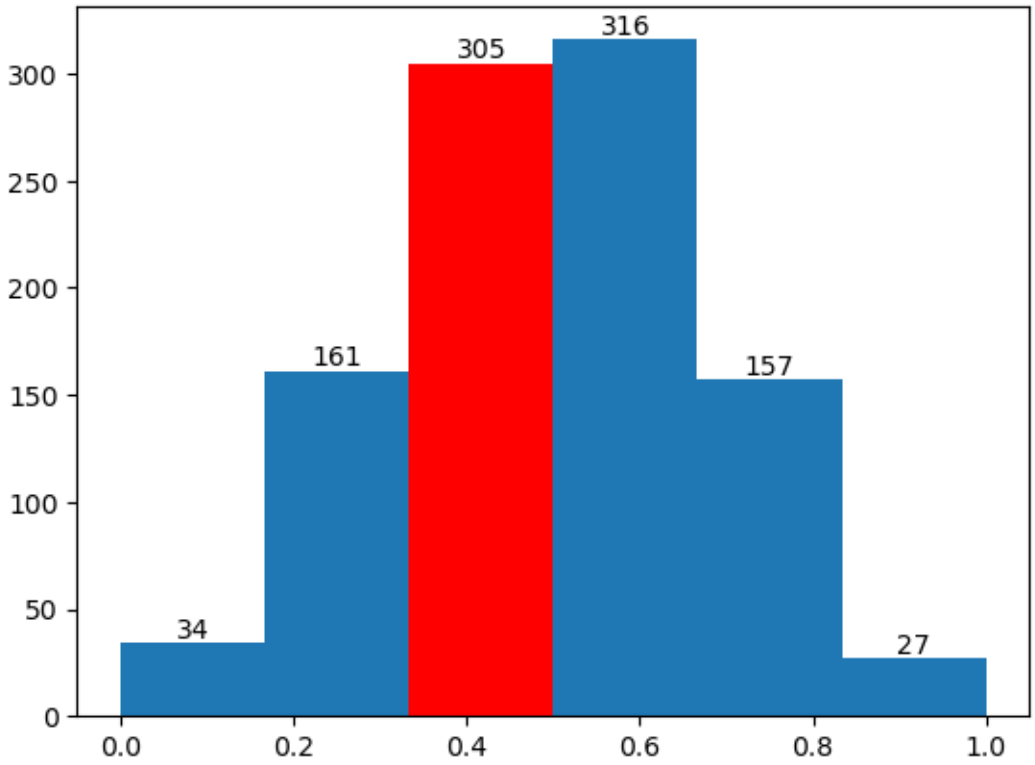
```
0.9948933904690602
0.9949064870035715
0.9949068871956697
0.994915349058596
0.9950529424049105
0.9950884780793511
0.9951055411402513
0.9951092562910379
0.995110428151002
```

```
0.9951619575414153
0.9951635360779449
0.9951713581394618
0.9951832008963568
0.9951849098624868
```

a) Hypothesis Testing Sandbox (follow along in class) Notes

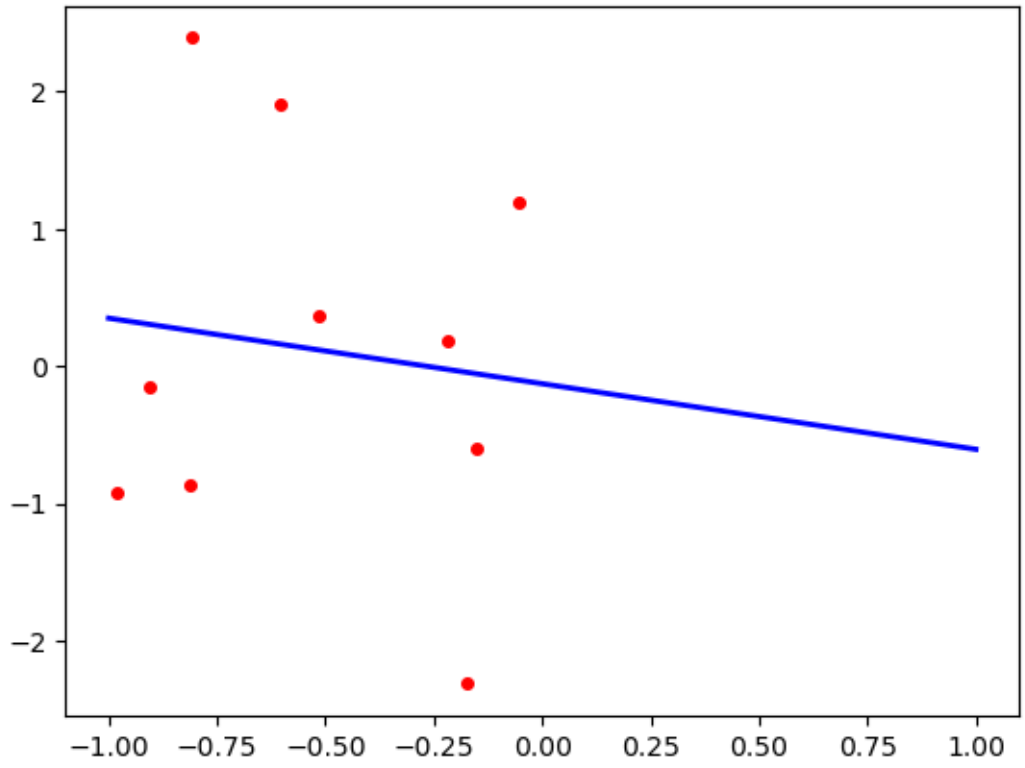
[2]:
```python
import numpy as np
from scipy.stats import binom
import matplotlib.pyplot as plt

flips = [1, 0, 0, 1, 0]

def num_successes(flips):
    return sum(flips)

print(binom.pmf(num_successes(flips), len(flips), 1/2))

SAMPLE_SIZE = 5
flips = [np.random.choice([0, 1]) for _ in range(SAMPLE_SIZE)]
print(flips)
print(binom.pmf(num_successes(flips), SAMPLE_SIZE, 1/2))

p_est = []

for _ in range(1000):
    flips = [np.random.choice([0, 1]) for _ in range(SAMPLE_SIZE)]
    p_est.append(sum(flips) / SAMPLE_SIZE)

fig, ax = plt.subplots()
_, bins, patches = ax.hist(p_est, bins=SAMPLE_SIZE + 1)
p = np.digitize([2/5], bins)
patches[p[0]-1].set_facecolor('r')
ax.bar_label(patches)
plt.show()
```

```
0.31249999999999983
[0, 0, 1, 0, 1]
0.31249999999999983
```



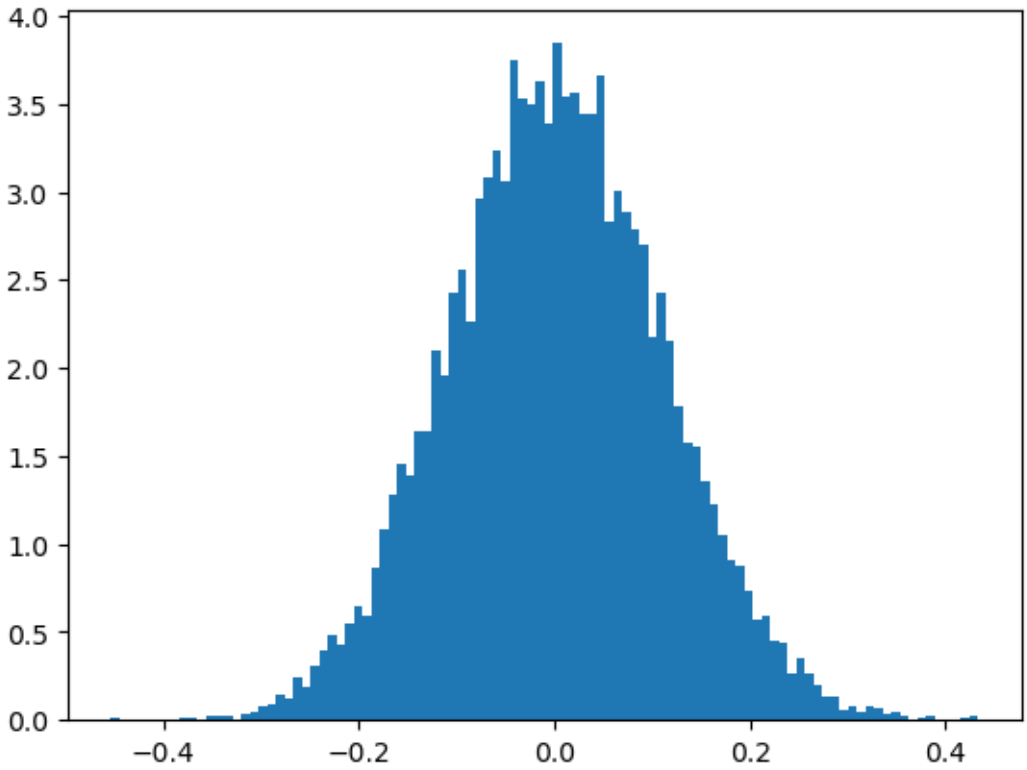
b) Plot a data set and fitted line through the point when there is no relationship between X and y.

[9]:
```
import numpy as np
import matplotlib.pyplot as plt

SAMPLE_SIZE = 10

xlin = -1.0 + 1.0 * np.random.random(SAMPLE_SIZE)
y = np.random.randn(SAMPLE_SIZE)

intercept = np.ones(np.shape(xlin)[0])
X = np.array([intercept, xlin]).T
beta = np.linalg.solve(X.T @ X, X.T @ y)

xplot = np.linspace(-1,1,20)
yestplot = beta[0] + beta[1] * xplot
plt.plot(xplot, yestplot,'b-',lw=2)
plt.plot(xlin, y,'ro',markersize=4)
plt.show()
```



c) Using the above code, plot a histogram of the parameter estimates for the slope after generating 1000 independent datasets. Comment on what the plot means. Increase the sample size to see what happens to the plot. Explain.

[4]:
```
SAMPLE_SIZE = 1000
NUM_TRIALS = 10000

beta_hist = []
for _ in range(NUM_TRIALS):
    xlin = -1.0 + 1.0 * np.random.random(SAMPLE_SIZE)
    y = np.random.randn(SAMPLE_SIZE)

    intercept = np.ones(np.shape(xlin)[0])
    X = np.array([intercept, xlin]).T
    beta = np.linalg.solve(X.T @ X, X.T @ y)
    beta_hist.append(beta[1])

fig, ax = plt.subplots()
ax.hist(beta_hist, bins=100, density=True)
plt.show()
```



The plot shows the distribution of estimated slopes for different independent datasets. As the number of samples increase, the estimated slopes become more concentrated around 0. This is because y was generated to have no relation to x, so the expected slope we get should be 0. As we get more samples, the law of large numbers implies that the estimates would get closer to the expected value of the slope which is 0.

d) We know that:

$$\hat{\beta} - \beta \sim \mathcal{N}(0, \sigma^2 (X^T X)^{-1})$$

thus for each component k of $\hat{\beta}$ (here there are only two - one slope and one intercept)

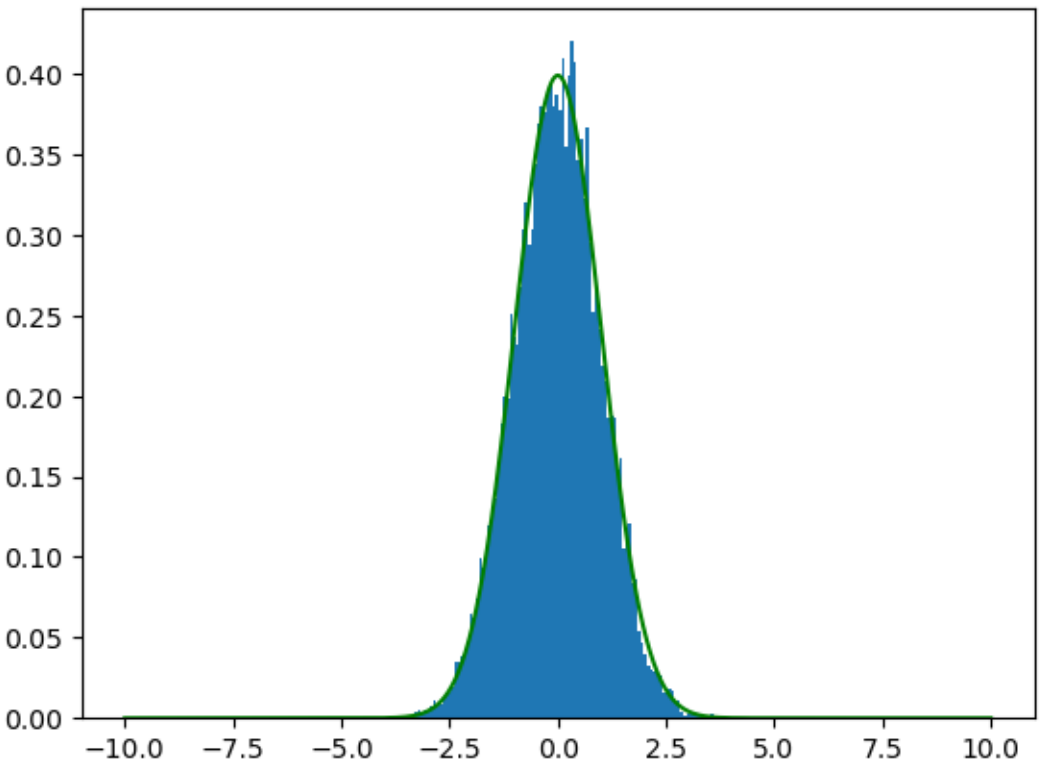
$$\hat{\beta}_k - \beta_k \sim \mathcal{N}(0, \sigma^2 S_{kk})$$

where S_{kk} is the k^{th} diagonal element of $(X^T X)^{-1}$. Thus, we know that

$$z_k = \frac{\hat{\beta}_k - \beta_k}{\sqrt{\sigma^2 S_{kk}}} \sim \mathcal{N}(0, 1)$$

Verify that this is the case through a simulation and compare it to the standard normal pdf by plotting it on top of the histogram.

[5]:
```
from scipy.stats import norm

SAMPLE_SIZE = 1000
NUM_TRIALS = 10000

beta_hist = []
for _ in range(NUM_TRIALS):
    xlin = -1.0 + 1.0 * np.random.random(SAMPLE_SIZE)
    y = np.random.randn(SAMPLE_SIZE)

    intercept = np.ones(np.shape(xlin)[0])
    X = np.array([intercept, xlin]).T
    inv_cov = np.linalg.inv(X.T @ X)
    beta = inv_cov @ X.T @ y

    beta_hist.append(beta[1] / np.sqrt(inv_cov[1,1]))

xs = np.linspace(-10,10,1000)
fig, ax = plt.subplots()
ax.hist(beta_hist, bins=100, density=True)
ax.plot(xs, norm.pdf(xs), color='green')
plt.show()
```

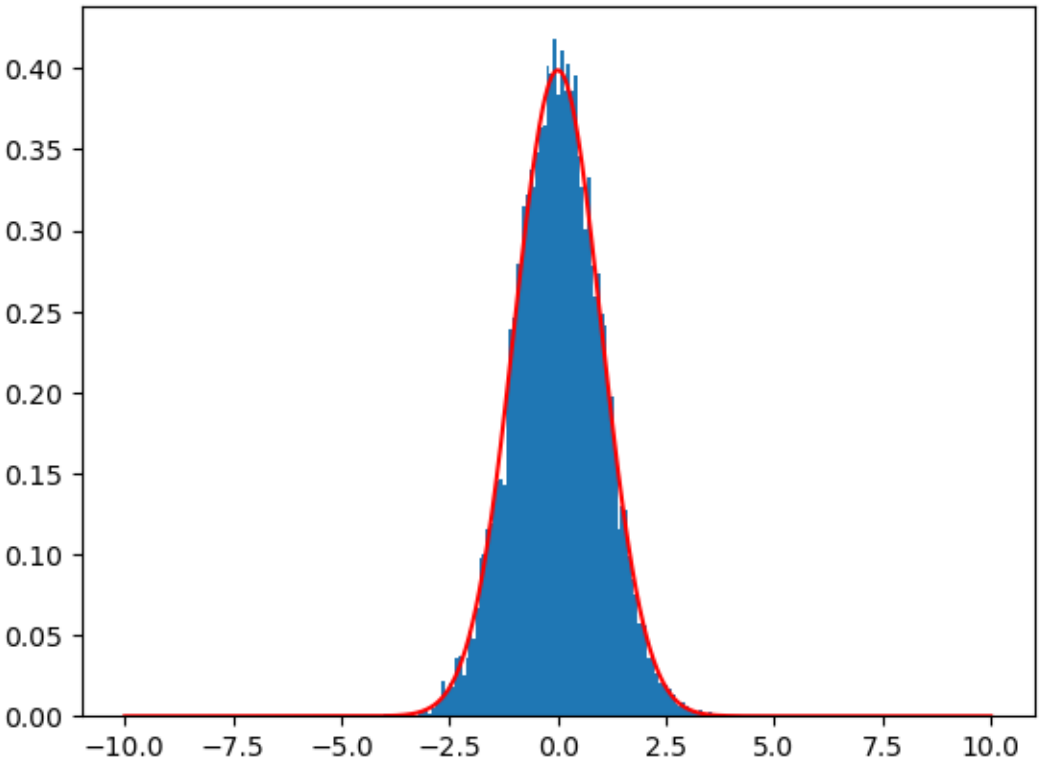


e) Above we normalized $\hat{\beta}$ by subtracting the mean and dividing by the standard deviation. While we know that the estimate of beta is an unbiased estimator, we don't know the standard deviation. So in practice when doing a hypothesis test where we want to assume that $\beta = 0$, we can simply use $\hat{\beta}$ in the numerator. However we don't know the standard deviation and need to use an unbiased estimate of the standard deviation instead. This estimate is the standard error `s`

$$s = \sqrt{\frac{RSS}{n-p}}$$

where p is the number of parameters beta (here there are 2 - one slope and one intercept). This normalized $\hat{\beta}$ can be shown to follow a t-distribution with `n-p` degrees of freedom. Verify this is the case with a simulation.

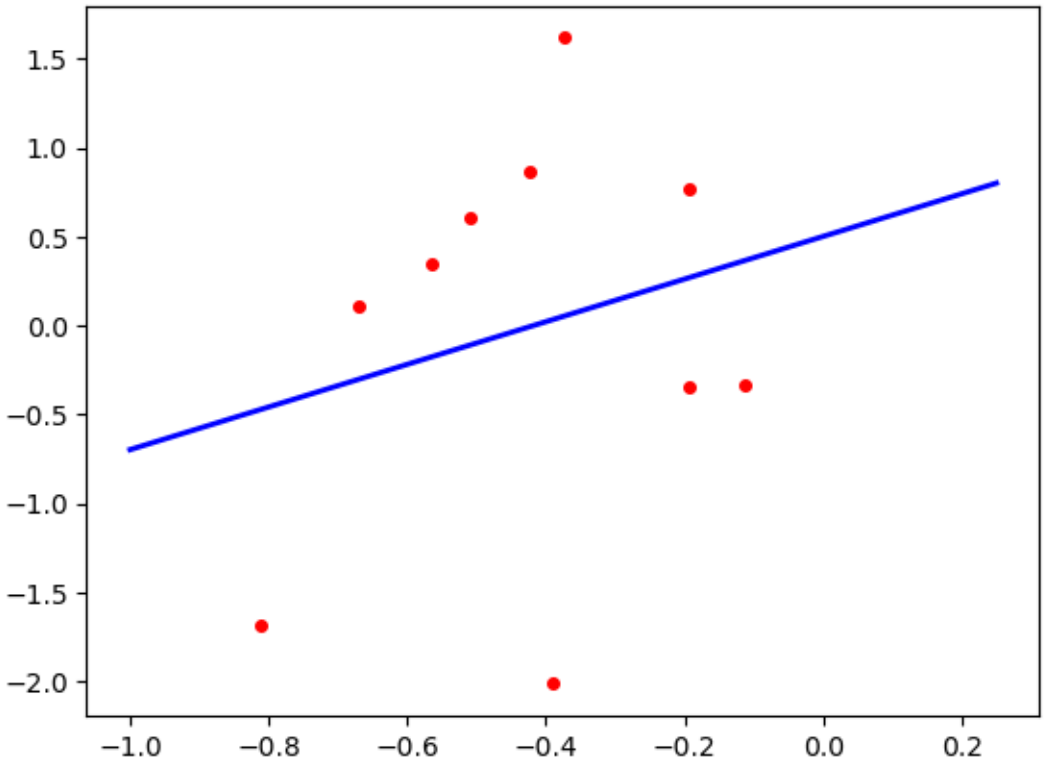
[6]:
```
from scipy.stats import t

SAMPLE_SIZE = 1000
NUM_TRIALS = 10000

def standard_error(ytrue, ypred):
    diff = ytrue - ypred
    rss = diff.T @ diff
    return np.sqrt(rss / (ytrue.shape[0] - 2))

beta_hist = []
for _ in range(NUM_TRIALS):
    xlin = -1.0 + 1.0 * np.random.random(SAMPLE_SIZE)
    y = np.random.randn(SAMPLE_SIZE) * 4

    intercept = np.ones(np.shape(xlin)[0])
    X = np.array([intercept, xlin]).T
    inv_cov = np.linalg.inv(X.T @ X)
    beta = inv_cov @ X.T @ y

    ypred = beta[0] + beta[1] * xlin
    s = standard_error(y, ypred)

    beta_hist.append(beta[1] / np.sqrt(s * s * inv_cov[1,1]))

xs = np.linspace(-10,10,1000)
fig, ax = plt.subplots()
ax.hist(beta_hist, bins=100, density=True)
ax.plot(xs, t.pdf(xs, SAMPLE_SIZE - 2), color='red')
plt.show()
```



f) You are given the following dataset:

[7]:
```
import numpy as np
import matplotlib.pyplot as plt

x = np.array([-0.1920605, -0.11290798, -0.56434374, -0.67052057, -0.19233284,␣
 ↪-0.42403586, -0.8114285, -0.38986946, -0.37384161, -0.50930229])
y = np.array([-0.34063108, -0.33409286, 0.34245857, 0.11062295, 0.76682389, 0.
 ↪86592388, -1.68912015, -2.01463592, 1.61798563, 0.60557414])

intercept = np.ones(np.shape(x)[0])
X = np.array([intercept, x]).T
beta_hat = np.linalg.inv(X.T @ X) @ X.T @ y

print(beta_hat)

xplot = np.linspace(-1,.25,20)
yestplot = beta_hat[0] + beta_hat[1] * xplot
plt.plot(xplot, yestplot,'b-',lw=2)
plt.plot(x, y,'ro',markersize=4)
plt.show()
```

```
[0.50155603 1.19902827]
```



what is the probability of observing a dataset at least as extreme as the above assuming $\beta = 0$?

[8]:
```
slope_pred = beta_hat[1]
y_pred = X @ beta_hat
s = standard_error(y, y_pred)
slope_score = slope_pred / (np.sqrt(s * s * np.linalg.inv(X.T @ X)[1,1]))
single_tail = 1 - t.cdf(slope_score, x.shape[0] - 2)
p_val = 2 * single_tail
p_val
```

[8]:
```
0.5131420720905755
```

The probability of seeing a dataset at least as extreme as the above is about 51%.