Robust principal component analysis? Some theory and some applications

Emmanuel Candès

Colloque inaugural Chaire Schlumberger, IHES, November 2010

First things first

Congratulations to George & Josselin!

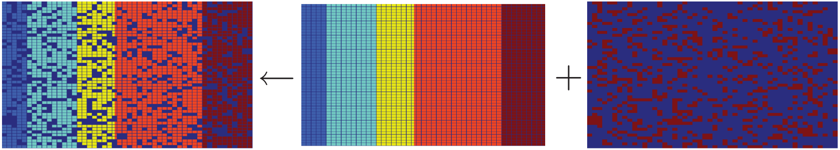
Collaborators

- Xiaodong Li (Stanford)
- Yi Ma (Microsoft Research Asia & UIUC)
- John Wright (Microsoft Research Asia)

Agenda

- A separation problem
- Computer vision applications

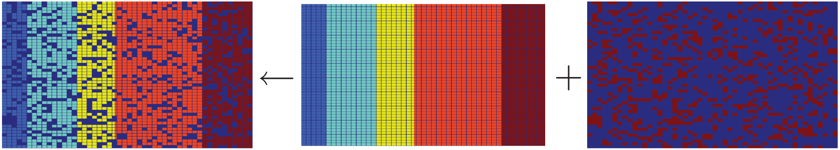
The separation problem



$$M = L_0 + S_0$$

- M: data matrix (observed)
- L_0: low-rank (unobserved)
- S_0: sparse (unobserved)

The separation problem



$$M = L_0 + S_0$$

- M: data matrix (observed)
- L_0: low-rank (unobserved)
- S_0: sparse (unobserved)

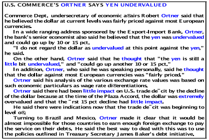
Problem: can we recover L_0 and S_0 accurately?

Seems daunting but solution would be really great!

Getting it right

Images

Videos



U.S. COMMERCE'S ORTNER SAYS YEN UNDERVALUED

Commerce Dept. undersecretary of economic a®airs Robert Ortner said that he believed the dollar at current levels was fairly priced against most European currencies.

In a wide ranging address sponsored by the Export-Import Bank, Ortner, the bank's senior economist also said he believed that the yen was undervalued and could go up by 10 or 15 pct.

"I do not regard the dollar as undervalued at this point against the yen," he said.

On the other hand, Ortner said that he thought that "the yen is still a little bit undervalued," and "could go up another 10 or 15 pct."

In addition, Ortner, who said he was speaking personally, said he thought that the dollar against most European currencies was "fairly priced."

Ortner said his analysis of the various exchange rate values was based on such economic particulars as wage rate di®erentiations.

Ortner said there had been little impact on U.S. trade de¯cit by the decline of the dollar because at the time of the Plaza Accord, the dollar was extremely overvalued and that the ¯rst 15 pct decline had little impact.

He said there were indications now that the trade de¯cit was beginning to level o®.

Turning to Brazil and Mexico, Ortner made it clear that it would be almost impossible for those countries to earn enough foreign exchange to pay the service on their debts. He said the best way to deal with this was to use the policies outlined in Treasury Secretary James Baker's debt initiative.

Text

Web data

How do we develop provably correct and efficient algorithms for recovery of low-dimensional linear structure from non-ideal observations?

Motivation

Classical PCA

$$M = L_0 + N_0$$

- L_0: low-rank (unobserved)
- N_0: (small) perturbation

Classical PCA

$$M = L_0 + N_0$$

- L_0: low-rank (unobserved)
- N_0: (small) perturbation

Dimensionality reduction (Schmidt 1907, Hotelling 1933)

$$\begin{array}{ll} \text{minimize} & \|M - L\| \\ \text{subject to} & \text{rank}(L) \le k \end{array}$$

Classical PCA

$$M = L_0 + N_0$$

- L_0: low-rank (unobserved)
- N_0: (small) perturbation

Dimensionality reduction (Schmidt 1907, Hotelling 1933)

$$\begin{array}{ll} \text{minimize} & \|M - L\| \\ \text{subject to} & \text{rank}(L) \le k \end{array}$$

Solution given by truncated SVD

$$M = U \Sigma V^* = \sum_i \sigma_i u_i v_i^* \quad \Rightarrow \quad L = \sum_{i \le k} \sigma_i u_i v_i^*$$

Classical PCA

$$M = L_0 + N_0$$

- L_0: low-rank (unobserved)
- N_0: (small) perturbation

Dimensionality reduction (Schmidt 1907, Hotelling 1933)

$$\begin{array}{ll} \text{minimize} & \|M - L\| \\ \text{subject to} & \text{rank}(L) \le k \end{array}$$

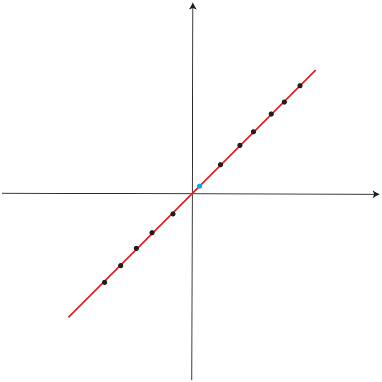
Solution given by truncated SVD

$$M = U \Sigma V^* = \sum_i \sigma_i u_i v_i^* \quad \Rightarrow \quad L = \sum_{i \le k} \sigma_i u_i v_i^*$$

Fundamental statistical tool: enormous impact

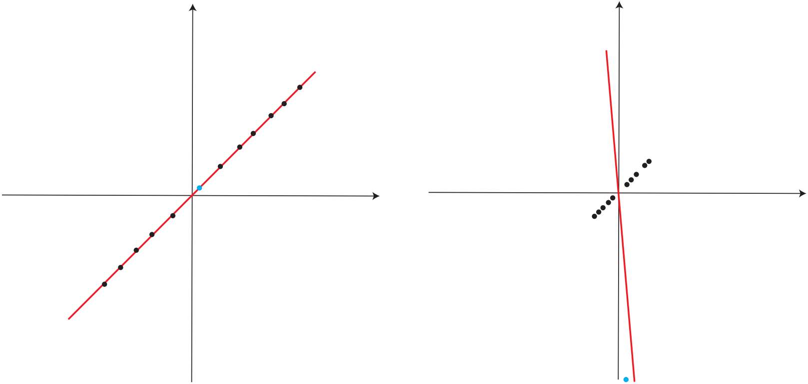
PCA and corruptions/outliers

PCA: very sensitive to outliers



PCA and corruptions/outliers

PCA: very sensitive to outliers



Breaks down with one (badly) corrupted data point

Robust PCA

Gross errors frequently occur in many applications

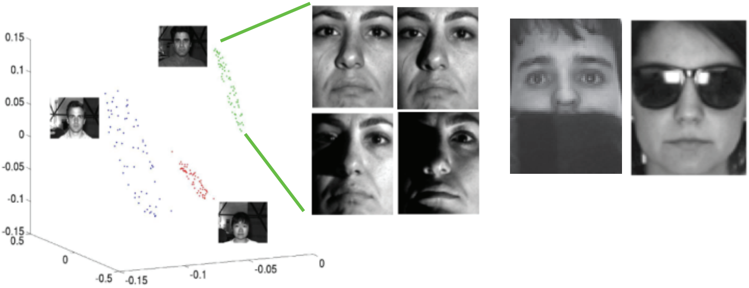
- Image processing
- Web data analysis
- Bioinformatics
- ...

- Occlusions
- Malicious tampering
- Sensor failures
- ...

Important to make PCA robust

- Influence function techniques: Huber; De La Torre and Black
- Multivariate trimming: Gnanadesikan and Kettenring
- Alternating minimization: Ke and Kanade
- Random sampling techniques: Fischler and Bolles
- ...

Occlusions in computer vision



An interesting separation problem

Recover low-rank L_0 and sparse S_0 from

$$M = L_0 + S_0$$

Many applications other than robust PCA: informative component may be

- L_0 (RPCA)
- S_0 (examples to follow)

Video surveillance

Sequence of video frames with a static background

Problem: detect any activity in the foreground

Video surveillance

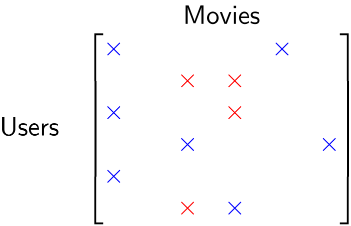
Sequence of video frames with a static background

Problem: detect any activity in the foreground

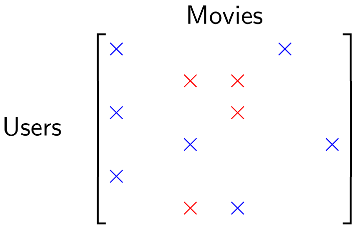
$M = L_0 + S_0$

This is a separation problem!

Ranking and collaborative filtering



Ranking and collaborative filtering



$$M = L_0 + S_0$$

- Available data $M_{ij} : (i,j) \in \Omega_{\text{obs}}$
- L_0 : all users' ratings (what we would like to know)
- S_0 : ratings that have been tampered with

Other applications

- Face recognition
- System identification
- Quantum-state tomography (Gross)
- Graphical modeling with latent variables (Chandrasekaran, Parrilo, Willsky)

Theoretical aspects

Principal Component Pursuit (PCP)

$$M = L_0 + S_0$$

- L_0 unknown (rank unknown)
- S_0 unknown ($\#$ of entries $\neq 0$, locations, magnitudes all unknown)

Principal Component Pursuit (PCP)

$$M = L_0 + S_0$$

- L_0 unknown (rank unknown)
- S_0 unknown ($\#$ of entries $\neq 0$, locations, magnitudes all unknown)

Recovery via (convex) PCP

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L + S = M \end{array}$$

See also Chandrasekaran, Sanghavi, Parrilo, Willsky ('09)

- nuclear norm: $\|L\|_* = \sum_i \sigma_i(L)$ (sum of sing. values)
- ℓ_1 norm: $\|S\|_1 = \sum_{ij} |S_{ij}|$ (sum of abs. values)

Principal Component Pursuit (PCP)

$$M = L_0 + S_0$$

- L_0 unknown (rank unknown)
- S_0 unknown ($\#$ of entries $\neq 0$, locations, magnitudes all unknown)

Recovery via (convex) PCP

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L + S = M \end{array}$$

See also Chandrasekaran, Sanghavi, Parrilo, Willsky ('09)

- nuclear norm: $\|L\|_* = \sum_i \sigma_i(L)$ (sum of sing. values)
- ℓ_1 norm: $\|S\|_1 = \sum_{ij} |S_{ij}|$ (sum of abs. values)

- Nuclear norm heuristics introduced in 90's
- ℓ_1 norm heuristics introduced in 50's

Surprise

$$M = L_0 + S_0$$

- L_0 unknown
- S_0 unknown

Recovery via

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L + S = M \end{array}$$

Surprise

$$M = L_0 + S_0$$

- L_0 unknown
- S_0 unknown

Recovery via

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L + S = M \end{array}$$

Under broad conditions, solution $(\hat{L}, \hat{S})$ obeys

$$\hat{L} = L_0, \quad \hat{S} = S_0!$$

When does separation make sense?

M cannot be low-rank and sparse

$$\boldsymbol{M} = \boldsymbol{e}_1 \boldsymbol{e}_n^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Low-rank component cannot be sparse

$$L_0 \in \mathbb{R}^{n \times n} = U\Sigma V^* = \sum_{1 \le i \le r} \sigma_i u_i v_i^* \quad r = \mathsf{rank}(L_0)$$

Coherence condition (C. and Recht, '08): $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$

$$\|U^* e_i\|^2 \le \frac{\mu r}{n} \quad \|V^* e_i\|^2 \le \frac{\mu r}{n}$$

and

$$|UV^*|_{ij}^2 \le \frac{\mu r}{n^2}$$

Roughly: singular vectors (PC's) are not sparse/spiky

What if the sparse component has low-rank?

Example: first column of S_0 is that of L_0

$$S_0 = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad M_0 = L_0 - S_0 = \begin{bmatrix} 0 & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & * & \cdots & * & * \end{bmatrix}$$

What if the sparse component has low-rank?

Example: first column of S_0 is that of L_0

$$S_0 = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad M_0 = L_0 - S_0 = \begin{bmatrix} 0 & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & * & \cdots & * & * \end{bmatrix}$$

Sparsity pattern will be assumed (uniform) random

Main result: $M = L_0 + S_0$

Theorem

- L_0 *is* $n \times n$ *of rank*$(L_0) \le \rho_r n\, \mu^{-1} (\log n)^{-2}$
- S_0 *is* $n \times n$*, random sparsity pattern of cardinality* $m \le \rho_s n^2$

Then with probability $1 - O(n^{-10})$*, PCP with* $\lambda = 1/\sqrt{n}$ *is exact:*

$$\hat{L} = L_0, \quad \hat{S} = S_0$$

Same conclusion for rectangular matrices with $\lambda = 1/\sqrt{\text{max dim}}$

- Exact
  - whatever the magnitudes of L_0!
  - whatever the magnitudes of S_0!
- No tuning parameter!

Can achieve stronger probabilities of success, e. g. $1 - O(n^{-\beta})$, $\beta > 0$

Connections with matrix completion (MC)

Recover a (low-rank) matrix from a subset of its entries

- C. and Recht ('08)
- C. and Tao ('09)
- Keshavan, Montanari and Oh ('09)
- Mazumder, Hastie and Tibshirani ('09)
- Different problem: Recht, Fazel and Parrilo ('07)
- Many others

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

Connections with matrix completion (MC)

$$\begin{array}{ll} \text{minimize} & \|L\|_* \\ \text{subject to} & L_{ij} = L^0_{ij} \ (i,j) \in \Omega_{\mathsf{obs}} \end{array}$$

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

Connections with matrix completion (MC)

$$\begin{array}{ll} \text{minimize} & \|L\|_* \\ \text{subject to} & L_{ij} = L^0_{ij} \ (i,j) \in \Omega_{\text{obs}} \end{array}$$

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

Theorem (C. and Tao '09 improving C. and Recht '08)

- *rank*$(L_0) = r$ *and* L_0 *as before*
- Ω_{obs} *random set of size* m

Solution to SDP is exact with probability at least $1 - n^{-10}$ *if*

$$m \gtrsim \mu\, nr \log^a n \quad a \le 6$$

Gross' near-optimal improvement

$$m \gtrsim \mu\, nr \log^2 n$$

Connections with matrix completion (MC)

Missing vs. corrupted data

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

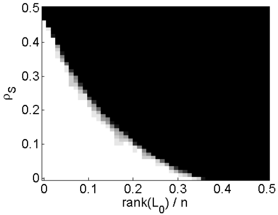
MC: missing

$$\begin{bmatrix} \times & \skull & \skull & \skull & \times & \skull \\ \skull & \skull & \times & \times & \skull & \skull \\ \times & \skull & \skull & \times & \skull & \skull \\ \skull & \skull & \times & \skull & \skull & \times \\ \times & \skull & \skull & \skull & \skull & \skull \\ \skull & \skull & \times & \times & \skull & \skull \end{bmatrix}$$

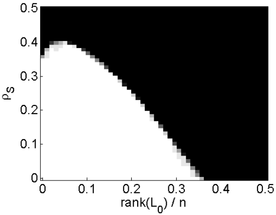
RPCA: corrupted

<u>Harder</u> to detect and correct than to fill in

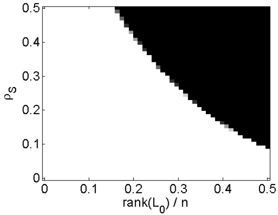
Phase transitions in probability of success



(a) PCP, Random Signs



(b) PCP, Coherent Signs



(c) Matrix Completion

$L_0 = XY^*$ is a product of independent $n \times r$ i.i.d. $\mathcal{N}(0, 1/n)$ matrices

Contemporary result: Chandrasekaran, Sanghavi, Parrilo, Willsky ('09)

Deterministic conditions for PCP to succeed

- $T(L_0)$: span of all matrices with row space included in that of L_0 or with col. space included in that of L_0

$$\xi(L_0) = \sup_{N \in T(L_0) : \|N\| \leq 1} \|N\|_\infty$$

- $\Omega(S_0)$: span of all matrices with support included in that of S_0

$$\nu(S_0) = \sup_{N \in \Omega(S_0) : \|N\|_\infty \leq 1} \|N\|$$

Contemporary result: Chandrasekaran, Sanghavi, Parrilo, Willsky ('09)

Deterministic conditions for PCP to succeed

- $T(L_0)$: span of all matrices with row space included in that of L_0 or with col. space included in that of L_0

$$\xi(L_0) = \sup_{N \in T(L_0) : \|N\| \leq 1} \|N\|_\infty$$

- $\Omega(S_0)$: span of all matrices with support included in that of S_0

$$\nu(S_0) = \sup_{N \in \Omega(S_0) : \|N\|_\infty \leq 1} \|N\|$$

Then PCP succeeds *for some* λ if

$$\xi(L_0)\, \nu(S_0) \leq 1/6$$

Comparison for random sparsity patterns

Corollary: correct recovery if

$$\text{max number of corruptions per col.} \times \sqrt{\mu r/n} < 1/12$$

so fraction of corrupted entries must obey

$$\rho_s \leq \frac{1}{12}\sqrt{\frac{1}{\mu n r}}$$

Accommodate only vanishing fractions – even for rank-1 matrices

Comparison for random sparsity patterns

Corollary: correct recovery if

$$\text{max number of corruptions per col.} \times \sqrt{\mu r/n} < 1/12$$

so fraction of corrupted entries must obey

$$\rho_s \leq \frac{1}{12}\sqrt{\frac{1}{\mu n r}}$$

Accommodate only vanishing fractions – even for rank-1 matrices

Significant differences

- models, proofs: not much in common
- selection of λ

Matrix completion from grossly corrupted data

Entries may be both corrupted and missing

Matrix completion from grossly corrupted data

Entries may be both corrupted and missing

$$\text{(PCP)} \qquad \begin{array}{ll} \text{minimize} & \|L\|_* + \lambda\|S\|_1 \\ \text{subject to} & L_{ij} + S_{ij} = M_{ij},\ (i,j) \in \Omega_{\mathsf{obs}} \end{array}$$

Ω_{obs} locations of observed entries

Matrix completion from grossly corrupted data

Entries may be both corrupted and missing

$$\text{(PCP)} \quad \begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L_{ij} + S_{ij} = M_{ij}, \ (i,j) \in \Omega_{\text{obs}} \end{array}$$

Ω_{obs} locations of observed entries

Theorem

- *L_0 is $n \times n$ as before, rank$(L_0) \le \rho_r n \, \mu^{-1} (\log n)^{-2}$*
- *Ω_{obs} random set of size[a] $m = 0.1 n^2$*
- *each observed entry is corrupted with probability $\tau \le \tau_s$*

Then with probability $1 - O(n^{-10})$, PCP with $\lambda = 1/\sqrt{0.1 n}$ is exact:

$$\hat{L} = L_0$$

Same conclusion for rectangular matrices with $\lambda = 1/\sqrt{0.1 \textit{max dim}}$

[a] missing fraction is arbitrary

Simultaneous completion and correction!

A cute thing

If no corruption $\rightarrow$ MC problem

- MC: perfect recovery via

$$\begin{array}{ll} \text{minimize} & \|L\|_* \\ \text{subject to} & L_{ij} = L^0_{ij}, \ (i,j) \in \Omega_{\text{obs}} \end{array}$$

- PCP

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \frac{1}{\sqrt{n}}\|S\|_1 \\ \text{subject to} & L_{ij} + S_{ij} = L^0_{ij}, \ (i,j) \in \Omega_{\text{obs}} \end{array}$$

A cute thing

If no corruption $\rightarrow$ MC problem

- MC: perfect recovery via

$$\begin{array}{ll} \text{minimize} & \|L\|_* \\ \text{subject to} & L_{ij} = L^0_{ij}, \ (i,j) \in \Omega_{\text{obs}} \end{array}$$

- PCP

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \frac{1}{\sqrt{n}}\|S\|_1 \\ \text{subject to} & L_{ij} + S_{ij} = L^0_{ij}, \ (i,j) \in \Omega_{\text{obs}} \end{array}$$

Same answer! $\hat{S} = 0$

Methods of Proof

Find a dual variable certifying that (L_0, S_0) is solution to PCP

Existence is a deep question in probability theory

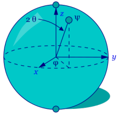
Methods of Proof

Find a dual variable certifying that (L_0, S_0) is solution to PCP

Existence is a deep question in probability theory

- Tools from Banach space theory (Rudelson's lemma, concentration of measure, noncommutative Khintchine inequality, ...)
- Arsenal of techniques developed for matrix completion (C. and Recht, 08)
- Important role played by Gross' golfing scheme ('09)

Quantum-state tomography



- k spin-$1/2$ system in an *unknown* quantum state $M \in \mathbb{C}^{n\times n}$ (density matrix)

$$n = 2^k, \quad \text{trace}(M) = 1, \quad M \succcurlyeq 0$$

- Quantum measurements (data)

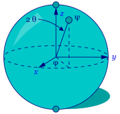
$$\mathbb{E}[\text{measurement with observable } A_j] = \langle A_j, M\rangle = \text{trace}(A_j^* M)$$

e.g. $\{A_j\}$: tensor Pauli matrices

Q? Can we reduce # measurements by using the structure of special classes of quantum states?

- pure state $\rightarrow$ rank$(M) = 1$
- interesting mixed states $\rightarrow$ (approx) low rank

Quantum-state tomography



- k spin-$1/2$ system in an *unknown* quantum state $M \in \mathbb{C}^{n \times n}$ (density matrix)

$$n = 2^k, \quad \text{trace}(M) = 1, \quad M \succcurlyeq 0$$

- Quantum measurements (data)

$$\mathbb{E}[\text{measurement with observable } A_j] = \langle A_j, M \rangle = \text{trace}(A_j^* M)$$

e.g. $\{A_j\}$: tensor Pauli matrices

Q? Can we reduce # measurements by using the structure of special classes of quantum states?

- pure state $\rightarrow$ rank$(M) = 1$
- interesting mixed states $\rightarrow$ (approx) low rank

A. Yes. Sample in proportion to the rank of the quantum state (Gross 09)

Computational aspects and simulations

Computational issues

Wish to solve the SDP

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L + S = M \end{array}$$

- Off-the-shelf algorithms (SDPT3, SeDuMi) need $n < 80, 100$
- Customized IPMs don't do much better

Have developed a simple and scalable algorithm via the Alternating Direction Method of Multipliers (ADMM)

Empirical performance II

n	$\text{rank}(L_0)$	$\|S_0\|_0$	$\text{rank}(\hat{L})$	$\|\hat{S}\|_0$	$\frac{\|\hat{L}-L_0\|_F}{\|L_0\|_F}$	# SVD	Time(s)
500	25	12,500	25	12,500	1.1×10^{-6}	16	2.9
1,000	50	50,000	50	50,000	1.2×10^{-6}	16	12.4
2,000	100	200,000	100	200,000	1.2×10^{-6}	16	61.8
3,000	250	450,000	250	450,000	2.3×10^{-6}	15	185.2

rank$(L_0) = 0.05 \times n$, $\|S_0\|_0 = 0.05 \times n^2$.

n	$\text{rank}(L_0)$	$\|S_0\|_0$	$\text{rank}(\hat{L})$	$\|\hat{S}\|_0$	$\frac{\|\hat{L}-L_0\|_F}{\|L_0\|_F}$	# SVD	Time(s)
500	25	25,000	25	25,000	1.2×10^{-6}	17	4.0
1,000	50	100,000	50	100,000	2.4×10^{-6}	16	13.7
2,000	100	400,000	100	400,000	2.4×10^{-6}	16	64.5
3,000	150	900,000	150	900,000	2.5×10^{-6}	16	191.0

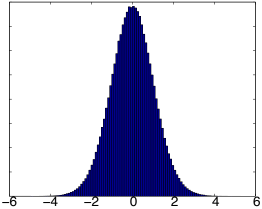
rank$(L_0) = 0.05 \times n$, $\|S_0\|_0 = 0.10 \times n^2$.

Computational cost higher than classical PCA but not by a large factor!

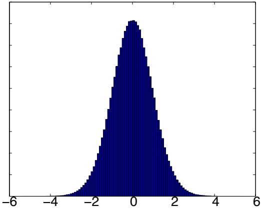
Empirical performance: Chiara's example

Rank-r matrix $L_0 = \frac{1}{\sqrt{r}} X_{n\times r} Y_{r\times n}$*:* X, Y independent $\mathcal{N}(0,1)$ entries

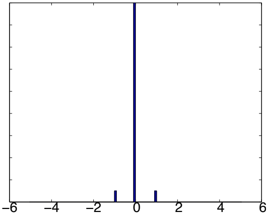
Sparse component S_0*:* random support + indep. symmetric ± 1 Bernoullis



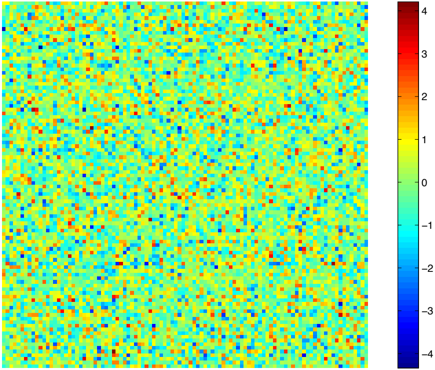
M



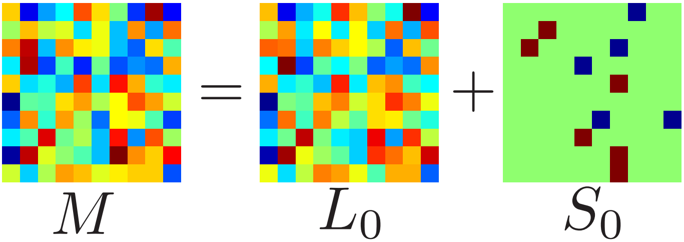
L_0

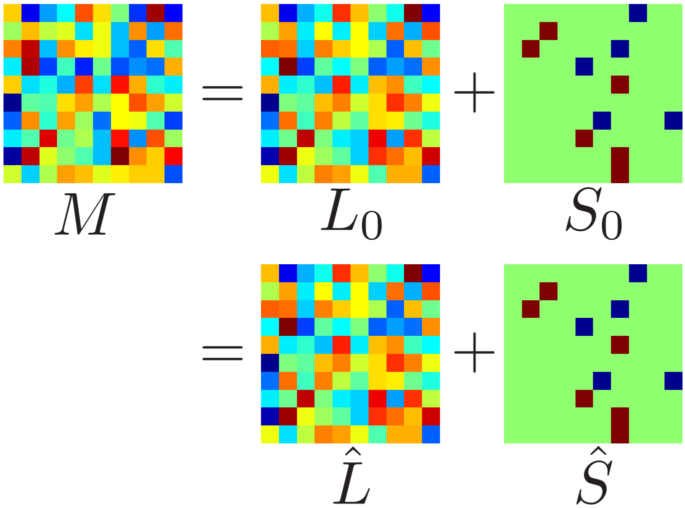


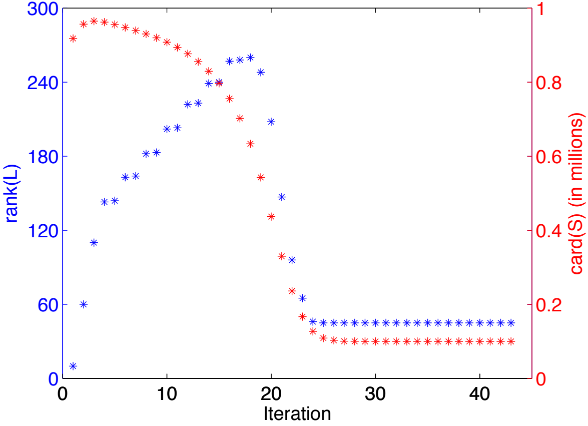
S_0



M


M
$=$
L_0
$+$
S_0


$M = L_0 + S_0$
$= \hat{L} + \hat{S}$



rank(L)
card(S) (in millions)
Iteration
300
240
180
120
60
0
1
0.8
0.6
0.4
0.2
0
10
20
30
40

Some applications

- Many applications
- Today, applications in computer vision

Application to video surveillance

Sequence of 200 video frames (144×172 pixels) with a static background

Problem: detect any activity in the foreground

Background modeling from surveillance video, I

(a) Original (b) Low-rank $\hat{L}$ (c) Sparse $\hat{S}$ (d) Low-rank $\hat{L}$ (e) Sparse $\hat{S}$

PCP Alternating minimization

Alternating minimization of an M-estimator (De La Torre and Black, '03)

Background modeling from surveillance video, II

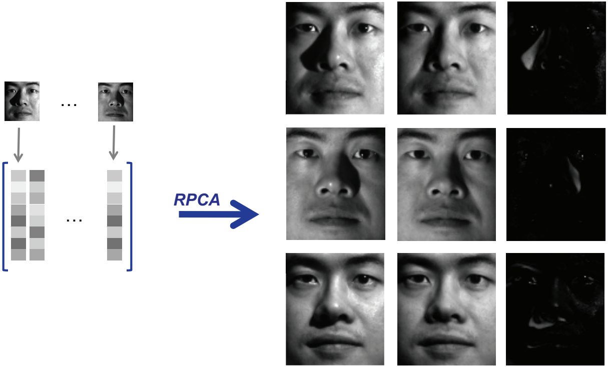
(a) Original (b) Low-rank $\hat{L}$ (c) Sparse $\hat{S}$ (d) Low-rank $\hat{L}$ (e) Sparse $\hat{S}$

PCP Alternating minimization

Three frames from a 250 frame sequence taken in a lobby, with varying illumination (Li et al., '04).

Removing shadows and specularities from face images

Sequence of 58 images (192×168) under different illumination conditions



Removing shadows and specularities from face images

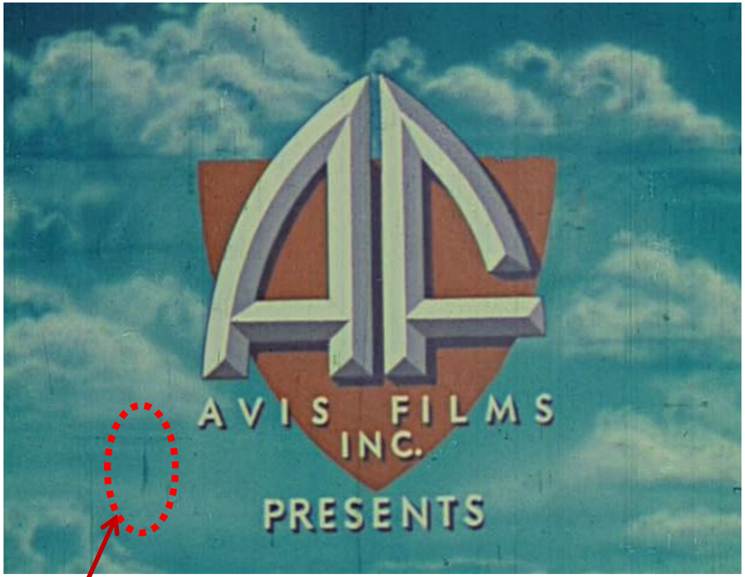
(a) M (b) $\hat{L}$ (c) $\hat{S}$

(a) M (b) $\hat{L}$ (c) $\hat{S}$

Corrections of specularities in the eyes, shadows, brightness saturation, ...

APPLICATIONS – *Repairing vintage movies*

Original D



Corruptions

Repaired A

Frame 1

480×620 pixels

APPLICATIONS – *Repairing vintage movies*

Original D

Corruptions

Repaired A

Frame 2

APPLICATIONS – *Repairing vintage movies*

Original D

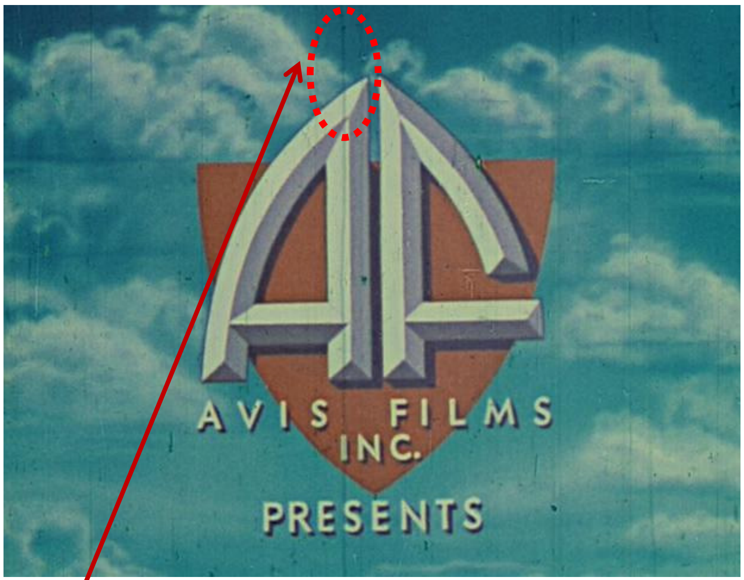
Corruptions

Repaired A

Frame 3

APPLICATIONS – *Repairing vintage movies*

Original D



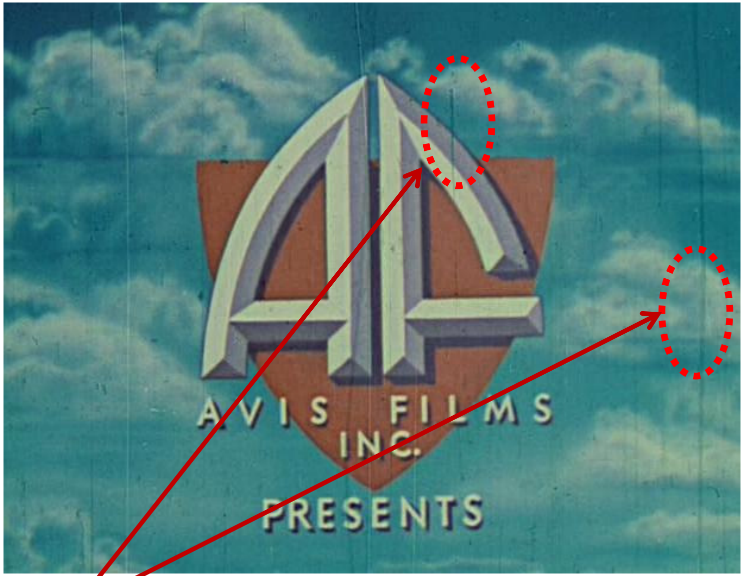
Corruptions

Repaired A

Frame 4

APPLICATIONS – *Repairing vintage movies*

Original D



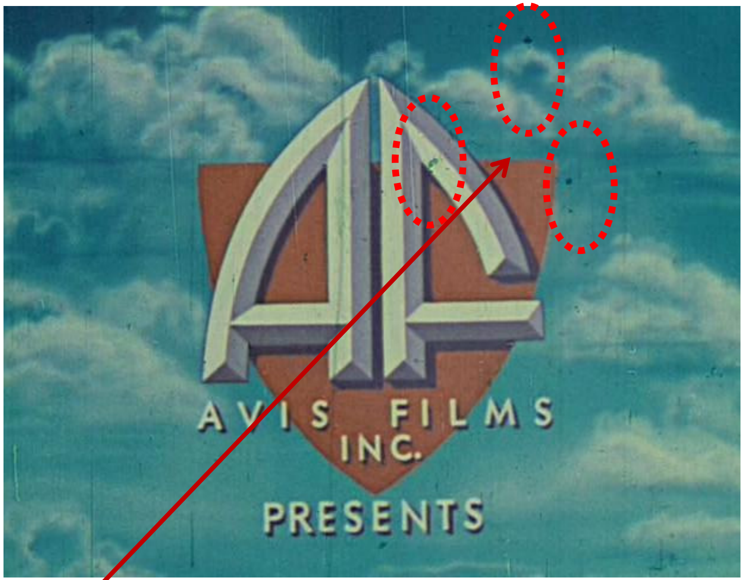
Repaired A

Corruptions

Frame 5

APPLICATIONS – *Repairing vintage movies*

Original D



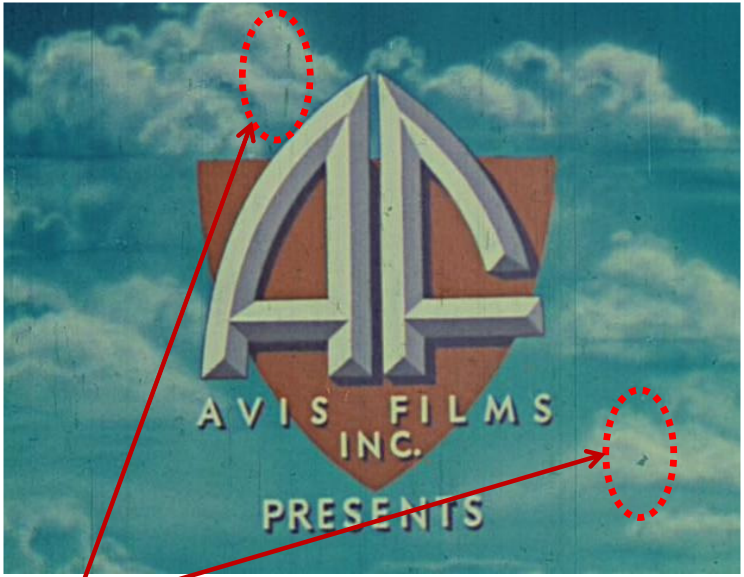
Repaired A

Corruptions

Frame 6

APPLICATIONS – *Repairing vintage movies*

Original D



Repaired A

Corruptions

Frame 7

Robust batch image alignment (Ma et al.)

- *Input*: $\boldsymbol{M}$ corrupted and misaligned batch of images (data)
- *Output*: $\boldsymbol{L}$ aligned low-rank images; $\boldsymbol{S}$ sparse errors

$$\text{(Model)} \qquad \boldsymbol{M} \circ \tau = \boldsymbol{L_0} + \boldsymbol{S_0}$$

τ: parametric deformation (rigid, affine, projective)

Robust batch image alignment (Ma et al.)

- *Input*: $\boldsymbol{M}$ corrupted and misaligned batch of images (data)
- *Output*: $\boldsymbol{L}$ aligned low-rank images; $\boldsymbol{S}$ sparse errors

$$\text{(Model)} \qquad \boldsymbol{M} \circ \tau = \boldsymbol{L_0} + \boldsymbol{S_0}$$

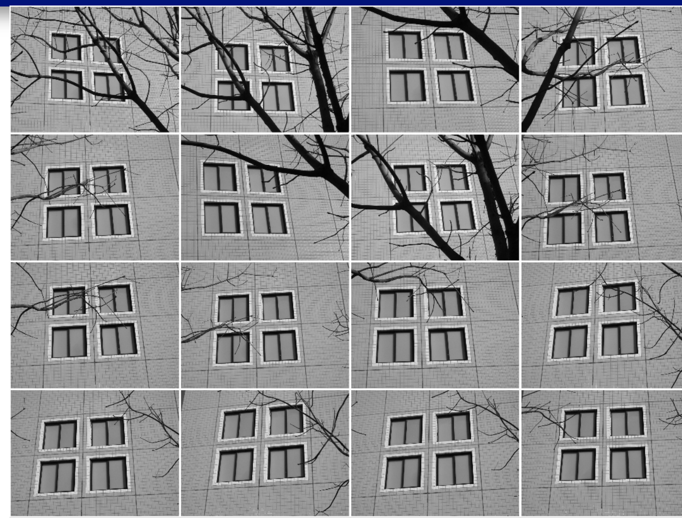
τ: parametric deformation (rigid, affine, projective)

Bootstrap: find $\boldsymbol{L}$ and $\boldsymbol{S}$ and τ solution to

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{L}\|_* + \lambda\|\boldsymbol{S}\|_1 \\ \text{subject to} & \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{M} \circ \tau \end{array}$$

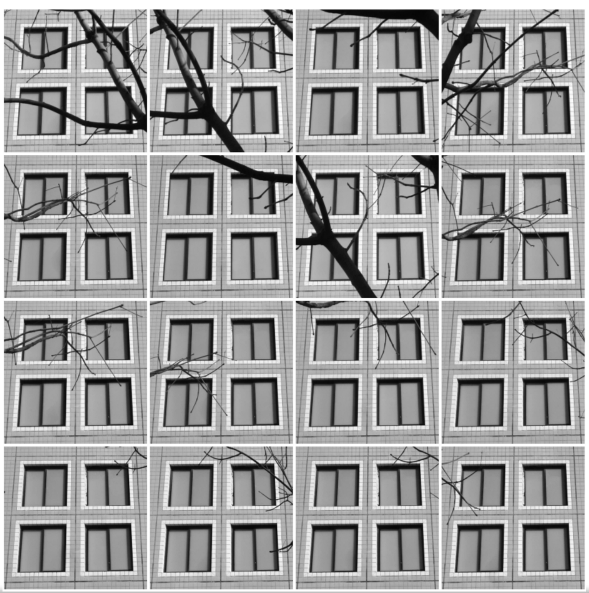
APPLICATIONS – *2D image matching and 3D modeling*

D

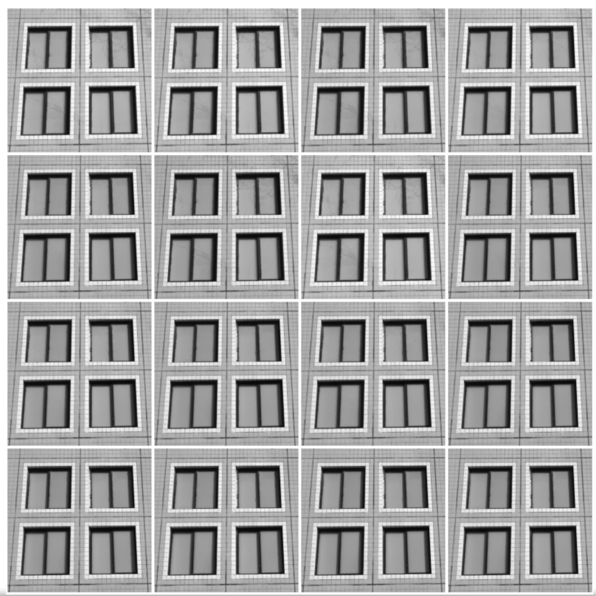


$\tau \in$ 2D homographies

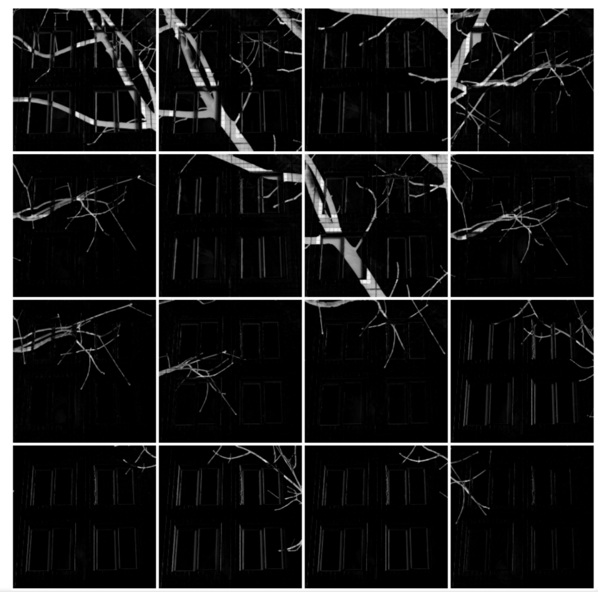
$D \circ \tau$



A



E



APPLICATIONS – *Video stabilization and enhancement*

Shaky video (D)
vs.
Aligned video ($D \circ \tau$)

Clean video (A)

Error video (E)

APPLICATIONS – *Aligning handwritten digits*

D

Learned-Miller PAMI'06

Vedaldi CVPR'08

$D \circ \tau$

A

E

Peng, Ganesh, Wright, Ma, submitted to CVPR'10.

APPLICATIONS – *Simultaneous Alignment and Repair*

D

$D \circ \tau$

A

E

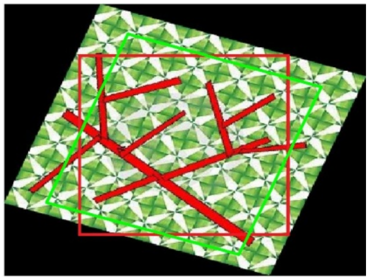
Peng, Ganesh, Wright, Ma, submitted to CVPR'10.

Transform Invariant Low-rank Textures (TILT)

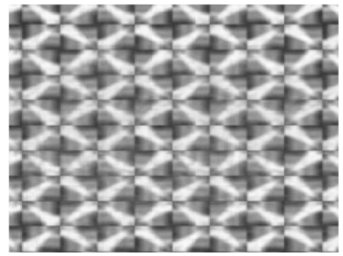
D– corrupted & deformed observation

A– rectified low-rank textures

E– sparse errors



$\circ\ \tau\ =$



$+$

Problem: Given $D \circ \tau\ =\ A_0 + E_0$, recover τ , A_0 and E_0.

Parametric deformations (affine, projective…)

Low-rank component

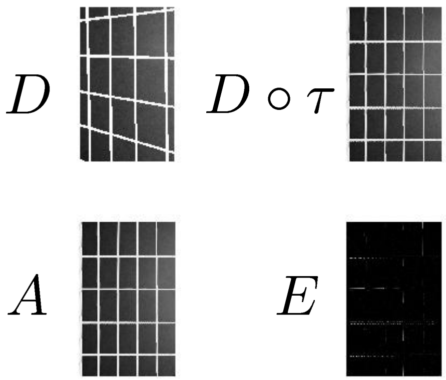
Sparse component

Solution: iteratively estimate the deformation and low-rank texture:

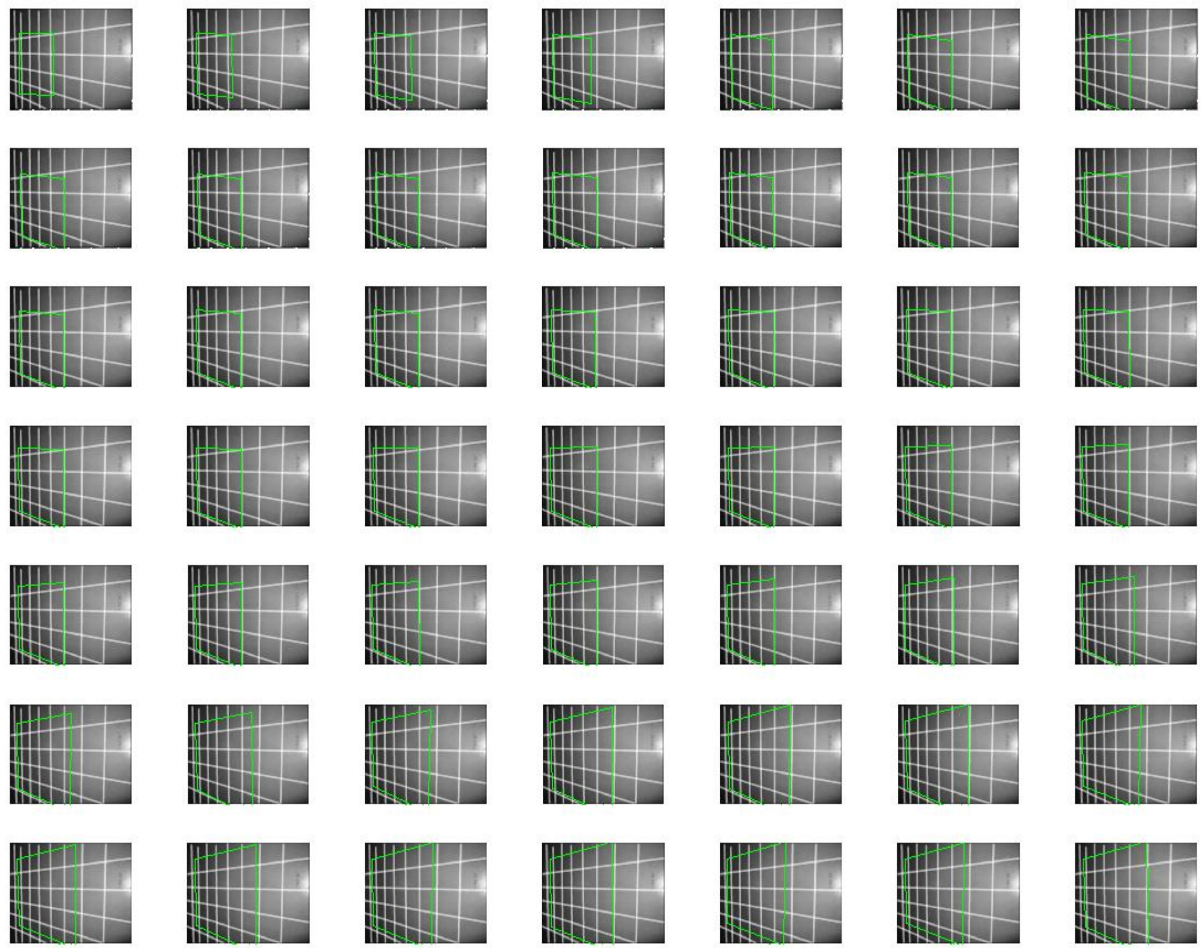
Iterate:

$$\min\ \|A\|_* + \lambda \|E\|_1 \quad \text{subj} \quad A + E = D \circ \tau_k + J \Delta\tau$$

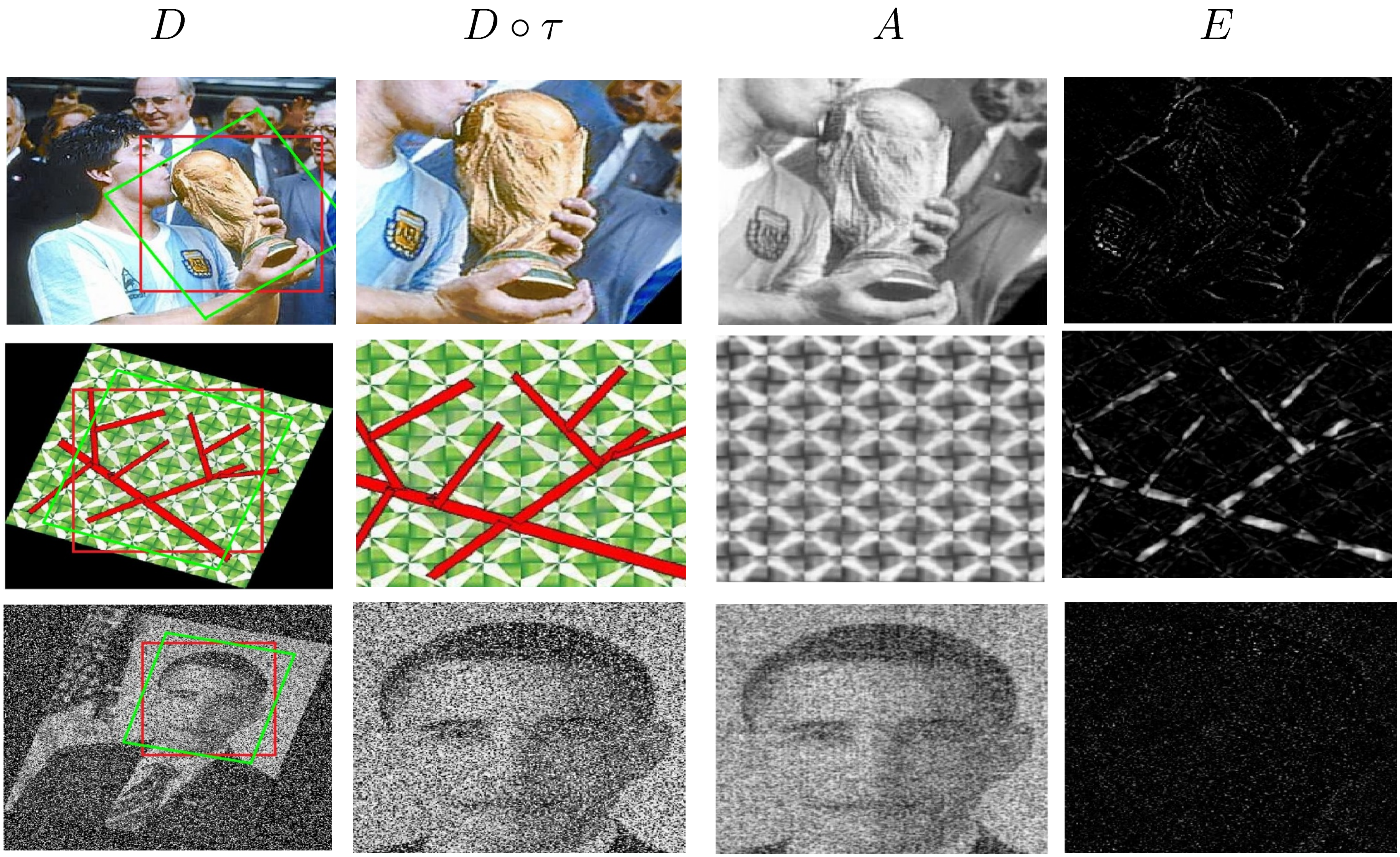
TILT via Iterative RPCA-Like Convex Optimization



Iteration Processes

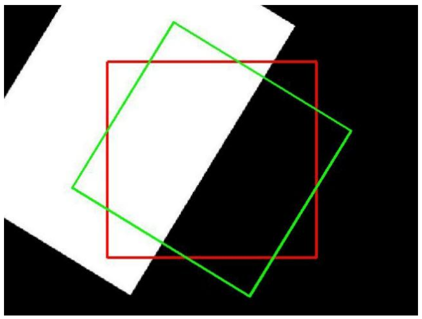


TILT – Robust to Background, Occlusion, and Corruption

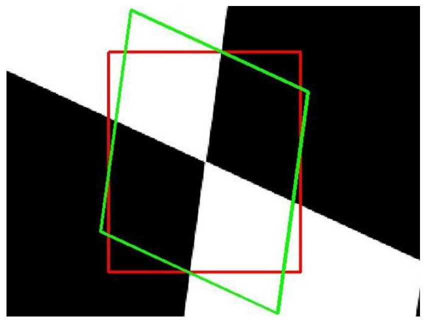


TILT: All Types of Regular Geometric Structures in Images

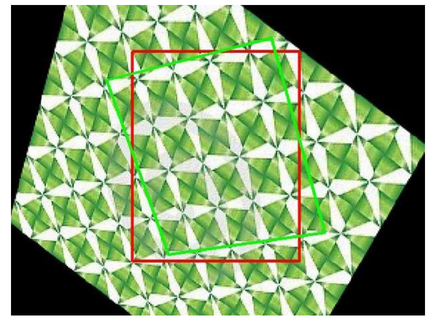
an edge



a corner

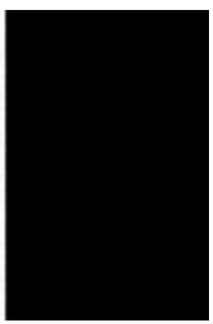


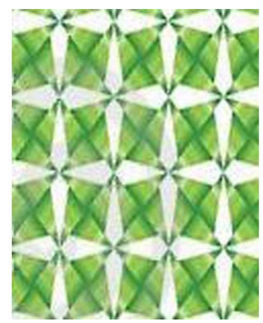
symmetry



regularity

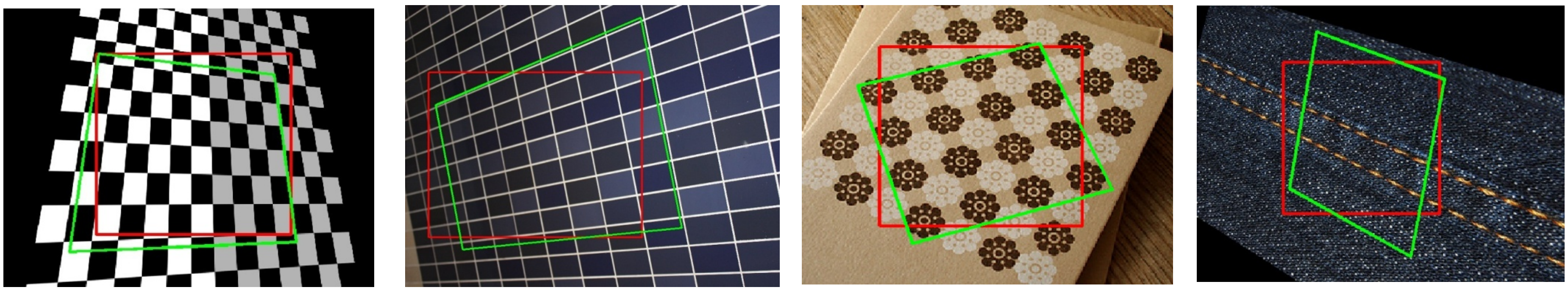
Un-Tilted Low-rank Textures



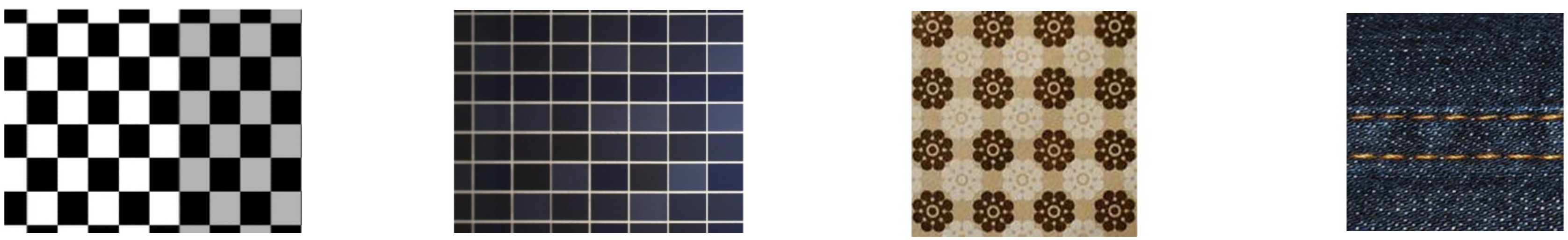


TILT: Examples of Symmetric Patterns and Textures

Input (red window)

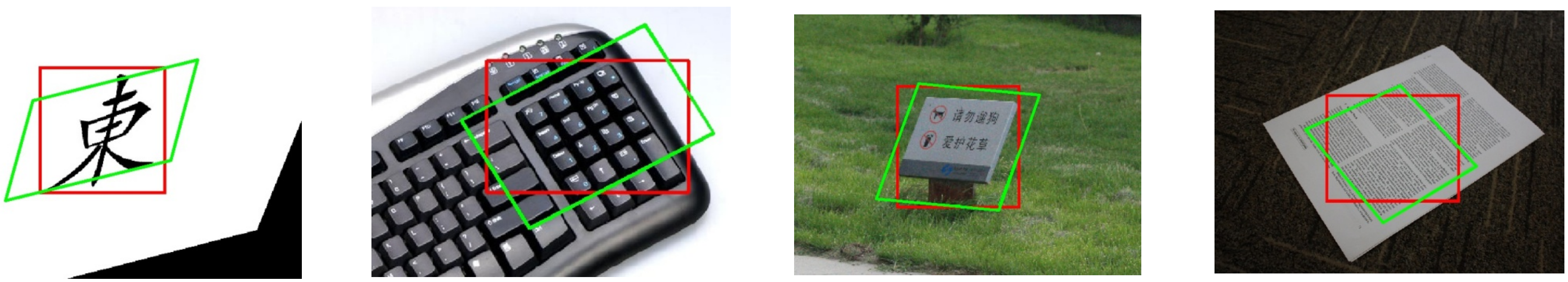


Output (rectified green window)



TILT: Examples of Characters, Signs, and Texts

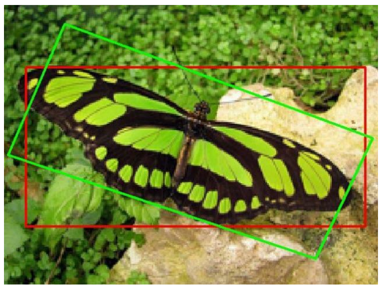
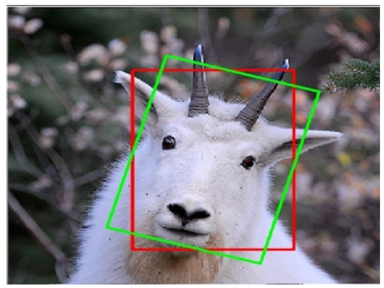
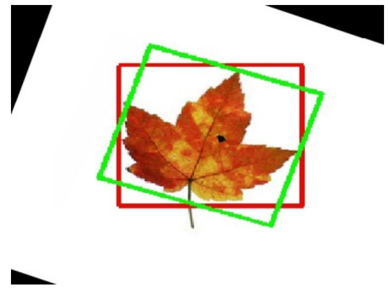
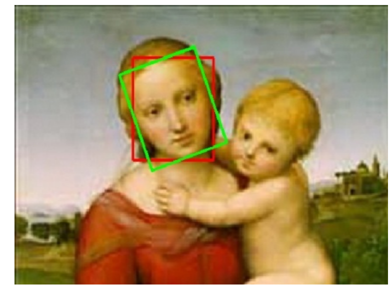
Input (red window)



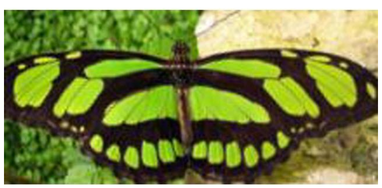
Output (rectified green window)

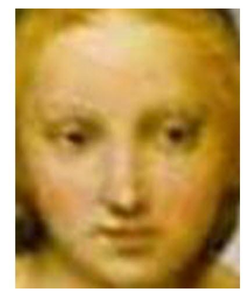
TILT: Examples of Natural Objects with Bilateral Symmetry

Input (red window)

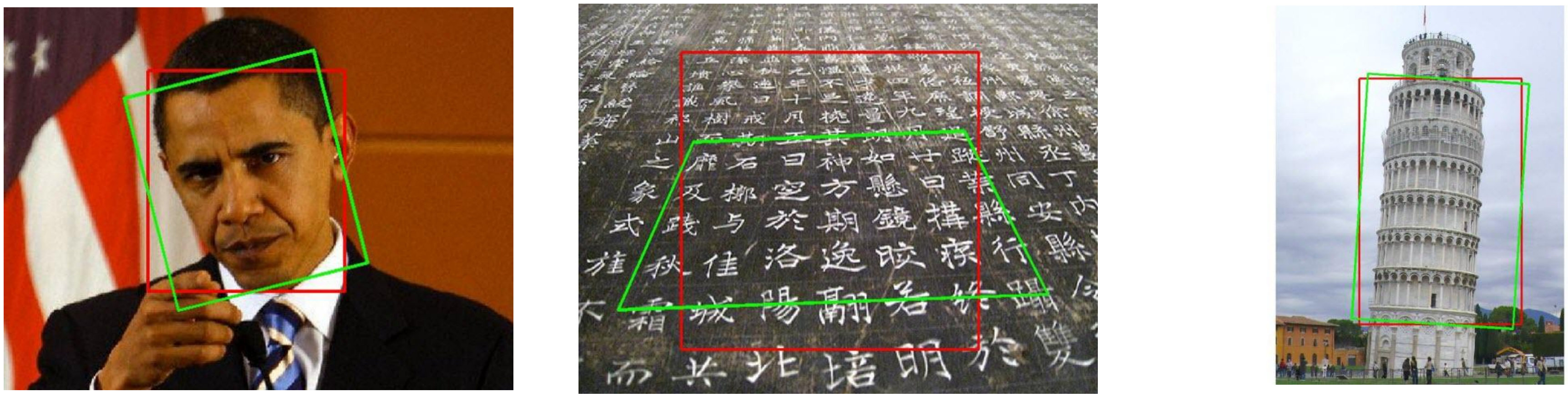
Output (rectified green window)



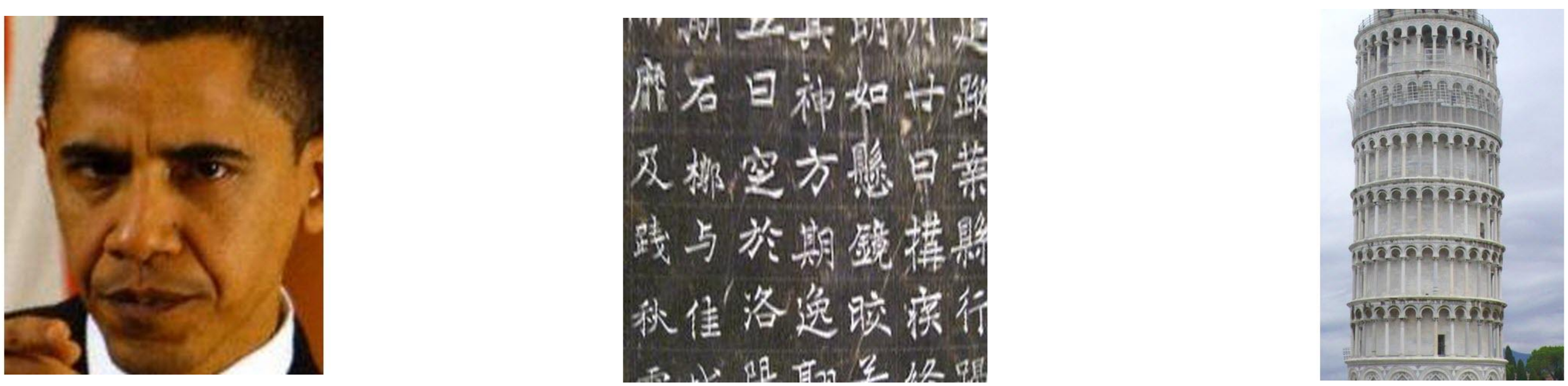



TILT: More Examples

Input (red window)

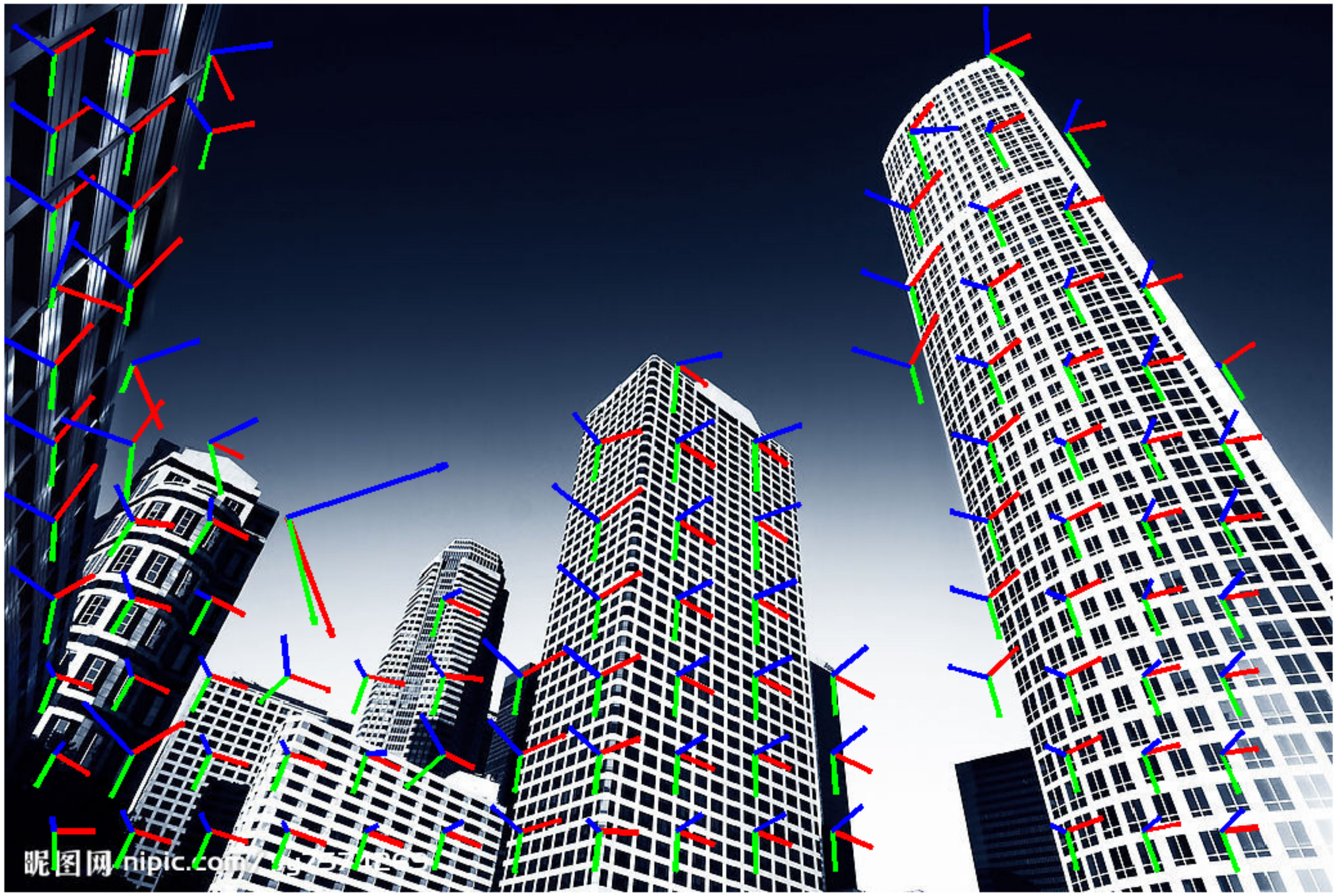


Output (rectified green window)

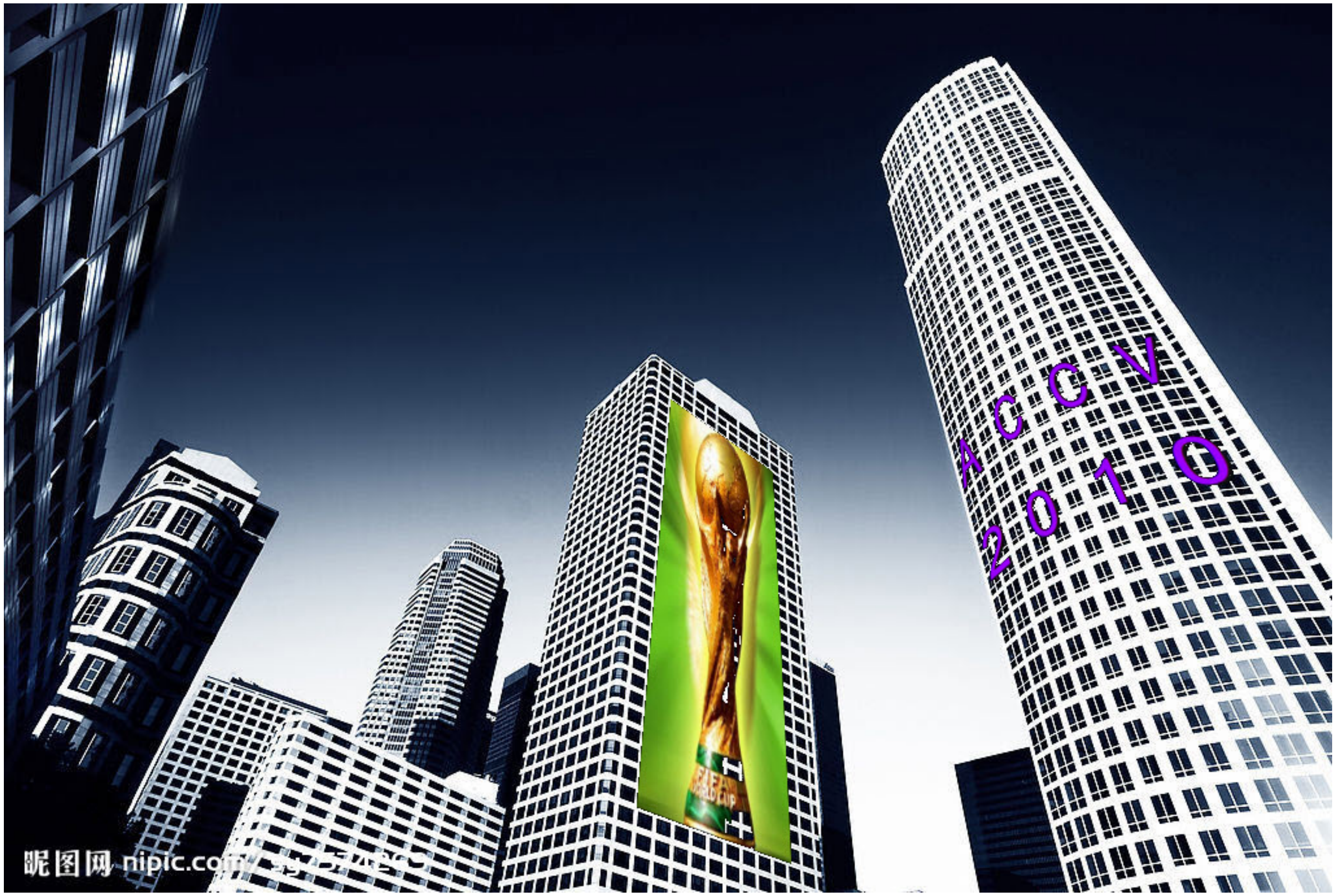


TILT – Local 3D Geometry from Low-rank Textures

Run TILT on a grid of 60x60 windows



TILT – Geometric Image Editing



Extensions

Robustness to noise (same people + Zhou)

- In reality: data matrix = low-rank + sparse + noise

$$M = L_0 + S_0 + Z_0, \quad \|Z_0\|_F \leq \delta$$

- Recovery via relaxed PCP

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & \|M - (L+S)\|_F \leq \delta \end{array}$$

- Reconstruction is stable

$$\frac{1}{n^2}\Big(\|\hat{L} - L_0\|_F^2 + \|\hat{S} - S_0\|_F^2) \leq O(\delta^2)$$

Extensions

Robustness to noise (same people + Zhou)

- In reality: data matrix = low-rank + sparse + noise

$$M = L_0 + S_0 + Z_0, \quad \|Z_0\|_F \leq \delta$$

- Recovery via relaxed PCP

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & \|M - (L + S)\|_F \leq \delta \end{array}$$

- Reconstruction is stable

$$\frac{1}{n^2}\left(\|\hat{L} - L_0\|_F^2 + \|\hat{S} - S_0\|_F^2\right) \leq O(\delta^2)$$

Dense correction (same people + Ganesh)

- Sparse component S_0 has random signs
- Fraction of nonzero entries in $S_0 \to 1$
- PCP still succeeds with high probability!

Summary

- Principled approach to Robust PCA
- Works well in theory and in practice
- Amenable to large scale problems – early effective algorithms
- Many applications
  - Computer vision
  - Signal processing
  - Data analysis
  - Many more (to come)
- *Interested in what you think!*

E. J. Candès, X. Li, Y. Ma, and J. Wright (2009). *Robust Principal Component Analysis?* Stanford Technical Report

Proof via dual certification

Find dual variable Y such that pair $(L_0, S_0; Y)$ obeys KKT optimality conditions

$$M = L_0 + S_0$$

Proof via dual certification

Find dual variable Y such that pair $(L_0, S_0; Y)$ obeys KKT optimality conditions

$$M = L_0 + S_0$$

- T: span of all matrices with row space <u>or</u> col. space included in that of L_0
- Ω: span of all matrices with support included in that of S_0

Proof via dual certification

Find dual variable Y such that pair $(L_0, S_0; Y)$ obeys KKT optimality conditions

$$M = L_0 + S_0$$

- T: span of all matrices with row space <u>or</u> col. space included in that of L_0
- Ω: span of all matrices with support included in that of S_0

Sufficient (and almost necessary) conditions

- $T \cap \Omega = \{0\}$
- There is $W \in T^\perp$ such that

  $$\|W\| < 1$$

  and $Y = UV^* + W$ obeys

  $$\begin{cases} Y_{ij} = \lambda[\mathrm{sgn}(S_0)]_{ij} & (i,j) \in \Omega \\ |Y_{ij}| < \lambda & \text{otherwise} \end{cases}$$

Augmented Lagrangian approach

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 + \frac{1}{2\tau}\|M - L - S\|_F^2 \\ \text{subject to} & L + S = M \end{array}$$

Lagrangian

$$\mathcal{L}(L, S; Y) = \|L\|_* + \lambda \|S\|_1 + \frac{1}{\tau}\langle Y, M - L - S\rangle + \frac{1}{2\tau}\|M - L - S\|_F^2$$

Basic algorithm (Usawa): dual gradient ascent

$$\begin{cases} (L_k, S_k) & = \arg\min_{L,S} \mathcal{L}(L, S; Y_{k-1}) \\ Y_k & = Y_{k-1} + \delta_k (M - L_k - S_k) \end{cases}$$

Sequential minimization

Scalar shrinkage: $\mathcal{S}_\tau[x] = \mathsf{sgn}(x)\max(|x| - \tau, 0)$

- Componentwise thresholding $\mathcal{S}_\tau(X)$
- Singular value thresholding $\mathcal{D}_\tau(X)$

$$\mathcal{D}_\tau(X) = U\mathcal{S}_\tau(\Sigma)V^* \qquad X = U\Sigma V^*$$

Sequential minimization

Scalar shrinkage: $\mathcal{S}_\tau[x] = \mathsf{sgn}(x)\max(|x| - \tau, 0)$

- Componentwise thresholding $\mathcal{S}_\tau(X)$
- Singular value thresholding $\mathcal{D}_\tau(X)$

$$\mathcal{D}_\tau(X) = U\mathcal{S}_\tau(\Sigma)V^* \qquad X = U\Sigma V^*$$

$$\mathcal{L}(L, S; Y) = \|L\|_* + \lambda\|S\|_1 + \frac{1}{\tau}\langle Y, M - L - S\rangle + \frac{1}{2\tau}\|M - L - S\|_F^2$$

Easy to minimize over L and S separately

$$\arg\min_L \mathcal{L}(L, S, Y) = \mathcal{D}_\tau(M - S + Y)$$
$$\arg\min_S \mathcal{L}(L, S, Y) = \mathcal{S}_{\lambda\tau}(M - L + Y)$$

PCP by alternating directions

initialize: S_0, Y_0 and $\tau > 0$
while not converged

1. $L_k = \mathcal{D}_\tau(M - S_{k-1} + Y_{k-1})$ (shrink singular values)
2. $S_k = \mathcal{S}_{\lambda\tau}(M - L_k + Y_{k-1})$ (shrink scalar entries)
3. $Y_k = Y_{k-1} + (M - L_k - S_k)$

end while
output: L, S

PCP by alternating directions

initialize: S_0, Y_0 and $\tau > 0$
while not converged

1. $L_k = \mathcal{D}_\tau(M - S_{k-1} + Y_{k-1})$ (shrink singular values)
2. $S_k = \mathcal{S}_{\lambda\tau}(M - L_k + Y_{k-1})$ (shrink scalar entries)
3. $Y_k = Y_{k-1} + (M - L_k - S_k)$

end while
output: L, S

All the computational work is in (1)

When iterates L_k have low rank

- Only need to compute few singular values (and vectors) at each step
- Lanczos iterations are very effective