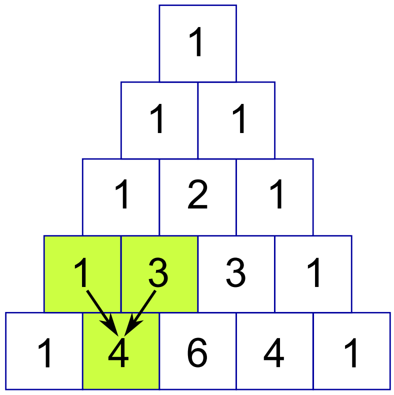
Pascal's Triangle

中文

O… ost interesting Number Patterns is Pascal's Triangle (n… *Blaise Pascal*, a famous French Mathematician and Philosopher).

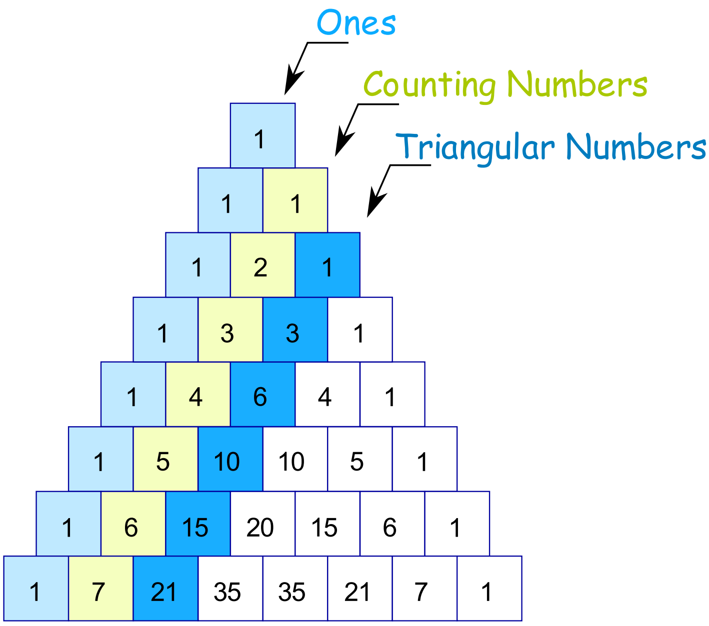


To build the triangle, start with "1" at the top, then continue placing numbers below it in a triangular pattern.

Each number is the numbers directly above it added together.

(Here I have highlighted that **1+3 = 4)**

Patterns Within the Triangle



Diagonals

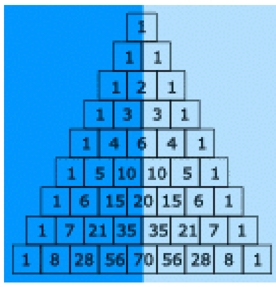
The first diagonal is, of course, just "1"s

The next diagonal has the Counting Numbers (1,2,3, etc).

The third diagonal has the triangular numbers

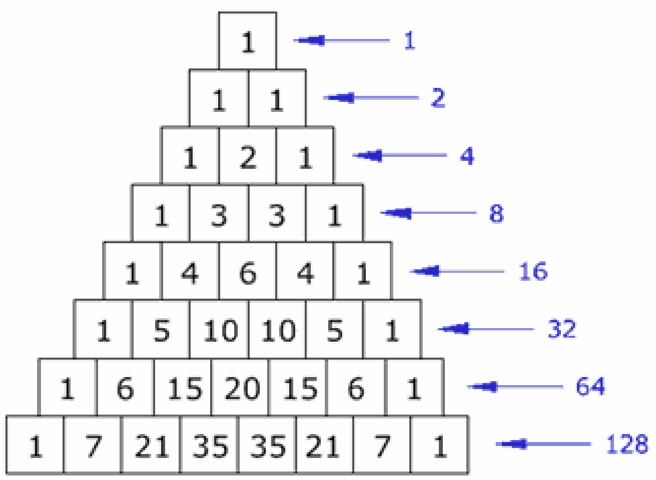
(The fourth diagonal, not highlighted, has the tetrahedral numbers.)

Symmetrical



The triangle is also symmetrical. The numbers on the left side have identical matching numbers on the right side, like a mirror image.

Horizontal Sums



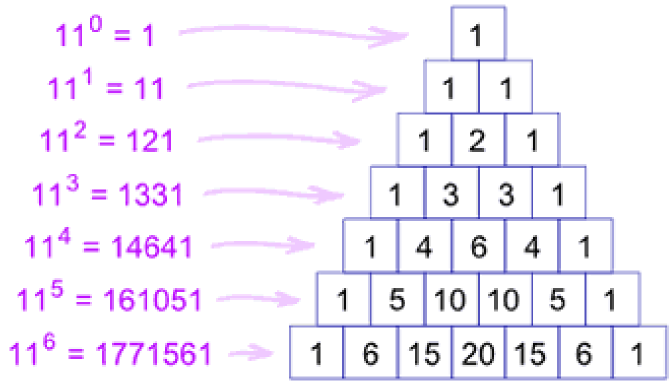
What do you notice about the horizontal sums?

Is there a pattern?

They **double** each time (powers of 2).

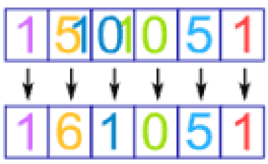
Exponents of 11

Each line is also the powers (exponents) of 11:

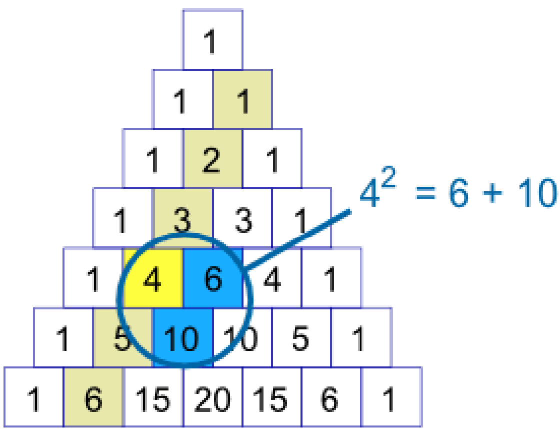


- 11^0=1 (the first line is just a "1")
- 11^1=11 (the second line is "1" and "1")
- 11^2=121 (the third line is "1", "2", "1")
- etc!

But what happens with 11^5 ? Simple! The digits just overlap, like this:



The same thing happens with 11^6 etc.



Squares

For the second diagonal, the square of a number is equal to the sum of the numbers next to it and below both of those.

Examples:

- $3^2 = 3 + 6 = 9$,
- $4^2 = 6 + 10 = 16$,

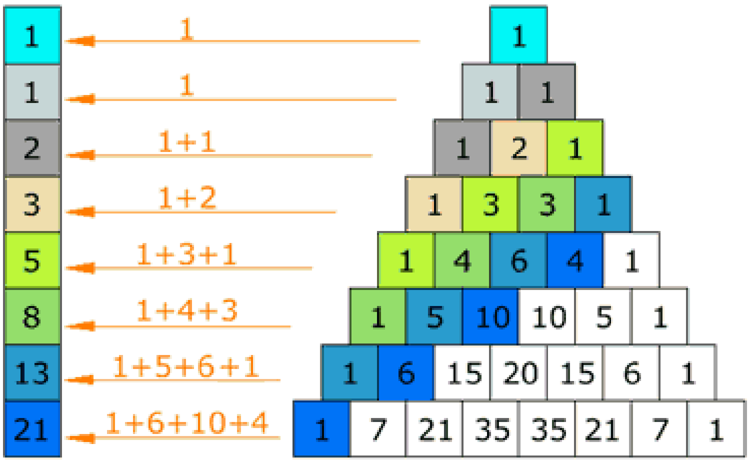
- 5^2 = 10 + 15 = 25,
- ...

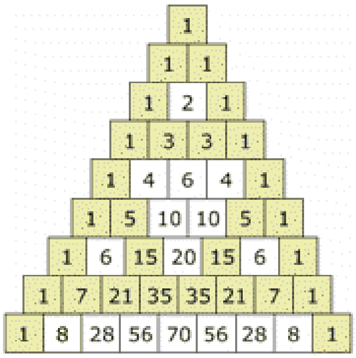
There is a good reason, too ... can you think of it? (Hint: 4^2=6+10, 6=3+2+1, and 10=4+3+2+1)

Fibonacci Sequence

Try this: make a pattern by going up and then along, then add up the values (as illustrated) ... you will get the Fibonacci Sequence.

(The Fibonacci Sequence starts "0, 1" and then continues by adding the two previous numbers, for example 3+5=8, then 5+8=13, etc)





Odds and Evens

If you color the Odd and Even numbers, you end up with a pattern the same as the Sierpinski Triangle

Using Pascal's Triangle

Heads and Tails

Pascal's Triangle can show you how many ways heads and tails can combine. This can then show you the probability of any combination.

For example, if you toss a coin three times, there is only one combination that will give you three heads (HHH), but there are three that will give two heads and one tail (HHT, HTH, THH), also three that give one head and two tails (HTT, THT, TTH) and one for all Tails (TTT). This is the pattern "1,3,3,1" in Pascal's Triangle.

Tosses	Possible Results (Grouped)	Pascal's Triangle
1	H T	1, 1
2	HH HT TH TT	1, 2, 1
3	HHH HHT, HTH, THH HTT, THT, TTH TTT	1, 3, 3, 1
4	HHHH HHHT, HHTH, HTHH, THHH HHTT, HTHT, HTTH, THHT, THTH, TTHH HTTT, THTT, TTHT, TTTH TTTT	1, 4, 6, 4, 1
	... etc ...	

Example: What is the probability of getting exactly two heads with 4 coin tosses?

There are 1+4+6+4+1 = 16 (or 2^4=16) possible results, and 6 of them give exactly two heads. So the probability is 6/16, or 37.5%

Combinations

The triangle also shows you how many Combinations of objects are possible.

Example: You have 16 pool balls. How many different ways could you choose just 3 of them (ignoring the order that you select them)?

Answer: go down to the start of row 16 (the top row is 0), and then along 3 places (the first place is 0) and the value there is your answer, **560**.

Here is an extract at row 16:

```
      1    14    91    364  ...
   1    15    105   455   1365  ...
1    16    120   560   1820  4368  ...
```

A Formula for Any Entry in The Triangle

In fact there is a formula from Combinations for working out the value at any place in Pascal's triangle:

It is commonly called "n choose k" and written like this: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Notation: "n choose k" can also be written **C(n,k)**, $^{n}C_{k}$ or even $_{n}C_{k}$.

The "!" is "factorial" and means to multiply a series of descending natural numbers. Examples:

- 4! = 4 × 3 × 2 × 1 = 24
- 7! = 7 × 6 × 5 × 4 × 3 × 2 × 1 = 5040
- 1! = 1

So Pascal's Triangle could also be an **"n choose k"** triangle like this one.

(Note how the top row is **row zero** and also the leftmost column is zero)

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

Example: Row 4, term 2 in Pascal's Triangle is "6" ...

... let's see if the formula works:

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot 2\cdot 1} = 6$$

Yes, it works! Try another value for yourself.

This can be very useful ... you can now work out any value in Pascal's Triangle **directly** (without calculating the whole triangle above it).

Polynomials

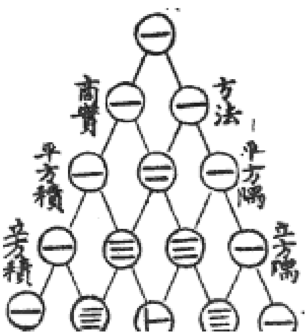
Pascal's Triangle can also show you the coefficients in binomial expansion:

Power	Binomial Expansion	Pascal's Triangle
2	$(x + 1)^2 = \mathbf{1}x^2 + \mathbf{2}x + \mathbf{1}$	1, 2, 1
3	$(x + 1)^3 = \mathbf{1}x^3 + \mathbf{3}x^2 + \mathbf{3}x + \mathbf{1}$	1, 3, 3, 1
4	$(x + 1)^4 = \mathbf{1}x^4 + \mathbf{4}x^3 + \mathbf{6}x^2 + \mathbf{4}x + \mathbf{1}$	1, 4, 6, 4, 1
	... etc ...	

The First 15 Lines

For reference, I have included row 0 to 14 of Pascal's Triangle

1														
1	1													
1	2	1												
1	3	3	1											
1	4	6	4	1										
1	5	10	10	5	1									
1	6	15	20	15	6	1								
1	7	21	35	35	21	7	1							
1	8	28	56	70	56	28	8	1						
1	9	36	84	126	126	84	36	9	1					
1	10	45	120	210	252	210	120	45	10	1				
1	11	55	165	330	462	462	330	165	55	11	1			
1	12	66	220	495	792	924	792	495	220	66	12	1		
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1

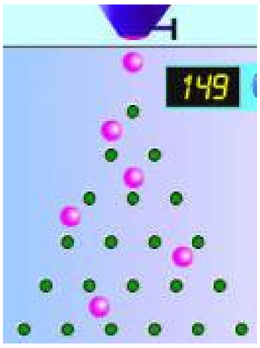


The Chinese Knew About It

This drawing is entitled "The Old Method Chart of the Seven Multiplying Squares". View Full Image

It is from the front of Chu Shi-Chieh's book "*Ssu Yuan Yü Chien" (Precious Mirror of the Four Elements)*, written in **AD 1303** (over 700 years ago, and more than 300 years before Pascal!), and in the book it says the triangle was known about more than two centuries before that.

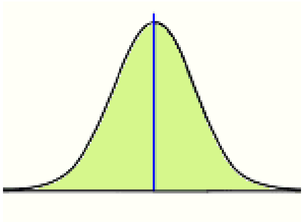
The Quincunx



An amazing little machine created by Sir Francis Galton is a Pascal's Triangle made out of pegs. It is called The Quincunx .

Balls are dropped onto the first peg and then bounce down to the bottom of the triangle where they collect in little bins.

At first it looks completely random (and it is), but then you find the balls pile up in a nice pattern: the Normal Distribution.



Question 1 Question 2 Question 3 Question 4 Question 5 Question 6 Question 7 Question 8 Question 9 Question 10 Question 11

Activity: Subsets