Seismic Waves

Abstract

In this essay we will investigate seismic waves and examine the equations that can be used to model these waves mathematically, using a technique called *ray paths.* We will begin with simple wave equations and manipulate them in this project into complex expressions, which finally can be used in a program to model seismic waves accurately.

1 Introduction

We know that the earth is a large ball with 4 main layers:

- **Crust** : 5 - 50 km thick which is where we live
- **Mantel** : 2,900 km thick and comprised of hot dense rock
- **Outer Core** : 3,500 km thick and made up of iron and nickel in a liquid state
- **Core** : 1,000 km radius ball of iron nickel, despite being 5,000°C due to extremely high pressure

When an earthquake occurs it releases a large amount of energy and seismic waves, often so large they can be recorded around the world on seismographs. There are 3 main types of wave; P - waves, S - waves and surface waves. In this assignment we will focus on only the two types of wave which travel through the earth:

- **P-waves**: (primary waves) are longitudinal and travel at the greatest velocity of the two waves, at the speed of sound (300 ms^{-}1).
- **S-waves**: (secondary waves) are transverse waves and cannot travel through air or water however have a much larger amplitude than P-waves so are much more destructive.

2 Ray Theory

This section is largely inspired by N. Rawlinson's lecture notes [1].

We will begin with a simple anstatz remembering that

$$\boldsymbol{\nabla}\phi \quad = \quad (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}), \tag{1}$$

both P and S waves satisfy the following wave equation in 3D, when V_s or V_p are both equal to V

$$\begin{aligned} \frac{1}{V^2}\frac{\partial^2 \phi}{\partial t^2} &= \boldsymbol{\nabla}^2 \phi \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}, \end{aligned}$$

Q1

When V is constant we obtain the following ansatz of an expression for ϕ which will satisfy (2)

$$\phi = A \exp\big(- i\omega(t - \mathbf{k}.\mathbf{x})\big), \tag{2}$$

where $\mathbf{k}$ is a constant vector such that $\mathbf{k}.\mathbf{x} = k_x\mathrm{x} + k_y\mathrm{y} + k_z\mathrm{z}$.
If we take the second derivative with respect to t we obtain

$$\frac{\partial^2\phi}{\partial t^2} = \exp(t - \mathbf{k}.\mathbf{x})\,, \tag{3}$$

and for the right hand side we can calculate

$$\boldsymbol{\nabla}^2\phi \quad = \quad (k_x + k_y + k_z)^2 \exp(t - \mathbf{k}.\mathbf{x})\,.$$

Note that $A\exp\big(-i\omega(t - \mathbf{k}.\mathbf{x})\big)$ can be expressed as $A\big(\exp(-i\omega) + \exp(t - \mathbf{k}.\mathbf{x})\big)$ where $A(\exp(-i\omega)$ is treated as a constant in both operations in (3) and (4) therefore disappears. Substitute (3) and (4) into (2) to give

$$\frac{1}{V^2}\exp(t - \mathbf{k}.\mathbf{x}) \quad = \quad (k_x + k_y + k_z)^2 \exp(t - \mathbf{k}.\mathbf{x})\,. \tag{4}$$

It is clear here that $V^2 = (\frac{1}{k_x+k_y+k_z})^2$ and this implies $V = \frac{1}{|\mathbf{k}|}$ which in practice is the inverse of the length of the vector $\mathbf{k}$. Therefore V is a function of the position of the wave when passing through rocks of different densities inside the Earth. Given period of a wave is $T = \frac{2\pi}{\omega}$ and that the speed of the wave will vary very little while passing through the earth over distances of order $\lambda = \frac{V}{T}$ we can use the following equations to approximate ϕ

$$\phi = A(\mathbf{x})\exp\big(-i\omega(t + T(\mathbf{x}))\big)\,, \tag{5}$$

Q2

Substituting (5) into (2) gives

$$\begin{aligned}\boldsymbol{\nabla}^2\phi =& \boldsymbol{\nabla}^2 A\exp\big(-i\omega(t + T(\mathbf{x}))\big)\\ &- i\omega\boldsymbol{\nabla}T(\mathbf{x}).\boldsymbol{\nabla}A\exp\big(-i\omega(t + T(\mathbf{x}))\big)\\ &- i\omega\boldsymbol{\nabla}A.\boldsymbol{\nabla}T\exp\big(-i\omega(t + T(\mathbf{x}))\big)\\ &- i\omega A\boldsymbol{\nabla}^2T\exp\big(-i\omega(t + T(\mathbf{x}))\big)\\ &- i\omega^2 A\boldsymbol{\nabla}T.\boldsymbol{\nabla}T\exp\big(-i\omega(t + T(\mathbf{x}))\big)\end{aligned} \tag{6}$$

,

$$\frac{\partial^2\phi}{\partial t^2} = -\omega^2 A\exp\big(-i\omega(t + T(\mathbf{x}))\big)$$

(7)

$$\boldsymbol{\nabla}^2A - i\omega\boldsymbol{\nabla}T.\boldsymbol{\nabla}A - i\omega\boldsymbol{\nabla}A.\boldsymbol{\nabla}T - i\omega\boldsymbol{\nabla}^2T - \omega^2A\boldsymbol{\nabla}T.\boldsymbol{\nabla}T + \frac{\omega^2A}{\alpha^2} = 0 \tag{8}$$

$$\boldsymbol{\nabla}^2A - \omega^2A\mid\boldsymbol{\nabla}T\mid^2 - i(2\omega\boldsymbol{\nabla}A.\boldsymbol{\nabla}T + \omega A\boldsymbol{\nabla}^2T) = \frac{-A\omega^2}{\alpha^2} \tag{9}$$

If we rearrange (2) to

$$\frac{1}{V^2}\frac{\partial^2\phi}{\partial t^2} - \boldsymbol{\nabla}^2\phi \quad = \quad 0\,,$$

and then substitute (5) we have (10)

$$2\omega \boldsymbol{\nabla} A \cdot \boldsymbol{\nabla} T + \omega A \boldsymbol{\nabla}^2 T = 0\,, \tag{10}$$

$$\boldsymbol{\nabla}^2 A - \omega^2 A \boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} T = -\frac{\omega^2}{V^2} A\,. \tag{11}$$

Dividing (11) by $-A\omega^2$ we obtain

$$-\frac{\boldsymbol{\nabla}^2 A}{A\omega^2} + |\boldsymbol{\nabla} T|^2 = \frac{1}{V^2}. \tag{12}$$

Our ansatz (5) assumes that V changes very little over a distance of the wavelength λ. Using a Taylor expansion, we can write

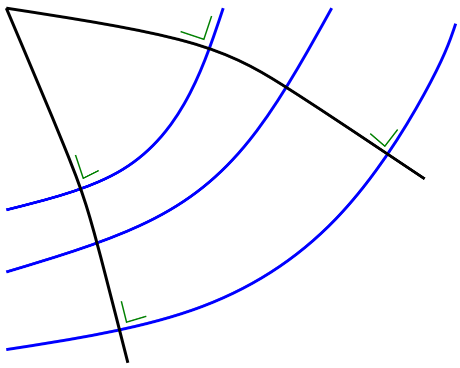
$$\frac{\mathrm{d}^2 A(x)}{\mathrm{d}x^2} \approx \frac{A(x+\mathrm{d}x) + A(x-\mathrm{d}x) - 2A(x)}{\mathrm{d}x^2}\,, \tag{13}$$

and take $\mathrm{d}x = \lambda$ so that

$$\begin{aligned} -\frac{\boldsymbol{\nabla}^2 A}{A\omega^2} &\approx \frac{A(x+\mathrm{d}x) + A(x-\mathrm{d}x) - 2A(x)}{\lambda^2} \frac{\tau^2}{A(x)4\pi^2} \\ &= \frac{A(x+\mathrm{d}x) + A(x-\mathrm{d}x) - 2A(x)}{A(x)} \frac{1}{4\pi^2 V^2} \ll \frac{1}{V^2}. \end{aligned} \tag{14}$$

where τ is the period of the wave. We use the fact that $V = \lambda/\tau$ and we can neglect the first term in (12) if A is nearly constant over a distance $\mathrm{d}x = \lambda$. Then (12) simplifies to

$$|\boldsymbol{\nabla} T|^2 = \frac{1}{V^2}. \tag{15}$$



[ht] Wave-front (blue) and ray path (black).

Note that in (5) $T(\mathbf{x}) = T_0$, where T_0 is a constant, defines a surface corresponding to a wave front. This is the surface corresponding to all parts of the wave which have travelled a time T_0 since a reference point, usually taken as the source of the wave. Ray paths are the lines which are perpendicular to those surfaces at all points.

The derivatives of T in the direction parallel to the wave-fronts is zero (as T is constant on it). Equivalently

$$\boldsymbol{\nabla} T \cdot \mathbf{q}_{\mathrm{wf}} = 0\,, \tag{16}$$

where $\mathbf{q}_{\mathrm{wf}}$ is a vector tangent/parallel to the wave-front.

Indeed: we can take a system of coordinates around a point $\mathbf{r}$ such that x and y are parallel to the wave-front and z perpendicular to it: $\partial T/\partial x = \partial T/\partial y = 0$. We have $\boldsymbol{\nabla} T \cdot \mathbf{q}_{\mathrm{wv}} = (\partial T/\partial x, \partial T/\partial y, \partial T/\partial z) \cdot (q_x, q_y, 0) = 0$ and as the result is independent of the system of coordinates, (16) holds.

A ray path can be parametrised as $\mathbf{r}(s)$ where s is a parameter satisfying $|\mathrm{d}\mathbf{r}/\mathrm{d}s| = 1$, and s is measured in the same units as $\mathbf{r}$. The vector $\mathrm{d}\mathbf{r}/\mathrm{d}s$ is tangent to a ray path: if we consider two close points $\mathbf{r}(s+\delta s)$ and $\mathbf{r}(s)$, the vector joining them is

$$\mathbf{b} = \mathbf{r}(s+\delta s) - \mathbf{r}(s) = \delta s \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} + \mathcal{O}(\mathrm{d}s^2)\,. \tag{17}$$

If we take δs very small then $\mathbf{b}$ is parallel to $\mathrm{d}\mathbf{r}/\mathrm{d}s$. We also notice that

$$T(\mathbf{r}+\mathbf{q}) - T(\mathbf{r}) \approx \mathbf{q}\cdot\boldsymbol{\nabla}T\,, \tag{18}$$

and if $\mathbf{q}$ is infinitesimal and parallel to the wave-front (i.e. proportional to $\mathbf{q}_{\mathrm{wf}}$), the right-hand side is zero as $\boldsymbol{\nabla}T$ is perpendicular to the wave-front. If $\mathbf{q}_{\mathbf{rp}}$ is parallel to the ray path, then $T(\mathbf{r}+\mathbf{q}_{\mathbf{rp}}) - T(\mathbf{r})$ is the time taken by the wave to travel from $\mathbf{r}$ to $\mathbf{r}+\mathbf{q}_{\mathbf{rp}}$ which is $|\mathbf{q}_{\mathbf{rp}}|/V$ and so $|\mathbf{q}_{\mathbf{rp}}|/V = \mathbf{q}_{\mathbf{rp}}\cdot\boldsymbol{\nabla}T = |\mathbf{q}_{\mathbf{rp}}||\boldsymbol{\nabla}T|$. We thus have

$$V|\boldsymbol{\nabla}T| = 1\,. \tag{19}$$

As $V\boldsymbol{\nabla}T$ and $\mathrm{d}\mathbf{r}/\mathrm{d}s$ are vectors perpendicular to the wave front and of norm 1 we have

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = V\boldsymbol{\nabla}T\,, \tag{20}$$

as one can always orient s so that the equality holds with a positive sign.

The time δT needed to go from a point s on the path to a close point $s+\delta s$ on the same path is $\delta T = T(s+\delta s) - T(s) = \delta s\frac{dT}{ds}$. As $\delta s/\delta T = V$, $\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{1}{V} = u$ where we have defined $u = 1/V$ which is called the slowness by geophysicists.

We now take the gradient of $\frac{\mathrm{d}T}{\mathrm{d}s} = u$ from both sides, we get

$$\frac{\mathrm{d}\boldsymbol{\nabla}T}{\mathrm{d}s} = \nabla u\,. \tag{21}$$

And inserting (20) into (21) we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[u\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right] = \nabla u\,. \tag{22}$$

Knowing the underlying structure of the Earth, implies knowing $u(\mathbf{r})$ and we can solve (22) for $\mathbf{r}(s)$ corresponding to the ray followed by the wave.

For simplicity we assume that the Earth is radially symmetric and that the speed of seismic waves only depends on the radial coordinate r measured from the centre of the Earth. We actually use the polar coordinates (r,Δ,ϕ) where we use the letter Δ instead of the more common θ to avoid ambiguity with the wave incident angle.

Q3

$$\frac{d}{ds}\left[\mathbf{r}\times\left(u\frac{d\mathbf{r}}{ds}\right)\right] \quad = \quad \frac{d\mathbf{r}}{ds}\times\left(u\frac{d\mathbf{r}}{ds}\right) + \mathbf{r}\times\frac{d}{ds}\left(u\frac{d\mathbf{r}}{ds}\right) = 0 \tag{23}$$

.

Substituting (22) into (23) we have

$$\frac{d}{ds}\left[\mathbf{r}\times\left(u\frac{d\mathbf{r}}{ds}\right)\right] \quad = \quad \frac{d\mathbf{r}}{ds}\times\boldsymbol{\nabla}u + \mathbf{r}\times\frac{d}{ds}\boldsymbol{\nabla}u = 0\,, \tag{24}$$

this is because the cross product of two parallel vectors is always 0. In (23) the first term is two identical vectors and the second term is two rays which are spherically symmetric.

Therefore using basic laws of calculus we can say that

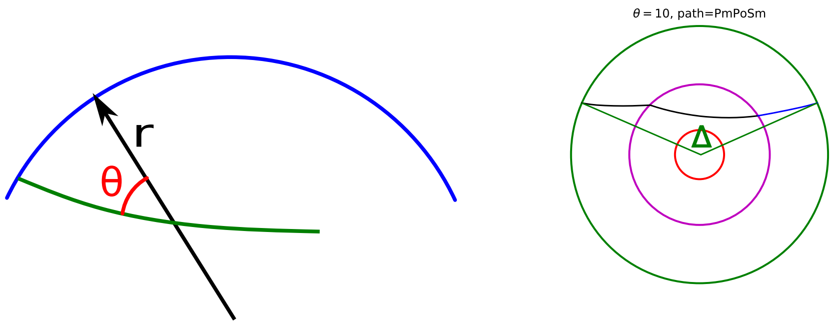


Figure 1: a) Incident angle θ between ray path (green) and radial direction (black). b) Ray path for a P wave propagating through the mantel and the inner core and re-entering the mantel as an S wave (path PmPoSm; the naming is explained in section **??**). Δ is the total deflection angle.

$$\mathbf{r} \times \left(u \frac{d\mathbf{r}}{ds} \right) = \int (0) ds = \mathbf{K} \tag{25}$$

where $\mathbf{K}$ is a constant vector.

We then write

$$\left| \mathbf{r} \times u \frac{d\mathbf{r}}{ds} \right| = |\mathbf{K}| = p = ru \sin(\theta_i) \tag{26}$$

where θ_i is the angle between the ray path direction $\frac{d\mathbf{r}}{ds}$ and the radial direction. Notice that p, called the ray parameter, is a constant. In other words, for any given path, p will remain the same for all times, even when the wave changes type.

When $\theta_i = \pi/2$, the ray path is perpendicular to the radial direction meaning it has reached its lowest point.

We use (19) to evaluate $c(\mathbf{r})^{-2} = u(\mathbf{r})^2$ by computing

$$\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} T = \left(\frac{\partial T}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial T}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin(\theta^2)} \left(\frac{\partial T}{\partial \phi} \right)^2 = u(\mathbf{r})^2. \tag{27}$$

For a radially symmetric Earth, waves travel along a great circle and the last term is zero. We then seek solution for T of the form

$$T(r, \theta) = f(\theta) + g(r) \tag{28}$$

and substitute it into (27) to get

$$r^2 \left[\frac{dg(r)}{dr} \right]^2 - r^2 u(r)^2 = - \left[\frac{df(\theta)}{d\theta} \right]^2 = -k^2. \tag{29}$$

Both sides of (29) must be the same constant, expressed here as $-k^2$, so that varying one of the independent variables (r on the right hand side and θ on the left hand side) will not affect the other. which we call $-k^2$. As $df/d\theta = k$ we have

$$f(\theta) = \int_0^\theta k d\phi = k\theta. \tag{30}$$

From the left hand side of (29)

$$\frac{dg(r)}{dr} = \pm \left(u(r)^2 - \frac{k^2}{r^2} \right)^{1/2}, \tag{31}$$

and we have

$$T(r,\theta) = k\theta \pm \int_0^r \left(u(x)^2 - \frac{k^2}{x^2}\right)^{1/2} dx. \tag{32}$$

Q4

Using (21) and (22) we obtain the expression

$$\boldsymbol{\nabla} T = u\frac{d\mathbf{r}}{ds}, \tag{33}$$

then we simply substitute (33) into (27) to get

$$\left(u\frac{d\mathbf{r}}{ds}\right)^2 = \left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \theta}\right)^2 + \frac{1}{r^2\sin(\theta^2)}\left(\frac{\partial T}{\partial \phi}\right)^2, \tag{34}$$

$$u\frac{d\mathbf{r}}{ds} = \left(\frac{\partial T}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial T}{\partial \theta}\right) + \frac{1}{r\sin(\theta)}\left(\frac{\partial T}{\partial \phi}\right). \tag{35}$$

Then given that $\boldsymbol{\nabla} = \left[\frac{\partial T}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right]$ and using the result (32) we have the following polar co-ordinates

$$u\frac{d\mathbf{r}}{ds} = \left[\pm\frac{1}{r}\left(r^2u(r)^2 - k^2\right)^{1/2}, \frac{k}{r}, 0\right] \tag{36}$$

then we divide through but $u(r)$ to get

$$\frac{d\mathbf{r}}{ds} = \left[\pm\frac{1}{ru(r)}\left(r^2u(r)^2 - k^2\right)^{1/2}, \frac{k}{u(r)r}, 0\right]. \tag{37}$$

As $|d\mathbf{r}/ds| = 1$, the angle θ between the incident ray path and the radial direction is

$$\sin(\theta) = \frac{k}{ru(r)} \tag{38}$$

and so $k = ru(r)\sin(\theta) = p$ (see (26)). Then $T(r,\theta)$ can be obtained by computing the integral

$$T(r,\theta) = p\theta \pm \int_{r_s}^r \frac{\sqrt{x^2u(x)^2 - p^2}}{x} dx. \tag{39}$$

with r_s the radius at the starting point. As T is a travelling time it must increase and so we take the $+$ sign for ascending waves, $dr > 0$, and the $-$ sign for descending waves, $dr < 0$.

It can be shown [1] that the wave travelling time along the path is a local minimum (Fermat's principle). Mathematically:

$$\frac{\partial T}{\partial p} = \theta \pm \int_{r_s}^r \frac{-p}{x\sqrt{r^2u(x)^2 - p^2}} dx = 0 \tag{40}$$

and so

$$\theta = \pm p\int_{r_s}^r \frac{1}{x\sqrt{r^2u(x)^2 - p^2}} dx. \tag{41}$$

Substituting this back into (39):

$$T(r,\theta) = \pm\int_{r_s}^r \left[\frac{p^2}{x\sqrt{x^2u(x)^2 - p^2}} + \frac{\sqrt{x^2u(x)^2 - p^2}}{x}\right] dx \tag{42}$$

$$= \pm\int_{r_s}^r \frac{x^2u(x)^2}{x\sqrt{x^2u(x)^2 - p^2}} dx \tag{43}$$

The total angle of propagation between 2 points can be determined from (41):

$$\Delta = \int_{r_s}^r \frac{p}{x\sqrt{x^2u(x)^2 - p^2}} dx. \tag{44}$$

3 Model

Using the formulas that we have derived in this essay we can combine them (see ray IASP91 v2.py) to generate a model of P S seismic waves an their paths as they travel through the centre of the earth. This can also reveal more characteristics of the waves which we have not included in our previous calculations such as; amplitude, wave length frequency.We can use models like this to predict the paths of waves produced in earthquakes.

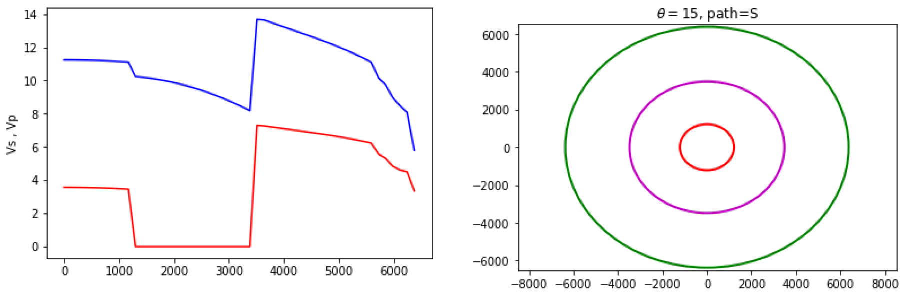


Figure 2:
a plot generated by ray IASP91v2.py showing the speed of waves P and S as they travel through the earth, the steps in velocity represent when the pass through the different layers, as illustrated in fig 3. S wave is in red, and P in blue. The S wave clearly flatlines through the mantle as it cannot travel through this liquid layer

Q5
Q6

4 Conclusions

We have shown in this essay that by building on what we fundamentally know about longitudinal and transverse waves, we can develop accurate models of how seismic waves travel through the earth. This is extremely useful for understanding more about the centre of the earth and predicting the global affects of earthquakes.

References

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