Traffic Jams

Abstract

In this essay we will investigate the nature of traffic jams and the flow of traffic by examining the graphs that can be used to model this mathematically and the results of changing variables. We will begin with an extremely simplistic model and develop it into more realistic models improving the accuracy of the output and increasing the number variables.

1 Introduction

Traffic congestion is unfortunately an increasingly common feature in our daily lives and is not only frustrating for drivers but has been proven to cause social, economic , safety and health issues as a result. In 2008 a study [2] calculated that the external costs of congestion alone, amount to 0.5% of Community GDP. Therefore the study and modelling of traffic flow is incredibly important to examine how councils and road engineers can optimize their use of congestion reducing measures such as:

- Reducing traffic speed
  - Traffic calming
  - Variable speed limits
- Reducing volume of traffic on road
  - Car pooling
  - Pedestrian only high streets
  - Increased parking charges
  - Improving public transport connections
- Improving drivers awareness
  - Courses
  - Improved road signs

We will also investigate the causes of traffic jams in free moving traffic (without traffic lights ect.) and how adjusting variables such as speed and density impact traffic jams. From experience we have a small intuitive knowledge of the nature of traffic jams, such as queues during rush hour, however this essay will explore the causes of this alongside other features and develop them further.This can be done by developing accurate models and interpreting the data they produce.

test

2 Basic Models

In all of the models in this assignment, we will model the cars on a circular road as in (1). Although this is very unrealistic, however makes the modeling much easier as when cars reach the end of the road they pass the detector and go to the beginning with the same velocity.

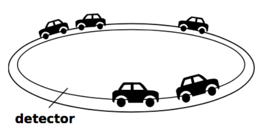

Figure 1: Road we will be modelling, with detector to count cars passing one loop

Most of our calculations in this essay will be based on the density and flow of cars on a circular road. Where ΔT is a fixed time interval, ΔN is number of cars passing the detector in that time, N is number of cars on the road and L is length of road (circumference of circular road) then the formulas for $flow, q$ and $density, \rho$ are as follows:

$$q = \frac{\Delta N}{\Delta T}, \tag{1}$$

$$\rho = \frac{N}{L}. \tag{2}$$

When considering units for flow are $\frac{cars}{time}$ we can rearrange (1) to give

$$\begin{aligned} q &= \frac{cars}{time}, &(3)\\ &= \frac{cars}{distance}\frac{distance}{time}, &(4)\\ &= velocity \cdot density. &(5) \end{aligned}$$

We will use these, along with other variables, to plot graphs to examine nature of traffic flow. The flow density graph for most recorded and modelled traffic flows will follow a similar shape to that shown in (2). This shape produced because when density is low the system is in 'free-flow' where drivers can control the cars mostly without constraint of other cars but there are not many cars on the road, so flow is low. However when density is high the cars are forced to decrease velocity due to others and the flow therefore decreases. At the extremes where density is 0 or 1 no cars pass the detector, so flow is zero as there are either no cars on the road or the cars are bumper to bumper so cannot move. In the middle we can observe the peak flow at what is referred to as 'critical density'. Research has suggested [4] that a common cause of traffic congestion is the cars breaking and the exaggerated reaction of cars behind it. We can construct a simple model to explain this phenomena, if we assume that a car travelling at speed v decelerates to stationary at acceleration a_{max}. In real life this represents a car jamming the breaks on suddenly in response to the car in front being to close. First we can calculate the time (Δt) for the car to be stationary from v as; **Q1**

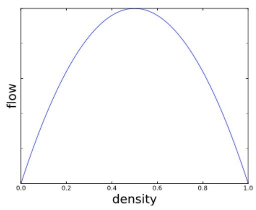

Figure 2: Example of a simple flow density graph

$$\begin{aligned} v &= u + a_{max}\Delta t &(6)\\ \Delta t &= \frac{v}{a_{max}}. &(7) \end{aligned}$$

Then the distance (Δx) travelled in this time can be calculated as;

$$\Delta x = v\Delta t - \frac{1}{2}a_{max}\Delta t, \tag{8}$$

then substituting in our values for Δt (7) and rearranging we have;

$$\Delta x = \frac{1}{2}\frac{v^2}{a_{max}}. \tag{9}$$

Assuming all cars on the circular road are travelling at a constant speed v_0 with each car occupying $\Delta L = \Delta l + \Delta x$, where Δl is the length of the individual car and Δx is the minimum

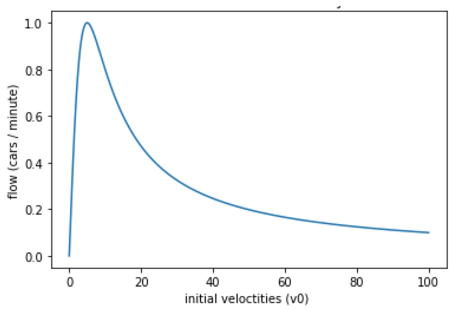

Figure 3: A plot of flow against initial velocity (13) for $a_{max} = 5m/s^{-2}$ and $l_c = 5m$.

distance needed to insure cars can break in time without crashing, as calculated in (9). So ΔL can also be expressed as;

$$\Delta L = \Delta l + \frac{v_0^2}{a_max}. \tag{10}$$

Using the result from (5) we can say that;

$$\begin{aligned} q &= v\frac{1}{\Delta L} && (11)\\ &= \frac{2v_0}{\frac{v_0^2}{a_{max}} + l_c} && (12)\\ &= \frac{2v_0 a_{max}}{v_0^2 + l_c a_{max}}. && (13) \end{aligned}$$

We can use this equation to plot a graph to show the change in flow depending on different initial velocities as see in fig.(3).This graph shows that as velocities increase, the flow increases until 'critical velocity' after which flow decreases. This is because when the cars are travelling at a greater velocity, more cars are catching the car in front and therefore having to stop suddenly as there is a greater range of velocities in the cars. This produces a ripple affect of cars breaking which eventually causes traffic jams, as reflected by the low flow.

3 Discrete models

From the previous simplistic model, our first step to improve it is to change it to a discrete plot. The road will be divided into individual segments and each segment is assigned a respective velocity. All the velocities are also discrete units, so for example v = 1 may be the same as v= 30 km/h. Also segments without a car have velocity of -1 as all cars have strictly positive velocity. We will still model this road as circular as show in fig(1) so when cars leave the end they return to the beginning with the same velocity.

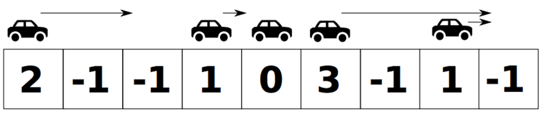

Figure 4: A discrete road divided into 9 sections with 5 cars each with corresponding velocities.

We will construct our model based on these four basic rules:

1. Each car accelerates until reaches v_{max} (speed limit)
2. If car is too close to the car infront it slows down (prevents crashes)

3. Each car slows randomly with probability $p_{slowdown}$ (very rough model of random behaviour of drivers)

4. Each car moves along discrete road the number of spaces equal to its velocity

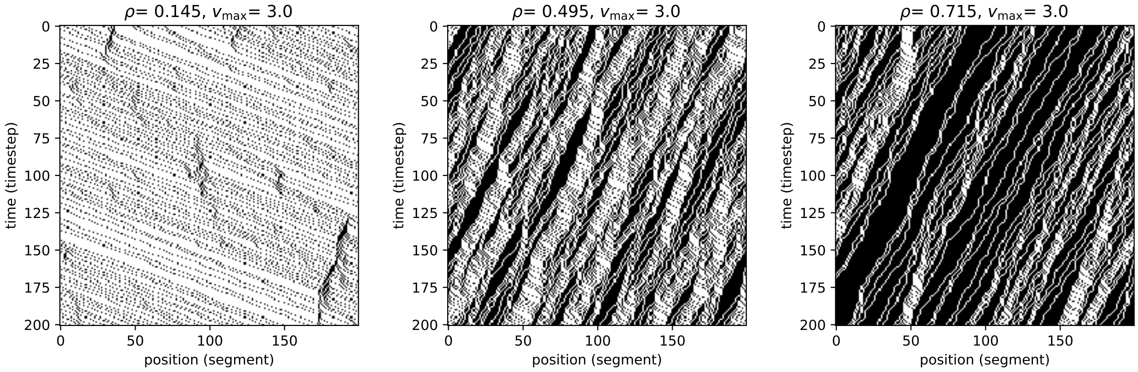

Figure 5:

Plots of cars positions (represented by black dots) against time using the discrete model and implementing the 4 basic rules. To generate the plots we have changed the density and max velocity to the values show above each individual graph, and $p_{slowdown}$ to 0.4 for all.

From the graphs fig(5) we can clearly see that as the density increases there are more thick black lines. This shows cars are grouping together in traffic jams as they having to slow down to avoid crashing. This concept is familiar as we observe on road in real life that in periods of high density such as rush hour, there is an increased likelihood of traffic jams.

Q3

The shape of fig(6a) shows cars average velocity is constant at v_{max} followed by a sudden decline where as fig(6b) describes a velocity decreasing at a decreasing rate as density increases with cars average velocity decreasing from v_{max} to 0. The difference between these curves highlights the impact of random slowing down compared to when the cars move in totally deterministic behaviour and follow $Rule184$. Many traffic models are based on $Rule184$ (also known as the 'traffic rule') which simulates traffic on a binary basis with cars stopping and starting in accordance to the others around them and no other variables.

As we investigated in part 2, a car braking causes a backlog cars behind it breaking in response. Therefore as shown in fig(6b) as the number of cars increase, the backlog of cars breaking due to others increases especially as headway will be reduced with more cars on the road. This produces the concave up decrease curve.

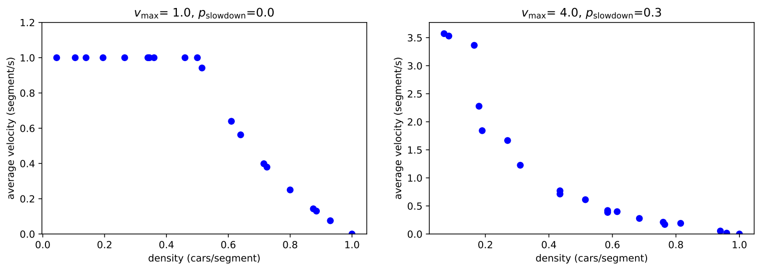

Figure 6:

Graphs showing how average velocity is affected by density. The difference between the two is a) has $p_{slowdown}$ = 0 , v_{max} = 1.0 b) $p_{slowdown}$ = 0.3 , v_{max} = 4.0

We can see clearly on both graphs in fig(7) regardless of the value of $p_{slowdown}$ and v_{max} density increases to a clear turning point, critical-density, the flow is greater for higher v_{max}. This is simply because more cars are travelling past the detector in a given time (see (1)). However after critical-density all v_{max} have the same flow, this is because the cars velocities are no longer determined by the speed limit, but are having to slow for other cars due to high density (see fig(6)). Furthermore the critical-density is greater for larger v_{max} for both values of $p_{slowdown}$ because with a higher velocity limit cars in the model will be starting at faster velocity and as shown by fig(3) the greater v_0, the lower the flow.

To understand the behavior of the cars displayed in fig(6a) we should consider equation(1) and fig(7). As we see in fig(6) the average velocities are constant as cars are in free-flow until critical-density in fig (7) where flow peaks. As explained previously,after which they have to decelerate to avoid other cars..

When $p_{slowdown} = 0$, cars follow a deterministic behaviour and therefore we can compare fig(7a) to fig(6a). In fig(7a) we observe an immediate change from a positive to negative gradient at critical density for all v_{max}, which can be explained as we know $flow \propto velocity$(5) and fig(6a) shows the average velocity decreases suddenly at critical density.However in fig(7b) we observe a different pattern as the cars with a curved graph with a gradual change from positive to negative gradient, it is a clear result of the decreasing curve in fig(6b). For both fig(6) and fig(7) the points in (a) have a weaker correlation than in (b) as the inclusion of $p_{slowdown}$ produces some random results with more spread which do not strictly fit with the curves.

We can also compare the flow density plots for the discrete model fig(7) to our initial simple flow density fig(2). The both exhibit a similar curve pattern however in fig(7) there is a clear left skew of the turning point, the most similar to fig(2) being fig(7b) for $v_{max} = 1$.

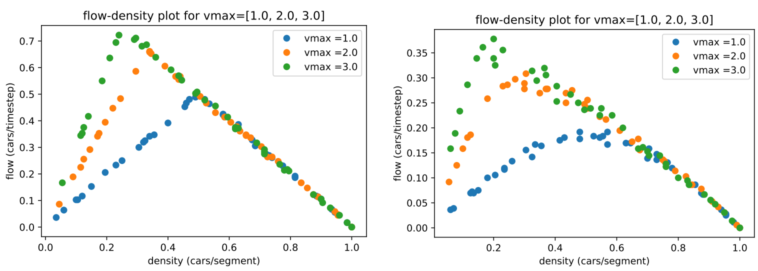

Figure 7:

Graphs showing how flow of cars is affected by density, for different velocities. They are sophisticated version of fig(2). The difference between the two is a) has $p_{slowdown} = 0$ b) has $p_{slowdown} = 0.3$

4 Deterministic ODE model

Our next step to develop the model is to take it from discrete to continuous, and consider our cars and their velocities as a function of t (time). We will continue to model the cars on a circular road as in fig(1) and in one direction of travel only with one lane.

Q4

In this model we will also continue to attempt to simulate drivers random actions and the consequences of them. However sometimes we may chose to produce purely deterministic models where we set $p_{slowdown}$= 0. For example in **part 2**, fig(6a) and fig(7). Although this is less accurate it may be helpful when we are studying other variables and the impact of random actions is not relevant. It may also help to simplify complex problems and allows us to compare the impact of randomness when comparing models with and without it included. Most importantly as seen in fig(6) and fig(7) randomness can make it more difficult to recognize patterns and make predictions due to the illogical nature of it.

Previously we did not consider the acceleration of cars, however in this model we will base our calculations on Newton's 2^{nd} law as a function of time:

$$F = ma, \tag{14}$$
$$F(t) = M\ddot{x}(t). \tag{15}$$

To make this more realistic we will assume that drivers with adjust their acceleration to avoid crashing into the car in front. If the distance between two consecutive cars front bumpers is represented by Δx we can calculate a cars acceleration using the *optimal velocity model* where v_{goal} is the *optimal velocity*:

$$v_0 = \frac{v_{max}}{1 - tanh[m(b_c - b_f)]}, \tag{16}$$
$$v_{goal}(\Delta x) = v_0\big(tanh[m(\Delta x - b_f)] - tanh[m(b_c - b_f)]\big), \tag{17}$$
$$\ddot{x}_n = s(v_{goal}(\Delta x) - v_n) \tag{18}$$

In (16 , 17, 18) b_f and m are parameters which tune v_{goal}, s represents the sensitivity of drivers, b_c is car length and v_0 is initial velocity. Note when h_n is headway between consecutive cars we can express Δx as:

$$\Delta x = h_n + b_c. \tag{19}$$
$$\tag{20}$$

Q5

By substituting (16) into (17) we obtain;

$$v_{goal}(\Delta x) = \frac{v_{max}\big(tanh[m(\Delta x - b_f)] - tanh[m(b_c - b_f)]\big)}{1 - tanh[m(b_c - b_f)]}, \tag{21}$$

For all Δx, $v_{goal}(\Delta x)$ is strictly positive so cars can never travel backwards as seen on the graph fig(8). We can also consider it algebraically as we know for all $x > 0$, $0 < tanh(x) < 1$ therefore the denominator of v_{goal} is strictly positive as is v_{max} and;

$$tanh[m(\Delta x - b_f)] - tanh[m(b_c - b_f)] > 0, \tag{22}$$
$$tanh[m(\Delta x - b_f)] > tanh[m(b_c - b_f)], \tag{23}$$
$$\Delta x - b_f > b_c - b_f, \tag{24}$$
$$h_n + b_c - b_f > b_c - b_f, \tag{25}$$
$$h_n > 0, \tag{26}$$
$$\tag{27}$$

so the numerator is strictly positive for all $h_n > 0$, so $v_{goal}(\Delta x)$ is positive for $\Delta x > b_c$. We can also show that as h_n tends toward 0, so does $v_{goal}(\Delta x)$;

$$v_{goal}(\Delta x) = \big(tanh[m(\Delta x - b_f)] - tanh[m(b_c - b_f)]\big)\frac{v_{max}}{1 - tanh[m(b_c - b_f)]}, \tag{28}$$
$$= \big(tanh[m(h_c + b_c - b_f)] - tanh[m(b_c - b_f)]\big)\frac{v_{max}}{1 - tanh[m(b_c - b_f)]}, \tag{29}$$
$$\lim_{h_n \to \infty} v_{goal}(\Delta x) = \big(tanh[m(b_c - b_f)] - tanh[m(b_c - b_f)]\big)\frac{v_{max}}{1 - tanh[m(b_c - b_f)]}, \tag{30}$$
$$= 0 \cdot \frac{v_{max}}{1 - tanh[m(b_c - b_f)]}, \tag{31}$$
$$= 0. \tag{32}$$

Therefore the cars cannot crash into on another, this is also shown on the graph fig(8). When $\Delta x = b_c$ the cars are effectively bumper to bumper on the circular road and therefore stationary, at this point $v_{goal} = 0$, however they can never reach this point from $v_0 > 0$ in our model.

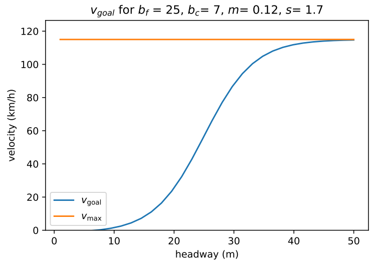

Figure 8:
Graph showing how v changes with *headway* compared to *vmax* with parameter of *vmax* set to values above graph.

Q6

If we assume all cars are at equal distance Δx, with velocity $v_{goal}(\Delta x)$ the according to (18), $\ddot{x_n} = 0$ and we can compute a stationary solution where cars are in free-flow so are moving at a constant velocity. We have show in (5) that;

$$q = velocity \cdot density. \tag{33}$$
$$\tag{34}$$

We also know that in this situation, $v = v_{goal}(\Delta x) = constant$ and given that Δx is also now constant, and equal to the distance between each cars front bumper we can say that

$$\begin{aligned} \Delta x &= \frac{L}{N}, && (35) \\ &= \rho^{-1}. && (36) \end{aligned}$$

Therefore substituting (36) and (5) into (1) we can say;

$$\begin{aligned} q &= v_{goal}\left(\frac{L}{N}\right)\frac{N}{L}, && (37) \\ &= v_{goal}(\rho^{-1})\rho, && (38) \end{aligned}$$

this enables us to produce flow density graphs for the continuous model such as fig(9). This graph shows a similar pattern to that observed in the discrete models flow density plots fig(7) and even our initial simple flow density plot in fig(2). There is a symmetry and defined turning point in fig(9) due to the restricted parameters we have used, where v is constant and the cars in free flow.

Q7

Comparing the two flow-density plots for the continuous model we can see that the free flow curve of fig(10) asymmetric, skewed to the left and curved compared to fig(9). This shows us the affect of non-stationary model, where cars are not travelling at constant velocity so traffic jams occur and fig(9) has a much we observe a higher critical density as cars are not braking for others. In fig(10) the step in the simulated curve arises as the traffic changes from free-flow to congested

and therefore no longer fits the free-flow curve. 7

Q8

We have briefly discussed previously when the homogeneous free flow is stable when $\ddot{x_n} = 0$ and all cars are at equal distance. This means that cars will not need to apply the breaks for

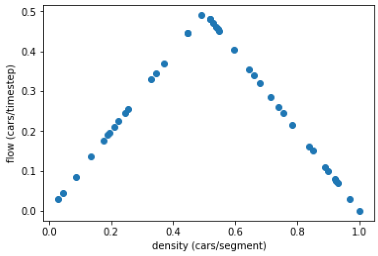

Figure 9:
Graph showing how flow changes with density for the continuous model in free flow where $\ddot{x} = 0$.

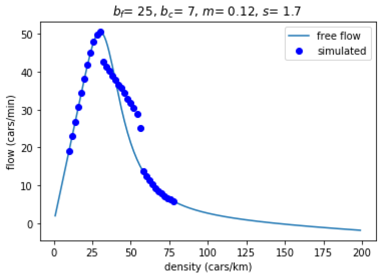

Figure 10:
Graph showing how flow changes with density for the continuous model for free flow and simulated for parameters shown above graph ($\neq 0$).

other cars as they are all moving at a constant velocity. It can be proven analytically the system is unstable when the following is true;

$$\frac{2v'_{goal}(h)}{s} < 1. \tag{39}$$

Therefore by solving the following for h we can find all the conditions for when the system is stable;

$$\frac{2v'_{goal}(h)}{s} = 1, \tag{40}$$

$$\frac{v_0}{cosh^2(m(h-b_f))} = \frac{s}{2} \tag{41}$$

$$cosh^2(m(h-b_f)) = \frac{2v_0}{s} \tag{42}$$

$$h = \frac{1}{m}arcosh\sqrt{\frac{2v_0}{s}} + b_f. \tag{43}$$

Any input variables which satisfy these conditions will result in homogeneous free flow and we have already discovered is true when $\ddot{x} = 0$, the cars are equally space and also have the same velocity. A trivial solution is also when there is one car on the road in free flow. This is a very important concept because when (40) is satisfied, there will be a reduction in traffic jams as they will only result from random human behaviour.

Q7

As shown in fig(9) critical density for free flow is at approximately $\rho = 0.5$, so as we compare figs(11) and figs(12) for $\rho = 0.4$ and $\rho = 0.6$ these are representative of cars behaviour, before critical density (free flow) and after, respectively. The thick blue bars in figs(11) represent cars clumping together in traffic jams. We can see that at a greater density there are more traffic jams as would be expected from the graph fig(9) as cars are forced to slow for one another. These traffic jams are also reflected in figs(12b) where the velocity of consecutive cars is 0 on 3 occasions, corresponding to the 3 traffic jams in fig(11b). Likewise the 1 traffic jam in fig(11a) corresponds to the 0 velocities on figs(12b). These outcomes are very similar to those shown in fig(5) for our discrete model and are to be expected for an increased considering what has been discussed in this essay.

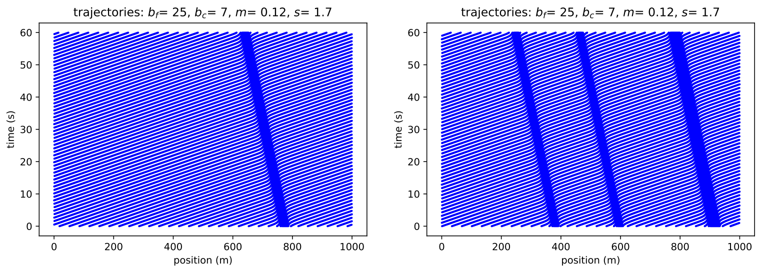

Figure 11:
Two plots showing the position of cars(m) over certain time steps with parameters equal to values shown above the graphs, a) has density = 40 cars / km b) has density = 60 cars / km.

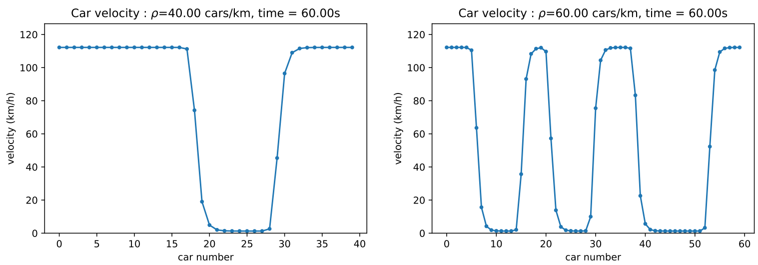

Figure 12:
Two plots of each cars velocity at time 40, as before a) has density = 40 cars / km b) has density = 60 cars / km

5 Conclusions

In this project we have produced a sophisticated model with many variables to simulate traffic flow, and used them to examine the types of traffic flow and the result of different inputs. Sometimes it is not necessary to include all of the possible variables and we can extract large amounts of information from simpler models such as in part 2. Our models developed on the common $Rule184$ traffic flow model which only adjusts velocity to proximity to other cars which is very unrealistic. We have shown that traffic jams often occur fundamentally as a result of cars decelerating, which has a domino effect onto other cars in the loop. This is caused by cars travelling at different velocities, random driver behavior and is exaggerated by high density. Furthermore we have shown that lowering speed limits increases critical density which is an incredibly useful principle and is often implemented on crowded motorways. The final model we produced is very powerful in modelling traffic flow and is very useful in predicting how traffic will react to certain changes and therefore how to reduce traffic jams.

References

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[2] Papageorgiou, M. and Diakaki, C. and Dinopoulou, V. and Kotsialos, A. and Wang, Y. (2003) 'Review of road tra c control strategies.', Proceedings of the IEEE., 91 (12). pp. 2043-2067.

[3] Awazu, A. (1999). Cellular automaton rule 184++C. A simple model for the complex dynamics of various particles flow. Physics Letters A, 261(5-6), pp.309-315.

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6 Question 2

Q2

In $flow\_density(self, density)$ we must return the density function, despite it already being passed through density as a parameter because $fill\_road\_randomly$ (as the name suggests) has a random element to it. Therefore the density of the cars on the road filled by this function, may not be the exact same as the density input. For example in (5) the input densities were (0.15 , 0.5 , 0.7) none of which are equal to the densities on the graph (denoted by the ρ above graphs) and were just approximations used by $fill\_road\_randomly$ function. Therefore it is important we use the actual road density produced from $fill\_road\_randomly$ in our $flow\_density$ function.