A Remedy for Proposal Failures in Metropolis Light Transport

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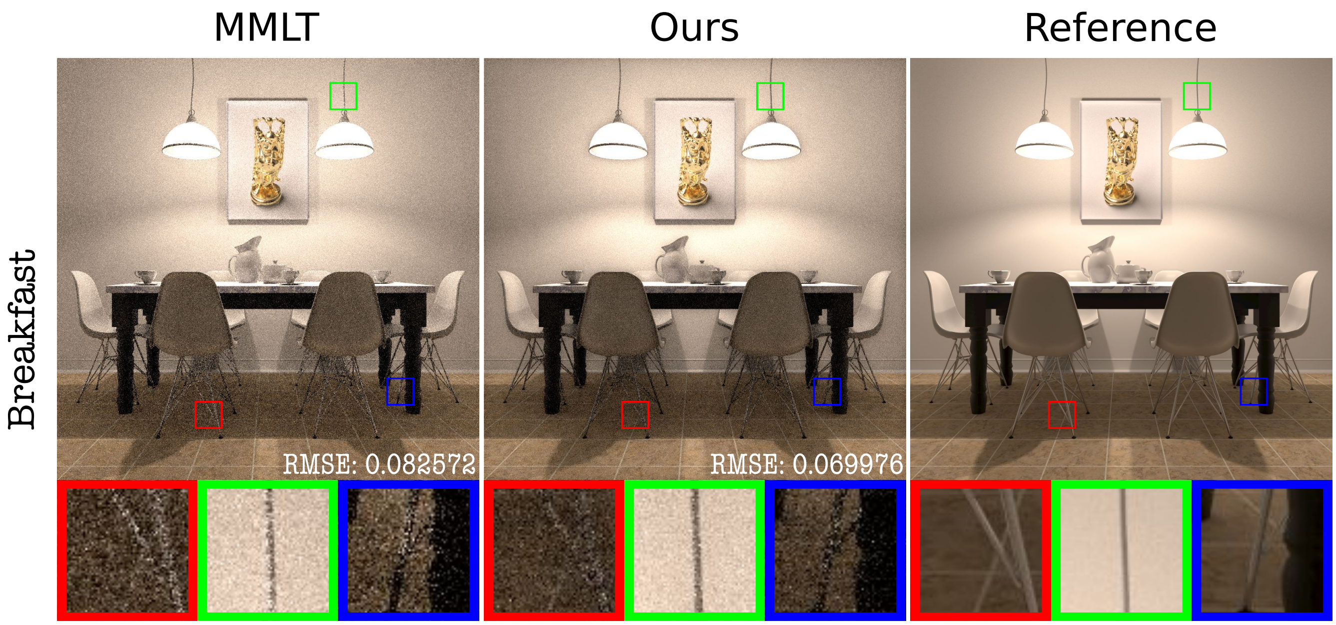

Figure 1: *The Breakfast scene.* Equal-time comparison. Left: Image rendered with MMLT shows suboptimal performance, because more than 37% of the traced paths are proposal failure paths. Middle: Excluding proposal failure paths from the states of Markov chain, our method produces higher quality results with the same computation. If our method is used to produce the same RMSE/quality result as MMLT does, in general, about a third of the computation will be saved.

ABSTRACT

We study MLT-type algorithms building on bidirectional path tracing. Let us define the failure of obtaining a complete mutation path, which may be caused by the failure of tracing required two sub-paths or the failure of connecting the two sub-paths, as *Proposal Failure*. All MLT-type algorithms don't distinguish proposal failure paths from normal proposed paths. Though proposal failure paths, as impossible values of the stationary distribution of Markov chain, don't affect the final equilibrium, they slow down the convergence rate of the Metropolis-Hastings sampler, which means that the related rendering algorithm produces suboptimal image with the same computation. To address this issue, based on MMLT, we propose a novel algorithm, Proposal Failure MLT (PFMLT), which distinguishes proposal failure paths from normal proposed paths and excludes them from the states of Markov chain. With various scenes of challenging lighting, geometry and material tested, PFMLT always produced better results than MMLT did. Furthermore, our method can be easily used to extend other MLT-type algorithms and make them work much more efficiently.

CCS CONCEPTS

• **Computing methodologies → Computer graphics**;

KEYWORDS

rendering

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1 INTRODUCTION

Physically-based rendering is extensively used now, but solving the rendering equation is still a laborious task, especially for scenes with challenging lighting, geometry or material.

[Veach 1997] introduced Metropolis-Hasting Sampling ([Hastings 1970; Metropolis et al. 1953]) into computer graphics and presented the original Metropolis Light Transport (MLT). The main advantage of MLT is that the path space is explored locally by making small mutations to the current path to generate proposal path, which makes MLT a substantially efficient rendering algorithm for particularly challenging scenes. However, MLT's mutation rules are complex and none of the rules is symmetric, which makes its implementation a significant undertaking.

[Kelemen et al. 2002] presented Primary Sample Space MLT (PSSMLT), whose mutation strategy in the unit cube of pseudo-random numbers is symmetric and easy to implement. [Hachisuka et al. 2014] presented an extension to PSSMLT named Multiplexed Metropolis Light Transport (MMLT) which combines MCMC methods with Multiple Importance Sampling [Veach 1997].

Some Adaptive mutation strategies working in path space or primary sample space have been developed, such as MEMLT [Jakob and Marschner 2012] for specular interactions, HSLT [Hanika et al. 2015] for rough materials, and GeoMLT [Otsu et al. 2018], which incorporates visibility into mutation strategies of MCMC rendering, and H2MC [Li et al. 2015], which is based on MMLT's approach and utilizes Hamiltonian Monte Carlo to adaptively control the shape of the transition kernels.

Also, several methods have been developed to bridge path space and primary sample space by using invertible mappings to transform samples between them, such as [Bitterli et al. 2018; Otsu et al. 2017; Pantaleoni 2017].

Unfortunately, the percentage of the paths that are used to reconstruct image carried no radiance in all of those algorithms is still high for challenging scenes. What's worse, all MLT-type algorithms don't distinguish the zero-contribution paths from normal proposed paths. We call the failure of obtaining a complete mutation path, which may be caused by the failure of tracing required two sub-paths or the failure of connecting the two sub-paths, as *Proposal Failure*. Though proposal failure paths, as impossible values of the stationary distribution of Markov chain, don't affect the final equilibrium, they slow down the convergence rate of the Metropolis-Hastings sampler, which means that the related rendering algorithm produces suboptimal image with the same computation. To address this issue, based on MMLT, we propose a novel algorithm, Proposal Failure MLT (PFMLT), which distinguishes proposal failure paths from normal proposed paths and excludes them from the states of Markov chain.

Our method mainly involves two steps. Firstly, during the process of tracing path, we check if the path fails to be a complete path. Then, when a path is detected as proposal path, instead of rejecting it and then repeatedly accumulating the contribution of the current path, our method excludes it from the states of Markov chain by switching the state counter backward and with no contribution accumulated.

With proposal failure paths excluded from the states of Markov chain, our method performs much better than MMLT. As exemplified in Figure 1, with more than 37% of the traced paths being proposal failure paths, the image rendered with MMLT shows suboptimal performance, while our method produces higher quality image with the same computation. If our method is used to produce the same RMSE/quality result as MMLT does, in general, about a third of the computation will be saved. Furthermore, our method can be easily used to extend other MLT-type algorithms and make them work much more efficiently.

In summary, the contributions of this paper are:

(1) Introduction of the concept of proposal failure.
(2) A novel extension of Metropolis sampling algorithm which simulates proposal failures and provides a remedy for them to verify our idea mathematically.
(3) Combination of the novel extension of Metropolis sampling with MMLT.

2 RELATED WORK

Light Transport Simulation. Light transport equation was first described by [Kajiya 1986]. [Veach 1997] presented that light transport simulation can be expressed as the integral:

$$I_j = \int_{\Omega} h_j(\bar{x}) f(\bar{x}) d\mu(\bar{x}) \tag{1}$$

, where I_j is the intensity of the j-th pixel of image, $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ is the space of light paths of all finite lengths k, $\bar{x}$ is a light path, $h_j()$ is the reconstruction filter that selects out all the light paths that contribute to the j-th pixel, f is the contribution of $\bar{x}$ and μ is the area measure.

Monte Carlo Integration. The integral in Equation (1) is infinite-dimensional because of the length of $\bar{x}$ can be infinite, which cannot be solved analytically. Monte Carlo numerical integration methods provide one solution to this kind of problem. The Monte Carlo integration estimator of Equation (1) is:

$$I_j \approx \hat{I}_j = \frac{1}{N} \sum_{i=1}^{N} \frac{h_j(\bar{x}^{(i)}) f(\bar{x}^{(i)})}{p(\bar{x}^{(i)})} \tag{2}$$

, where $\bar{x}^{(i)}$ is the i-th independently sampled light path, $p(\bar{x}^{(i)})$ is the probability density of the sample and N is the total number of samples.

In order to decrease the variance of the estimator, $p(\bar{x}^{(i)})$ should be approximately proportional to $f(\bar{x}^{(i)})$. Unfortunately, $f(\bar{x}^{(i)})$ is a spectrally valued function, and thus there is no unambiguous notion of what it means to generate samples proportional to $f(\bar{x}^{(i)})$. To address this issue, a scalar contribution function f^* is defined as the luminance of the path contribution f. Intuitively, $p(\bar{x}^{(i)})$ can be $f^*(\bar{x}^{(i)})/b$ (where b is the normalization constant $\int_{\Omega} f(\bar{x}) d\mu(\bar{x})$). So, Equation (2) can be presented as:

$$I_j \approx \hat{I}_j = \frac{b}{N} \sum_{i=1}^{N} \frac{h_j(\bar{x}^{(i)}) f(\bar{x}^{(i)})}{f^*(\bar{x}^{(i)})} \tag{3}$$

Now, we face two problems: calculating b and geting samples from $p(\bar{x}^{(i)})$ that is $f^*(\bar{x}^{(i)})/b$. The value b is typically estimated with another independent rendering algorithm such as bidirectional path tracing ([Lafortune and Willems 1993, 1996; Veach and Guibas 1994]). Then, the only problem is how to do the sampling.

Path-Space MLT.. The remaining problem in Equation (3) is how to get sample $\bar{x}^{(i)}$ in path space Ω with probability density $p(\bar{x}^{(i)})$ proportional to it's function value $f(\bar{x}^{(i)})$, which is so-called *importance sampling*. Veach[Veach and Guibas 1997] originally introduced Metropolis-Hastings (MH) sampling ([Hastings 1970; Metropolis et al. 1953]) into computer graphics and proposed a novel rendering algorithm, Metropolis Light Transport (MLT).

The mathematical foundation of MH sampling is building a Markov chain whose stationary distribution is exactly the target distribution. After reaching its stationary distribution, every subsequent mutation state of the Markov chain is a valid sample of the target distribution. However, it's never easy to know when the stationary is reached. MH sampling provides a practically way to solve this problem. It generates a set of samples from a non-negative function f that is distributed proportionally to $f's$ value without waiting for the stationary and whithout requiring the evaluation of b in two steps:

- *Proposal.* A new proposal path $\bar{y}$ is obtained from current path $\bar{x}$ by a mutation kernel $Q(\bar{y}|\bar{x})$.
- *Acceptance-Rejection.* Acceptance rate α is calculated:

$$\alpha(\bar{y}|\bar{x}) = min\{1, \frac{f^*(\bar{y})Q(\bar{x}|\bar{y})}{f^*(\bar{x})Q(\bar{y}|\bar{x})}\} \tag{4}$$

. Then accept the proposal path $\bar{y}$ with the probability α, otherwise reject it.

Mutation kernel Q is the key factor increasing acceptance rate and computation efficiency. In rendering algorithms, mutation kernel Q actually is the strategy to get proposal paths. The original MLT designs three mutation strategies targeting specific families of light paths, such as caustics or paths containing sequences of specular-diffuse-specular interactions. Some other mutation strategies have been developed in variants of path-space MLT, such as MEMLT [Jakob and Marschner 2012] for specular interactions, HSLT [Hanika et al. 2015] for rough materials, and GeoMLT [Otsu et al. 2018], which incorporates visibility into mutation strategies of MCMC rendering.

Primary Sample Space MLT.. Mutating in path space is not symmetric, which means that $Q(\bar{x}|\bar{y}) \mathrel{!=} Q(\bar{y}|\bar{x})$. This makes implementing these rendering algorithms a significant undertaking and makes debugging a notorious task. [Kelemen et al. 2002] presented a new MLT-type algorithm, Primary Sample Space MLT (PSSMLT), whose mutation strategy works in the unit cube of pseudo-random numbers, which is symmetric and easy to implement. The unit cube is called Primary Sample Space U.

PSSMLT converts random numbers $\bar{u}$ into light path $\bar{x}$ by path sampling. $\bar{x}$ can be thought of as being mapped from $\bar{u}$ by the inverse cumulative distribution function $P^{-1}(\bar{u})$, where $P(\bar{u}) = \int_0^{\bar{u}} p(\bar{u}')d\bar{u}'$. That is: $\bar{x} = P^{-1}(\bar{u})$. For the brevity of notation, we define:

$$C(\bar{x}) = \frac{f(\bar{x})}{p(\bar{x})}$$
$$\hat{C}(\bar{u}) = C(P^{-1}(\bar{u})) = C(\bar{x})$$
$$\hat{p}(\bar{u}) = p(P^{-1}(\bar{u})) = p(\bar{x})$$
$$\hat{h}_j(\bar{u}) = h_j(P^{-1}(\bar{u})) = h_j(\bar{x})$$

Then, in primary sample space, Equation (1) can be expressed as:

$$I_j = \int_U \hat{h}_j(\bar{u})\hat{C}(\bar{u})d\bar{u} \tag{5}$$

Like Equation (3), the Monte Carlo integration estimator of Equation (5) can be presented as:

$$I_j \approx \hat{I}_j = \frac{b}{N}\sum_{i=1}^{N} \frac{\hat{h}_j(\bar{u}^{(i)})\hat{C}(\bar{u}^{(i)})}{\hat{C}^*(\bar{u}^{(i)})} \tag{6}$$

, where $\hat{C}^*$ is the luminance of the importance function $\hat{C}$. Mutating in primary sample space is symmetric. That is $Q(\bar{u}|\bar{v}) = Q(\bar{v}|\bar{u})$, which makes implementing PSSMLT much easier, and which also avoids the computation of the transition probabilities altogether. So, the calculation of acceptance rate can be simplified as:

$$\alpha(\bar{v}|\bar{u}) = min\{1, \frac{\hat{C}^*(\bar{v})}{\hat{C}^*(\bar{u})}\} \tag{7}$$

, where $\bar{v}$ is the proposal state of $\bar{u}$ in primary sample space.

PSSMLT makes the integrand much flatter, and increases the average acceptance rate and therefore reduces the variance. The generality of PSSMLT has led to a lot of applications, such as [Gruson et al. 2017; Hachisuka and Jensen 2011; Hoberock and Hart 2010; Kitaoka et al. 2009; Šik et al. 2016]. As mentioned before, the method of conversion from random numbers $\bar{u}$ into light path $\bar{x}$ is path sampling. Actually, PSSMLT adopts bidirectional path tracing to do the path sampling. Bidirectional path tracing produces many paths from a single primary sample by using different connection strategies to connect camera sub-path and light sub-path. And then, the connected path of maximum luminance is selected as the final proposal light path. Unfortunately, the process of finding the path of maximum luminance is not efficient.

Multiplexed MLT.. In order to solve the efficient problem of PSSMLT, [Hachisuka et al. 2014] presented an extension to PSSMLT called Multiplexed Metropolis Light Transport (MMLT) which combines MCMC algorithm with Multiple Importance Sampling [Veach 1997]. Instead of always implementing all BDPT connection strategies and finding the path of maximum luminance, the algorithm chooses a single strategy according to an extra random number and returns its contribution scaled by the inverse discrete probability of the choice. The additional random number used for strategy selection can be mutated in the same way as the other components of $\bar{u}$ in primary sample space. The Monte Carlo Integration estimator is generalized as:

$$I_j \approx \hat{I}_j = \sum_{i=1}^{M} \frac{b}{N_t} \sum_{i=1}^{N_t} \frac{\hat{h}_j(\bar{u}^{(i,t)})\hat{w}_t(\bar{u}^{(i,t)})\hat{C}(\bar{u}^{(i,t)})}{\hat{C}^*(\bar{u}^{(i,t)})} \tag{8}$$

, where, t is determined by the extra state dimension and is used to choose sample technique, and $\hat{w}_t$ is a weighting function for any given path. The extension of the extra state dimension for choosing sample technique results that every mutation may change both random numbers from $\bar{u}$ to $\bar{v}$ and sample technique from t to t'. So, based on Equation (7), the calculation of acceptance rate is generalized:

$$\alpha((\bar{v}, t')|(\bar{u}, t)) = min\{1, \frac{\hat{w}_{t'}(\bar{v})\hat{C}^*_{t'}(\bar{v})}{\hat{w}_t(\bar{u})\hat{C}^*_t(\bar{u})}\} \tag{9}$$

This generalization produces the practical effect that the Metropolis sampler mainly focus on more effective strategies that leads to greater MIS-weighted contributions to the final image. Furthermore, the individual iterations of every Markov chain are much more efficient since they only execute a single connection strategy. Based on MMLT's approach, some developments have been made, such as H2MC[Li et al. 2015], which utilizes Hamiltonian Monte Carlo to adaptively control the shape of the transition kernels.

Bridging Path-Space and Primary-Sample-Space. In order to combine the flexibility of mutation strategies of path space MLT with the simplicity and efficiency of primary sample space MLT, several methods have been developed to bridge the two state spaces by using invertible mappings to transform samples between them, such as [Bitterli et al. 2018; Otsu et al. 2017; Pantaleoni 2017].

Analysis and Problem Statement. Generally, the improvements in MLT-type rendering algorithms can be classified into three categories. First, developments of mutation strategies in path space, such as MEMLT [Jakob and Marschner 2012] for specular interactions, HSLT [Hanika et al. 2015] for rough materials, and GeoMLT [Otsu et al. 2018], which incorporates visibility into mutation strategies. Second, developments of mutation strategies in primary sample space, such as MMLT and H2MC [Li et al. 2015]. Third, bridging the two spaces, so that the developments of mutation strategies in both spaces can be used in the same algorithm framework, such as [Bitterli et al. 2018; Otsu et al. 2017; Pantaleoni 2017].

All of these efforts are put on variant mutation strategies to increase the acceptance rate and to reduce the variance of their different aim scene scenarios. However, the ratio of proposal failures in traced paths is still a serious problem. For MMLT-related algorithms, things get worse, because they run many independent Markov chains with fixed depth of path for each chain, which increases the chance of confronting proposal failure paths.

3 OVERVIEW

The main idea of our method is distinguishing proposal failure paths from normal proposal paths and then excluding them from the states of Markov chain, so that speed up the convergence rate of the Metropolis-Hastings sampler.

First (Section 4), we verify the idea mathematically, and present a novel MCMC, Proposal Failure MCMC (PFMCMC), which simulates proposal failures by setting function value $f(X_t)$ to 0 with failure probability p_f and provides a special remedy for proposal failures. In the example based on normal distribution function, we can assign different value to p_f to simulate scenes with different ratio of proposal failure and to compare the results sampled with or without the remedy available.

Second (Section 5), we combine PFMCMC with MMLT and present a novel light transport algorithm, Proposal Failure Metropolis Light Transport (PFMLT). In order to exhibit the advantages of PFMLT over MMLT, we test several different scenes with challenging lighting, geometry and material. PFMLT always produces much better results than MMLT does in the tests, as shown in Figure 1 and Figure 3.

4 METHOD

An adaptive MCMC algorithm would automatically tune the parameters of its proposal distribution, [Brooks et al. 2011; Shaby and Wells 2010]. Some adaptive MCMC algorithms have been developed for various problems in different fields, such as [Giordani and Kohn 2006; Haario et al. 2001; Roberts and Rosenthal 2009; Vihola 2012]. All of these algorithms consider that proposals never be impossible values of the states of the stationary distribution of Markov chain, which is not true for MLT-type rendering algorithms. For example, Multiplexed Metropolis Light Transport (MMLT) [Hachisuka et al. 2014] has trouble searching light-carrying paths in Breakfast scene (As exemplified in Figure 1): about 37 percent of the total generated paths are proposal failure paths, which are impossible values of the stationary distribution of any Markov chain. Though proposal failure paths, as impossible values of the stationary distribution of Markov chain, don't affect the final equilibrium, they slow down the convergence rate of the Metropolis-Hastings sampler. So, exclusion of proposal failure paths could be effective to speed up the convergence rate of the Metropolis-Hastings sampler.

In this section, we will verify the idea mathematically. First, we present a novel MCMC, Proposal Failure MCMC (PFMCMC), which simulates proposal failures by setting function value $f(X_t)$ to 0 with failure probability p_f and provides a special remedy for proposal failures. Then, we take a simple example to show that exclusion of proposal failures is truly helpful to speed up the convergence rate and produce better results with the same computation.

4.1 Math

Our algorithm, PFMCMC, restricts focus to the case of proposal failure. PFMCMC is an extension of Metropolis Hastings (MH) [Hastings 1970; Metropolis et al. 1953], one of the most popular, classical MCMC algorithms. PFMCMC proceeds according to Algorithm 1. We aggregate some important terms of our notation in Table 1 for reference.

Table 1: Table of notation

Symbol	Explanation
X	Sample result set
X_0	Initial sample
X^*	Proposal sample
X_t	Current sample
X_{t+1}	Next sample
$f(X^*)$	Function value of proposal sample
$f(X_t)$	Function value of current sample
$sTotalNum$	Set number of total proposal
$rTotalNum$	Real number of total proposal
$TotalRep$	Total number of repetition for proposal failure
j	Proposal counter
p_f	Probability of proposal failure
α	Acceptance rate

From Algorithm 1, we can see that the basic structure of PFMCMC is similar to MH's. The major modification is that PFMCMC introduces probability p_f to simulate proposal failure, which is corresponding to the red part (line 4 to 7) in the pseudocode of Algorithm 1, and provides a remedy for the failure situation in the blue part (line 8 to 11).

Require: initial values X_0, $sTotalNum$, and failure possibility p_f.
Ensure: output sample set X

1: **for** $j = 0$ to $sTotalNum$ **do**
2: Draw a proposal value X^* from X_t
3: Calculate $f(X^*)$
4: Draw $v \sim U(0, 1)$.
5: **if** $v < p_f$ **then**
6: $f(X^*) \leftarrow 0$
7: **end if**
8: **if** $(f(X^*) == 0)$ **then**
9: $j - -$
10: **continue**
11: **end if**
12: Calculate acceptance rate α. $\alpha = \min\{1, \frac{f(X^*)}{f(X_t)}\}$
13: Draw $u \sim U(0, 1)$.
14: **if** $u < \alpha$ **then**
15: $X_{t+1} \leftarrow X^*$
16: **else**
17: $X_{t+1} \leftarrow X_t$
18: **end if**
19: **end for**
20: **return** X

Algorithm 1: *Proposal Failure MCMC (PFMCMC).* This is an extension of MH sampling, which simulates proposal failures by setting function value $f(X_t)$ to 0 with failure probability p_f (as shown in the red part) and gives a special remedy to proposal failures (as shown in blue part).

Proposal Failure Simulation. $f(X_t)$ is the function of sample X_t (we think of samples with function values being zero as proposal failures). p_f indicates the overall probability of proposal's random failure. Generally, the percentage of zero-contribution paths in MMLT for our test scenes, like Breakfast scene in Figure 1, is between 35% and 65%. So, usually, reasonably, users can set p_f at a value in $[0.35, 0.65]$ to verify the effectiveness of PFMCMC.

Proposal Failure Remedy. The basic idea of the remedy for proposal failures is excluding them from the states of Markov chain. To be specific, as shown in the blue part (line 8 to 11), the first step is detecting proposal failures, and the second step is shifting the sample counter j backward to exclude proposal failures.

4.2 Example

In order to illustrate the effect of PFMCMC, we'll show how it can be used to sample a 1D model based on a normal distribution function with a probability p_f of proposal failure. The effect comparison of PFMCMC and MCMC is based on equal sampling time. Considering different systems, such Mac, Windows and Linux, have different performance, we use the total number of proposal $rTotalNum$ as an indicator of sampling overhead, which is more reliable because of the elimination of influences of system performance. For MCMC, $rTotalNum$ equals to $sTotalNum$. For PFMCMC, $rTotalNum$ is the additive result of $sTotalNum$ and $TotalRep$ which is related to p_f.

Parameter Setting. We are assuming that, for the normal distribution function, the mean is 3 and the standard deviation is 2. Considering that the percentage of proposal failure for most of scenes rendered with MMLT locates in the interval $[0.4, 0.6]$, we test two cases with p_f set at 0.4 and 0.6 respectively. For MCMC, $sTotalNum$ is set at 1600. For PFMCMC, in order to make $rTotalNum$ close at 1600, $sTotalNum$ setting is more complicate, which is affected by p_f: when p_f is 0.4, we set $sTotalNum$ at 960; when p_f is 0.6, we set $sTotalNum$ at 690.

Results Analysis. Figure 2 demonstrates the sample results of MCMC and PFMCMC. (a) and (d) show reference sample results sampled with MCMC in the condition of zero-p_f, which is the case of normal MH sampling; (b) and (e) show the results sampled with the extension of MCMC, which includes failure simulation and no remedy; (c) and (f) show the results sampled with the extension of MCMC, which is our PFMCMC including both failure simulation and remedy. From (b) and (c), we can see that PFMCMC gets better results than MCMC does when p_f being set to 0.4. From (e) and (f), we can see that PFMCMC also gets better results than MCMC does when p_f being set to 0.6.

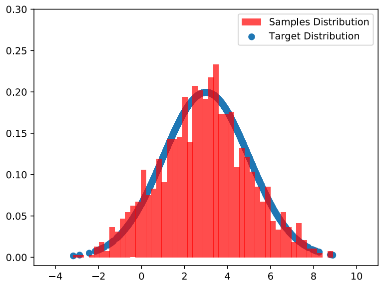
Table 2: Table of average result of 100 times

Reference (p_f==0, MCMC)	$rTotalNum$: 1600 RMSE: 0.004983	
	p_f==0.4	**p_f==0.6**
MCMC	$rTotalNum$: 1600 RMSE: 0.006834	$rTotalNum$: 1600 RMSE: 0.008760
PFMCMC, (similar $rTotalNum$)	$rTotalNum$: 1584 RMSE: 0.006370	$rTotalNum$: 1589 RMSE: 0.007728
PFMCMC, (similar RMSE)	$rTotalNum$: 1383 RMSE: 0.006787	$rTotalNum$: 1262 RMSE: 0.008686

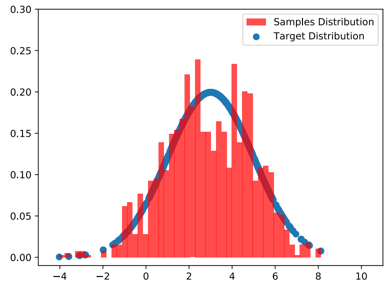
Because of the use of correlated samples, a single run of an MCMC integrator may not be representative. We test every case for 100 times and average the corresponding results, which are recorded in Table 2. Row "PFMCMC, (similar $rTotalNum$)" of the table shows that PFMCMC produces higher quality results than MCMC with similar overhead ($sTotalNum$), which supports the results shown in Figure 2 well. Less RMSE means higher quality. But, RMSE numbers may not be intuitive enough. In order to fully exhibit the superiority of PFMCMC, we tune its parameters so that it produces similar quality results to MCMC, which is shown in Row "PFMCMC, (similar RMSE)" of the table. We can see that PFMCMC pays much less overheads to achieve results similar to MCMC's. For example, in the case "p_f==0.4", PFMCMC only pays 1383/1600($\approx$ 0.8643) of MCMC's overheads, and in the case "p_f==0.6", the ratio is 1262/1600($\approx$ 0.7888).

5 IMPLEMENTATION

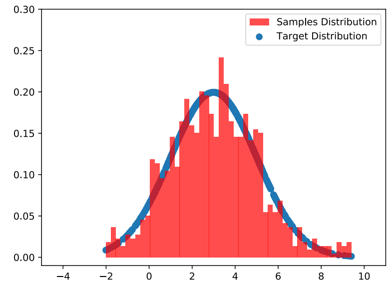
Although we can integrate PFMCMC framework with any current MLT-type algorithms with mutation in primary sample space or path space, in this section, we will describe our rendering algorithm, Proposal Failure Metropolis Light Transport (PFMLT), by recasting MMLT ([Hachisuka et al. 2014]) in the framework of PFMCMC. MMLT improves rendering efficiency by selecting seed path



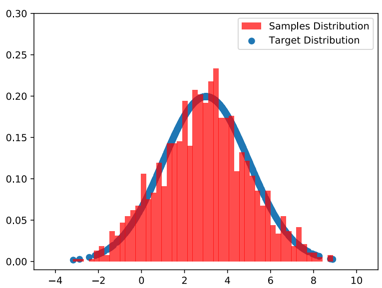
(a) $p_f = 0$, MCMC, RMSE = 0.004006



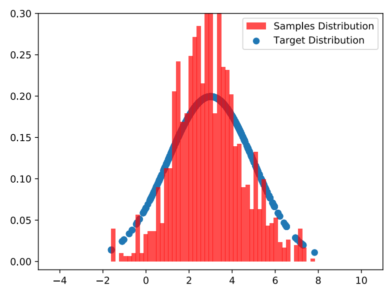
(b) $p_f = 0.4$, MCMC, RMSE = 0.007173



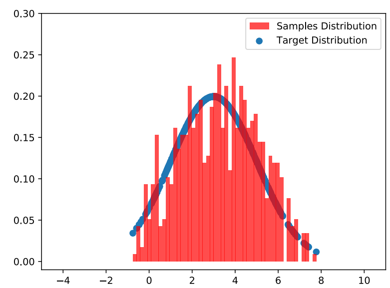
(c) $p_f = 0.4$, PFMCMC, RMSE = 0.004836



(d) $p_f = 0$, MCMC, RMSE = 0.004006



(e) $p_f = 0.6$, MCMC, RMSE = 0.008756



(f) $p_f = 0.6$, PFMCMC, RMSE = 0.006296

Figure 2: *PFMCMC is used to sample 1D normal distribution model with proposal failure probability p_f.* The total number of proposal is used as an indicator of sampling overhead. All of these results are produced with equal number of total proposal, 1600. (a) and (d) show reference sample results sampled with MCMC in the condition of zero-p_f, which is the case of normal MH sampling; (b) and (e) show the results sampled with the extension of MCMC, which includes failure simulation and no remedy; (c) and (f) show the results sampled with the extension of MCMC, which is our PFMCMC including both failure simulation and remedy. From (b) and (c), we can see that PFMCMC gets better results than MCMC does when p_f being set to 0.4. From (e) and (f), we can see that PFMCMC also gets better results than MCMC does when p_f being set to 0.6.

with probability proportion to its contribution to the final image. Fusing MMLT with the idea of proposal failure remedy of PFMCMC makes PFMLT even much more efficient than MMLT. For example, rendering Breakfast scene (As exemplified in Figure 1) with PFMLT saves about 30 percent of time to produce the same quality image as MMLT does. Algorithm 2 shows the pseudocode of PFMLT.

Initialization. First, the same as MMLT, we sample a seed path $\bar{s}$ from Initial Path Set and calculate it's depth k and contribution $f(\bar{x}_s)$. The depth k should be emphasized here, like MMLT, all of subsequent proposal paths are of the same depth as this seed path. Seed path $\bar{s}$ should be assigned to current path $\bar{x}$. $sTotalNum$ can be obtained by dividing total proposal number by total number of Markov chains. Then, some special variables for implementing the core idea of PFMCMC should be initiated. Proposal failure flag $IsFailure$ is initiated as 0.

Random Numbers Proposal. For MMLT, the same as PSSMLT, two main steps are needed to draw a proposal $\bar{y}$ from $\bar{x}$ in path space. Firstly, draw a proposal vector of random numbers $\bar{v}$ from $\bar{u}$ which is the primary sample space counterpart of $\bar{x}$. Secondly, obtain $\bar{y}$ by path sampling in path space using the proposal vector of random numbers $\bar{v}$. As to the first step, note that we apply normally distributed perturbations to each component of the vector of random numbers. The advantage of sampling with a normal distribution like this is that it naturally tries a variety of mutation sizes. It preferentially makes small mutations that remain close to the current state, which help locally explore the path space in small areas of high contribution.

Path Sampling and Potential Failures. As mentioned in "Random Numbers Proposal" part, the second step of path proposal is path sampling. It is this step where proposal failures occur. The proposal vector of random numbers $\bar{v}$ has four main uses in path sampling. First, one random number, v_s, of $\bar{v}$ is used to choose sample technique, so that t and s are determined. Second, a sub-vector, $\bar{v}_{camera}$, of $\bar{v}$, along with t, is used to sample a camera sub-path $\bar{y}_{camera}$ with depth exactly being t. Third, a sub-vector, $\bar{v}_{light}$, of $\bar{v}$, along with s, is used to sample a light sub-path $\bar{y}_{light}$ with depth exactly being s. Fourth, a sub-vector, $\bar{v}_{connect}$, of $\bar{v}$ is used to connect $\bar{y}_{camera}$

Ensure: Accumulation of Path Contributions

1: Sample a seed path $\bar{s}$ from Initial Path Set
2: Calculate the depth of $\bar{s}$, k and path contribution $f(\bar{x}_s)$
3: $\bar{s}$ is used as current path $\bar{x}$: $\bar{x} \leftarrow \bar{s}$, $f(\bar{x}) \leftarrow f(\bar{x}_s)$
4: $IsFailure \leftarrow 0$
5: **for** $j = 0$ to $sTotalNum$ **do**
6: Draw a proposal vector of random numbers $\bar{v}$ from $\bar{u}$.
7: $t \leftarrow \text{int}((k+2)v_s)$
8: $s \leftarrow (k+1) - t$
9: **if** !($\bar{y}_{camera} \leftarrow$ SampleCameraSubpath($\bar{v}_{camera}, t$)) **then**
10: $IsFailure \leftarrow 1$
11: **end if**
12: **if** !($IsFailure$) **then**
13: **if** !($\bar{y}_{light} \leftarrow$ SampleLightSubpath($\bar{v}_{light}, s$)) **then**
14: $IsFailure \leftarrow 1$
15: **end if**
16: **end if**
17: **if** !($IsFailure$) **then**
18: **if** !($f(\bar{y}) \leftarrow$ Connect($\bar{y}_{camera}, \bar{y}_{light}, \bar{v}_{connect}$)) **then**
19: $IsFailure \leftarrow 1$
20: **end if**
21: **end if**
22: **if** ($IsFailure$) **then**
23: $j--$
24: REJECT()
25: $IsFailure \leftarrow 0$
26: **Continue**
27: **end if**
28: $IsFailure \leftarrow 0$
29: Calculate acceptance rate α. $\alpha = \min\{1, \frac{L(f(\bar{y}))}{L(f(\bar{x}))}\}$
30: Draw $u \sim U(0,1)$.
31: **if** $u < \alpha$ **then**
32: $\bar{x} \leftarrow \bar{y}$
33: ACCEPT()
34: **else**
35: REJECT()
36: **end if**
37: AccumulatePathContribution($f(\bar{x}), f(\bar{y})$)
38: **end for**

Algorithm 2: *PFMLT (Proposal Failure MLT).* The blue part is the simple remedy for proposal failures. If proposal failure paths are detected, we exclude them from the states of Markov chain by shifting sample counter j backward before getting into the step of calculating acceptance rate

and $\bar{y}_{light}$ to make a complete proposal path $\bar{y}$ and to calculate proposal path contribution $f(\bar{y})$.

Except for the first stage, the other three are of potential failures. In the sampling sub-path stages, the depth of camera sub-path $\bar{y}_{camera}$ may not be t, or the depth of light sub-path $\bar{y}_{light}$ may not be s, so failures happen. In the connecting stage, the connection between the two sub-paths, $\bar{y}_{camera}$ and $\bar{y}_{light}$, may be blocked, which is a very high probability event. The existence of these failures is the very reason why PFMLT is much more efficient than MMLT.

Failure Remedy. $IsFailure$ is used to indicate whether any failure has happened. If any failure is detected, the flag $IsFailure$ is set to 1. Note that if a failure has been detected earlier, the subsequent path sampling programs are not necessary to run.

If proposal failure paths are detected, we exclude them from the states of Markov chain by shifting sample counter j backward before getting into the step of calculating acceptance rate, which is shown in Line 22 to Line 27 of Algorithm 2.

Accept Probability. Considering that e used the same mutation function for getting the vector of random numbers $\bar{v}$ from $\bar{u}$ as MMLT. Since these mutations are all symmetric, transition probability density functions are not needed to evaluate. The acceptance probability α is simply the ratio of the luminosities of proposal path contribution $f(\bar{y})$ and current path contribution $f(\bar{x})$.

Contribution Accumulation. The same as MMLT, the scaling by the reciprocal of the discrete probability density of the selected path length as noted before, we also need to scale each contribution by the number of techniques $k+2$. This scaling corresponds to the fact that the chain explores $k+2$ different sub-spaces.

6 RESULTS

This section includes two subsections, "Main Results" and "Extra Results". In the "Main Results", we implement our method by extending the system of PBRT, and we compare against MMLT implemented in the same system. In the "Extra Result", we show that our method can be easily used to extend other systems and extend both RJMLT and MMLT in Tungsten [Bitterli 2017], a renderer that is a much simpler and faster than PBRT, and we compare against MMLT and RJMLT implemented in Tungsten.

6.1 Main Results

We implement our method by extending the system of PBRT, and we compare against MMLT implemented in the same system. Five scenes — Breakfast (1024 × 1024), Villa (1200 × 580), Bathroom (1280 × 720), Bidir (768 × 576), Living Room (1280 × 720) — with different geometry, lighting and material configurations are rendered using Mac pro with Intel Core i5 at 2.7GHz. Villa scene reference is rendered using MMLT in PBRT and the references of the other four scenes are rendered with BDPT in PBRT. Rendering each of those references costs several days of computation. We set the maximum path length at 9 for Living Room scene and at 5 for the other four scenes.

Equal-time Comparison. We show the image comparison results in Figure 1 and Figure 3. To make equal-time comparisons between MMLT and our method, we rendered all the five scenes. Because of extra overheads for failure remedy in our method, the parameter of mutations per pixel should be set at a smaller value than in MMLT, as show Column "Mutations Per Pixel" of Table 3.

In order to obtain the statistic data of these comparisons, we define some counters in Algorithm 2: $TotalPaths$ is the total number of paths traced; $RemedyPaths$ is the number of paths traced for the remedy of proposal failures, which is the extra overhead of our method; $FinalFailurePaths$ is the number of final failure paths after remedy; $AcceptancePaths$ is the number of paths accepted. All of these counters should be initialized to 0 before the beginning of rendering (Line 5 of Algorithm 2). In the Algorithm 2, we add some other code for statistics: "$TotalPaths{+}{+}$;" at Line 6; "$RemedyPaths{+}{+}$;" at Line 23; "$if(\alpha < 0.000001) FinalFailurePaths{+}{+}$;" at Line 29; "$AcceptancePaths{+}{+}$;" at Line 32. We use the following equations to calculate $FailureRate$ and $AcceptanceRate$.

$$FailureRate = \frac{FinalFailurePaths}{(TotalPaths - RemedyPaths)}$$
$$AcceptanceRate = \frac{AcceptancePaths}{(TotalPaths - RemedyPaths)}$$

These statistics are also recorded in Table 3.

Scenes	Algorithms	Mutations Per Pixel	Failure Rate	Acceptance Rate	RMSE
Breakfast	MMLT	150	37.37%	49.38%	0.082572
	Ours	100	0%	77.84%	0.069976
Villa	MMLT	200	62.82%	32.39%	0.083772
	Ours	80	0%	75.26%	0.070404
Bathroom	MMLT	280	40.00%	57.00%	0.120891
	Ours	150	0%	89.01%	0.103564
Bidir	MMLT	75	37.40%	57.82%	0.067776
	Ours	45	0%	90.18%	0.056806
Living Room	MMLT	670	51.12%	45.07%	0.129880
	Ours	310	0%	78.35%	0.102854

Table 3: Table of statistics of equal-time comparison (2 hours for Breakfast, 1.5 hours for Villa, 1 hour for Bathroom, 25 minutes for Bidir, and 110 minutes for Living Room). Because of extra overheads for failure treatment in our method, the parameter of mutations per pixel should be set to a smaller value than in MMLT. We also show the comparisons of failure rate, acceptance rate and RMSE for these scenes. From the table, we can see that our method always produce better results (smaller RMSE) than MMLT does.

Breakfast Scene. Figure 1 shows an equal-time (2 hours) comparison on the Breakfast Scene which is illuminated by two lamps. Part of the scene, like the objects on the top of the desk, get direct illumination, however, the lighting for objects under the desk or over the lamps is much more complicated. From Figure 1 and Table 3, we can see that MMLT shows suboptimal performance because about 37.37% of the paths that were used to reconstruct image carried no radiance. Giving special treatment to zero-contribution paths, our method reduces the percentage to around 1.54% and exhibits much better results with smaller RMSE. Our method distinguishes itself in those difficult settings where a large fraction of all of the possible proposal paths fail to carry any radiance in MMLT rendering, as shown in the three insets of Figure 1.

Villa Scene. The first part of Figure 3 shows an equal-time (1.5 hours) comparison on the Villa scene with complex materials and a difficult geometry configuration lit by outside environment daylight. This is a challenging scene because of the hard-to-find specular-diffuse-specular (SDS) light paths between the villa interior and the near-specular glass windows. The reference rendered by MMLT with 10000 mutations per pixel is still slightly noisy after several days of computation. Also, from Figure 3 and Table 3, we can see that MMLT produces a lot of noises because more than 62.81% of the paths that were used to reconstruct image were zero-contribution. Our method considers zero-contribution proposal as failure and gives special treatment to it. As a result, the percentage was reduced to about 14.47 and higher quality image was obtained. If our method is used to render the image with the same RMSE of 0.083772 like MMLT does, more than 37% of the time will be saved.

Bathroom Scene. The second part of Figure 3 shows an equal-time (1 hour) comparison on the Bathroom scene which contains several different materials including diffuse, specular, and glossy. The scene is illuminated with a large area light source directly visible from the camera. Unlike the Villa scene, the major part of the scene is directly illuminated by the light source. Even in such a simple lighting situation, our method can still produce a better image than MMLT does in equal time. If our method is used to render the image with the same RMSE of 0.120981 like MMLT does, more than 45% of the time will be saved.

Bidir Scene. The third part of Figure 3 shows an equal-time (25 minutes) comparison on the Bidir scene which is very classical. The illumination resembles the Breakfast scene whose part of objects get direct lighting and other parts not. If our method is used to render the image with the same RMSE of 0.067776 like MMLT does, about 28% of the time will be saved.

Living Room Scene. The last part of Figure 3 shows an equal-time (110 minutes) comparison on the Living Room scene. Light coming from outside environment enter the room through glass widows like the Villa scene does. Again, this kind of lighting makes rendering the scene a challenging task. What's more, the materials of the Living Room scene is more complex. The floor, the table and the paneling are made of substrate material, a layered model that varies between glossy specular and diffuse reflection depending on the viewing angle. The cups and the bottle on the table are glass. Other objects like the big mirror and the brushed stainless steel lampshades also contribute to the complication of materials of the scene. All of this result that more than 51.12% of all paths that were used to reconstruct image in MMLT don't carry any radiance. Again, our method is even more efficient in rendering scenes with complex material and lighting. If our method is used to render the image with the same RMSE of 0.129880 like MMLT does, about 47% of the time will be saved.

Convergence Comparison. To make our method more convincing, we did a sequence of comparisons with different computations and compared the convergence of MMLT and our method. We used ABCDEF and OPQRST to number the resulting images rendered by MMLT and our method respectively. The mutations per pixel of the images rendered with similar time were set as Table 4. The resulting comparison images were shown in Figure 4-(a).

MMLT	A	B	C	D	E	F
	20	40	85	170	340	670
Ours	O	P	Q	R	S	T
	10	20	35	75	150	310

Table 4: Table of parameter setting of mutations per pixel for convergence comparison.

Based on the comparison images and their error images in Figure 4-(a), we can see that our method converges much faster than

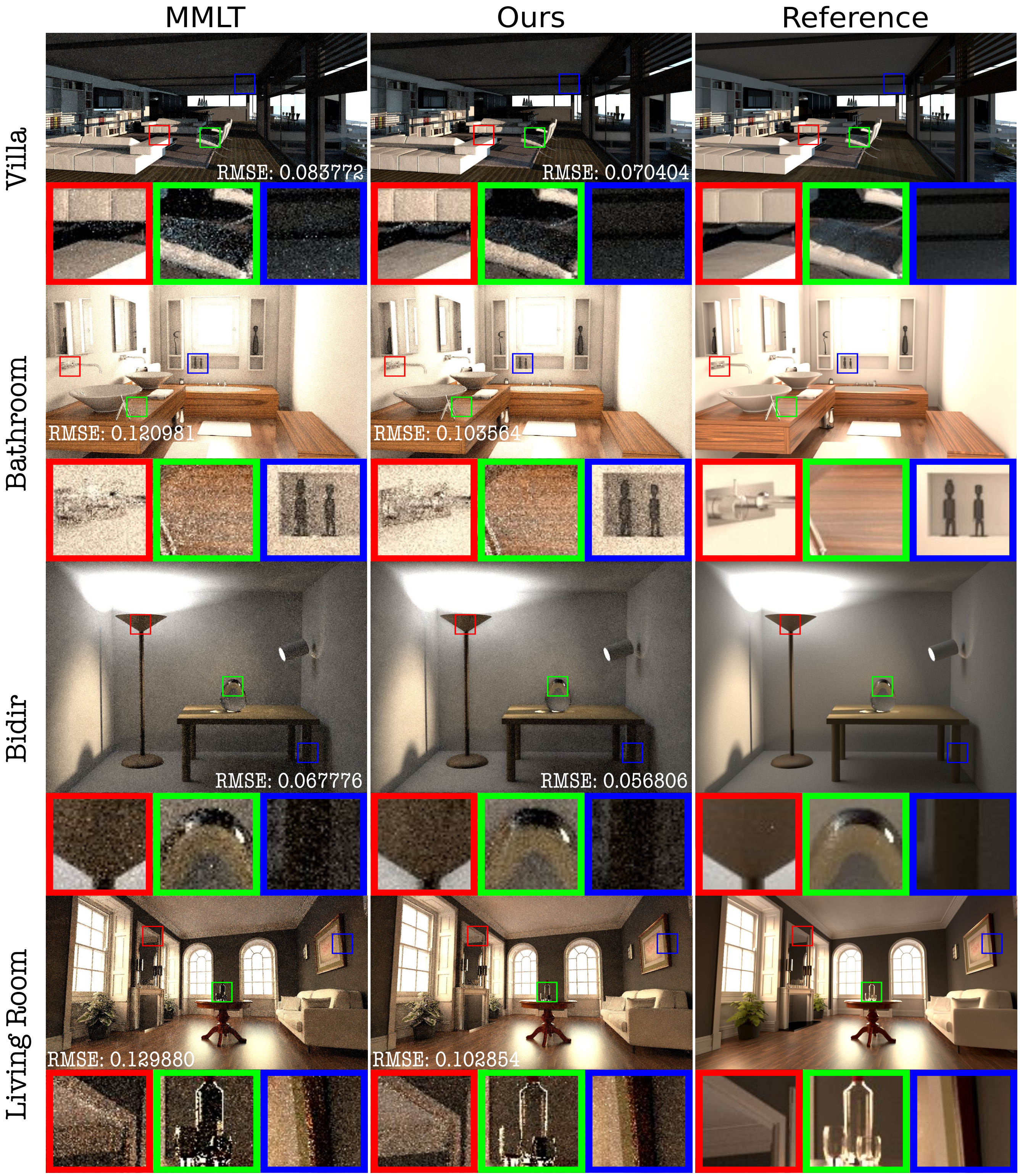
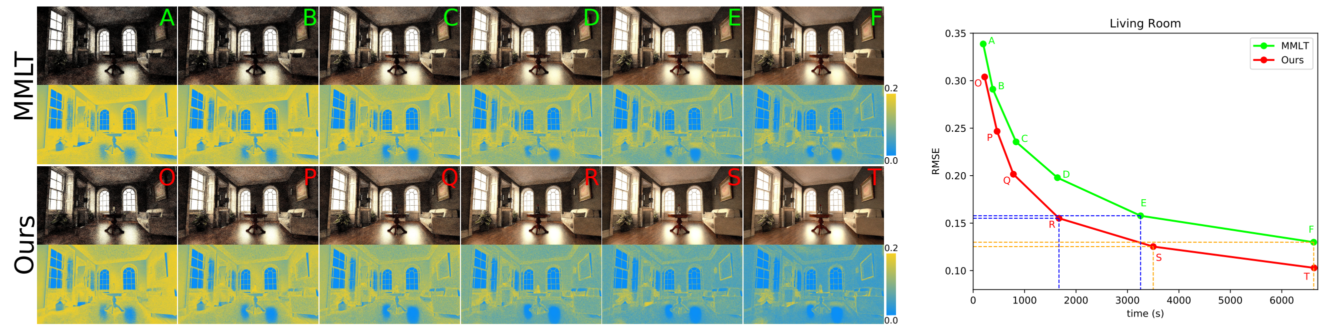


Figure 3: ***Equal-time comparisons.*** **These challenging scenes, Villa, Bathrooom, Bidir and Living Room, are of various complex material, lighting and geometry. RMSE is used as the indicator of image quality. Our method always produce better results, with smaller RMSE, than MMLT does.**



(a) Images comparisons **(b) Convergence plot**

Figure 4: *Convergence comparisons for the Living Room scene.* Based on the comparison images and their error images in (a), we can see that our method converges much faster than MMLT does. The RMSEs of each of these images are exhibited in line chart as (b). As the blue dash lines show, the RMSE of image R rendered with our method is slightly smaller than the RMSE of image E rendered with MMLT, but the rendering time of image R is just about half of image E. Also, as the orange dash lines show the comparison the RMSE of image S and image F, our method saves even bigger ratio of time.

MMLT does. To make this statement clearer, we calculated the RMSEs of each of these images and exhibited them in line chart as shown in Figure 4-(b).

As the blue dash lines mark in Figure 4-(b), the RMSE of image R rendered with our method is slightly smaller than the RMSE of image E rendered with MMLT, but the rendering time of image R is just about half of image E. Also, as the orange dash lines show the comparison the RMSE of image S and image F, our method saves even bigger ratio of time. So, we can deduce that: *the higher same-quality of images rendered by MMLT and our method, the bigger ratio of time will be saved by our method.*

6.2 Extra Results

In order to further verify the effectiveness of our method and the fact that other MLT-type algorithms can be easily extended with our method, we now do some comparison tests with RJMLT [Bitterli et al. 2018] with our method. We extend both RJMLT and MMLT in Bitterli's original source code, Tungsten [Bitterli 2017], a renderer that is a much simpler and faster than PBRT, and get two new algorithms, RJMLT+PF and MMLT+PF ("PF" means Proposal Failure). Considering that the random seeds used in RJMLT are not fixed, the result of any single run of RJMLT may not be representative, so average behavior of many times of rendering with exactly same settings is used to do comparison test with other algorithms, which is the very reason that we don't include RJMLT in all of tests of the prior scenes as shown in Figure 3.

Here, we'll compare equal-time images of the Glass-Of-Water scene rendered with the four algorithms in Tungsten, MMLT, RJMLT, MMLT+PF, RJMLT+PF. We set the maximum path length at 15. Expected results should be: MMLT+PF is better than MMLT, and RJMLT+PF is better than RJMLT. First, we produce the two five-minute images of MMLT and MMLT+PF. By the way, five-minute images are of relatively high quality because of the fact that Tungsten is much simpler and faster than PBRT. Second, we find the parameters of RJMLT and RJMLT+PF to produce five-minute images. Third, we run RJMLT and RJMLT+PF with the parameters 50 times to produce 50 images respectively. Fourth, calculate the average RMSEs of RJMLT and RJMLT+PF with their own 50 images respectively. The Results are shown in Figure 5. For RJMLT and RJMLT+PF, while the two images are specially chosen from their own 50 images, which may not be representative, the RMSEs are the average of 50 results, which is reliable. In this scene, based on the RMSEs, we can see that: both RJMLT and MMLT+PF are better than MMLT; RJMLT+PF is the best (of course better than RJMLT); in particular, we get a bonus: *MMLT+PF is better than RJMLT*, which means that PF makes MMLT not just better than MMLT, but also better than RJMLT.

7 CONCLUSION

We first came up with the concept of *Proposal Failure*, and then presented a novel extension of MCMC, Proposal Failure MCMC (PFMCMC). A general example shown the effectiveness of PFMCMC. Furthermore, we applied PFMCMC to light transport simulation, getting a novel rendering algorithm, PFMLT. With 5 challenging scenes tested, PFMLT was verified to be much more efficient than MMLT. In order to further demonstrate that our algorithm is easy to implement and can be used to extend other MLT-type rendering algorithms, we extended MMLT and RJMLT in Tungsten renderer. With the test on the Glass-Of-Water scene, we exhibited that our method made RJMLT more efficient than the original RJMLT, and even made MMLT more efficient than the original RJMLT. All of these confirmed the general effectiveness of our method, the remedy for proposal failures.

7.1 Limitations and Future Work

As a MLT-type rendering algorithm, generally, our method is also only good at rendering challenging scenes which contain complex materials and lighting. Like other adaptive MLT-type algorithms, such as *MEMLT* [Jakob and Marschner 2012], H^2MC [Li et al. 2015],

Figure 5: *Equal-time comparisons with RJMLT.* "MMLT+PF" and "RJMLT+PF" are Proposal Failure extensions of MMLT and RJMLT respectively. Considering that the random seeds used in RJMLT are not fixed, the result of any single run of RJMLT may not be representative, so average behavior of many times of rendering with exactly same settings is used to do comparison test with other algorithms. For RJMLT and RJMLT+PF, while the two images are specially chosen from their own 50 images, which may not be representative, the RMSEs are the average of 50 results, which is reliable. In this scene, based on the RMSEs, we can see that: both RJMLT and MMLT+PF are better than MMLT; RJMLT+PF is the best (of course better than RJMLT); in particular, we get a bonus: *MMLT+PF is better than RJMLT*, which means that PF makes MMLT not just better than MMLT, but also better than RJMLT.

HSLT [Hanika et al. 2015] and *GeoMLT* [Otsu et al. 2018], extra computations are needed to address challenging parts of scenes. So, the number of mutations per pixel that can be finished in equal time as non-adaptive algorithms decreases, which may slightly affect the rendering of simple parts of the same scene. Our method cannot change/improve the original nature of the MLT-type algorithm that is extended with our method. For example, RJMLT+PF inherited the nature that the random seeds used are not fixed from RJMLT, which means that the result of any single run of RJMLT+PF may also not be representative, so average behavior of many times of rendering with exactly same settings must be used to show its effectiveness.

Providing remedies to minimize the impact of proposal failure without modifying mutation strategies is a new direction to improve the efficiency of MLT-type algorithms. We just presented a simple and crude remedy in this paper, and we believe that many more efficient and sophisticated remedies can be developed to further decrease the serious impact caused by proposal failures in the future research. Considering that MCMC is widely used in various fields, if necessary, the idea of remedy for proposal failures can be effectively introduced into these fields.

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