# 第二次

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**Problem 1. 《数值分析基础 (第二版) (关治, 陆金甫)》P226. 1.**

设 $f(x) = e^x, \ x \in [0, 2]$

1. $x_0 = 0.0$, $x_1 = 0.5$, 构造一次 Lagrange 插值多项式 $L_1$, 并计算 $L_1(0.25)$.
2. $x_0 = 0.5$, $x_1 = 1.0$, 构造一次 Lagrange 插值多项式 $L_1$, 并计算 $L_1(0.75)$.
3. $x_0 = 0.0$, $x_1 = 1.0$, $x_2 = 2.0$, 构造二次 Lagrange 插值多项式 $L_2$ 并计算 $L_2(0.25), L2(0.75)$.

*Solution.*

1.

$$
\begin{aligned}
L_{10} &= \frac{x - x_1}{x_0 - x_1} \\
L_{11} &= \frac{x - x_0}{x_1 - x_0} \\
L_1 &= f(x_0)L_{10}(x) + f(x_1)L_{11}(x) \\
&= e^{0.0} \cdot \frac{x - 0.5}{0.0 - 0.5} + e^{0.5} \cdot \frac{x - 0.0}{0.5 - 0.0} \\
&= 1 + (2e^{0.5} - 2)x \\
L_1(0.25) &= 1 + (2e^{0.5} - 2) \cdot 0.25 \approx 1.324
\end{aligned}
$$

2.

$$
\begin{aligned}
L_1 &= e^{0.5} \cdot \frac{x - 1.0}{0.5 - 1.0} + e^{1.0} \cdot \frac{x - 0.5}{1.0 - 0.5} \\
&= 2e^{0.5} - e + (2e - 2e^{0.5})x \\
L_1(0.75) &= 2e^{0.5} - e + (2e - 2e^{0.5}) \cdot 0.75 \approx 2.184
\end{aligned}
$$

3.

$$
\begin{aligned}
L_{20} &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\
L_{21} &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\
L_{22} &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}
\end{aligned}
$$

$$
\begin{aligned}
L_2 &= f(x_0)L_{20}(x) + f(x_1)L_{21}(x) + f(x_2)L_{22}(x) \\
&= e^0 \cdot \frac{(x-1)(x-2)}{(0-1)(0-2)} + e^{0.5} \cdot \frac{x(x-2)}{(1-0)(1-2)} + e^1 \cdot \frac{x(x-1)}{(2-0)(2-1)} \\
&= \frac{1}{2}(x-1)(x-2) - ex(x-2) + \frac{e^2}{2}x(x-1) \\
& L_2(0.25) \approx 1.153, \ L_2(0.75) \approx 2.012
\end{aligned}
$$

**Problem 2.** 《数值分析基础 (第二版) (关治, 陆金甫)》P226. 4.

设 $l_0, l_1, \cdots, l_n$ 是以节点 $x_0, x_1, \cdots, x_n$ 的 $n$ 次 Lagrange 插值基函数, 试证明

$$\sum_{k=0}^{n} l_k(0) x_k^j = \begin{cases} 0, & j = 1, 2, \cdots, n, \\ (-1)^n x_0 x_1 \cdots x_n, & j = n+1. \end{cases}$$

*Proof.* 令 $f(x) = x^j$, 则 $n$ 次插值多项式为 $L_n(x) = \sum_{k=0}^{n} l_k(x) x_k^j$. 当 $j = 1, 2, \cdots, n$ 时, 由插值多项式的性质可知, $\sum_{k=0}^{n} l_k(0) x_k^j = L_n(0) = f(0) = 0$. 当 $j = n+1$ 时, 对于 $\bar{x} = 0$, 存在 $\xi \in I[x_0, x_1, \cdots, x_n, 0]$, 使得

$$\sum_{k=0}^{n} l_k(0) x_k^j = f(0) - L_n(0) = \frac{f^{n+1}(\xi)}{(n+1)!}(0 - x_0) \cdots (0 - x_n) = (-1)^n x_0 x_1 \cdots x_n$$

□

**Problem 3.** 《数值分析基础 (第二版) (关治, 陆金甫)》P226. 5.

设 $f(x) = x^5 + 4x^3 + 1$, 试求均差 $f[0, 1, 2]$, $f[0, 1, 2, 3, 4, 5]$, $f[0, 1, 2, 3, 4, 5, 6]$.

*Solution.*

| | $k = 0$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $x_0 = 0$ | 1 | | | | | | |
| $x_1 = 1$ | 6 | 5 | | | | | |
| $x_2 = 2$ | 65 | 59 | 27 | | | | |
| $x_3 = 3$ | 352 | 287 | 114 | 29 | | | |
| $x_4 = 4$ | 1281 | 929 | 321 | 69 | 10 | | |
| $x_5 = 5$ | 3626 | 2345 | 708 | 129 | 15 | 1 | |
| $x_6 = 6$ | 8641 | 5015 | 1335 | 209 | 20 | 1 | 0 |

$$f[0, 1, 2] = 27$$
$$f[0, 1, 2, 3, 4, 5] = 1$$
$$f[0, 1, 2, 3, 4, 5, 6] = 0$$

**Problem 4.** 《数值分析基础 (第二版) (关治, 陆金甫)》P227. 12.

求次数不高于三次的多项式 $P$ 使其满足 $p(0) = p'(0) = 0$, $p(1) = 1$, $p(2) = 1$, 并写出其 Newton 形式的余项.

*Solution.*

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| | $p(0) = 0$ | | | |
| | $p(0) = 0$ | 0 | | |
| | $p(1) = 1$ | 1 | 1 | |
| | $p(2) = 1$ | 0 | $-\frac{1}{2}$ | $-\frac{3}{4}$ |

$$P(x) = x^2 - \frac{3}{4}x^2(x-1)$$
$$R_3(x) = f[0, 0, 1, 2, x]x^2(x-1)(x-2)$$

**Problem 5.** 《数值分析基础（第二版）(关治, 陆金甫)》P227. 13.

求次数不超过 4 次的多项式 $P$, 使其满足

$$p(1)=p(3)=0,\ p(2)=1,\ p'(1)=0,\ p''(1)=8;$$

并写出其 Newton 形式的余项.

*Solution.*

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p(1)=0$ | | | | |
| $p(1)=0$ | $p'(1)=0$ | | | |
| $p(1)=0$ | $p'(1)=0$ | $p''(1)=8$ | | |
| $p(2)=1$ | 1 | 1 | -7 | |
| $p(3)=0$ | -1 | -1 | -1 | 3 |

$$P_4(x)=8(x-1)^2-7(x-1)^3+3(x-1)^3(x-2)$$

$$R_4(x)=f[1,1,1,2,3,x](x-1)^3(x-2)(x-3)$$

**Problem 6.** 《数值分析基础（第二版）(关治, 陆金甫)》P227. 14.

设 $f(x)=\frac{1}{a-x}$, 证明

1. $f[x_0,x_1,\cdots,x_n]=\prod_{j=0}^{n}\left(\frac{1}{a-x_j}\right)$.
2. $\frac{1}{a-x}=\frac{1}{a-x_0}+\frac{1}{(a-x_0)(a-x_1)}(x-x_0)+\cdots+\frac{1}{(a-x_0)\cdots(a-x_n)}(x-x_0)\cdots(x-x_{n-1})+\frac{1}{(a-x_0)\cdots(a-x_n)(a-x)}(x-x_0)\cdots(x-x_n)$.

(提示: item 1 用归纳法; item 2 Newton 插值多项式)

*Proof.*

1. 当 $n=0$ 时,

$$f[x_0]=f(x_0)=\frac{1}{a-x_0}$$

假设当 $n=k$ 时, $f[x_0,x_1,\cdots,x_k]=\prod_{j=0}^{k}\left(\frac{1}{a-x_j}\right)$ 成立, 则当 $n=k+1$ 时,

$$\begin{aligned}
f[x_0,x_1,\cdots,x_{k+1}] &= \frac{f[x_1,\cdots,x_{k+1}]-f[x_0,\cdots,x_k]}{x_{k+1}-x_0}\\
&= \frac{1}{x_{k+1}-x_0}\left(\prod_{j=1}^{k+1}\left(\frac{1}{a-x_k}\right)-\prod_{j=0}^{k}\left(\frac{1}{a-x_k}\right)\right)\\
&= \frac{1}{x_{k+1}-x_0}\left(\frac{1}{a-x_{k+1}}-\frac{1}{a-x_0}\right)\prod_{j=1}^{k}\left(\frac{1}{a-x_k}\right)\\
&= \prod_{j=0}^{k+1}\left(\frac{1}{a-x_k}\right)
\end{aligned}$$

由数学归纳法可知, $\forall n\in\mathbb{N}$, $f[x_0,x_1,\cdots,x_n]=\prod_{j=0}^{n}\left(\frac{1}{a-x_j}\right)$.

2. Newton 插值多项式为

$$
\begin{aligned}
P_n(x) &= f[x_0] + f[x_0,x_1](x-x_0) + \cdots + f[x_0,x_1,\cdots,x_n](x-x_0)\cdots(x-x_{n-1}) \\
&= \frac{1}{a-x_0} + \frac{1}{(a-x_0)(a-x_1)}(x-x_0) + \cdots + \frac{1}{(a-x_0)\cdots(a-x_n)}(x-x_0)\cdots(x-x_{n-1})
\end{aligned}
$$

余项为

$$R_n(x) = f[x_0,x_1,\cdots,x_n,x](x-x_0)\cdots(x-x_n)$$

由 item 1 可知, $f[x_0,x_1,\cdots,x_n,x] = \prod_{j=0}^{n}\left(\frac{1}{a-x_j}\right)\cdot\frac{1}{a-x} = \frac{1}{(a-x_0)\cdots(a-x_n)(a-x)}$. 故

$$R_n(x) = \frac{1}{(a-x_0)\cdots(a-x_n)(a-x)}(x-x_0)\cdots(x-x_n)$$

由 $f(x)=\frac{1}{a-x}$ 可知, $\frac{1}{a-x} = P_n(x)+R_n(x)$.

□

**Problem 7. 《数值分析基础 (第二版) (关治, 陆金甫)》P227. 15.**

设 $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$ 有 $n$ 个不同的实根 $x_1,x_2,\cdots,x_n$, 试证明

$$\sum_{j=1}^{n}\frac{x_j^k}{f'(x_j)} = \begin{cases}0, & 0\leqslant k\leqslant n-2,\\ a_n^{-1}, & k=n-1.\end{cases}$$

*Proof.* 不妨设 $f(x)=a_n(x-x_1)(x-x_2)\cdots(x-x_n)$. 令 $P(x)=x^k$, $w_n(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$. 则存在 $\xi\in I(x_0,x_1,\cdots,x_n)$, 使得

$$P[x_1,x_2,\cdots,x_n] = \frac{P^{(n)}(\xi)}{n!} = P[x_1,x_2,\cdots,x_n] = \sum_{j=1}^{n}\frac{P(x_j)}{w_n'(x_j)} = \frac{1}{a_n}\sum_{j=1}^{n}\frac{x_j^k}{f'(x_j)}$$

当 $0\leqslant k\leqslant n-2$ 时, $P^{(n)}(\xi)=0$, $\sum_{j=1}^{n}\frac{x_j^k}{f'(x_j)}=0$.
当 $k=n-1$ 时, $P^{(n)}(\xi)=n!$, $\sum_{j=1}^{n}\frac{x_j^k}{f'(x_j)}=a_n^{-1}$. □