

# Negligible Obstructions & Turán Exponents

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# Turán Number

$\text{ex}(n, F) = \max$  number of edges in an  $n$ -vertex graph  
  $\uparrow$   
forbidden subgraph      contains no  $F$  as a subgraph.

8. Now I return to ordinary graphs, i.e. ( $r = 2$ ). The results of Stone, Simonovits and myself show that the most interesting open problems are if  $G$  is bipartite. I first state some of our favourite conjectures with Simonovits.

Is it true that for every bipartite  $G$  there is a rational  $\alpha_1 = \alpha(G_1) \leq \alpha < 2$  for which

$$\lim_{n \rightarrow \infty} T(n; G)/n^\alpha = c(G), \quad 0 < c(G) < \infty \quad (1)$$

exists? Is it true that for every rational  $\alpha$ ,  $1 \leq \alpha \leq 2$  there is a bipartite  $G$  for which the limit (1) exists? At present these conjectures which are about 20 years old are beyond our reach, and in fact we have no real evidence for their truth. We have no idea of the possible value of  $c(G)$ , it would perhaps be reasonable to assume that  $c(G)$  is algebraic.

CONJ

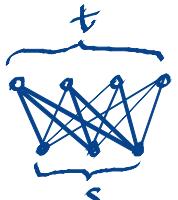
For every bipartite graph  $F$ , there exists  $r \in \mathbb{Q}$  st  $\text{ex}(n, F) = \Theta(n^r)$ .

[Erdős 1988]

CONJ (Rational Exponents).  $\forall$  bipartite  $F \exists r \in \mathbb{Q}$ :  $\text{ex}(n, F) = \Theta(n^r)$ .

## Classical Results

$$F = K_{s,t}.$$

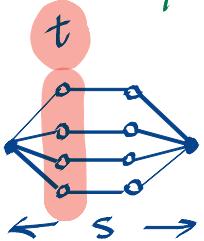


[Kővári - Sós - Turán]

$$\text{ex}(n, K_{s,t}) = O(n^{2-\frac{1}{s}}).$$

[Kollar - Rónyai - Szabó]  $\text{ex}(n, K_{s,t}) = \Omega(n^{2-\frac{1}{s}})$  when  $t \geq t_0(s)$ .

$$F = \Theta_{s,t}.$$



[Faudree - Simonovits].

$$\text{ex}(n, \Theta_{s,t}) = O(n^{1+\frac{1}{s}})$$

[Conlon 2014]

$$\text{ex}(n, \Theta_{s,t}) = \Omega(n^{1+\frac{1}{s}}),$$

when  $t \geq t_0(s)$ .

## Open Problems

$$\text{ex}(n, K_{4,4}) = \Theta(n^?)$$

$$\text{ex}(n, \Theta_{4,2}) = \Theta(n^?)$$

"beyond our reach"

CONJ (Realizability)  $\forall r \in \mathbb{Q} \cap (1, 2) \exists$  bipartite  $F$ :  $\text{ex}(n, F) = \Theta(n^r)$ .

Breakthrough [Bukh, Conlon 2015].

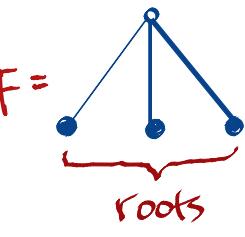
$\forall r \in \mathbb{Q} \cap (1, 2), \exists F$ :  $\text{ex}(n, F) = \Theta(n^r)$

a finite family of forbidden graphs

DEF A rooted graph  $F$  is a graph  $F$  equipped with  $R(F) \subseteq V(F)$

The density of  $F$ :  $\beta_F = \frac{|E(F)|}{\# \text{ of non-roots}}$ .  $\uparrow$   
root set.

The  $p$ -th power of  $F$ :  $F^p = p$  disjoint copies of  $F$ ,

EG  $F =$    $\rightarrow$    $= F^3$ .  
 $f = 3$

CONJ (Realizability)  $\forall r \in \mathbb{Q} \cap (1, 2) \exists$  bipartite  $F$ :  $\text{ex}(n, F) = \Theta(n^r)$ .

THM [Bukh - Conlon].

$\forall$  "balanced" rooted tree  $F$ ,  $\exists p \in N^+$ :  $\text{ex}(n, F^p) = \Omega(n^{2-\frac{1}{g_F}})$ .

a rooted graph, also a tree.  $F = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$   $g_F = 3$ .  $\text{ex}(n, F^p) = \Omega(n^{2-\frac{1}{3}})$   
 $\stackrel{k_{3,p}}{=}$   $p$  large

CONJ  $\forall g \in \mathbb{Q} \cap (1, \infty), \exists$  "balanced" rooted tree  $F$  with density  $g$ :

( $g$  is a BC-density)  $\forall p \in N^+ \text{ ex}(n, F^p) = O(n^{2-\frac{1}{g}})$ .

The Bukh - Conlon CONJ  $\forall$  "balanced" rooted tree  $F$ ,  $\forall p \in N^+$   
 $\text{ex}(n, F^p) = O(n^{2-\frac{1}{g}})$ .

RMK "balanced" condition is necessary in the above 2 conjectures.

CONT  $\forall g \in Q \cap (1, \infty)$ .  $\exists$  "balanced" rooted tree  $F$  with density  $g$ :

( $g$  is a BC-density)  $\forall p \in N^+$   $\text{ex}(n, F^p) = O(n^{2-\frac{1}{s}})$ .

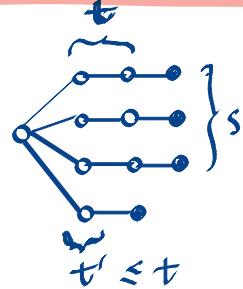
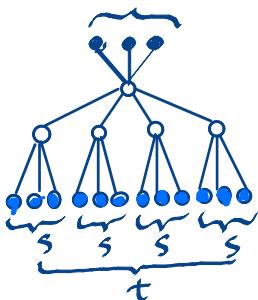
Known



$$g = s.$$

$$g = \frac{s}{s-1}$$

$$\leftarrow s \rightarrow$$



$$\beta_F = \frac{e(F)}{\# \text{ of non-roots.}}$$

Léth [Kang-Kim-Liu, Erdős-Simonovits]   
 If  $g$  is a BC density, so is  $g + m$  ( $\forall m \in N$ ).

{Jiang, Ma, Yeganyan, Kang, Kim, Liu, Coulon, Janzer, Lee, Qiu}.

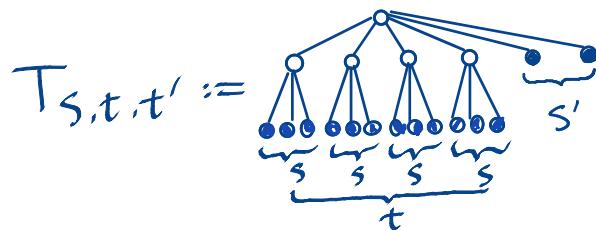
THM [Jiang-I.-Ma].  $\frac{b}{a} = \frac{21}{8}$  or  $\frac{27}{10}$

$\forall g = \frac{b}{a} > 1$ , if  $\lfloor \frac{b}{a} \rfloor^3 \leq a \leq \frac{b}{\lfloor \frac{b}{a} \rfloor + 1} + 1$  then  $g$  is a BC density.

New BC densities:  $m + \frac{5}{8}, m + \frac{7}{10}$ . ( $m \geq 2$ )

THM [Tjiang-J.-Ma].

$\forall g = \frac{b}{a} > 1$ , if  $\lfloor b/a \rfloor^3 \leq a \leq \frac{b}{\lfloor b/a \rfloor + 1} + 1$  then  $g$  is a BC density.



THM  $\forall s, t \in \mathbb{N}^+, s' \in \mathbb{N}$ , with  $t \geq s'^3 - 1$ .

if  $F := T_{s,t,s'}$  is balanced, then  $\text{ex}(n, F^P) = O(n^{2-\frac{1}{8F}})$

# Framework & Application

Given "balanced" rooted tree,  $F$

$$\text{Ex } F = \bullet - \circ - \circ - \circ - \bullet$$

Goal:  $\text{ex}(n, F^p) = O(n^{2-\frac{1}{\beta_F}})$  for all  $p$ .

$$\beta_F = 4/3.$$

THOUGHT EXPERIMENT ①  $G$  is a regular  $n$ -vertex graph.

② degree  $d$  of  $G = \omega(n^{1-\frac{1}{\beta_F}})$ .

Goal Find  $F^p$  in  $G$ .  $\Leftrightarrow$  Find  $p$ -ample embedding  $F \hookrightarrow G$  Rmk {embeddings from  $F$  to  $G$ }  $=: \{F \hookrightarrow G\}$ .

DEF An embedding  $\eta$  from  $F$  to  $G$  is

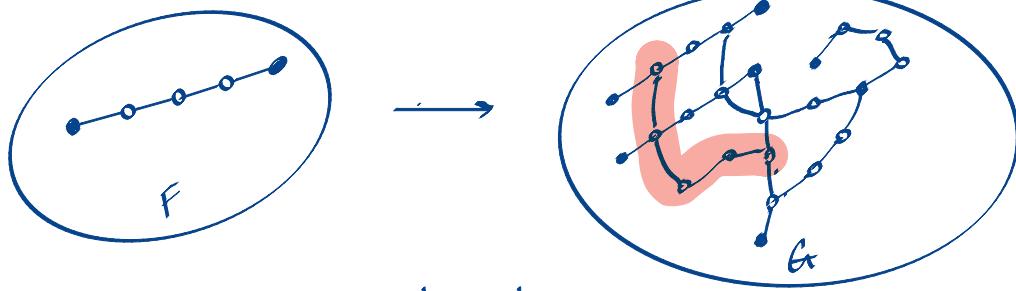
an injection  $\eta: V(F) \rightarrow V(G)$  s.t.

$$\#\{F \hookrightarrow G\} \approx F\text{-subgraph counts. in } G.$$

$\forall u_1 \sim u_2 \text{ in } F. \quad \eta(u_1) \sim \eta(u_2) \text{ in } G.$

OBS  $\#\{F \hookrightarrow G\} = \sum_{F \hookrightarrow G} (nd)^{e(F)} = \omega(n^{1+(1-\frac{1}{\beta_F})e(F)}) = \omega(n^{|R(F)|})$

DEF An embedding  $\eta$  is  $C$ -ample if  $\exists \eta_1, \dots, \eta_C$  s.t. they are identical on  $R(F)$ , but images of non-roots are pairwise disjoint

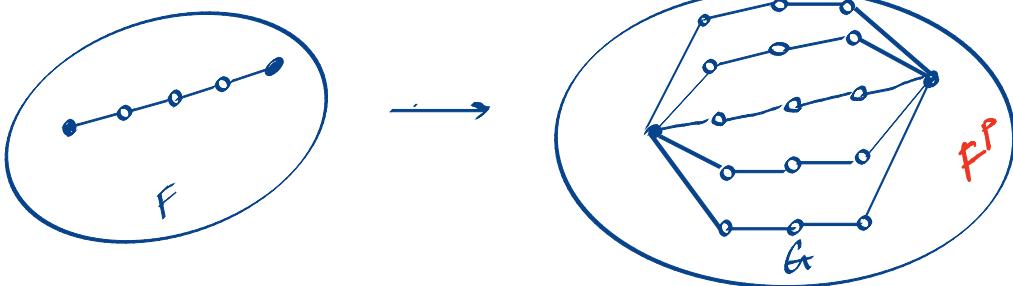


OBS.  $\#\{F \hookrightarrow G\} = \omega(n^{|R(F)|})$

$\Rightarrow \exists \sigma: R(F) \rightarrow V(G): \#\{F \hookrightarrow G \mid \sigma\} = \omega(1)$

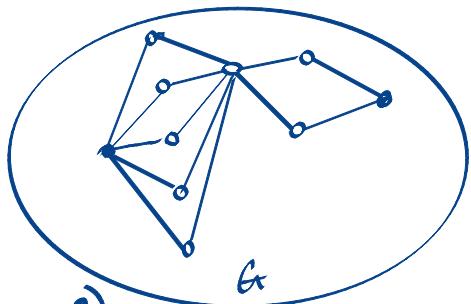
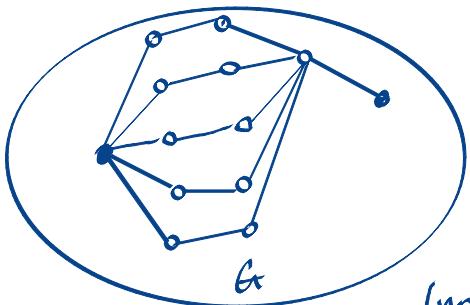
$\uparrow$   
embeddings from  $F$  to  $G$  "agreeing" with  $\sigma$

Ideally



Images of non-roots are pairwise disjoint.

## Possible ways to go wrong



↙ (none of  
which is  $\bullet \bullet$ )

DEF A family  $F_0$  of subtrees of  $F$  is an **obstruction family** for  $F$  if  $\nexists U \subseteq \{\text{non-roots}\}$ , with  $U \neq \emptyset$ . after adding  $U$  to the root set  $R(F)$ , the resulting rooted graph contains

a member of  $F_0$  as a rooted subgraph. ( $F_0 \subseteq F$   
 $R(F_0) \subseteq R(F)$ ).

Eg

$$F = \bullet - o - o - o - \bullet$$

$$F_0 = \left\{ \bullet - o - o - \bullet, \bullet - o - \bullet \right\}.$$

is obstruction family for  $F$

Additional assumption Give an obstruction family  $F_o$  for  $F$ .

$$\forall F_o \in F_o \quad \# \{ F_o \xrightarrow{\omega(1)} G \} = O(nd^{e(F_o)}).$$

$\omega(1)$ -ample embeddings from  $F_o$  to  $G$

Consider  $I := \{F \hookrightarrow G\} \setminus \bigcup_{F_o \in F_o} \{ \text{those } \eta: F \hookrightarrow G \text{ that "extends" } \eta_o: F_o \hookrightarrow G \}$ .

$$\exists \eta': F_o \hookrightarrow F \text{ s.t. } \begin{array}{ccc} F_o & \xrightarrow{\eta'} & F \\ & \downarrow \eta_o & \downarrow \eta \\ & G & \end{array}$$

OBS

$$\# \{ \text{extension of } F_o \xrightarrow{\omega(1)} G \} = O(nd^{e(F_o)} \cdot d^{e(F) - e(F_o)}) \\ = O(nd^{e(F)}).$$

$$\Rightarrow |I| = \omega(nd^{e(F)}) = \omega(n^{|R(F)|}) \quad \text{Find } F^P \text{ in } G.$$

$$\Rightarrow \exists \sigma: R(F) \rightarrow V(G): \# \{ I | \sigma \} = \omega(1) \quad \text{Cannot go wrong.}$$

## Upshot of Thought Experiment

- ①  $G_t$  is regular graph.
- ② degree  $d = \omega(n^{1 - 1/\beta_F})$ .
- ③ "additional assumption" on  $F_0$  (obstruction family for  $F$ )

Then can find  $F^P$  in  $G_t$ .

DEF Given  $F_0$  and  $F$ , we say  $F_0$  is **negligible** for  $F$ , if

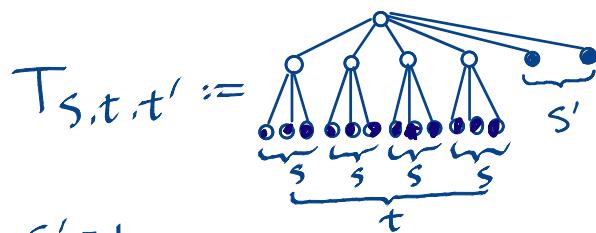
$\forall p \in \mathbb{N}^t, q > 0, \exists c_0 > 0$  and  $M \in \mathbb{N}$  s.t.

$\forall c > c_0$  and  $n$ -vertex graph  $G_t$  if degrees of  $G_t$  are between  $c n^\alpha$  and  $5^{4/\alpha} c n^\alpha$  ( $\alpha = 1 - 1/\beta_F$ )

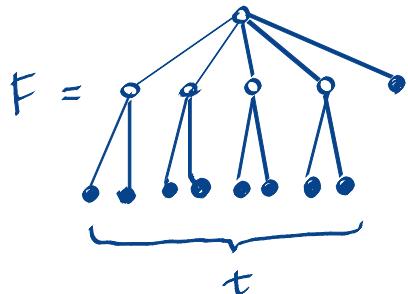
and  $\{F \xrightarrow{P} G_t\} = \emptyset$ , then  $\#\{F_0 \xrightarrow{M} G_t\} \leq (q + o(1)) n d^{e(F_0)}$

LEM (Negligibility lemma). Given  $F_0$ . If every member of  $F_0$  is neg. for  $F$  then  $ex(n, F^P) = O(n^{2 - 1/\beta_F}) \quad \forall p \in \mathbb{N}^t$

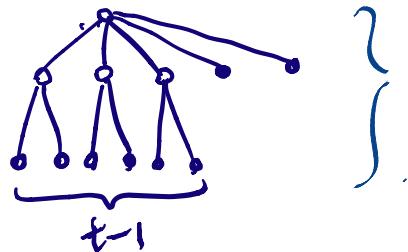
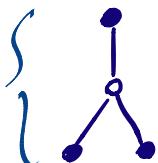
# Framework & Application



Consider  $s=2, s'=1$ .



Obstruction family:



LEM [Kang-Kim-Liu, Erdős-Simonovits]

If  $\mathfrak{g}$  is a BC density, so is  $\mathfrak{g} + m$  ( $\forall m \in \mathbb{N}$ ).

CONJ:  $\forall s, a \in \mathbb{N}, s < a, \exists m \in \mathbb{N}^+: m + \frac{s}{a}$  is BC density.

CONJ  $\uparrow$

~~THM~~ [Jiang-I.-Ma]:

$\forall g = \frac{b}{a} > 1$ , if  $\lfloor \frac{b}{a} \rfloor^3 \leq a \leq \frac{b}{\lfloor \frac{b}{a} \rfloor + 1} + 1$  then  $\mathfrak{g}$  is a BC density.



$$T_{s,t,s'} := \begin{array}{c} \text{Diagram of a full binary tree: } \\ \text{Root node connected to 2 children, which are connected to } 2^3 = 8 \text{ leaf nodes.} \\ \text{The root is labeled } s, \text{ the level below it is labeled } t, \text{ and the leaves are labeled } s'. \end{array} \Rightarrow g = \frac{(s+1)t + s'}{t+1}.$$

CONJ

~~THM~~

$\forall s, t \in \mathbb{N}^+, s' \in \mathbb{N}$  (with  $t \geq s^3 - 1$ )

if  $F := T_{s,t,s'}$  is balanced, then  $\text{ex}(n, F^P) = O(n^{2-\frac{1}{g_F}})$