Beyond classification theorem of Cameron, Goethals, Seidel and Shult (CGSS).

<u>Fundamental problem</u>: Characterization of graphs with limited eigenvalues.

$\mathcal{G}(\lambda) = \{$ graphs G with smallest eigenvalue $\lambda_1(G) \geq -\lambda \}$ (refer to adjacency matrix).

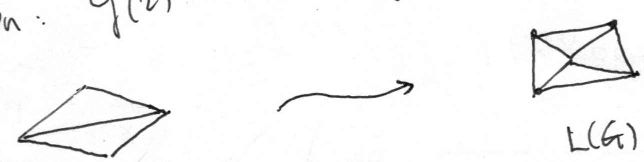
~~History of $\mathcal{G}(2)$~~

- Review of $\mathcal{G}(2)$
- ~~New results~~ Classification theorem of $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$. ~~where $\lambda^* = 2.0198\ldots$~~
- Beyond $\mathcal{G}(\lambda^*)$

<u>Review of $\mathcal{G}(2)$</u>:

Well known: $\mathcal{G}(2) \supseteq \{$ line graphs $\}$.

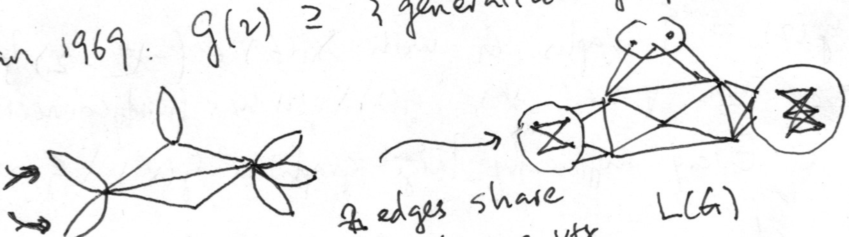
no 



G L(G)

edge-vertex incidence matrix B: adjacency matrix $A = BB^T - 2I$.

Hoffman 1969: $\mathcal{G}(2) \supseteq \{$ generalized line graphs $\}$.



petals

G

edges share exactly one vtx

L(G)

... Note $\mathcal{G}(\lambda)$ is closed under taking disjoint union.

1

CGSS (1976): Obs: ~~For $G \in \mathcal{G}(2)$, $\frac{1}{2}A_G + I$ is the Gram matrix~~ ~~of unit vectors in~~ whose pairwise ~~inner prod~~ angle is $60°$ or $90°$. (geometric)

$G \in \mathcal{G}(2) \iff \frac{1}{2}A_G + I \succeq 0 \iff$ unit vectors.

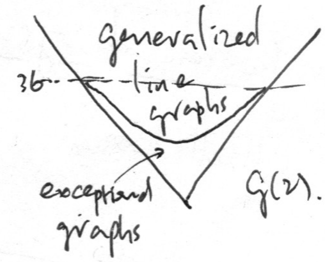
$\begin{pmatrix} 1 & & 0 \\ \frac{1}{2} & \ddots & \\ & & 1 \end{pmatrix}$ pairwise angle $60°$ or $90°$

~~$\iff$ repre~~ $\nearrow$ representation theory of semisimple Lie algebra.

Classification thm: ~~1976~~

$\{$connected graphs in $\mathcal{G}(2)\}$ = $\{$generalized line graphs$\}$ $\cup$ $\{$exceptional graphs$\}$

$\nwarrow$ represented by subset of E_8 root system. at most 36 vertices



- Classification theorem of $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$.

$\lambda^* = 2.0198008871\ldots$

$\lambda_1($  $E_n) \searrow -\lambda^*$.

$|\leftarrow n \rightarrow|$

λ^* not totally real. cannot be graph e.v.

(some conjugates are not real)

$\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2) = \{$graphs G with $\lambda_1(G) \in (-\lambda^*, -2)\}$

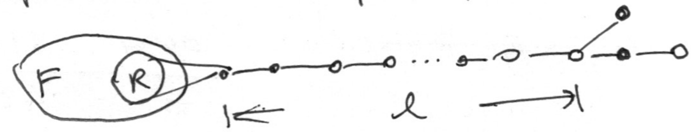
~~Thm~~ THM: $\forall \lambda \in (2, \lambda^*)$. $\mathcal{G}(\lambda) \setminus \mathcal{G}(2)$ has finitely connected graphs

~~Main result~~. "Every sufficiently large graph in $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$

~~Part 2~~: looks more or less like E_n".

DEF: (~~Rooted graph~~ a. A rooted graph F_R is a graph F equipped with nonempty subset R of vertices

The augmented path extension (ape) $(F_R, \ell, \;)$ of F_R is defined by



Part 1: Every sufficiently large connected $G \in \mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$ is an ape of a rooted graph.

↑ classify this ?

~~Part 2: There exists~~

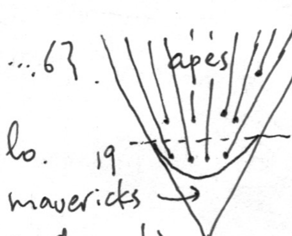
DEF: A single-rooted graph H_r is a rooted graph H with a single root r. The line graph $L(H_r)$ is the rooted graph F_R where $F = L(H)$ and R ~~=~~ $\{$edges of H incident to $r\}$

Part 2: There exists a finite family $\mathcal{F}$ of rooted graphs s.t.

① every F_R in $\mathcal{F}$ is $L(H_r)$ for some connected bipartite single rooted H_r.

② every connected ape in $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$ is an ape of some ~~rooted graph~~ F_R in $\mathcal{F}$.

③ for every F_R in $\mathcal{F}$, there exists $l_0 \in \{0, \dots, 6\}$ s.t. $\lambda_1(F_R, \ell, \;) \in (-\lambda^*, -2)$ iff $\ell \geq l_0$.



Quantative version: enum ~~list~~ of 794 ~~those~~ H_r. (794 total)

(48 maximal)

Computer assisted.

DEF: A maverick graph is a connected graph in $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$ that is not an ape.

Part 3: ~~The~~ Enumeration of 4752 maverick graphs.

at most 19 vertices. (computer assisted).

~~Cor: For every connected~~

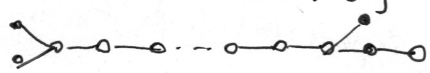
order	17	18	19
#	42	13	3

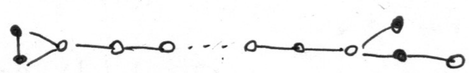
Key linear algebraic lemma: ~~Cor~~ (mention after Part 2).

~~$\lambda_1(F_R, l, \cdot) > -\lambda^* \iff \lambda_1(F_R, 0, \cdot)$~~

$(F_R, l, \cdot) \in \mathcal{G}(\lambda^*) \iff (F_R, 0, \cdot) \in \mathcal{G}(\lambda^*)$.

~~Example~~ ~~Context~~: Consider two cases $F_R \in \{ \cdot, \cdot \}$

$(: , l, \cdot) =$  $\lambda_1 \downarrow -\lambda^*$

$(| , l, \cdot) =$  $\lambda_1 \uparrow -\lambda^*$

COR: If $\lambda_1(F_R, l, \cdot) > -\lambda^*$, then $F[R]$ is complete.

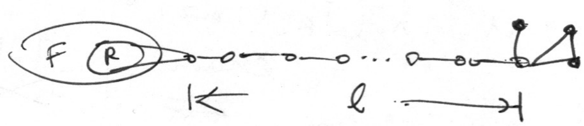
Pf: ....

app

Beyond $\mathcal{G}(\lambda^*)$.

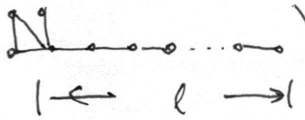
A notable portion of maverick graphs look alike

DFE: Twisted path extension (tpe).

$(F_R, l, \cdot) =$ 

Thm: There're 1161 twisted maverick graphs.

order	17	18	19
#	40	13	3

$\lambda_1\big($  $\big) \searrow -\lambda'$. $\lambda' \approx 2.02124$.

<u>Generalization of Part 1</u>.

$\forall \lambda \in (\lambda^*, \lambda')$ ~~N~~ every sufficiently large graph in $\mathcal{G}(\lambda) \setminus \mathcal{G}(2)$ is an ape.

<u>Nogo thm for Part 2</u>: $\forall \lambda > \lambda^*$, finite $\mathcal{F}$ of rooted graphs, $N \in \mathbb{N}$. $\exists G$ on $> N$ vertices in $\mathcal{G}(\lambda) \setminus \mathcal{G}(2)$ that is not an ape of $\in \mathcal{F}$.

<u>App.</u>: Every connected graph G on ≥ 18 vertices in $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$ contains a unique leaf u s.t. $G - u$ is the line graph of a bipartite graph.