Forbidden subgraphs & Spherical two-distance sets

Forbidden subgraphs

* All subgraphs are induced.

Joint with Alexandr Polyanskii

Families of graphs

** Refer to adjacency matrix.

- $F'(\lambda) = \{$ graphs with spectral radius** $\leq \lambda \}$.
- $F(-\lambda) = \{$ graphs with smallest eigenvalue** $\geq -\lambda \}$.

Problems

- Classification of graphs in $F'(\lambda)$ or $F(-\lambda)$.
- Define $F'(\lambda)$ or $F(-\lambda)$ by a finite set of forbidden subgraphs?

(Observation: Cauchy interlacing $\Rightarrow F'(\lambda)$ and $F(-\lambda)$ are closed under taking subgraphs.)

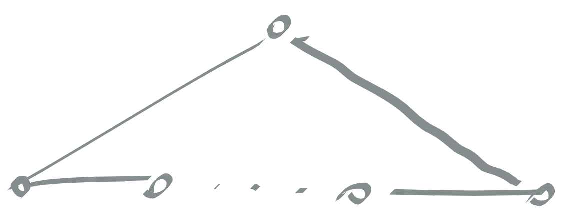
i.e. find finite $\mathcal{G}$ of graphs s.t. $\underline{\underline{F}} = \{ G : $ no member of $\mathcal{G}$ is a subgraph of $G. \}$.

$F'(\lambda) = \{\text{graphs with spectral radius} \leq \lambda\}$

$F(-\lambda) = \{\text{graphs with smallest e.v.} \geq -\lambda\}$

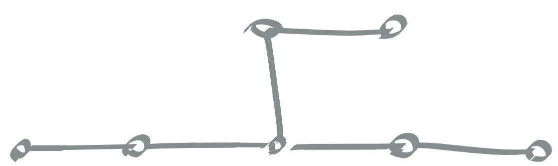
Results on $\lambda = 2$.

Maximal connected graphs in $F'(2)$

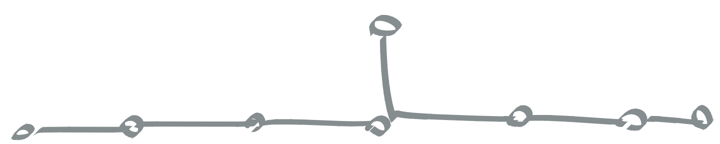


$\tilde{A}_n$ $(n \geq 2)$

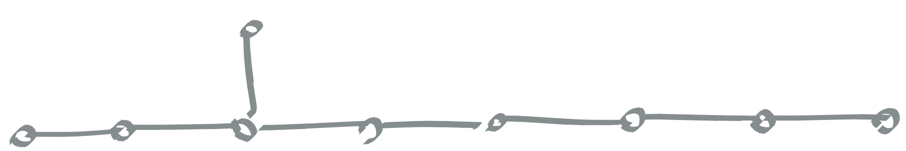
$\tilde{D}_n$ $(n \geq 4)$



$\tilde{E}_6$



$\tilde{E}_7$.



$\tilde{E}_8$. [Smith 70].

$F'(2)$ can be defined by 18 forbidden subgraphs.

$F(-2)$ is much richer.

- $F(-2) \supseteq F'(2)$.
- $F(-2) \supseteq \{\text{line graphs}\}$

Pf: Line graph $L(G)$.

adj. mat. of $L(G)$

$= \underline{B^T B} - 2I$ $\square$

B is incidence mat of G.

1976

[Cameron, Goethals, Seidel, Shult]

Connected graphs in $\mathcal{F}(-2)$ are either "generalized line graphs" or "representable by E_8 root system"* finitely many exceptions

[Bussemaker, Neumaier 1992]

$\mathcal{F}(-2)$ can be defined by 1812 forbidden subgraphs.

(computer assisted).

A result beyond $\lambda = 2$

[Cvetković, Doob, Gutman 1982].

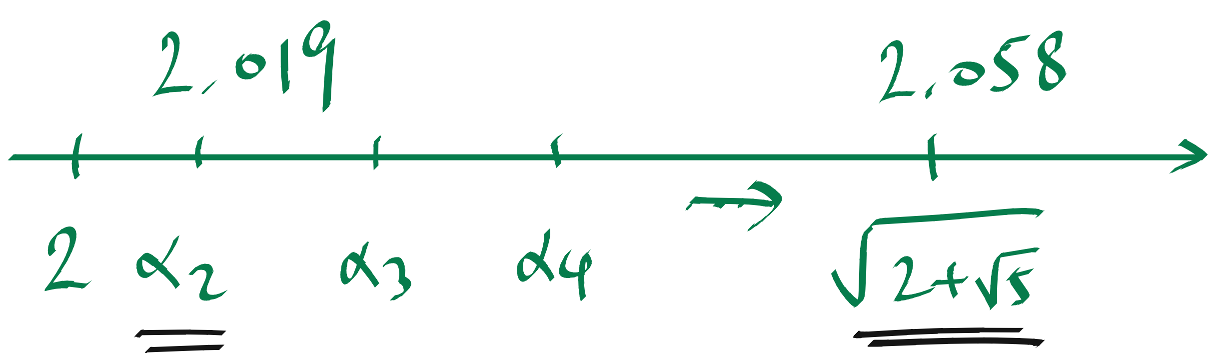
Classification of graphs in $\mathcal{F}'(\sqrt{2+\sqrt{5}})$

≈ 2.058

[BN92]: It would be interesting to know the set of λ s.t. $\mathcal{F}'(\lambda)$ or $\mathcal{F}(-\lambda)$ can be defined by finite set of forbidden subgraphs; "however, these seem to be very difficult".

THM [J. Polyanskii 2020].

$\mathcal{F}'(\lambda)$ can be defined by a finite set of forbidden subgraphs iff $\lambda < \sqrt{2+\sqrt{5}}$ and $\lambda \notin \{\alpha_2, \alpha_3, \dots\}$



silver threshold

golden threshold

$$\sqrt{2+\sqrt{5}} = \varphi^{1/2} + \varphi^{-1/2} \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

$$\alpha_2 = \beta^{1/2} + \beta^{-1/2}. \qquad \beta^3 = 1+\beta.$$

THM [J. Polyanskii. in progress].

$\mathcal{F}(-\lambda)$ can be defined by a finite set of forbidden subgraphs iff $\lambda < \alpha_2$.

Question: Characterize all graphs in $\mathcal{F}(-\alpha_2) \setminus \mathcal{F}(-2)$.

Generalization to signed graphs

$\mathcal{F}^{\pm}(\lambda) = \{$ signed graphs with largest e.v. $\le \lambda \}$.

THM $\mathcal{F}^{\pm}(\lambda) \cdots$ iff $\lambda < \alpha_2$.

Spherical two-distance sets

Joint work with Jonathan Tidor, Yuan Yao, Shengtong Zhang, & Yufei Zhao.

Spherical two-distance set in $\mathbb{R}^d$

$$= \{ v_1, v_2, \cdots, v_N \in \mathbb{R}^d : \forall i, \ \|v_i\| = 1, \ \& \ \forall i \neq j, \ \langle v_i, v_j \rangle = \underline{\alpha \text{ or } \beta} \}.$$

"angles"

Focus on $-1 \leq \beta < 0 \leq \alpha < 1$.

Remark: $\beta = -\alpha \Leftrightarrow$ equiangular lines

$N_{\alpha,\beta}(d)$ = max size of spherical two-distance sets in $\mathbb{R}^d$ with angles α, β.

Prop [JTYZZ 20+].

$$N_{\alpha,\beta}(d) \geq \frac{k_p(\lambda)\, d}{k_p(\lambda) - 1} + O_{\alpha,\beta}(1).$$

where $\lambda = \frac{1-\alpha}{\alpha-\beta}$, $p = \lfloor \frac{-\alpha}{\beta} \rfloor + 1$

and $k_p(\lambda) = \inf \{ \frac{|G^{\pm}|}{\text{mult}(\lambda, G^{\pm})} : \underset{\uparrow \text{ chromatic \#}}{\underline{\chi(G^{\pm})}} \leq p \text{ and } \underset{\uparrow \text{ largest e.v.}}{\underline{\lambda'}}(G^{\pm}) = \lambda \}.$

Conj: Lower bound is tight.

($\underline{\lambda}$ and $\underline{p}$ governs $N_{\alpha,\beta}(d)$)

Conj $N_{\alpha,\beta}(d) = \dfrac{kp(\lambda)\, d}{kp(\lambda)-1} + O(1)$

where $\lambda = \dfrac{1-\alpha}{\alpha-\beta}$. $p = \left\lfloor \dfrac{-\alpha}{\beta} \right\rfloor + 1$.

THM [JTYZZ 20+].

Conj holds when

$p \le 2$, or $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$.

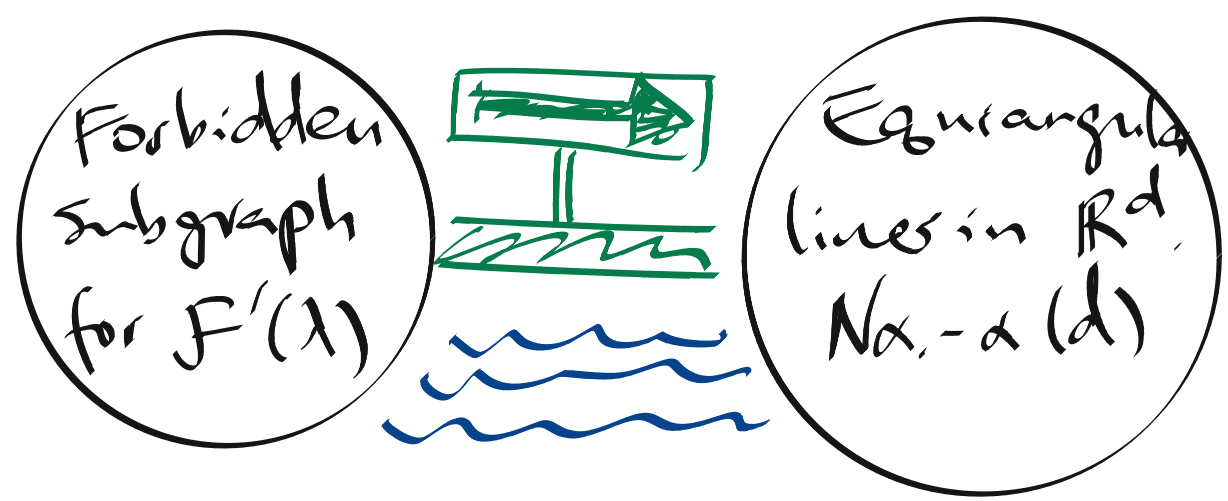
Tool: 2nd eigenvalue multiplicity

Tool fails for signed graphs

Equiangular lines $\beta = -\alpha$ (or $p = 2$.)

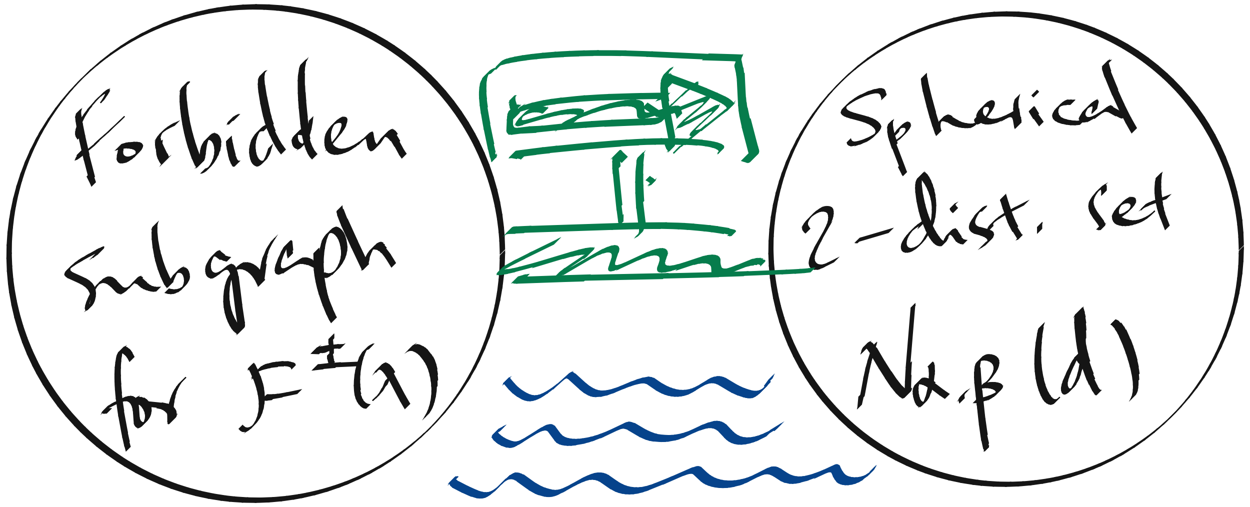
Recent work: Bukh, BDSK...

[J. Polyanskii 20]. If $F'(\lambda)$ can be defined by a finite set of forb. subgraph. then conj holds when $\dfrac{1-\alpha}{\alpha-\beta} = \lambda$ and $p = 2$.



[JTYZZ 19+] Conj holds when:

$\beta = -\alpha$.



[JTYZZ 20+]. If $F^{\pm}(\lambda)$ can be defined by a finite set of forbidden subgraphs, then conj holds for $\frac{1-\alpha}{\alpha-\beta} = \lambda$.

Cor [J. Polyanskii] Conj holds for $p \le 2$ or $\lambda < \alpha_2 \approx 2.019$.

Recall. $F^{\pm}(\lambda) = \{$ signed graphs with largest e.v. $\le \lambda \}$.

Cor. $Q(x_1, \cdots, x_n)$ $\quad c_i \in \{0, 1\}$

$$= \sum_i x_i^2 + \sum_{i<j} c_i x_i x_j .$$

$c_i \in \mathbb{Z}$. If $Q(x) < 0 \ \exists x$. then $\exists x$ supported on 10 coordinates s.t. $Q(x) < 0$.