Eigenvalue multiplicity & Equiangular lines

Joint with Alexandr Polyanskii
arxiv: 1708.02317.

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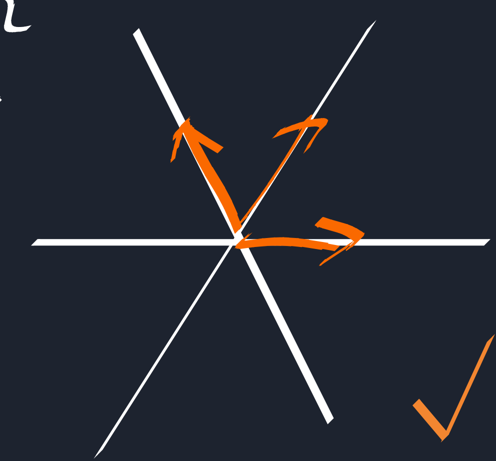
arxiv: 1907.12466

Lines in $\mathbb{R}^n$ (through O)
pairwise separated by same angle

$\mathbb{R}^2$



$\mathbb{R}^3$.

6 lines.

<u>Question</u> max size of equi. lines in $\mathbb{R}^n$?

Question max size of equi. lines in $\mathbb{R}^n$?

n	2	3-4	5	6	7-14	...
max	3	6	10	16	28	...

$$\frac{23-41}{276.}$$

$$cn^2 \leq \max \leq \binom{n+1}{2}$$

de Caen 2000. Gerzon 1973

2018. Balla, Dräxler, Sudakov, Keevash

$E_\alpha(n) \leq 1.93n$ if $n \geq n_0(\alpha)$, unless $\alpha = 1/3$

Question What if the angle is fixed?

$E_\alpha(n)$ = max size of equi. lines in $\mathbb{R}^n$ with angle $= \arccos\alpha$.

1973 Lemmens–Seidel

$E_{1/3}(n) = 2(n-1)$ $n \geq 15$.

1989. Neumaier for $n \geq n_0$

$E_{1/5}(n) = \lfloor \frac{3}{2}(n-1) \rfloor$ ✓

1973. Neumann. $E_\alpha(n) \leq 2n$ unless $\alpha = 1/3, 1/5, 1/7, \ldots$

2016. Bukh. $E_\alpha(n) \leq C_\alpha n$.

Conj 1 (Bukh) $E_{1/7}(n) \simeq \frac{4}{3}n$

$$E_{\frac{1}{2k-1}}(n) \simeq \frac{k}{k-1}n.$$

Conj 2 (J.-Polyanskii).

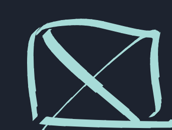
$$E_\alpha(n) \simeq \frac{k}{k-1}n, \quad \text{where}$$

$$k = k(\lambda), \quad \lambda = \frac{1-\alpha}{2\alpha}.$$

Spectral radius order

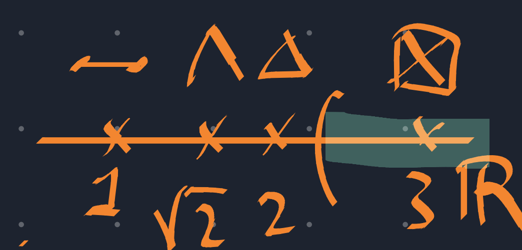
$k(\lambda)$ = smallest k s.t. $\exists$ k-vertex graph G s.t. $\lambda_1(G) = \lambda$

$\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_k(G)$ eigenvalues of adjacency matrix of G.

α	λ	G	k	$E_\alpha(n) \simeq$
$1/3$	1		2	$2n$
$1/5$	2		3	$\frac{3}{2}n$
$1/7$	3		4	$\frac{4}{3}n$
$1/(1+2\sqrt{2})$	$\sqrt{2}$.		3	$\frac{3}{2}n$.

THM (JP). Conj 2 holds for $\lambda \leq \sqrt{2+\sqrt{5}}$.

↑ Barrier



THM (JTYZZ).

$$E_\alpha(n) = \left\lfloor \frac{k}{k-1}(n-1) \right\rfloor$$

for all $n \geq n_0(\alpha)$,

where $k = k(\lambda)$, $\lambda = \frac{1-\alpha}{2\alpha}$

Goal. Given n, find largest m s.t. $\exists$ m-vertex graph G with (PSD) + (RANK).

Alternative goal. Given m, find smallest n s.t. ... (PSD) + (RANK).

$\iff$ minimize $\operatorname{rank}(\lambda I - A)$

$\iff$ maximize $\operatorname{mult}(\lambda, A)$.

Equiangular lines in $\mathbb{R}^n$.

V = set of unit vectors (each vector represents a line).

$\langle v_1, v_2 \rangle = \pm\alpha$.

Gram matrix $(\langle v_i, v_j \rangle)_{i,j} \succeq 0$.

rank (Gram matrix) $\leq n$.

m-vertex graph G.

V - vertex set.

$v_i \sim v_j \iff \langle v_i, v_j \rangle < 0$.

$$\underset{\lambda = \frac{1-\alpha}{2\alpha}}{\lambda} I - \underset{\text{adj. mat. of } G}{A} + \frac{1}{2}\underset{\text{all-ones mat.}}{J} \succeq 0 \quad \text{(PSD)}$$

$$\operatorname{rank}(\lambda I - A + \tfrac{1}{2}J) \leq n \quad \text{(RANK)}$$

Alt goal Given m, find m-vertex G satisfying

(PSD): $\lambda I - A + \frac{1}{2}J \succeq 0$.

that maximizes $\text{mult}(\lambda, A)$

(PSD) $\xRightarrow{\text{Weil's inequality}}$

(Completely reducible):

$G = G_1 \cup \cdots \cup G_c$ where

each connected component G_i

satisfies $\lambda_1(G_i) \leq \lambda$.

$$\text{mult}(\lambda, G) = \sum_{i=1}^{c} \text{mult}(\lambda, G_i) \leq c.$$

Best to choose $|G_i| = k(\lambda)$

$=$ smallest k s.t. $\exists$ k-vtx G

s.t. $\lambda_1(G) = \lambda$.

$\frac{m}{k(\lambda)}$

(Irreducible): G is connected,

but $\lambda_2(G) = \lambda$.

$$\text{mult}(\lambda, G) = o(m).$$

Prop [BDSK]. There is

a switching of G s.t.

max deg of G is bounded by

a costant $\Delta = \Delta(\alpha)$.

THM (JTYZZ). For every n-vertex connected graph G with max deg $\le \Delta$. If $\lambda = \lambda_2(G)$, then

$$\mathrm{mult}(\lambda, G) \le \frac{cn}{\log\log n} = o(n).$$

???

Perron-Frobenius

Near-miss examples.

$\Delta \quad \Delta \quad \cdots \quad \Delta \qquad n/3.$

Strongly regular graph.

$\mathrm{mult}(0, G) = \frac{n}{3}.$

LEM 1. Every connected n-vertex graph G has an **r-net** of size $\lceil \frac{n}{r+1} \rceil$.

$V_0 \subseteq V$ s.t. $\forall v \in V$, $\exists u \in V_0$, $\mathrm{dist}(u, v) \le r$. ← vertex set of G.

Pf. WLOG. G is tree.

LEM 2. If $H = G - (\text{an } r\text{-net of } G)$,

then $\lambda_1(H)^{2r} \le \lambda_1(G)^{2r} - 1$.

$\Uparrow$

pf: $A_H^{2r} \le A_G^{2r} - I$

LEM 3: $\displaystyle\sum_{i=1}^{|H|} \lambda_i(H)^{2r} \le \sum_{v \in V(H)} \lambda_1(H_r(v))^{2r}$

$H_r(v) =$

$\uparrow$

r-nbhd of v in H.

Pf. $\text{LHS} = \operatorname{tr}(A_H^{2r})$.

$= \displaystyle\sum_{v \in V(H)}$ # of closed walks of length $2r$ starting at v.

$= \displaystyle\sum_{v \in V(H)} \mathbb{1}_v^T A_{H_r(v)}^{2r} \mathbb{1}_v$.

$\le \displaystyle\sum_{v \in V(H)} \lambda_1(H_r(v))^{2r}$. $\square$

Pf. $\lambda = \lambda_2(G)$.

$r = r_1 + r_2$, $\quad r_1 = c \log\log n$

$\quad\quad\quad\quad\quad\quad r_2 = \underline{\underline{c}} \log n$.

<u>Case 1</u>. Assume $\exists\, v$.

$$\lambda_1(G_r(v)) > \lambda.$$

Then $\lambda_1(G - \underline{G_{r+1}(v)}) < \lambda$.

By Cauchy interlacing.

$$\mathrm{mult}(\lambda, G) \le |G_{r+1}(v)|$$

$$\le \underline{\Delta^{r+1}}.$$

$$= o(n).$$

<u>Case 2</u>. Assume $\forall\, v$.

$$\lambda_1(G_r(v)) \le \lambda.$$

Let V_0 be a small $\underline{\underline{r_1}}$- net of G.

$$H = G - V_0.$$

$$\text{LEM 2} \Rightarrow \lambda_1(H_{r_2}(v))^{2r_1}$$

$$\le \lambda_1(G_{\underline{r}}(v))^{2r_1} - 1.$$

$$\le \underline{\underline{\lambda^{2r_1} - 1}}$$

$$\sum \lambda_i(H)^{2r_2}$$

LEM3 $\Rightarrow \operatorname{mult}(\lambda, H) \cdot \lambda^{2r_2}$

$$\leq \sum_{i=1}^{|H|} \lambda_i(H)^{2r_2}$$

$$\leq \sum_{v \in V(H)} \lambda_1(H_{r_2}(v))^{2r_2}$$

$$\leq |H| \cdot \left(\lambda^{2r_1} - 1\right)^{r_2/r_1}$$

$$|H| \leq n$$

$\Rightarrow \operatorname{mult}(\lambda, H) = o(n)$.

Cauchy interlacing $\Rightarrow$

$$\operatorname{mult}(\lambda, G) \leq \operatorname{mult}(\lambda, H) + |V_0| = o(n).$$