Math 307: Homework 2
Part 1

1. Calculate the inner products, 1-norms, 2- norms, and infinity-norms for the following vectors:
   (a) the real vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$,
   (b) the complex vectors $\begin{bmatrix} 1+i \\ 3-i \\ 2+2i \\ 6-3i \end{bmatrix}$ and $\begin{bmatrix} 2-2i \\ 4+3i \\ 6-i \\ 1 \end{bmatrix}$.

2. Plot the location of the complex numbers $z_k = e^{2\pi ik/5}$, $k = 0, 1, 2, 3, 4$ in the complex plane. Show that these numbers are fifth roots of unity, that is, they satisfy $z^5 = 1$.
   (a) What is z_0?
   (b) Find a polynomial (with real coefficients) whose roots are precisely $z_0, \dots, z_5$.
   (c) Show that $z_k^4 + z_k^3 + z_k^2 + z_k + 1 = 0$ for each $k = 1, 2, 3, 4$.
   (d) Consider the Nth roots of unity, given by $w_k = e^{2\pi ik/N}$. Show that $\cos\big(2\pi k/N\big) = \frac{1}{2}(w_{-k} + \overline{w_k})$ for each k. Find a similar formula for $\sin\big(2\pi k/N\big)$.

3. Show that any 2×2 orthogonal matrix is either a rotation matrix or a reflection matrix.

4. Show that if A is an $n \times n$ square matrix and each column sums to c, then c is an eigenvalue of A. *Hint: if you cannot show this in a few lines, try another approach.*

5. (a) What can you say about the diagonal elements of a Hermition matrix?
   (b) Show that if A is an $n \times n$ matrix such that $\langle \mathbf{v}, A\mathbf{w} \rangle = \langle A\mathbf{v}, \mathbf{w} \rangle$ then A is Hermitian.

6. Show that if A is any matrix then $\overline{A}^T \overline{A}$ and $\overline{A}\,\overline{A}^T$ are Hermitian with non-negative eigenvalues.

7. [**Minimum 2-norm solution of underdetermined systems**] In this problem we will consider overdetermined systems $Ax = b$ where A is $m \times n$ with $m < n$ with the additional assumption that A is full-rank, i.e., $\operatorname{rank}(A) = m$ (as large as it can be).
   (a) Find the set $\mathcal{S}$ of all solutions of the system $Ax = b$ if
   $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$
   (b) **[Regularization.]** As we observe above, this system has infinitely many solutions. In many applications, we would like to "regularize", i.e., pick one of these solutions that satisfies some additional constraint. A common regularization method is to go for the solution with the smallest

2-norm or the so-called "minimum 2-norm solution". That is, find the minimizer of the optimization problem

$$\text{minimize } \|x\|_2 \quad \text{subject to: } \; x \in \mathcal{S}$$

or equivalently

$$\text{minimize } \|x\|_2 \quad \text{subject to: } \; Ax = b$$

Now, use your solution from part (a) to come with a function that give the square of the 2-norm of the solutions in $\mathcal{S}$ as a function of a single parameter. Use elementary calculus to calculate the minimizer of this function and use this solution to determine the minimum 2-norm solution of this system.

(c) Next, we will develop a more systematic approach to obtain the minimum 2-norm solution of such a system $Ax = b$ where A is $m \times n$ with $m < n$ and $\text{rank}(A) = m$ in several steps.

  (i) Let y be any (fixed) solution of the system, i.e., $Ay = b$. Show that z is a solution of the system $Ax = b$ if and only if $z - y \in \mathcal{N}(A)$ ($\mathcal{N}(A)$ denotes the nullspace of A).

  (ii) Explain why the previous step implies that for any solution z, $P(z)$, where P is the projector onto $\mathcal{N}(A)^\perp$, the orthogonal complement of $\mathcal{N}(A)$ is also a solution.

  (iii) Suppose z_1 and z_2 are both solutions. Then explain why we have $P(z_1) = P(z_2)$.

  (iv) Set $x^* = P(z)$ where P is as above and z is any solution. Show that x^* is the minimum-2-norm solution.

  (v) Finally, we recall that $\mathcal{N}(A)^\perp$ is the row space of A, i.e., the range of A^T. Accordingly, there exists w such that $x^* = A^T w$. Subsitute this into the equation $Ax = b$ and obtain an explicit formula for x^* in terms of A, A^T, and b.

(d) Find the minimum-2-norm solution of the system give part (a), this time using the formula you obtained above.

8. The Fibonacci sequence is a well known sequence in mathematics that is obtained following a simple rule: set the first two entries of the sequence to be 0 and 1, and then, inductively, find the next entry by adding two previous entries. This results in the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots.$$

It is natural to ask whether one can find a formula for the nth Fibonacci number (i.e., the nth entry of this sequence) without having to compute all previous entries.

Such rules are called recursions and in this problem we will develop a method to analyze recursion relations.

(a) Set F_0=0, $F_1 = 1$, and let F_n denote the nth Fibonacci number (with the convention that 0 is the $0th$ Fibonacci number). Then the sequence of Fibonacci numbers can be obtained by running the recursion

$$F_n = F_{n-1} + F_{n-2}; \quad F_0 = 0, \; F_1 = 1.$$

Now, set $v_n = [F_n, F_{n-1}]^T$. Identify a matrix A such that the recursion above can be written, equivalently, as

$$v_{n+1} = Av_n; \quad v_0 = [F_1, F_0]^T$$

(b) Using the matrix recursion above A, obtain a (non-recursive) formula for v_{n+1} in terms of powers of A and v_0.

(c) Let $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$. Diagonalize A and give a formula for each entry of A^n in terms of λ_1 and λ_2.

(d) Use your answer from the previous part to obtain a formula for F_n in terms of λ_1 and λ_2.

(e) Show that $\lim_{n\to\infty} F_{n+1}/F_n = \lambda_1$. (Note that λ_1 is the famous "golden ratio".)

9. **[Trigonometric Interpolation and DFT]** Suppose that we are given data: $(x_j, y_j) \in \mathbb{R}^2$, $j = 0, 1, \ldots, m-1$. We wish to interpolate the data using "trigonometric polynomials", i.e., functions in

$$\mathcal{T}_N^T := \{f(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k t/T} : \quad c_k \in \mathbb{C}\}$$

where $c = [c_{-N}, \ldots, c_N]^T$ is the vector of coefficients of f. Note that all functions in $\mathcal{T}_N^T$ are T-periodic. So in an application, we would set T to be the assumed period of the process (i.e. function) that has generated the data. **In the rest of the problem we will assume $T = 1$ and $\mathcal{T}_N$ will denote $\mathcal{T}_N^1$. Furthermore, we will assume m is an odd positive integer.**

(a) For the given data $(x_j, y_j) \in \mathbb{R}^2$, $j = 0, 1, \ldots, m-1$, find $\tau^*(x) \in \mathcal{T}_N$ that minimizes the ℓ_2-mismatch $\mathcal{E} = \sum_{j=0}^{m} |\tau(x_j) - y_j|^2$ over all $\tau(x) \in \mathcal{T}_N$. [Hint: (1) This is similar to polynomial interpolation. So, you should obtain a matrix equation that needs to be solved for c^*, the coefficients of τ^*, in some sense (i.e., exactly, least squares, via regularization) to obtain the coeeficient vector $c = [c_{-N}, \ldots, c_N]^T$. (2) For simplicity of notation, you may set $e_k(t) := e^{2\pi i k t}$.]

(b) In the setting described above, suppose $m = 2N + 1$ and $x_j = j/m$. Find a formula for c^*, the coefficients of τ^*, in terms of the DFT or inverse DFT of the data vector $y = [y_{-N}, \ldots, y_N]^T$.

(c) We now reverse the previous question. Suppose you compute the m-point DFT of a vector $y = [y_0, \ldots, y_{m-1}]^T$ and call the resulting vector c. State an interpolation problem whose solution can be calculated using this c. Express the solution in terms of c. [You may assume that m is odd.]

(d) In the interpolation problem of part (a), suppose that $m = 2M + 1 > 2N + 1$ and set $x_j = j/m$, $j = 0, \ldots, 2M$.

   i. State the least squares equation that will lead to c^*. Explain why this equation has a unique solution.

   ii. Find a formula for c^* in terms of the DFT or the inverse DFT of y.

(e) Now consider the following "underdetermined" case: Similar to above, in the interpolation problem of part (a), suppose $m < 2N + 1$ with $x_j = j/(2N + 1)$, $j = 0, 1, \ldots, m-1$. This time the matrix equation that will lead to c^* is "underdetermined", i.e., it has infinitely many solutions.

   i. State this matrix equation, say $Ac = y$, and show that A is a submatrix of the $(2N + 1) \times (2N + 1)$ DFT matrix after shuffling the positions of some columns.

   ii. Find the minimum-2-norm solution of the system of equation you obtained above. (Use the formula you obtained in Problem 7.(c) above for the minimum-2-norm solution together with properties of DFT matrices.)