Math 307 Homework 1 due Oct 14

1 Norms and Condition Numbers

When $1 \leq p < \infty$, we defined the p norm for vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ to be

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

When $p = \infty$ we define

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

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1. Show that $\|\mathbf{x}\|_\infty = \lim_{p \to \infty} \|\mathbf{x}\|_p$.
2. Use the Cauchy Schwarz inequality

$$|\mathbf{y}^T \mathbf{x}| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2$$

to show that

$$\|\mathbf{x}\|_2 = \max_{\mathbf{y} \neq 0} \frac{|\mathbf{y}^T \mathbf{x}|}{\|\mathbf{y}\|_2}$$

The generalization of the Cauchy-Schwarz inequality to p norms is called the Hölder inequality.

To state the general case, define the conjugate exponent q to p so that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then Hölder's inequality states

$$|\mathbf{y}^T \mathbf{x}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$$

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3. Prove the special case

$$|\mathbf{y}^T\mathbf{x}| \leq \|\mathbf{x}\|_\infty \|\mathbf{y}\|_1$$

4. Suppose you are not sure if this is true, and before sinking a lot of time into proving it, you decide to test the inequality on a bunch of random vectors in $\mathbb{R}^5$ using python. Show how you would do that for $p = 3$.

We defined the matrix norm to be

$$\|A\| = \max_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}.$$

5. Show that for a diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ the matrix norm is equal to

$$\|A\| = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}$$

Based on this resutl you might guess that

$$\|A\| = \max\{|\lambda| : \lambda \in \text{eigenvalues}(A)\}$$

6. Use python to show that this is not correct, but might be for *real, symmetric* matrices A

We will see later in the course that for a real matrix A, $\|A\|^2$ is the largest eigenvalue of $A^T A$.

7. Use this to find a formula for the norm of any 2×2 matrix.

2   Balancing chemical reactions

8. Read section 1.6 in Keith Nicholson's text and do problems 1.6.3 and 1.6.4.

3 Jupyter notebook problems

9–14. There are 3 Jupyter notebooks with problems: Do these five problems in the notebooks, highlighting your solutions.

a. homework1LU.ipynb (2 problems)

b. homework1_chebyshev.ipynb (1 long answer problem)

c. homework1_BVP.ipynb together with 3 files d.png, d2.png and d3.png to be placed in a subdirectory called img (2 problems)