International Journal of Computer Mathematics
Vol. 84, No. 12, December 2007, 1805–1817

Condition numbers for linear systems and Kronecker product linear systems with multiple right-hand sides

HAIMING ZHANG†, HUA XIANG‡|| and YIMIN WEI*¶§

†Institute of Mathematics, Fudan University, Shanghai 200433, P.R. China
‡School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P.R. China
¶School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China
§Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education, P.R. China

(Received 17 December 2006; revised version received 06 February 2007; accepted 12 March 2007)

In this paper we investigate linear systems with multiple right-handed sides in the form of $AX = B$ and $(A \otimes B)X = C$. We derive normwise, mixed and componentwise condition numbers for these linear systems. Examples are given to evaluate the tightness of the first-order perturbation bounds.

Keywords: Componentwise condition number; Kronecker product linear system; Mixed condition number; Multiple right-handed sides; Normwise condition number

AMS Subject Classifications: 15A06; 65F99; 65G99

1. Introduction

We need the concepts of backward errors and condition numbers when considering the following linear systems [1, 2]

$$Ax = b. \tag{1}$$

Backward errors can answer the question of how close is the problem actually solved to the one we want to solve. Condition numbers express the worst-case sensitivity of the solution of a problem to small perturbations in both data A and data b [3]. The product of condition number and backward error provides a first-order of upper bound on the error in a computational solution. To make these concepts precise we must specify what kinds of perturbations are allowed and how they are to be measured.

*Corresponding author. Email: ymwei@fudan.edu.cn
||Current address: INRIA Futurs, Parc Club Orsay Université, 4 rue Jacques Monod—Bat G, 91893 Orsay Cedex France.

International Journal of Computer Mathematics
ISSN 0020-7160 print/ISSN 1029-0265 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207160701332390

The normwise condition number [4] is defined as follows:

$$\kappa_{E,f}(A,x) := \lim_{\epsilon\to 0}\sup\left\{\frac{\|\triangle x\|}{\epsilon\|x\|} : (A+\triangle A)(x+\triangle x) = b+\triangle b,\ \|\triangle A\| \leqslant \epsilon\|E\|,\ \|\triangle b\| \leqslant \epsilon\|f\|\right\}. \tag{2}$$

It follows that

$$\kappa_{E,f}(A,x) := \frac{\|A^{-1}\|\|f\|}{\|x\|} + \|A^{-1}\|\|E\|, \tag{3}$$

where $\|\cdot\|$ denotes any vector norm and its corresponding subordinate matrix norm, and the matrix E and the vector f are arbitrary. For the choice $E = A$ and $f = b$, we have $\kappa(A) \leqslant \kappa_{E,f}(A,x) \leqslant 2\kappa(A)$ where $\kappa(A) = \|A\|\|A^{-1}\|$.

The componentwise condition number [4]

$$\mathrm{Cond}_{E,f}(A,x) := \lim_{\epsilon\to 0}\sup\left\{\frac{\|\Delta x\|_\infty}{\epsilon\|x\|_\infty} : (A+\Delta A)(x+\Delta x) = b+\Delta b,\quad |\Delta A| \leqslant \epsilon E,\ |\Delta b| \leqslant \epsilon f\right\} \tag{4}$$

is given by

$$\mathrm{Cond}_{E,f}(A,x) = \frac{\left\||A^{-1}|(E|x|+f)\right\|_\infty}{\|x\|_\infty}, \tag{5}$$

where $|\cdot|$ denotes the componentwise absolute value. This condition number depends on x, or equivalently on the right-hand side b. A worst case measure of sensitivity approximate to all x is

$$\mathrm{Cond}_{E,f}(A) = \max_x \mathrm{Cond}_{E,f}(A,x).$$

In practice we can take any convenient approximation to the maximum that is correct to within a constant factor. For the special case $E = |A|$ and $f = |b|$, we have the condition numbers introduced by Skeel [5]

$$\mathrm{Cond}(A,x) := \frac{\left\||A^{-1}|\,|A|\,|x|\right\|_\infty}{\|x\|_\infty},$$

$$\mathrm{Cond}(A) := \mathrm{Cond}(A,e) = \||A^{-1}||A|\|_\infty,$$

where $e = [1, 1, \ldots, 1]^{\mathrm{T}}$.

In addition, the componentwise condition number introduced by Rohn [6] is

$$c(A,b) := \max_i \lim_{\epsilon\to 0}\sup\left\{\frac{|\Delta x_i|}{\epsilon|x_i|} : (A+\Delta A)(x+\Delta x) = b+\Delta b, |\Delta A| \leqslant \epsilon|A|, |\Delta b| \leqslant \epsilon|b|\right\}$$
$$= \max_i \frac{(|A^{-1}||A||x| + |A^{-1}||b|)_i}{|x|_i}. \tag{6}$$

Kronecker product linear systems (see [7–13]) take the form

$$(A\otimes B)x = d, \tag{7}$$

where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$ and $x, d \in \mathbb{R}^{mn}$. Since A and B are non-singular, so there exists a unique solution $x = (A\otimes B)^{-1}d$. In [14] the authors have investigated normwise and componentwise perturbation bounds. There, the normwise condition number of (7) is defined as

$$\kappa_{E,F,f}(A \otimes B, x) := \lim_{\epsilon \to 0} \sup \left\{ \frac{\|\Delta x\|}{\epsilon \|x\|} : [(A + \Delta A) \otimes (B + \Delta B)](x + \Delta x) = d + \Delta d, \right.$$
$$\left. \|\Delta A\| \leqslant \epsilon \|E\|, \|\Delta B\| \leqslant \epsilon \|F\|, \|\Delta d\| \leqslant \epsilon \|f\| \right\}.$$

The componentwise condition number of (7) is defined as

$$\mathrm{Cond}_{E,F,f}(A \otimes B, x) := \lim_{\epsilon \to 0} \sup \left\{ \frac{\|\Delta x\|_\infty}{\epsilon \|x\|_\infty} : [(A + \Delta A) \otimes (B + \Delta B)](x + \Delta x) \right.$$
$$\left. = d + \Delta d, |\Delta A| \leqslant \epsilon E,\ |\Delta B| \leqslant \epsilon F,\ |\Delta d| \leqslant \epsilon f \right\}.$$

There is another condition number defined by

$$c(A \otimes B, d) := \max_i \lim_{\epsilon \to 0} \sup \left\{ \frac{|\Delta x_i|}{\epsilon |x_i|} : (A + \Delta A) \otimes (B + \Delta B)(x + \Delta x) = d + \Delta d, \right.$$
$$\left. |\Delta A| \leqslant \epsilon |A|, |\Delta B| \leqslant \epsilon |B|, |\Delta d| \leqslant \epsilon |d| \right\}.$$

The following results [14] provide the upper bounds of these condition numbers.

$$\kappa_{E,F,f}(A \otimes B, x) \leqslant \frac{\|A^{-1}\| \|B^{-1}\| \|f\|}{\|x\|} + \|A^{-1}\| \|E\| + \|B^{-1}\| \|F\|, \tag{8}$$

$$\mathrm{Cond}_{E,F,f}(A \otimes B, x) \leqslant \frac{\|(|A^{-1}| \otimes |B^{-1}|)[(E \otimes |B| + |A| \otimes F)|x| + f]\|_\infty}{\|x\|_\infty}, \tag{9}$$

and

$$c(A \otimes B, d) \leqslant \max_i \frac{\left\{2\left[(|A^{-1}|\,|A|) \otimes (|B^{-1}|\,|B|)\right]|x| + (|A^{-1}| \otimes |B^{-1}|)|d|\right\}_i}{|x|_i}. \tag{10}$$

A multiple right-hand side version of (7) is of the form $(A \otimes B)X = C$, and the linear equations (1) can be generalized to linear systems with multiple right-hand sides in the form $AX = B$, which arises in many applications, such as quasi-Newton methods for solving nonlinear equations with multiple secant equations [15, 16], inverse ODE problems [17], wave-scattering problems [18], and structure mechanics problems. There exist many algorithms for solving $AX = B$, for example block CG [19], seed methods [18, 20], block GMRES, hybrid block GMRES [21], block QMR [22] and block EN [23]. See [24] for a review. In this paper we will focus on deriving the corresponding condition numbers, which characterize the sensitivity towards perturbations of such linear systems with multiple right-hand sides.

Before our discussion, some properties of the Kronecker product are needed. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$; the following results can be found in [11, 25–28]:

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \tag{11}$$

$$\|A \otimes B\| = \|A\|\|B\|, \tag{12}$$

$$(A \otimes C)(B \otimes D) = (AB) \otimes (CD), \tag{13}$$

$$\mathrm{vec}(AXB) = (B^{\mathrm{T}} \otimes A)\mathrm{vec}(X), \tag{14}$$

$$\mathrm{vec}(A \otimes B) = (I_p \otimes K_{nm} \otimes I_q)[\mathrm{vec}(A) \otimes \mathrm{vec}(B)], \tag{15}$$

where $\mathrm{vec}(A)$ is defined as $\mathrm{vec}(A) = [a_{11}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn}]^{\mathrm{T}}$, and K_{nm} is the permutation matrix, $K_{nm}\mathrm{vec}(C) = \mathrm{vec}(C^{\mathrm{T}})$, for $C \in \mathbb{R}^{n\times m}$.

The outline of this paper is as follows. In section 2, we investigate the condition numbers of general linear systems with multiple right-hand sides. In section 3, we examine Kronecker product linear systems with multiple right-hand sides. In section 4, we give some numerical examples. Final remarks are given in section 5.

2. Condition number of linear systems with multiple right-hand sides

In this section we first use the concept of matrix derivative [25, 29] and deduce the normwise, mixed and componentwise condition numbers. Consider the function $\Phi : A \longmapsto X = \Phi(A)$, where A is the input data of the problem. Let $a := \mathrm{vec}(A)$, $x := \mathrm{vec}(X)$. Then we have the vector representation $x = \varphi(a) = \mathrm{vec} \circ \Phi(A)$ of $X = \Phi(A)$.

Following the definitions in [1, 30], we introduce the definition of the normwise condition number $\kappa(\Phi, A)$, mixed relative condition number $m(\Phi, A)$ and componentwise condition number $c(\Phi, A)$ as follows:

$$\kappa(\Phi, A) := \lim_{\epsilon\to 0} \sup_{\|\Delta A\|\leqslant\epsilon\|A\|} \frac{\|\delta x\|/\|x\|}{\|\delta a\|/\|a\|},$$

$$m(\Phi, A) := \lim_{\epsilon\to 0} \sup_{|\Delta A|\leqslant\epsilon|A|} \frac{\|\delta x\|_\infty/\|x\|_\infty}{\|\delta a/a\|_\infty},$$

$$c(\Phi, A) := \lim_{\epsilon\to 0} \sup_{|\Delta A|\leqslant\epsilon|A|} \frac{\|\delta x/x\|_\infty}{\|\delta a/a\|_\infty},$$

where $\delta a := \mathrm{vec}(\Delta A)$, $\delta x := \mathrm{vec}(\Delta X)$. $\|b/a\|_\infty$ is defined by $\|b/a\|_\infty := \min\{\gamma \geqslant 0 : |b_i| \leqslant \gamma|a_i|\}$.

In order to derive the explicit expression of the above defined condition number, we need the following lemma [31, 32].

LEMMA 2.1

(a) *Let $F: \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a continuous mapping defined on an open set $D_F \subset \mathbb{R}^p$ such that $0 \notin D_F$. For a given vector $a \in D_F$ such that $F(a) \neq 0$ and $\epsilon > 0$ small enough such that $B(a, \epsilon) = \{x \in \mathbb{R}^p | \, \|x - a\| \leq \epsilon\}$, the normwise condition number of F at a can be* *defined as*

$$\kappa(F, a) = \lim_{\epsilon \to 0} \sup_{\substack{x \in B(a,\epsilon) \\ x \neq a}} \frac{\|F(x) - F(a)\|}{\|x - a\|} \frac{\|a\|}{\|F(a)\|}.$$

If F is Fréchet differentiable at a, then

$$\kappa(F, a) = \frac{\|F'(a)\| \|a\|}{\|F(a)\|}. \tag{16}$$

(b) *As stated in* (a), *let $B^0(a, \epsilon) = \{x : |x_i - a_i| < \epsilon |a_i|,\ i = 1, 2, \ldots, p\} \subset D_F$, define*

$$m(F, a) = \lim_{\epsilon \to 0} \sup_{\substack{x \in B^0(a,\epsilon) \\ x \neq a}} \frac{\|F(x) - F(a)\|_\infty}{\|F(a)\|_\infty} \frac{1}{d(x, a)},$$

where $d(x, a) = \max_{\substack{i=1,2,\ldots,p \\ a_i \neq 0}} \{(|x_i - a_i|/|a_i|)\}$, if F is Fréchet differentiable at a, then the mixed condition number is

$$m(F, a) = \frac{\|F'(a) D_a\|_\infty}{\|F(a)\|_\infty}, \tag{17}$$

where $D_a = \mathrm{diag}(a_1, a_2, \ldots, a_p)$.

(c) *Suppose $F(a) = (f_1(a), f_2(a), \ldots, f_q(a))$ be such that $f_j(a) \neq 0,\ \ j = 1, 2, \ldots, q$. Then the componentwise condition number of the mapping F at the point a is*

$$c(F, a) = \lim_{\epsilon \to 0} \sup_{\substack{x \in B^0(a,\epsilon) \\ x \neq a}} \frac{\mathrm{d}(F(x), F(a))}{\mathrm{d}(x, a)},$$

where $d(F(x), F(a)) = \max_{j=1,2,\ldots,q}\{(|f_j(x) - f_j(a)|/|f_j(a)|)\}$, if F is Fréchet differentiable at a, then

$$c(F, a) = \|D_{F(a)}^{-1} F'(a) D_a\|_\infty. \tag{18}$$

According to Lemma 2.1, we can obtain

$$\kappa(\Phi, A) = \frac{\|\varphi'(a)\| \cdot \|a\|}{\|\varphi(a)\|}, \tag{19}$$

$$m(\Phi, A) = \frac{\|\varphi'(a) D_a\|_\infty}{\|\varphi(a)\|_\infty}, \tag{20}$$

$$c(\Phi, A) = \|D_x^{-1} \varphi'(a) D_a\|_\infty, \tag{21}$$

where $D_a := \mathrm{diag}(a)$, $D_x = \mathrm{diag}(x)$ and $\varphi'(\cdot)$ denotes the derivative of φ.

We now consider the multiple right-hand side linear systems

$$AX = B, \tag{22}$$

where $A \in \mathbb{R}^{n \times n}$ is non-singular and $B \in \mathbb{R}^{n \times p}$. The perturbed systems are as follows:

$$(A + \Delta A)(X + \Delta X) = B + \Delta B, \tag{23}$$

where ΔA, ΔX and ΔB have the same sizes as A, X and B.

We first define normwise condition number $\kappa(A, X)$, mixed condition number $m(A, X)$, and componentwise condition number $c(A, X)$ in the following notation.

DEFINITION 2.2

$$\kappa(A, X) = \lim_{\epsilon \to 0} \sup_{\|[\Delta A, \Delta B]\|_F \leqslant \varepsilon \|[A, B]\|_F} \frac{\|\Delta X\|_F}{\varepsilon \|X\|_F}, \tag{24}$$

$$m(A, X) = \lim_{\epsilon \longrightarrow 0} \sup_{|\Delta A| \leqslant \varepsilon |A|, |\Delta B| \leqslant \varepsilon |B|} \frac{\|\Delta X\|_{\max}}{\epsilon \|X\|_{\max}}, \tag{25}$$

$$c(A, X) = \lim_{\epsilon \longrightarrow 0} \sup_{|\Delta A| \leqslant \varepsilon |A|, |\Delta B| \leqslant \varepsilon |B|} \frac{1}{\varepsilon} \left\| \frac{\Delta X}{X} \right\|_{\max}, \tag{26}$$

where $\|\cdot\|_{\max}$ *denotes* $\|A\|_{\max} := \max_{i,j} |a_{ij}|$, (B/A) *is an entry-wise division,* $(B/A) := ((b_{ij}/a_{ij}))$*, or* $B./A$ *in MATLAB notation.*

The following theorem presents explicit expressions for these three condition numbers.

THEOREM 2.3 *In the notation above, we have*

$$\kappa(A, X) = \frac{\|[-(X^{\mathrm{T}} \otimes A^{-1}), I \otimes A^{-1}]\|_2 \|[A, B]\|_F}{\|X\|_F}, \tag{27}$$

$$m(A, X) = \||A^{-1}||A||X| + |A^{-1}||B|\|_{\max} / \|X\|_{\max}, \tag{28}$$

$$c(A, X) = \max_{i,j} \frac{(|A^{-1}||A||X| + |A^{-1}||B|)_{ij}}{(|X|)_{ij}}. \tag{29}$$

Proof We define the function $\psi : (A, B) \longmapsto X$ as $X = \psi(A, B) = A^{-1}B$. Let $b = \mathrm{vec}(B)$. We can have the differential of the linear systems $AX = B$ as follows:

$$(dA)X + A(dX) = dB.$$

Using (14), we have

$$(X^{\mathrm{T}} \otimes I)\mathrm{vec}(dA) + (I \otimes A)\mathrm{vec}(dX) = \mathrm{vec}(dB).$$

That is

$$(X^{\mathrm{T}} \otimes I)da + (I \otimes A)dx = db,$$

and

$$dx = -(I \otimes A)^{-1}(X^{\mathrm{T}} \otimes I)da + (I \otimes A)^{-1}db \tag{30}$$

$$= [-(X^{\mathrm{T}} \otimes A^{-1}), I \otimes A^{-1}][da^{\mathrm{T}}, db^{\mathrm{T}}]^{\mathrm{T}}. \tag{31}$$

Hence

$$\psi' = [-(X^{\mathrm{T}} \otimes A^{\mathrm{T}}), I \otimes A^{-1}].$$

According to (19), we get

$$\kappa(A, X) = \frac{\|\psi'\|_2 \|[a^{\mathrm{T}}, b^{\mathrm{T}}]^{\mathrm{T}}\|_2}{\|x\|_2} = \frac{\|[-(X^{\mathrm{T}} \otimes A^{-1}), I \otimes A^{-1}]\|_2 \|[A, B]\|_F}{\|X\|_F}. \tag{32}$$

With (20) and (14), we have

$$
\begin{aligned}
m(A, X) &= \frac{\left\| \psi' \begin{bmatrix} D_A & 0 \\ 0 & D_B \end{bmatrix} \right\|_\infty}{\|X\|_{\max}} \\
&= \frac{\left\| [-X^{\mathrm{T}} \otimes A^{-1}, I \otimes A^{-1}] \begin{bmatrix} D_A & 0 \\ 0 & D_B \end{bmatrix} \right\|_\infty}{\|X\|_{\max}} \\
&= \frac{\left\| \left| [-X^{\mathrm{T}} \otimes A^{-1}, I \otimes A^{-1}] \begin{bmatrix} D_A & 0 \\ 0 & D_B \end{bmatrix} \right| e \right\|_\infty}{\|X\|_{\max}} \\
&= \frac{\left\| |[X^{\mathrm{T}} \otimes A^{-1}, I \otimes A^{-1}]| \begin{bmatrix} \mathrm{vec}(|A|) \\ \mathrm{vec}(|B|) \end{bmatrix} \right\|_\infty}{\|X\|_{\max}} \\
&= \frac{\|\mathrm{vec}(|A^{-1}||A||X|) + \mathrm{vec}(|A^{-1}||B|)\|_\infty}{\|X\|_{\max}} \\
&= \frac{\||A^{-1}||A||X| + |A^{-1}||B|\|_{\max}}{\|X\|_{\max}}.
\end{aligned} \tag{33}
$$

It follows from (21) and (14) that we can deduce the expression of $c(A, X)$ similarly,

$$
\begin{aligned}
c(A, X) &= \left\| D_X^{-1} \psi' \begin{bmatrix} D_A & 0 \\ 0 & D_B \end{bmatrix} \right\|_\infty \\
&= \left\| \left| D_X^{-1} [-(X^{\mathrm{T}} \otimes A^{-1}), I \otimes A^{-1}] \begin{bmatrix} D_A & 0 \\ 0 & D_B \end{bmatrix} \right| e \right\|_\infty \\
&= \left\| |D_X^{-1}| \left[|X^{\mathrm{T}} \otimes A^{-1}|, |I \otimes A^{-1}| \right] | \begin{bmatrix} \mathrm{vec}(|A|) \\ \mathrm{vec}(|B|) \end{bmatrix} \right\|_\infty \\
&= \left\| |D_X|^{-1} [\mathrm{vec}(|A^{-1}||A||X|) + \mathrm{vec}(|A^{-1}||B|) \right\|_\infty \\
&= \max_{i,j} \frac{(|A^{-1}||A||X| + |A^{-1}||B|)_{ij}}{(|X|)_{ij}},
\end{aligned}
$$

where $D_X = \mathrm{diag}(\mathrm{vec}(X))$, $D_A = \mathrm{diag}(\mathrm{vec}(A))$ and $D_B = \mathrm{diag}(\mathrm{vec}(B))$. ■

3. Condition numbers of Kronecker product linear systems with multiple right-hand sides

In this section we consider Kronecker product linear systems with multiple right-hand sides

$$
(A \otimes B)X = C, \tag{34}
$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{mn \times p}$ and $X \in \mathbb{R}^{mn \times p}$. We still assume that A and B are both non-singular. So there exists a unique solution $X = (A \otimes B)^{-1}C$. The perturbed system is as follows:

$$[(A + \Delta A) \otimes (B + \Delta B)] (X + \Delta X) = C + \Delta C, \tag{35}$$

where ΔA, ΔX and ΔB have the same size as A, B and C. Then we will present the new condition numbers for the systems (34).

Define the mapping $\Phi : (A, B, C) \longmapsto X$, where X is the unique solution of (34) such that $(A \otimes B)X = C$. We first define the normwise condition number $\kappa(\Phi; X)$, the mixed condition number $m(\Phi; X)$, and the componentwise condition number $c(\Phi; X)$ of this problem.

DEFINITION 3.1

$$\kappa(\Phi; X) := \lim_{\epsilon \to 0} \sup_{\substack{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2} \\ \leqslant \epsilon \sqrt{\|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2}}} \frac{\|\Delta X\|_F}{\epsilon \|X\|_F}, \tag{36}$$

$$m(\Phi; X) := \lim_{\epsilon \to 0} \sup_{\substack{|\Delta A| \leqslant \epsilon |A| \\ |\Delta B| \leqslant \epsilon |B| \\ |\Delta C| \leqslant \epsilon |C|}} \frac{\|\Delta X\|_{\max}}{\epsilon \|X\|_{\max}}, \tag{37}$$

$$c(\Phi; X) := \lim_{\epsilon \to 0} \sup_{\substack{|\Delta A| \leqslant \epsilon |A| \\ |\Delta B| \leqslant \epsilon |B| \\ |\Delta C| \leqslant \epsilon |C|}} \frac{1}{\epsilon} \left\| \frac{\Delta X}{X} \right\|_{\max}. \tag{38}$$

The following theorem gives the explicit expression of the normwise condition number.

THEOREM 3.2 *In the notation above*, *we have*

$$\kappa(\Phi; X) = \left\| P^{-1} \left[Q(I_{m^2} \otimes \mathrm{vec}(B)), Q(\mathrm{vec}(A) \otimes I_{n^2}), I_{mnp} \right] \right\|_2 \frac{(\|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2)^{1/2}}{\|X\|_F},$$

where $P = I_p \otimes (A \otimes B)$, $Q = (X^{\mathrm{T}} \otimes I_{mn})(I_m \otimes K_{nm} \otimes I_n)$. *Here* K_{nm} *is the permutation matrix.*

Proof It is easy to obtain from (34) that

$$(dA \otimes B + A \otimes dB)X + (A \otimes B)dX = dC.$$

Applying (14), we obtain

$$(X^{\mathrm{T}} \otimes I_{mn})\mathrm{vec}(dA \otimes B) + (X^{\mathrm{T}} \otimes I_{mn})\mathrm{vec}(A \otimes dB) + [I_p \otimes (A \otimes B)]\mathrm{vec}(dX) = \mathrm{vec}(dC).$$

Since

$$\begin{aligned} \mathrm{vec}(dA \otimes B) &= (I_m \otimes K_{nm} \otimes I_n)(\mathrm{vec}(dA) \otimes \mathrm{vec}(B)) \\ &= (I_m \otimes K_{nm} \otimes I_n)(I_{m^2} \otimes \mathrm{vec}(B))\mathrm{vec}(dA), \\ \mathrm{vec}(A \otimes dB) &= (I_m \otimes K_{nm} \otimes I_n)(\mathrm{vec}(A) \otimes \mathrm{vec}(dB)) \\ &= (I_m \otimes K_{nm} \otimes I_n)(\mathrm{vec}(A) \otimes I_{n^2})\mathrm{vec}(dB), \end{aligned}$$

and noting the definition of P, Q, we have

$$Q(I_{m^2} \otimes \mathrm{vec}(B))da + Q(\mathrm{vec}(A) \otimes I_{n^2})db + Pdx = dc.$$

Since A and B are both non-singular, $I_p \otimes (A \otimes B)$ is also non-singular. So we have

$$dx = -P^{-1}\left[Q(I_{m^2} \otimes \mathrm{vec}(B)),\, Q(\mathrm{vec}(A) \otimes I_{n^2}),\, I_{mnp}\right][da^{\mathrm{T}}, db^{\mathrm{T}}, dc^{\mathrm{T}}]^{\mathrm{T}}. \tag{39}$$

From (39), we can get

$$\begin{aligned}&\kappa(\Phi; X)\\ &= \left\|P^{-1}\left[Q(I_{m^2} \otimes \mathrm{vec}(B)),\, Q(\mathrm{vec}(A) \otimes I_{n^2}),\, I_{mnp}\right]\right\|_2 \frac{(\|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2)^{1/2}}{\|X\|_F}.\end{aligned}$$

■

In the following theorem we obtain explicit expressions for the mixed condition number and the componentwise condition number similarly.

THEOREM 3.3 *In the notation above*, *we have*

$$m(\Phi; X) = \frac{\begin{array}{c}\||P^{-1}Q(I_{m^2} \otimes \mathrm{vec}(B))|\mathrm{vec}(|A|) + |P^{-1}Q(\mathrm{vec}(A) \otimes I_{n^2})|\mathrm{vec}(|B|)\\ + |P^{-1}|\mathrm{vec}(|C|)\|_\infty\end{array}}{\|X\|_{\max}},$$

$$\begin{aligned}c(\Phi; X) = \|D_X^{-1}[&|P^{-1}Q(I_{m^2} \otimes \mathrm{vec}(B))|\mathrm{vec}(|A|) + |P^{-1}Q(\mathrm{vec}(A) \otimes I_{n^2})|\mathrm{vec}(|B|)\\ &+ |P^{-1}|\mathrm{vec}(|C|)]\|_\infty.\end{aligned}$$

Proof It follows from (39) that

$$\begin{aligned}\mathrm{d}x &= -P^{-1}\left[Q(I_{m^2} \otimes \mathrm{vec}(B)),\, Q(\mathrm{vec}(A) \otimes I_{n^2}),\, I_{mnp}\right]\begin{bmatrix} D_A & & \\ & D_B & \\ & & D_C\end{bmatrix}\begin{bmatrix} D_A^{-1}\,\mathrm{d}a \\ D_B^{-1}\,\mathrm{d}b \\ D_C^{-1}\,\mathrm{d}c\end{bmatrix}\\ &= -P^{-1}\left[Q(I_{m^2} \otimes \mathrm{vec}(B))D_A,\, Q(\mathrm{vec}(A) \otimes I_{n^2})D_B,\, D_C\right]\begin{bmatrix} D_A^{-1}\,\mathrm{d}a \\ D_B^{-1}\,\mathrm{d}b \\ D_C^{-1}\,\mathrm{d}c\end{bmatrix}.\end{aligned}$$

Applying (20) and (21), we can deduce that

$$\begin{aligned}m(\Phi; X) &= \frac{\|-P^{-1}[Q(I_{m^2} \otimes \mathrm{vec}(B))D_A,\, Q(\mathrm{vec}(A) \otimes I_{n^2})D_B,\, D_C]\|_\infty}{\|X\|_{\max}}\\ &= \frac{\||P^{-1}[Q(I_{m^2} \otimes \mathrm{vec}(B))D_A,\, Q(\mathrm{vec}(A) \otimes I_{n^2})D_B,\, D_C]|e\|_\infty}{\|X\|_{\max}}\\ &= \frac{\||P^{-1}Q(I_{m^2} \otimes \mathrm{vec}(B))D_A|e + |P^{-1}Q(\mathrm{vec}(A) \otimes I_{n^2})D_B|e + |P^{-1}D_C|e\|_\infty}{|X\|_{\max}}\\ &= \frac{\begin{array}{c}\||P^{-1}Q(I_{m^2} \otimes \mathrm{vec}(B))|\mathrm{vec}(|A|) + |P^{-1}Q(\mathrm{vec}(A) \otimes I_{n^2})|\mathrm{vec}(|B|)\\ + |P^{-1}|\mathrm{vec}(|C|)\|_\infty\end{array}}{\|X\|_{\max}},\end{aligned}$$

and

$$\begin{aligned}
c(\Phi; X) &= \| -D_X^{-1} P^{-1} [Q(I_{m^2} \otimes \mathrm{vec}(B)) D_A,\, Q(\mathrm{vec}(A) \otimes I_{n^2}) D_B,\, D_C] \|_\infty \\
&= \| |D_X^{-1} P^{-1} [Q(I_{m^2} \otimes \mathrm{vec}(B)) D_A,\, Q(\mathrm{vec}(A) \otimes I_{n^2}) D_B,\, D_C]| e \|_\infty \\
&= \| D_{|X|}^{-1} [|P^{-1} Q(I_{m^2} \otimes \mathrm{vec}(B))| D_{|A|} e + |P^{-1} Q(\mathrm{vec}(A) \otimes I_{n^2})| D_{|B|} e \\
&\quad + |P^{-1}| D_{|C|} e] \|_\infty \\
&= \| D_X^{-1} [|P^{-1} Q(I_{m^2} \otimes \mathrm{vec}(B))| \mathrm{vec}(|A|) + |P^{-1} Q(\mathrm{vec}(A) \otimes I_{n^2})| \mathrm{vec}(|B|) \\
&\quad + |P^{-1}| \mathrm{vec}(|C|)] \|_\infty,
\end{aligned}$$

where $e = [1, 1, \dots, 1]^{\mathrm{T}}$ and has proper dimension to be conformable for the matrix-vector product. ■

Using the properties of Kronecker products, we can derive an upper bound for the mixed condition number:

$$\begin{aligned}
m(\Phi; X) &\leqslant \frac{\begin{array}{c}\| |P^{-1}|\, |Q| [(I_{m^2} \otimes \mathrm{vec}(|B|)) \mathrm{vec}(|A|) + (\mathrm{vec}(|A|) \otimes I_{n^2}) \mathrm{vec}(|B|)] \\ + |P^{-1}| \mathrm{vec}(|C|) \|_\infty\end{array}}{\|X\|_{\max}} \\
&= \frac{\| 2|P^{-1}| \mathrm{vec}((|A| \otimes |B|)|X|) + |P^{-1}| \mathrm{vec}(|C|) \|_\infty}{\|X\|_{\max}} \\
&= \frac{\| 2[(|A^{-1}|\, |A|) \otimes (|B^{-1}|\, |B|)] |X| + (|A^{-1}| \otimes |B^{-1}|) |C| \|_{\max}}{\|X\|_{\max}}.
\end{aligned}$$

Similarly, we can obtain an upper bound for the componentwise condition number:

$$c(\Phi; X) \leqslant \max_{ij} \frac{(2[(|A^{-1}|\, |A|) \otimes (|B^{-1}|\, |B|)] |X| + (|A^{-1}| \otimes |B^{-1}|) |C|)_{ij}}{|X|_{ij}}.$$

4. Numerical examples

In this section, we will consider some examples to show the sharpness of our new perturbation bounds. All the numerical results are carried out using MATLAB 6.5 with machine epsilon $\epsilon \approx 2.2 \times 10^{-16}$.

4.1 *General linear systems with multiple right-hand sides*

Let X, ΔX satisfy (22) and (23), respectively. We will compare the perturbation bounds given by (27), (28) and (29). Consider the linear system (22) with

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 12 & 11 \\ 11 & 15 & 11 \\ 9 & 10 & 19 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Suppose the perturbations are $\Delta A = 10^{-j} \times \mathrm{rand}(3)$, $\Delta B = 10^{-j} \times \mathrm{rand}(3)$, where $\mathrm{rand}(\cdot)$ is the MATLAB function.

Table 1. Comparison of the perturbation bounds.

j	8	10	12	14
γ_κ	2.1371e−009	6.2431e−011	4.7042e−013	6.4820e−015
$\epsilon_1\kappa$	2.3717e−008	3.0987e−010	3.5005e−012	3.2187e−014
γ_m	2.9339e−009	7.5451e−011	1.9806e−013	2.1464e−015
$\epsilon_2 m$	7.8863e−008	6.1061e−010	9.2289e−012	9.4111e−014
γ_c	1.1395e−008	1.1556e−010	1.3378e−012	1.0880e−014
$\epsilon_2 c$	1.9311e−007	3.6839e−009	3.3622e−011	2.9027e−013

Let $\gamma_\kappa := \|\Delta X\|_F/\|X\|_F$, $\gamma_m := \|\Delta X\|_{\max}/\|X\|_{\max}$, $\gamma_c := \|(\Delta X/X)\|_{\max}$ and define

$$\epsilon_1 := \frac{\|[\Delta A, \Delta B]\|_F}{\|[A, B]\|_F},$$

$$\epsilon_2 := \min\{\epsilon : |\Delta A| \leqslant \epsilon|A|, |\Delta B| \leqslant \epsilon|B|\}.$$

We list the computed pertubation bounds in table 1.

Note that κ, m, c in table 1 denote the normwise, mixed and componentwise condition numbers, respectively. From table 1, it can be seen that the computed bounds are only one order of magnitude larger than the actual forward errors. We can conclude that the first-order upper bounds of these three kinds of condition numbers are almost tight.

4.2 *Kronecker product linear systems with multiple right-hand sides*

Let X, ΔX satisfy (34) and (35), respectively. We will compare the perturbation bounds by using the condition numbers presented in Theorems 3.2 and 3.3. Consider the linear system (34) with

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 12 & 23 & 32 & 40 & 46 & 48 \\ 18 & 34 & 49 & 61 & 69 & 75 \\ 15 & 29 & 40 & 50 & 58 & 60 \\ 12 & 22 & 28 & 32 & 35 & 36 \\ 18 & 32 & 44 & 50 & 54 & 57 \\ 15 & 28 & 35 & 40 & 44 & 45 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}.$$

We also assume the perturbations are $\Delta A = 10^{-j} \times \mathrm{rand}(2)$, $\Delta B = 10^{-j} \times \mathrm{rand}(3)$, $\Delta C = 10^{-j} \times \mathrm{rand}(6)$. Here γ_κ, γ_m and γ_c are defined as before, and

$$\epsilon_1 := \frac{\|[\Delta A, \Delta B, \Delta C]\|_F}{\|[A, B, C]\|_F},$$

$$\epsilon_2 := \min\{\epsilon : |\Delta A| \leqslant \epsilon|A|, |\Delta B| \leqslant \epsilon|B|, |\Delta C| \leqslant \epsilon|C|\}.$$

We list the computed perturbation bounds in table 2.

From table 2, we can see that the normwise and componentwise bounds are only one order of magnitude larger than the actual ones. We can come to the conclusion that our theorems about the condition numbers give reasonable numerical results.

Table 2. Comparison of the perturbation bounds.

j	8	10	12	14
γ_κ	1.1692e−008	1.6850e−010	1.3145e−012	5.4330e−014
$\epsilon_1\kappa$	4.6262e−007	4.6818e−009	4.6047e−011	3.7876e−013
γ_m	1.4059e−008	2.9377e−010	5.0370e−012	5.3291e−014
$\epsilon_2 m$	1.6899e−006	1.6850e−008	1.6381e−010	1.5629e−012
γ_c	1.0985e−007	2.3251e−009	1.7783e−011	1.1613e−013
$\epsilon_2 c$	7.1049e−006	6.4601e−008	6.3849e−010	7.4406e−012

5. Conclusion and future work

In this paper, we have derived the normwise, mixed and componentwise condition numbers of linear systems $AX = B$ and $(A \otimes B)X = C$. It would be of interest to extend our results to singular [33] and Kronecker product linear systems with multiple right hands, where a generalized inverse [34] is needed.

Acknowledgements

The authors would like to thank two referees for their very detailed comments on this paper.

This project is supported by the National Natural Science Foundation of China under Grant 10471027 and the Shanghai Education Committee under Grant 06FZ024.

References

[1] Higham, N.J., 2002, *Accuracy and Stability of Numerical Algorithms* (2nd edn) (Philadelphia: SIAM).
[2] Jin, X. and Wei, Y., 2004, *Numerical Linear Algebra and its Applications* (Beijing: Science Press).
[3] Rice, J.R., 1966, A theory of condition. *SIAM Journal on Numerical Analysis*, **3**, 287–310.
[4] Higham, N.J., 1994, A survey of componentwise perturbation theory in numerical linear algebra. In: *Proceedings of the Symposium on Applied Mathematics*, Vol. 48, pp. 49–77.
[5] Skeel, R.D., 1979, Scaling for numerical stability in Gaussian elimination. *Journal of the Association for Computing Machinery*, **26**, 494–526.
[6] Rohn, J., 1989, New condition numbers for matrices and linear systems. *Computing*, **41**, 167–169.
[7] Chan, R.H., 1992, Numerical solutions for the inverse heat problems in $\mathbf{R}^N$. *Southeast Asian Bulletin of Mathematics*, **16**, 97–105.
[8] Hackbusch, W., Khoromskij, B.N. and Tyrtyshnikov, E., 2005, Hierarchical Kronecker tensor-product approximations. *Journal of Numerical Mathematics*, **13**, 119–156.
[9] Hansen, P.C., Nagy, J.G. and O'Leary, D.P., 2006, *Deblurring Images: Matrices, Spectra, and Filtering* (Philadelphia: SIAM).
[10] Jin, X., 2002, Developments and applications of block Toeplitz iterative solvers. In: *Combinatorics and Computer Science*, Vol. 2 (Dordrecht, The Netherlands: Kluwer Academic/Beijing: Science Press).
[11] Langville, A.N. and Stewart, W.J., 2004, The Kronecker product and stochastic automata networks. *Journal of Computational Applied Mathematics*, **167**, 429–447.
[12] Olshevsky, V., Oseledets, I. and Tyrtyshnikov, E., 2006, Tensor properties of multilevel Toeplitz and related matrices. *Linear Algebra and its Applications*, **412**, 1–21.
[13] Nagy, J.G., Ng, M.K. and Perrone, L., 2003, Kronecker product approximations for image restoration with reflexive boundary conditions. *SIAM Journal on Matrix Analysis and Applications*, **25**, 829–841.
[14] Xiang, H., Diao, H. and Wei, Y., 2005, On perturbation bounds of Kronecker product linear systems and their level-2 condition numbers. *Journal of Computational Applied Mathematics*, **183**, 210–231.
[15] Dennis, J.E., Jr and Schnabel, R.B., 1983, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations* (Englewood Cliffs, NJ: Prentice-Hall).
[16] Higham, D.J. and Higham, N.J., 1992, Componentwise perturbation theory for linear systems with multiple right-hand sides. *Linear Algebra and its Applications*, **174**, 111–129.
[17] Allen, R.C. and Pruess, S.A., 1981, An analysis of an inverse problem in ordinary differential equations. *SIAM Journal on Scientific and Statistical Computing*, **2**, 176–185.
[18] Smith, C., Peterson, A. and Mittra, R., 1989, A conjugate gradient algorithm for the treatment of multiple incident electromagnetic fields. *IEEE Transactions on Antennas and Propagation*, **37**, 1490–1493.

[19] O'Leary, D.P., 1980, The block conjugate gradient algorithm and related methods. *Linear Algebra and its Applications*, **29**, 293–322.

[20] Chan T.F. and Wan, W.L., 1997, Analysis of projection methods for solving linear systems with multiple right-hand sides. *SIAM Journal on Scientific Computing*, **18**, 1698–1721.

[21] Simoncini, V. and Gallopoulos, E., 1996, A hybrid block GMRES method for nonsymmetric systems with multiple right-hand sides. *Journal of Computational Applied Mathematics*, **66**, 457–469.

[22] Freund, R.W. and Malhotra, M., 1997, A block QMR algorithm for non-Hermitian linear systems with multiple right-hand sides. *Linear Algebra and its Applications*, **254**, 119–157.

[23] Gu, G. and Wu, H., 1999, A block EN algorithm for nonsymmetric linear systems with multiple right-hand sides. *Linear Algebra and its Applications*, **299**, 1–20.

[24] Simoncini, V. and Gallopoulos, E., 1995, An iterative method for nonsymmetric systems with multiple right-hand sides. *SIAM Journal on Scientific Computing*, **16**, 917–933.

[25] Graham, A., 1981, *Kronecker Products and Matrix Calculus with Applications* (Chichester, UK: Ellis Horwood).

[26] Horn, R.A. and Johnson, C.R., 1991, *Topics in Matrix Analysis* (Cambridge, UK: Cambridge University Press).

[27] Meyer, C.D., 2000, *Matrix Analysis and Applied Linear Algebra* (Philadephia, PA: SIAM).

[28] Van Loan, C.F., 2000, The ubiquitous Kronecker product. *Journal of Computational Applied Mathematics*, **123**, 85–100.

[29] Magnus, J.R. and Neudecker, H., 1988, *Matrix Differential Calculus with Applications in Statistics and Econometrics* (New York: John Wiley & Sons).

[30] Konstaintinov, M., Gu, D., Mehrmann, V. and Petkov, P., 2003, *Perturbation Theory for Matrix Equations* (Amsterdam: Elsevier).

[31] Gohberg, I. and Koltracht, I., 1990, On the inversion of Cauchy matrices. In: *Proceedings of the International Symposium MTNS-89*, Vol. III (Boston, MA: Birkhäuser), pp. 381–293.

[32] Gohberg, I. and Koltracht, I., 1993, Mixed, componentwise, and structured condition numbers. *SIAM Journal on Matrix Analysis and Applications*, **14**, 688–704.

[33] Xu, W., Wei, Y. and Qiao, S., 2006, Condition numbers for structured least squares problems. *BIT*, **46**, 203–225.

[34] Wang, G., Wei, Y. and Qiao, S., 2004, *Generalized Inverse: Theory and Computations* (Beijing: Science Press).

[35] Xiang, H., Wei, Y. and Diao, H., 2006, Perturbation analysis of generalized saddle point systems. *Linear Algebra and its Applications*, **419**, 8–23.

[36] Xiang, H. and Wei, Y., 2007, On normwise structured backward errors for saddle point systems. *SIAM Journal on Matrix Analysis and Applications*, **29**, 838–849.

Copyright of International Journal of Computer Mathematics is the property of Taylor & Francis Ltd and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.