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Entrywise perturbation theory for diagonally dominant M-matrices with applications

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Summary. This paper introduces a new perspective on the study of computational problems related to diagonally dominant M-matrices by establishing some new entrywise perturbation results. If a diagonally dominant M-matrix is perturbed in a way that each off-diagonal entry and its row sums (i.e. the quantities of diagonal dominance) have small relative errors, we show that its determinant, cofactors, each entry of the inverse and the smallest eigenvalue all have small relative errors. The error bounds are given and they do not depend on any condition number. Applying this result to the studies of electrical circuits and tail probabilities of a queue whose embedded Markov chains is of GI/M/1 type, we discuss the relative sensitivity of the operating speed of circuits and of the percentile of the queue length, respectively.

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1 Introduction

Diagonally dominant M-matrices arise in a large variety of application problems and have been studied extensively in the literature, see [2]. Besides the inverse, the smallest eigenvalue (corresponding to the Perron root of the inverse) is of particular interest since it typically characterizes certain properties of corresponding physical systems. For example, it represents decay rates of signals in linear electrical circuits (see Sect. 3 or [16]) or of steady state distributions in infinite Markov chains (see Sect. 4). Since the smallest eigenvalue could be very small relative to the norm of the matrix, understanding its relative perturbation and computing it with a small relative error are important in practice. In this paper, we are concerned with a perturbation analysis for the inverse and the smallest eigenvalue of a (weakly) diagonally dominant M-matrix under entrywise perturbations. We shall study algorithms that compute the inverse and the smallest eigenvalue to high relative accuracy in a separate work [1].

There has been considerable interest in the entrywise perturbation theory and great advances have been made in recent years in both analysis and algorithms (see [4,5,9,10,15,17,18] for example). For many interesting problems, by restricting perturbations to those that preserve certain sparsity structure and are small entrywise, the perturbation bounds are usually strengthened and thus even some small quantities can be guaranteed to have small relative perturbations. For example, the smallest singular value of a bidiagonal matrix [4] and the steady state distribution of a Markov chain [14] are known to have such properties.

For an M-matrix with entrywise perturbations, some relative perturbation bounds for its inverse and the smallest eigenvalue have been obtained in [7], [19], and [20], which show that $(1-\gamma)^{-1}$ as the condition number measuring the sensitivity of the perturbation, where γ is the spectral radius of the Jacobi iteration matrix for the M-matrix. In the case of a diagonally dominant M-matrix $A = (a_{ij})$, γ is related to the maximal ratio of the diagonal dominance $\sum_{j\neq i} |a_{ij}|/a_{ii}$, and thus the dependence of the perturbation bounds on γ suggest that the relative perturbation will be very sensitive when the diagonal dominance is weak. This is expected because in this case $|A|\,|A^{-1}|$ could be very large and the quantity of minimal diagonal dominance $\min_i(a_{ii} - \sum_{j\neq i} |a_{ij}|)$, which bounds the smallest eigenvalue away from 0, is very sensitive to entrywise perturbations. We observe, however, that if we restrict perturbations to those that also have small relative errors in the quantities of diagonal dominance

$$d_i \equiv a_{ii} - \sum_{j\neq i} |a_{ij}|,$$

which often have physical significance themself in applications (see Sect. 3), the lower bound $\min_i d_i$ will have small relative perturbation, and it turns out that the same is true for the smallest eigenvalue and hence the inverse.

Indeed, the main result of this paper states that if $A = (a_{ij})$ and $\widetilde{A} = (\widetilde{a}_{ij})$ are two $n \times n$ (weakly) diagonally dominant M-matrices with $|a_{ij} - \widetilde{a}_{ij}| \leq \epsilon |a_{ij}|$ (for $i \neq j$) and $|d_i - \widetilde{d}_i| \leq \epsilon |d_i|$ (where $\widetilde{d}_i = \widetilde{a}_{ii} - \sum_{j \neq i} |\widetilde{a}_{ij}|$), then

$$\frac{(1-\epsilon)^n}{(1+\epsilon)^{n-1}} A^{-1} \leq \widetilde{A}^{-1} \leq \frac{(1+\epsilon)^n}{(1-\epsilon)^{n-1}} A^{-1} \tag{1}$$

and

$$\frac{(1-\epsilon)^{n-1}}{(1+\epsilon)^n} \lambda \leq \widetilde{\lambda} \leq \frac{(1+\epsilon)^{n-1}}{(1-\epsilon)^n} \lambda, \tag{2}$$

where λ and $\widetilde{\lambda}$ are the smallest eigenvalues of A and $\widetilde{A}$ respectively. With these bounds independent of any condition number, the entries of A^{-1} and λ are determined to the same order of relative accuracy as in its off diagonal entries and its diagonal dominant parts d_i. This in particular suggests that the computations of A^{-1} and λ should be based on these data rather than all entries of A (see [1]). We also note that the off-diagonal entries and the diagonal dominant parts are frequently the physical parameters in applications, while the diagonal entries are functions of them (see Sect. 3 for example).

Our interest in this type of perturbation analysis is motivated by the studies of electrical circuits and GI/M/1 type Markov chains for modeling queuing systems. For a digital electrical circuit, our results directly imply how perturbations in the physical parameters affect the operating speed of the circuit. On the other hand, in considering entrywise perturbations to each block of an embedded Markov chain, we are led to a perturbation of a diagonally dominant M-matrix that satisfies the above conditions. We can then apply our results to study how the perturbation affects the percentile of queue length through the smallest eigenvalue of the M-matrix.

The paper is organized as follows. In Sect. 2 we present the relative error bounds for the determinant of A and each algebraic cofactor, from which we obtain the main result (1), (2). We also give a slight improvement of the entrywise perturbation theorem for finite Markov chains. In Sect. 3, we describe applications of our result to an electrical circuit network, and in Sect. 4 to the study of GI/M/1 type queues.

2 Main perturbation bounds

In this section, we present our main results on the perturbation bounds for the determinant, the cofactors, the inverse and the smallest eigenvalue of a diagonally dominant M-matrix A. First, we define our notation and terminology and give some preliminary results.

Definition. *A matrix A is called an M-matrix [2] if it can be expressed in the form*

$$A = sI - B, \quad s \geq 0,\ B \geq 0$$

with $s \geq \rho(B)$, the Perron root of B. A matrix $A = (a_{ij})$ is said to be diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

for all i.

We note that this definition of the M-matrix does not require that it be invertible. Also the diagonally dominant matrix defined here is sometimes called a weakly diagonally dominant matrix in the literature.

It is easy to show that if $A = (a_{ij})$ is diagonally dominant with $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$, then A is an M-matrix [2]. In this case, the row sum $\sum_j a_{ij} = a_{ii} - \sum_{j \neq i} |a_{ij}|$ is the diagonal dominant part for the i-th row.

Throughout this work, $\mathbf{1}$ denotes the vector of ones, i.e.,

$$\mathbf{1} = [1, 1, \cdots, 1]^{\mathrm{T}}\ .$$

Given subscripts

$$(3) \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq n, \quad \textit{and} \quad 1 \leq j_1 < j_2 < \cdots < j_p \leq n,$$

we denote the submatrix of Q obtained by deleting rows i_k and columns j_k listed in (3) by

$$Q^c \begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_p \end{pmatrix}.$$

Let Q_{ij} denote the algebraic cofactor of Q to the (i,j) entry, i.e.,

$$Q_{ij} = (-1)^{\mathrm{i}+j} \det Q^c \begin{pmatrix} i \\ j \end{pmatrix}.$$

In entrywise perturbation analysis for a matrix A, one typically considers perturbations that have small relative errors entrywise. Then, as pointed out in the introduction, the perturbations on the inverse and eigenvalues would depend on certain condition number. We suggest that it is more appropriate to consider perturbations that have small errors on the off-diagonal entries and on the diagonal dominant parts (i.e. the row sums $A\mathbf{1}$), and as we will see, this leads to stronger results. The following lemma shows that such perturbations also have small relative errors for the diagoanl entries and thus for all entries. The converse, however, is not true, and this is why it is important to consider perturbations to the diagonal dominant parts.

Lemma 2.1 *Let $A = (a_{ij})$ and $\widetilde{A} = (\widetilde{a}_{ij})$ be diagonally dominant M-matrices. If $|a_{ij} - \widetilde{a}_{ij}| \le \epsilon |a_{ij}|$ for all $i \ne j$ and $|A\mathbf{1} - \widetilde{A}\mathbf{1}| \le \epsilon A\mathbf{1}$, then*

$$|A - \widetilde{A}| \le \epsilon |A|.$$

Proof. We only need to show that $(1-\epsilon)a_{ii} \le \widetilde{a}_{ii} \le (1+\epsilon)a_{ii}$ for all i. This follows from

$$\widetilde{a}_{ii} = \sum_{j=1}^{n} \widetilde{a}_{ij} - \sum_{j \ne i} \widetilde{a}_{ij} = (\widetilde{A}\mathbf{1})_i + \sum_{j \ne i} (-\widetilde{a}_{ij}),$$

and the assumptions for $(\widetilde{A}\mathbf{1})_i$ and $-\widetilde{a}_{ij}$ ($j \ne i$). $\square$

Now, we give an identity relating cofactors by expansions.

Lemma 2.2 *Let A and $B = A^c\begin{pmatrix} j \\ j \end{pmatrix}$. Then, for $i \ne j$,*

$$A_{ij} = \sum_{p<j} (-a_{jp}) B_{i-1,p} + \sum_{p>j} (-a_{jp}) B_{i-1,p-1}. \tag{4}$$

Proof. We consider the case $i > j$. Expanding the cofactor to element a_{ij} along its jth row, we have

$$\det A^c\begin{pmatrix} i \\ j \end{pmatrix} = \sum_{p<j} (-1)^{j+p} a_{jp} \det A^c\begin{pmatrix} j & i \\ p & j \end{pmatrix}$$
$$+ \sum_{p>j} (-1)^{j+p-1} a_{jp} \det A^c\begin{pmatrix} j & i \\ j & p \end{pmatrix}.$$

Since

$$A^c\begin{pmatrix} j & i \\ p & j \end{pmatrix} = B^c\begin{pmatrix} i-1 \\ p \end{pmatrix}, \quad \text{for} \quad j > p$$

and

$$A^c\begin{pmatrix} j & i \\ j & p \end{pmatrix} = B^c\begin{pmatrix} i-1 \\ p-1 \end{pmatrix}, \quad \text{for} \quad j < p,$$

we have

$$\det A^c\begin{pmatrix} j & i \\ p & j \end{pmatrix} = (-1)^{\mathrm{i}+p-1} B_{i-1,p} \quad \text{for} \quad j > p$$

and

$$\det A^c\begin{pmatrix} j & i \\ j & p \end{pmatrix} = (-1)^{\mathrm{i}+p} B_{i-1,p-1} \quad \text{for} \quad j < p,$$

which give

$$
\begin{aligned}
A_{ij} &= (-1)^{\mathrm{i}+j} \det A^c \begin{pmatrix} i \\ j \end{pmatrix} \\
&= \sum_{p<j} (-a_{jp}) B_{i-1,p} + \sum_{p>j} (-a_{jp}) B_{i-1,p-1}.
\end{aligned}
$$

□

Using the identity just established, we can prove some perturbation bounds for the determinant of A and its algebraic cofactors. Note that for the determinants, we do not require that A be nonsingular or irreducible.

Theorem 2.3 *Let A and $\widetilde{A}$ be two diagonally dominant M-matrices. If for all $i \neq j$*

$$|a_{ij} - \widetilde{a}_{ij}| \leq \epsilon |a_{ij}| \quad \textit{and} \quad |A\mathbf{1} - \widetilde{A}\mathbf{1}| \leq \epsilon A\mathbf{1},$$

then

$$(1-\epsilon)^n \det A \leq \det \widetilde{A} \leq (1+\epsilon)^n \det A, \tag{5}$$

$$(1-\epsilon)^{n-1} A_{ij} \leq \widetilde{A}_{ij} \leq (1+\epsilon)^{n-1} A_{ij}. \tag{6}$$

Proof. We prove this theorem by induction on n. It is easy to check that it holds for $n = 2$. Now suppose it holds for $n - 1 \geq 2$ and consider the case of n. We first prove (6) for $i \geq j$. The proof for $i < j$ is analogous. Let

$$B = A^c \begin{pmatrix} j \\ j \end{pmatrix} \quad \textit{and} \quad \widetilde{B} = \widetilde{A}^c \begin{pmatrix} j \\ j \end{pmatrix}.$$

Obviously B and $\widetilde{B}$ are diagonally dominant M-matrices and it can be checked that they satisfy

$$|B - \widetilde{B}| \leq \epsilon |B| \quad \textit{and} \quad |B\mathbf{1} - \widetilde{B}\mathbf{1}| \leq \epsilon B\mathbf{1}.$$

Thus, by the induction assumption,

$$(1-\epsilon)^{n-1} \det B \leq \det \widetilde{B} \leq (1+\epsilon)^{n-1} \det B,$$

i.e. (6) holds for $i = j$. Furthermore,

$$(1-\epsilon)^{n-2} B_{lk} \leq \widetilde{B}_{lk} \leq (1+\epsilon)^{n-2} B_{lk}.$$

Thus, for $i > j$, by Lemma 2.1,

$$A_{ij} = \sum_{p<j} (-a_{jp}) B_{i-1,p} + \sum_{p>j} (-a_{jp}) B_{i-1,p-1},$$

and

$$\begin{aligned}\widetilde{A}_{ij} &= \sum_{p<j}(-\widetilde{a}_{jp})\widetilde{B}_{i-1,p} + \sum_{p>j}(-\widetilde{a}_{jp})\widetilde{B}_{i-1,p-1} \\ &\leq \sum_{p<j}(1+\epsilon)(-a_{jp})(1+\epsilon)^{n-2}B_{i-1,p} \\ &\quad + \sum_{p>j}(1+\epsilon)(-a_{jp})(1+\epsilon)^{n-2}B_{i-1,p-1} \\ &\leq (1+\epsilon)^{n-1}A_{ij},\end{aligned}$$

where we note that each term in the expansion is nonnegative. Similarly $(1-\epsilon)^{n-1}A_{ij} \leq \widetilde{A}_{ij}$.

To prove (5), we write

$$A\mathbf{1} = (d_1, \cdots, d_n)^{\mathrm{T}} \quad and \quad \widetilde{A}\mathbf{1} = (\widetilde{d}_1, \cdots, \widetilde{d}_n)^{\mathrm{T}}.$$

Adding all columns of A to its first column and then expanding $\det A$ along the first column, we obtain

$$\det A = \sum_{i=1}^{n} d_i A_{i1}.$$

Similarly,

$$\det \widetilde{A} = \sum_{i=1}^{n} \widetilde{d}_i \widetilde{A}_{i1},$$

since A_{i1}, $\widetilde{A}_{i1}$, d_i and $\widetilde{d}_i$ are all nonnegative, (5) follows from (6) and $(1-\epsilon)d_i \leq \widetilde{d}_i \leq (1-\epsilon)d_i$. $\square$

This theorem shows that no matter how small $\det A$ is, the relative error will be small. This is the case even when $\det A = 0$. Furthermore, a direct application leads to the following Corollary.

Corollary 2.4 *Let P and $\widetilde{P}$ be two stochastic matrices with respectively stationary distributions $\pi = [\pi_1, \cdots, \pi_n]$ and $\widetilde{\pi} = [\widetilde{\pi}_1, \cdots, \widetilde{\pi}_n]$. If $|P - \widetilde{P}| \leq \epsilon P$, then*

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)^{n-1} \pi_i \leq \widetilde{\pi}_i \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)^{n-1} \pi_i, \quad 1 \leq i \leq n.$$

The proof follows from applying Theorem 2.2 to $A = I - P$ and $\widetilde{A} = I - \widetilde{P}$ and using $\pi_i = A_{ii}/(\sum_{j=1}^{n} A_{jj})$. We note that entrywise perturbation bounds of the above type was originally obtained by O'Cinneide [14] and improved by Xue [21]. Corollary 2.4 slightly improves these bounds.

If A is nonsingular, then the following theorem is a straightforward result of Lemma 2.3 because

$$A^{-1} = \frac{1}{\det A}(A_{ji})_{i,j=1}^{n}.$$

Theorem 2.5 *Let A and $\widetilde{A}$ be nonsingular diagonally dominant M-matrices. If for all $i \neq j$*

$$|a_{ij} - \widetilde{a}_{ij}| \leq \epsilon |a_{ij}| \quad and \quad |A\mathbf{1} - \widetilde{A}\mathbf{1}| \leq \epsilon A\mathbf{1},$$

then

$$\frac{(1-\epsilon)^n}{(1+\epsilon)^{n-1}} A^{-1} \leq \widetilde{A}^{-1} \leq \frac{(1+\epsilon)^n}{(1-\epsilon)^{n-1}} A^{-1}.$$

This theorem demonstrates that the relative error for each entry of A^{-1}, regardless of its magnitude, is of order ϵ. We can use this theorem to obtain the relative error bound for the smallest eigenvalue of A.

Theorem 2.6 *Let A and $\widetilde{A}$ be diagonally dominant M-matrices and λ and $\widetilde{\lambda}$ are the smallest eigenvalues of A and $\widetilde{A}$ respectively. If for all $i \neq j$*

$$|a_{ij} - \widetilde{a}_{ij}| \leq \epsilon |a_{ij}| \quad and \quad |A\mathbf{1} - \widetilde{A}\mathbf{1}| \leq \epsilon A\mathbf{1},$$

then

$$\frac{(1-\epsilon)^{n-1}}{(1+\epsilon)^n} \lambda \leq \widetilde{\lambda} \leq \frac{(1+\epsilon)^{n-1}}{(1-\epsilon)^n} \lambda.$$

Proof. If A is singular, from Lemma 2.3, $\widetilde{A}$ is also singular, then $\lambda = \widetilde{\lambda} = 0$. If A is nonsingular, then $1/\lambda$ is the spectral radius of A^{-1}. From Theorem 1 in [6] (i.e. the monotonicity of the spectral radius of a nonnegative matrix) and Theorem 2.5 here, we have

$$\frac{(1-\epsilon)^n}{(1+\epsilon)^{n-1}} \frac{1}{\lambda} \leq \frac{1}{\widetilde{\lambda}} \leq \frac{(1+\epsilon)^n}{(1-\epsilon)^{n-1}} \frac{1}{\lambda},$$

which yields the bound for $\widetilde{\lambda}$. □

Remark. From the theorem, we have $|\lambda - \widetilde{\lambda}|/\lambda \leq (2n-1)\epsilon + O(\epsilon^2)$. However, if the perturbation to A is confined to a single row only (say row k), i.e. $a_{ij} - \widetilde{a}_{ij} = 0$ for $i \neq k$, then it can be easily proved that the bounds for the determinants in Theorem 2.3 reduces to $(1-\epsilon)\det A \leq \det \widetilde{A} \leq (1+\epsilon)\det A$ and $(1-\epsilon)A_{ij} \leq \widetilde{A}_{ij} \leq (1+\epsilon)A_{ij}$. Thus the bounds for A^{-1} and λ can be strengthened. In particular, $|\lambda - \widetilde{\lambda}|/\lambda \leq 2\epsilon + O(\epsilon^2)$. Indeed, it is our conjecture that this stronger bound for the eigenvalue is true even for the general perturbations.

3 Application to electrical circuits

In this section, we briefly describe a direct application of our results to digital electrical circuits. We consider a general RC network with n nodes, in which the ith node is capacitively grounded by $c_i > 0$ and resistively grounded by a conductance $g_{i_0} \geq 0$. Also, each node i is connected by a conductance $-g_{ij} \geq 0$ with the node j (see [16] for an illustration of the network). Then, if we denote by $v = [v_1(t), v_2(t), ..., v_n(t)]^{\mathrm{T}}$ the vector of node voltages and by $v^\infty \in \mathbf{R}^n$ the stationary voltage vector, the transient evolution of this circuit is described by the equation [11, 16]

$$C\frac{dx}{dt} = -G\,x$$

where $x(t) = v(t) - v^\infty$, $C = diag(c_1, c_2, ..., c_n)$ and $G = (g_{ij})$ with its diagonal entries defined by the given physical parameters as follows

$$g_{ii} = g_{i_0} + \sum_{j \neq i} |g_{ij}|.$$

Clearly, the matrix $A = C^{-1}G$ is a diagonally dominant M-matrix and the diagonal dominant part is given by the vector

$$A\mathbf{1} = [g_{1_0}/c_1, \cdots, g_{n_0}/c_n]^{\mathrm{T}}.$$

Furthermore, the smallest eigenvalue λ of A is the (exponential) rate of decay for the solution $x(t)$, and the operating speed of this digital circuit is proportional to $1/\lambda$. It is naturally desirable in circuit design that a small perturbation to each component does not lead to significant change in the operating speed (i.e. λ).

Clearly, an order ϵ perturbation to the physical parameters c_i, g_{i_0}, g_{ij} results in an order ϵ perturbation to the entries of A and the diagoanl dominant part $A\mathbf{1}$, which by Theorem 2.6 leads to an order ϵ perturbation to the operating speed of the circuit. The following theorem states this result.

Theorem 3.1 *Let $A = C^{-1}G$ and $\widetilde{A} = \widetilde{C}^{-1}\widetilde{G}$ be defined by the physical parameters c_i, g_{i_0}, g_{ij} and $\widetilde{c}_i, \widetilde{g}_{i_0}, \widetilde{g}_{ij}$ respectively as above. If $|c_i - \widetilde{c}_i| \leq \epsilon|c_i|$, $|g_{i_0} - \widetilde{g}_{i_0}| \leq \epsilon|g_{i_0}|$, and $|g_{ij} - \widetilde{g}_{ij}| \leq \epsilon|g_{ij}|$, then*

$$\frac{|\lambda - \widetilde{\lambda}|}{\lambda} \leq 2(2n-1)\epsilon + O(\epsilon^2)). \tag{7}$$

where λ and $\widetilde{\lambda}$ are the smallest eigenvalues of A and $\widetilde{A}$ respectively.

We note that if there are perturbations at only one node of the circuit, it results in perturbations to one row of the matrix A and thus $2n-1$ in the bound can be reduced to 2 (see the remark after Theorem 2.6).

4 Application to GI/M/1 type Markov chain

In this section, we study in details an application of our results to a queuing model. The transition probability matrix P of a Markov chain of GI/M/1 type is an infinite matrix of the form

$$P = \begin{pmatrix} B_0 & A_0 & & & & \\ B_1 & A_1 & A_0 & & & \\ B_2 & A_2 & A_1 & A_0 & & \\ B_3 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{8}$$

in which all elements are $n \times n$ nonnegative matrices. In queuing theory, the embedded Markov chains of a large class of queues, such as GI/PH/1 queues, are of this type. In most applications, $A = \sum_{k=0}^{\infty} A_k$ is stochastic and irreducible.

4.1 Steady-state queue length

It turns out that the steady-state of P can be determined by the minimal nonnegative solution R to the nonlinear matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$ (see [12]). Furthermore, the spectral radius of R is the unique solution η in $[0, 1)$ to the equation

$$z = \chi(z) \quad \text{with} \quad \chi(z) = \rho(A(z)) = \rho\left(\sum_{k=0}^{\infty} z^k A_k\right),$$

where $\rho(A(z))$ denotes the spectral radius of the matrix $A(z) = \sum_{k=0}^{\infty} z^k A_k$.

The quantity η, often a function of system parameters and called caudal characteristic by Neuts [13], is the decay rate of the tail probabilities of the steady-state queue length. Indeed, the probability of the steady-state queue length L being at least k_0 is

$$P(L \geq k_0) = x_0 \sum_{k=k_0}^{\infty} R^k \mathbf{1} \approx \frac{\alpha \eta^{k_0}}{1-\eta}. \tag{9}$$

where α is a constant. On the other hand, the $(100p)$-th percentile of queue length (see [3]) for a given p, i.e. the length k_0 such that $P(L \geq k_0) = 1-p$, can be approximated by

$$k_0(p) \approx \frac{-\log \beta + \log(1-p)}{\log \eta}, \tag{10}$$

where $\beta = \alpha/(1-\eta)$. For a number of complex queuing models, the computation of R, and therefore of x_0 and α, is very difficult. However, for a broad class of such models, the spectral radius η can be computed by elementary algorithm. It is desirable to study the queue through η.

In modeling a queuing system, one is interested in how sensitive the probability of the steady-state queue length $P(L \geq k_0)$ or the percentile of queue length $k_0(p)$ is to perturbations of system parameters. From (9) and (10), we see that this depends on η and $1-\eta$ (noting that $\log \eta \sim 1-\eta$ when $\eta \sim 1$). For this, we study in the next subsection the relative perturbations of η and $1-\eta$. We note that it is necessary to study $1-\eta$ separately because η in applications could be very close to 1.

4.2 Sensitivity of the queue length

Consider two GI/M/1 type Markov chains P and $\widetilde{P}$ as defined by A_i and $\widetilde{A}_i$ respectively. Then $A = \sum_{k=0}^{\infty} A_k$ and $\widetilde{A} = \sum_{k=0}^{\infty} \widetilde{A}_k$ are stochastic. We assume for all k,

$$|A_k - \widetilde{A}_k| \leq \epsilon A_k.$$

Let η and $\widetilde{\eta}$ be respectively the solutions to

$$z = \chi(z) \quad \textit{and} \quad z = \widetilde{\chi}(z)$$

where $\widetilde{\chi}(z) = \rho(\widetilde{A}(z)) = \rho(\sum_{k=0}^{\infty} z^k \widetilde{A}_k)$. To consider the perturbations of $\delta = 1-\eta$ and $\widetilde{\delta} = 1-\widetilde{\eta}$, we first study the perturbations of $1-\chi(z)$ and $1-\widetilde{\chi}(z)$.

Theorem 4.1 *For* $0 \leq z < 1$, $B(z) = I - A(z)$ *and* $\widetilde{B}(z) = I - \widetilde{A}(z)$ *are diagonally dominant M-matrices satisfying*

$$|B(z) - \widetilde{B}(z)| \leq \epsilon |B(z)| \quad \textit{and} \quad |B(z)\mathbf{1} - \widetilde{B}(z)\mathbf{1}| \leq \epsilon B(z)\mathbf{1}.$$

$1-\chi(z)$ *and* $1-\widetilde{\chi}(z)$ *are the smallest eigenvalues of* $B(z)$ *and* $\widetilde{B}(z)$ *resp.*

Proof. Since for $0 \leq z < 1$, $A(z) \leq A$, $A(z)$ is substochastic. Then $B(z) = I - A(z)$ is a diagonally dominant M-matrix. The same holds for $\widetilde{B}(z)$. Because

$$B(z)\mathbf{1} = \sum_{k=1}^{\infty}(1-z^k)A_k\mathbf{1} \quad \textit{and} \quad \widetilde{B}(z)\mathbf{1} = \sum_{k=1}^{\infty}(1-z^k)\widetilde{A}_k\mathbf{1},$$

we have

$$|B(z)\mathbf{1} - \widetilde{B}(z)\mathbf{1}| \leq \sum_{k=1}^{\infty}(1-z^k)|A_k - \widetilde{A}_k|\mathbf{1} \leq \epsilon B(z)\mathbf{1}.$$

Writing $A(z) = (a_{ij}(z))$ and $B(z) = (b_{ij}(z))$. Then,

$$|A(z) - \widetilde{A}(z)| \le \sum_{k=1}^{\infty} z^k |A_k - \widetilde{A}_k| \le \epsilon A(z).$$

Thus, for $i \neq j$, $|b_{ij}(z) - \widetilde{b}_{ij}(z)| \le \epsilon |b_{ij}(z)|$ since $b_{ij}(z) = -a_{ij}(z)$. By Proposition 2.7, this leads to

$$|B(z) - \widetilde{B}(z)| \le \epsilon B(z).$$

Clearly, $1 - \chi(z)$ and $1 - \widetilde{\chi}(z)$ are the smallest eigenvalues of $B(z)$ and $\widetilde{B}(z)$ resp. $\square$

The theorem shows that small perturbations to the physical parameters result in the same order perturbations to $B(z)$ and its diagonal dominant part $B(z)\mathbf{1}$. Note that $B(z)$ defines $\delta = 1 - \eta$. We now can apply our results in Sect. 2 to the above to obtain the following bounds.

Lemma 4.2 *For* $0 \le z < 1$, *we have*

$$|\widetilde{\chi}(z) - \chi(z)| \le \epsilon \chi(z)$$

and

$$|(1 - \widetilde{\chi}(z)) - (1 - \chi(z))| \le ((2n-1)\epsilon + O(\epsilon^2))(1 - \chi(z)).$$

Using Lemma 3.2, we can bound the relative errors in η and $\delta = 1 - \eta$.

Theorem 4.3 *Let conditions hold as above. Then*

$$\left| \frac{\eta - \widetilde{\eta}}{\eta} \right| \le \frac{\epsilon}{1 - \chi'(\eta)} + O(\epsilon^2) \tag{11}$$

and

$$\left| \frac{\delta - \widetilde{\delta}}{\delta} \right| \le \frac{(2n-1)\epsilon}{1 - \chi'(\eta)} + O(\epsilon^2), \tag{12}$$

where $\chi'(\eta) = \sum_{k=1}^{\infty} k\eta^{k-1} u^{\mathrm{T}} A_k v$ *and* u *and* v *are the respective right and left eigenvectors associated with* $\chi(\eta)$ *of* $A(\eta)$ *with normalization* $u^{\mathrm{T}} v = 1$.

Proof. We only prove (11). The proof for (12) is similar.

Obviously δ and $\widetilde{\delta}$ are the respective roots to the equations

$$z = \Delta(z) \quad \textit{and} \quad z = \widetilde{\Delta}(z) \quad z > 0,$$

where

$$\Delta(z) = 1 - \chi(1 - z) \quad \textit{and} \quad \widetilde{\Delta}(z) = 1 - \widetilde{\chi}(1 - z).$$

It follows from Lemma 3.2 that

$$|\Delta(z) - \widetilde{\Delta}(z)| \leq ((2n-1)\epsilon + O(\epsilon^2))\Delta(z).$$

Noting

$$\Delta'(\delta) = \chi'(\eta) \quad and \quad \lim_{\epsilon \to 0}(\widetilde{\delta} - \delta) = 0,$$

we have

$$\begin{aligned}\widetilde{\delta} - \delta &= \widetilde{\Delta}(\widetilde{\delta}) - \Delta(\delta) \\ &\leq [1 + (2n-1)\epsilon](\Delta(\widetilde{\delta}) - \Delta(\delta)) + (2n-1)\epsilon\delta + O(\epsilon^2) \\ &= (1 + (2n-1)\epsilon)\chi'(\eta)(\widetilde{\delta} - \delta) + (2n-1)\epsilon\delta + O(\epsilon^2) \\ &\quad + O((\widetilde{\delta} - \delta)^2)\end{aligned}$$

which implies

$$\frac{\widetilde{\delta} - \delta}{\delta} \leq \frac{(2n-1)\epsilon}{1 - \chi'(\eta)} + O(\epsilon^2).$$

Similarly,

$$\frac{\widetilde{\delta} - \delta}{\delta} \geq -\frac{(2n-1)\epsilon}{1 - \chi'(\eta)} + O(\epsilon^2),$$

which completes the proof. □

Our results show that the sensitivity of the tail probability or the percentile of the queue length depends on $1 - \chi'(\eta)$. If $\chi'(\eta)$ is not near to one, small entrywise perturbations in A_i only causes small relative error in the percentile. However, $\chi'(\eta)$ is in general not computable and it would be interesting to find some easily computable bounds for it.

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