Example

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1 Problem 1

Consider the ring homomorphism $\hat{\varphi} : \mathbb{C}[x, y] \to \mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}], x \mapsto (x, 0), y \mapsto (0, y)$.

Claim: $\ker \hat{\varphi} = (xy)$.

Proof: every polynomials f in $\mathbb{C}[x, y]$ can be written into the form $c + xP(x) + yP(y) + xyR(x, y)$, where $c \in \mathbb{C}, P \in \mathbb{C}[x], Q \in \mathbb{C}[y], R \in \mathbb{C}[x, y]$, and we have $\hat{\varphi}(f) = (c + xP(x), c + yQ(y))$, so we see that $f \in \ker \hat{\varphi} \Leftrightarrow f \in (xy)$.

$\blacksquare$

Thus $\hat{\varphi}$ induces an injective homomorphism $\varphi : \mathbb{C}[x, y]/(xy) \to \mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}]$. From this we can view $\mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}]$ as a $\mathbb{C}[x, y]/(xy)$-algebra.

And we have $\varphi(\overline{x + y}) = \varphi(\bar{x}) + \varphi(\bar{y}) = (x, y)$ has inverse (x^{-1}, y^{-1}) (in $\mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}]$).

Thus by proposition in lecture 9, $\exists$homomorphism $\varphi' : (\mathbb{C}[x, y]/(xy))_{x+y} \to \mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}]$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}[x, y]/(xy) \arrow[d, "\iota"] \arrow[dr, "\varphi"] & \\
(\mathbb{C}[x, y]/(xy))_{x+y} \arrow[r, dotted, "\varphi'"] & \mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}]
\end{tikzcd}$$

And from the construction in lecture 9, φ' is given by

$$\varphi'\left(\frac{\overline{P(x,y)}}{\overline{x+y}^n}\right) = \varphi\left(\overline{P(x,y)}\right)\cdot(x^{-n},y^{-n})$$

Claim: φ' is bijective, hence an isomorphism.

Proof: injectivity: suppose $\varphi'\left(\frac{\overline{P(x,y)}}{\overline{x+y}^n}\right) = 0$, then $\varphi\left(\overline{P(x,y)}\right)\cdot(x^{-n},y^{-n}) = 0$, and because (x^{-n},y^{-n}) is not a zero divisor, we see $\varphi\left(\overline{P(x,y)}\right) = 0$, but φ is injective, hence $\overline{P(x,y)} = 0 \Rightarrow \frac{\overline{P(x,y)}}{\overline{x+y}^n} = 0$.

Surjectivity: $\forall\left(\frac{P(x)}{x^n},\frac{Q(y)}{y^m}\right) \in \mathbb{C}[x^{\pm1}]\times\mathbb{C}[y^{\pm1}]$, pick $N \in \mathbb{Z}^+ > \max(n,m)$, then

$$\left(\frac{P(x)}{x^n},\frac{Q(y)}{y^m}\right) = \left(\frac{P(x)x^{N-n}}{x^N},\frac{Q(y)y^{N-m}}{y^N}\right) = \varphi'\left(\frac{\overline{P(x)x^{N-n}+Q(y)y^{N-m}}}{\overline{x+y}^N}\right)$$

Thus φ' is surjective.

■

So φ' gives the desired isomorphism.

2 Problem 2

Put $\lambda = \sqrt{-5}$.

2.1 a

$1/1 = \frac{2}{2} \in I_2$, which is the identity of A_2, hence because I_2 is an ideal in A_2 we see $I_2 = A_2$, hence I_2 is obvious free module of rank 1 because it is generated by $1/1 \in A_2$.

2.2 b

Claim: $I_3 = \frac{1+\lambda}{1}A_3$.

Proof: $1+\lambda \in I \Rightarrow \frac{1+\lambda}{1} \in I_3 \Rightarrow \frac{1+\lambda}{1}A_3 \subset I_3$.

And pick any $\frac{2x+(1+\lambda)y}{3^n} \in I_3 (n \in \mathbb{Z}_{\geq 0})$, we have $\frac{2x+(1+\lambda)y}{3^n} = \frac{(1-\lambda)x+3y}{3^{n+1}}\frac{1+\lambda}{1}$, this shows $I_3 \subset \frac{1+\lambda}{1}A_3$.

$\blacksquare$

Consider the (A_3-module) homomorphism $A_3 \to I_3, x \to \frac{1+\lambda}{1}x$.

It is surjective because $I_3 = \frac{1+\lambda}{1}A_3$, it is injective because $\frac{1+\lambda}{1}\frac{a}{3^n} = 0(a \in A, n \in \mathbb{Z}_{\geq 0}) \Rightarrow \exists m \in \mathbb{Z}_{\geq 0}$ $s.t.$ $3^m(1+\lambda)a = 0 \Rightarrow [A \text{ is domain}]a = 0 \Rightarrow \frac{a}{3^n} = 0$.

3 Problem 3

Put $\iota : A \to A_{\mathfrak{p}}, a \to a/1$ as usual. Put $S = A - \mathfrak{p}$.

Claim: for prime ideal $\mathfrak{q} \subset A$, TFAE:

(1) $\mathfrak{q} \subset \mathfrak{p}$.

(2) $sa \in \mathfrak{q} \Rightarrow a \in \mathfrak{q}$ $\forall s \in S, a \in A$.

Proof: $(1) \Rightarrow (2) : \mathfrak{q} \subset \mathfrak{p} \Rightarrow S \cap \mathfrak{q} = \varnothing \Rightarrow [\mathfrak{q} \text{ prime}] \Rightarrow a \in \mathfrak{q}$.

$(2) \Rightarrow (1)$: if $\mathfrak{q} \not\subset \mathfrak{p}$, then $\mathfrak{q} \cap S \neq \varnothing$.

Let $s \in \mathfrak{q} \cap S$, then because $\mathfrak{q}$ is prime, $\exists a \in A - \mathfrak{q}$, then $sa \in \mathfrak{q}$ but $a \notin \mathfrak{q}$, contradiction!

$\blacksquare$

From lecture 10, $\exists$a (set) bijection

$\Psi : \{\text{ideal } I \subset A : sa \in A \Rightarrow a \in I \ \forall s \in S, a \in A\} \xrightarrow{\sim} \{\text{ideal } I' \subset A_{\mathfrak{p}}\}$.

And $\Psi(I) = I_{\mathfrak{p}}, \Psi^{-1}(I') = \iota^{-1}(I')$.

Claim: if $\mathfrak{q} \subset \mathfrak{p}$ is a prime ideal, then $\mathfrak{q}_{\mathfrak{p}} \subset A_{\mathfrak{p}}$ is also a prime ideal.

Proof: $\mathfrak{q}_{\mathfrak{p}} \neq A_{\mathfrak{p}}$ because $\mathfrak{q} \neq A$ and Ψ is a bijection.

Suppose $\frac{a}{s}\frac{b}{t} \in \mathfrak{q}_{\mathfrak{p}}$ $(a, b \in A, s, t \in S)$, then $\exists q \in \mathfrak{q}, r \in S$ $s.t.$ $\frac{a}{s}\frac{b}{t} = \frac{q}{r}$.

Thus $\exists l \in S$ $s.t.$ $lrab = lstq \in \mathfrak{q}$.

But $l, r \in S \Rightarrow l, r \notin \mathfrak{q}$, thus because $\mathfrak{q}$ prime, we see $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$.

So $\frac{a}{s} \in \mathfrak{q}_{\mathfrak{p}}$ or $\frac{b}{t} \in \mathfrak{q}_{\mathfrak{p}}$, therefore $\mathfrak{q}_{\mathfrak{p}}$ is a prime ideal.

$\blacksquare$

Claim: if $\mathfrak{q}' \subset A_{\mathfrak{p}}$ is a prime ideal, then $\iota^{-1}(\mathfrak{q}') \subset A$ is a prime ideal.

Proof: $\iota^{-1}(\mathfrak{q}') \neq A$ because $\mathfrak{q}' \neq A_{\mathfrak{p}}$ and Ψ^{-1} is a bijection.

Suppose $ab \in \iota^{-1}(\mathfrak{q}')(a, b \in A)$, then $\frac{ab}{1} \in \mathfrak{q}' \Rightarrow \frac{a}{1}\frac{b}{1} \in \mathfrak{q}'$

$\Rightarrow \frac{a}{1} \in \mathfrak{q}'$ or $\frac{b}{1} \in \mathfrak{q}' \Rightarrow a \in \iota^{-1}(\mathfrak{q}')$ or $b \in \iota^{-1}(\mathfrak{q}')$.

$\blacksquare$

Therefore from these claims we see that the restriction of Ψ on $\{\text{prime ideal } \mathfrak{q} \subset \mathfrak{p}\}$, $\Psi|_{\{\text{prime ideal } \mathfrak{q} \subset \mathfrak{p}\}}$: $\{\text{prime ideal } \mathfrak{q} \subset \mathfrak{p}\} \to \{\text{prime ideal } \mathfrak{q}' \subset A_{\mathfrak{p}}\}$ is a bijection.

4 Problem 4

Consider the map $\psi : (M \oplus N)_S \to M_S \oplus N_S, (m, n)/s \to (m/s, n/s)$
and the map $\iota : M_S \oplus N_S \to (M \oplus N)_S, (m/s, n/t) \to (mt, ns)/(st)$.
ψ well-defined: $(m, n)/s = (m', n')/s' \Rightarrow \exists t \in S$ $s.t.$ $ts'(m, n) = ts(m', n') \Rightarrow ts'm = tsm'$ & $ts'n = tsn' \Rightarrow (m/s, n/s) = (m'/s', n'/s')$.
ι well-defined: $(m/s, n/t) = (m'/s', n'/t') \Rightarrow \exists r, l \in S$ $s.t.$ $rms' = rm's$ & $lnt' = ln't \Rightarrow (rl)mts't' = (rl)m't'st$ & $(rl)nss't' = (rl)n's'st \Rightarrow (mt, ns)/(st) = (m't', n's')/(s't')$.
It's easy to see that both ψ and ι are A-module homomorphisms.
$\iota \circ \psi = \mathrm{id} : \iota \circ \psi((m, n)/s) = (ms, ns)/s^2 = (m, n)/s$.
$\psi \circ \iota = \mathrm{id} : \psi \circ \iota((m/s, n/t)) = (mt/(st), ns/(st)) = (m/s, n/t)$.
Therefore ψ gives the desired isomorphism $(M \oplus N)_S \xrightarrow{\sim} M_S \oplus N_S$.

5 Problem 5

6 Problem 6

7 Problem 7

8 Problem 8

For $X \xrightarrow{f} Y, Z \xrightarrow{g} W$, we call the following commutative diagram D1:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\kappa_X"] & F(Y) \arrow[d, "\kappa_Y"] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

And call the following commutative diagram D2:

$$\begin{tikzcd} G(Z) \arrow[r, "G(g)"] \arrow[d, "\eta_Z"] & G(W) \arrow[d, "\eta_W"] \\ G'(Z) \arrow[r, "G'(g)"] & G'(W) \end{tikzcd}$$

8.1 a

We construct functor morphism $\widetilde{\kappa} : FG \Rightarrow F'G$ where $\widetilde{\kappa}_X = \kappa_{G(X)}$.

Check: let $X \xrightarrow{f} Y$ (which induces $G(X) \xrightarrow{G(f)} G(Y)$), consider

$$\begin{tikzcd} F(G(X)) \arrow[r, "F(G(f))"] \arrow[d, "\widetilde{\kappa}_X=\kappa_{G(X)}"] & F(G(Y)) \arrow[d, "\widetilde{\kappa}_Y=\kappa_{G(Y)}"] \\ F'(G(X)) \arrow[r, "F'(G(f))"] & F'(G(Y)) \end{tikzcd}$$

Then this diagram is the same diagram as D1 if we take $X = G(X), Y = G(Y), f = G(f)$, thus commute. This shows $\widetilde{\kappa}$ is indeed functor morphism.

Similarly we can define $\widetilde{\kappa}' : FG' \Rightarrow F'G'$.

Now we construct functor morphism $\widetilde{\eta} : FG \Rightarrow FG'$ where $\widetilde{\eta}_X = F(\eta_X)$.

Check: let $X \xrightarrow{f} Y$ (which induces $G(X) \xrightarrow{G(f)} G(Y)$), consider

$$\begin{tikzcd} F(G(X)) \arrow[r, "F(G(f))"] \arrow[d, "\widetilde{\eta}_X=F(\eta_X)"] & F(G(Y)) \arrow[d, "\widetilde{\eta}_Y=F(\eta_Y)"] \\ F(G'(X)) \arrow[r, "F(G'(f))"] & F(G'(Y)) \end{tikzcd}$$

Route $\rightarrow\downarrow$: $F(\eta_Y) \circ F(G(f)) = F(\eta_Y \circ G(f))$.

Route $\downarrow\rightarrow$: $F(G'(f)) \circ F(\eta_X) = F(G'(f) \circ \eta_X)$.

From D2 we see $\eta_Y \circ G(f) = G'(f) \circ \eta_X$, which shows that the above diagram is commute, i.e. $\widetilde{\eta}$ is indeed functor morphism.

Similarly we can define $\widetilde{\eta}' : F'G \Rightarrow F'G'$.

8.2 b

We claim the following diagram commutes:

$$\begin{tikzcd} FG \arrow[r, Rightarrow, "\widetilde{\eta}"] \arrow[d, Rightarrow, "\widetilde{\kappa}"] & FG' \arrow[d, Rightarrow, "\widetilde{\kappa}'"] \\ F'G \arrow[r, Rightarrow, "\widetilde{\eta}'"] & F'G' \end{tikzcd}$$

We want to show that $\widetilde{\kappa}' \circ \widetilde{\eta} = \widetilde{\eta}' \circ \widetilde{\kappa}$, so we only need to show $\forall X, \widetilde{\kappa}'_X \circ \widetilde{\eta}_X = \widetilde{\eta}'_X \circ \widetilde{\kappa}_X$.

This is equivalent to $\kappa_{G'(X)} \circ F(\eta_X) = F'(\eta_X) \circ \kappa_{G(X)}$, which is equivalent to the following diagram commutes:

$$\begin{tikzcd} FG(X) \arrow[r, "F(\eta_X)"] \arrow[d, "\kappa_{G(X)}"] & FG'(X) \arrow[d, "\kappa_{G'(X)}"] \\ F'G(X) \arrow[r, "F'(\eta_X)"] & F'G'(X) \end{tikzcd}$$

This is just D1 with $X = G(X), Y = G'(X), f = \eta_X$, thus commutes.

9 Problem 9

9.1 a

Recall that in lecture 12, $\sigma_{X,F}$ is given by $\sigma_{X,F}(\tau) = \tau_X(1_X)$, where $\tau \in \mathrm{Hom}_{Fun}(F_X, F)$.

$$\begin{tikzcd} \mathrm{Hom}_{Fun}(F_X, F) \arrow[r, "\sigma_{X,F}"] \arrow[d, "\eta\circ?"] & F(X) \arrow[d, "\eta_X"] \\ \mathrm{Hom}_{Fun}(F_X, G) \arrow[r, "\sigma_{X,G}"] & G(X) \end{tikzcd}$$

Now take any $\kappa \in \mathrm{Hom}_{Fun}(F_X, F)$.

Route $\rightarrow\downarrow$: $\kappa \mapsto \eta_X(\sigma_{X,F}(\kappa)) = \eta_X(\kappa_X(1_X))$.

Route $\downarrow\rightarrow$: $\kappa \mapsto \sigma_{X,F}(\eta \circ \kappa) = (\eta \circ \kappa)_X(1_X) = \eta_X \circ \kappa_X(1_X)$.

And they are obviously equal.

9.2 b

From lecture 12, we have $f_Y^* : \mathrm{Hom}_{\mathcal{C}}(Y,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Y), \psi \mapsto \psi \circ f$.

$$\begin{array}{ccc} \mathrm{Hom}_{Fun}(F_X, F) & \xrightarrow{\sigma_{X,F}} & F(X) \\ \downarrow{\scriptstyle ?\circ f^*} & & \downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}_{Fun}(F_Y, F) & \xrightarrow{\sigma_{Y,F}} & F(Y) \end{array}$$

Now take any $\kappa \in \mathrm{Hom}_{Fun}(F_X, F)$.

Route $\rightarrow\downarrow$: $\kappa \mapsto F(f)(\sigma_{X,F}(\kappa)) = F(f)(\kappa_X(1_X))$.

Route $\downarrow\rightarrow$: $\kappa \mapsto \sigma_{Y,F}(\kappa \circ f^*) = (\kappa \circ f^*)_Y(1_Y) = \kappa_Y(f_Y^*(1_Y)) = \kappa_Y(1_Y \circ f) = \kappa_Y(f)$.

Because $\kappa : F_X \Rightarrow F$ is a morphism of functors, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,X) & \xrightarrow{f\circ ?} & \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ \downarrow{\scriptstyle \kappa_X} & & \downarrow{\scriptstyle \kappa_Y} \\ F(X) & \xrightarrow{\quad F(f) \quad} & F(Y) \end{array}$$

Route $\rightarrow\downarrow$: $1_X \mapsto \kappa_Y(f \circ 1_X) = \kappa_Y(f)$.

Route $\downarrow\rightarrow$: $1_X \mapsto F(f)(\kappa_X(1_X))$.

Thus we have $F(f)(\kappa_X(1_X)) = \kappa_Y(f)$, i.e. the original diagram commutes.

10 Problem 10

10.1 a

Let $\mathcal{C} = A$-Mod.

We will construct a functor isomorphism $\kappa : F_A \Rightarrow F$.

Let $M \in Ob(\mathcal{C})$, then we define $\kappa_M : \mathrm{Hom}_{\mathcal{C}}(A, M) \to M, \psi \to \psi(1)$ (as sets).

It is bijective because ψ is uniquely determined by the value of it on 1.

Check: $\forall X \xrightarrow{f} Y, \psi \in \mathrm{Hom}_{\mathcal{C}}(A, X)$, we have $(f \circ \psi)(1) = f(\psi(1))$, i.e. the diagram is commute. Hence κ is indeed morphism of functors. And because each κ_M is bijective, we see κ is isomorphism of functors.

Thus we have an isomorphism $\mathrm{End}_{\mathcal{C}}(A) \xrightarrow{\sim} \mathrm{End}_{Fun}(F_A) \xrightarrow{\sim} \mathrm{End}_{Fun}(F)$.

It sends an element $g \in \mathrm{End}_{\mathcal{C}}(A)$ to $\eta^g \in \mathrm{End}_{Fun}(F_A)$ and then to $\kappa \circ \eta^g \circ \kappa^{-1} \in \mathrm{End}_{Fun}(F)$. And we have $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}}(A), a \mapsto (g_a : 1 \mapsto a)$.

Thus we get an isomorphism $A \xrightarrow{\sim} \mathrm{End}_{Fun}(F), a \mapsto \iota^a$, where $\iota^a_M(m) = am$.

And the composition is given by $\iota^a \circ \iota^b = \iota^{ab}$.

10.2 b

Let $\mathcal{C} = \text{Rings}$.

We will construct a functor isomorphism $\kappa : F_{\mathbb{Z}[x]} \Rightarrow F$.

Let $M \in Ob(\mathcal{C})$, then we define $\kappa_M : \mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}[x], M) \to M, \psi \to \psi(x)$ (as sets).

It is bijective because ψ is uniquely determined by the value of it on x (since $\psi(1)$ must be 1).

Check: $\forall X \xrightarrow{f} Y, \psi \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}[x], X)$, we have $(f \circ \psi)(x) = f(\psi(x))$, i.e. the diagram is commute. Hence κ is indeed morphism of functors. And because each κ_M is bijective, we see κ is isomorphism of functors.

Thus we have an isomorphism $\mathrm{End}_{\mathcal{C}}(\mathbb{Z}[x]) \xrightarrow{\sim} \mathrm{End}_{Fun}(F_{\mathbb{Z}[x]}) \xrightarrow{\sim} \mathrm{End}_{Fun}(F)$.

It sends an element $g \in \mathrm{End}_{\mathcal{C}}(\mathbb{Z}[x])$ to $\eta^g \in \mathrm{End}_{Fun}(F_{\mathbb{Z}[x]})$ and then to $\kappa \circ \eta^g \circ \kappa^{-1} \in \mathrm{End}_{Fun}(F)$. And we have $\mathbb{Z}[x] \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}}(\mathbb{Z}[x]), P \mapsto (g_a : x \mapsto P)$.

Thus we get an isomorphism $\mathbb{Z}[x] \xrightarrow{\sim} \mathrm{End}_{Fun}(F), P \mapsto \iota^P$, where $\iota^P_R(r) = P(r)$.

And the composition is given by $\iota^P \circ \iota^Q = \iota^{P \circ Q}$.