# [Ross] 4 squares

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## 0 Clarification

I really learnt a lot from Ross program, one of the most impressive things is the proof for Lagrange 4-squares theorem.

## 1 Statement

**Theorem 1.1** (Lagrange) $\forall n \in \mathbb{Z}^+, \exists a, b, c, d \in \mathbb{Z}\ s.t.\ n = a^2 + b^2 + c^2 + d^2$

## 2 Lemmas

We use $\mathbb{P}$ to denote all prime numbers.

**Lemma 2.1** $(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+w^2) = (-ax+by+cz+dw)^2 + (az-bw+cx+dy)^2 + (aw+bz-cy+dx)^2 + (ay+bx+cw-dz)^2$, thus we only need to consider $n = p \in \mathbb{P}$

**Proof:** Straight forward.

$\square$

**Lemma 2.2** $\forall p \in \mathbb{P}, \exists s, t\ s.t.\ s^2 + t^2 \equiv -1 \pmod{p}$

**Proof:** Consider $S \triangleq p + \sum\limits_{t \in \mathbb{F}_p} \left(\frac{-1-t^2}{p}\right)$

Then use standard method we get $S = p - \left(\frac{-1}{p}\right) \geq p - 1$

$\square$

**Lemma 2.3** (Minkowski) In $\mathbb{R}^d$, Let $\Lambda$ be a lattice, $\mathscr{F}$ be a convex, O-symmetric area, if $\mathrm{Vol}(\mathscr{F}) > 2^d \mu(\Lambda)$, then $\mathscr{F} \cap \Lambda - \vec{0} \neq \varnothing$

**Proof:** This is well-known.

□

# 3 Proof

We want to use Minkowski theorem.

So we need to first construct a lattice $\Lambda \subset \mathbb{R}^4$ *s.t.*

$$a^2 + b^2 + c^2 + d^2 \equiv 0 \pmod p \ \forall (a, b, c, d) \in \Lambda$$

To do this, we consider the equation $a^2 + b^2 \equiv -1 \cdot (c^2 + d^2)$

This remind us to discuss in $\mathbb{Z}[i]$

From *L2.2* we know that $\exists s + it$ *s.t.* $N(s + it) \equiv -1 \pmod p$

We have

$$\alpha(s + it) \equiv \beta \Rightarrow \bar{\alpha}(s - it) \equiv \bar{\beta}$$

$$\Rightarrow N(\alpha) \equiv -1 \cdot N(\beta)$$

So we only need to consider $(a + ib)(s + it) \equiv (c + id)$

What's good about this equation is that it is linear: any linear combination of solutions is again a solution.

The left work is to find 4 (linear independent) solutions, with det as small as possible.

$$v_1^T = (1, 0, s, t)$$

$$v_2^T = (0, 1, -t, s)$$

$$v_3^T = (0, 0, p, 0)$$

$$v_4^T = (0, 0, 0, p)$$

$$\Lambda = \sum^{4} v_i \mathbb{Z}$$

We have $\mu(\Lambda) = |\det(v_i)| = p^2$

Next follow the standard method we construct $\mathscr{F}$ :

$$\mathscr{F} \triangleq \{x \in \mathbb{R}^4 : \|x\|^2 \leq r^2\} \text{ with } r^2 = 1.99p$$

We have

$$\mathrm{Vol}(\mathscr{F}) = \frac{r^4\pi^2}{2} = \frac{1.99^2 p^2 \pi^2}{2} \approx 19.54p^2 > 2^4 \mu(\Lambda)$$

So $\exists (a_0, b_0, c_0, d_0) \in \mathscr{F} \cap \Lambda - \vec{0}$

Now $0 < a_0^2 + b_0^2 + c_0^2 + d_0^2 \leq 1.99p$ and $p \mid a_0^2 + b_0^2 + c_0^2 + d_0^2$

$\Rightarrow a_0^2 + b_0^2 + c_0^2 + d_0^2 = p$

$\square$