

HW5

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November 4, 2025

1 Collaborators

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2 Problem 1

Suppose that the metrics g and \tilde{g} are conformal with $\tilde{g} = u g$ for a positive function u on an n -dimensional manifold M .

- (a) Find a formula in terms of u and its derivatives that relates the two Laplacians.
- (b) Show that when $n = 2$, then a conformal change keeps a harmonic function harmonic.

2.1 a

Pick an orthonormal frame e_i for g , then $u^{-1/2}e_i$ is an orthonormal frame for \tilde{g} . We have (here $\tilde{\nabla}_{e_i}e_j = \sum_k \tilde{\Gamma}_{ij}^k e_k$) (here $\tilde{\Gamma}$ is a letter)

$$\begin{aligned}\tilde{\Gamma}_{ij}^k &= \frac{1}{2}\tilde{g}^{k7}(\partial_i\tilde{g}_{7j} + \partial_j\tilde{g}_{7i} - \partial_7\tilde{g}_{ij}) \\ &= \frac{1}{2}\left(\delta_j^k\frac{u_i}{u} + \delta_i^k\frac{u_j}{u} - \delta_{ij}\frac{u_k}{u}\right).\end{aligned}$$

For $\varphi \in C(M)$, we have

$$\begin{aligned}\Delta\varphi &= \sum_i \langle \nabla_{e_i}\nabla\varphi, e_i \rangle \\ &= \sum_i e_i e_i(\varphi) - \nabla_{e_i}e_i(\varphi) \\ &= \sum_i \varphi_{ii}\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Delta}\varphi &= \sum_i \tilde{g}(\tilde{\nabla}_{u^{-1/2}e_i} \tilde{\nabla}\varphi, u^{-1/2}e_i) \\
&= \sum_i \left(u^{-1/2}e_i\right) \tilde{g}(\tilde{\nabla}\varphi, u^{-1/2}e_i) - \tilde{g}(\tilde{\nabla}\varphi, \tilde{\nabla}_{u^{-1/2}e_i} u^{-1/2}e_i) \\
&= \sum_i \left(u^{-1/2}e_i\right) \left(u^{-1/2}e_i(\varphi)\right) - \tilde{\nabla}_{u^{-1/2}e_i} (u^{-1/2}e_i)(\varphi) \\
&= u^{-1/2} \sum_i e_i(u^{-1/2})\varphi_i + \frac{1}{u} \sum_i \varphi_{ii} - u^{-1/2} \sum_i e_i(u^{-1/2})\varphi_i - \frac{1}{u} \sum_i \left(\tilde{\nabla}_{e_i} e_i\right)(\varphi) \\
&= \frac{1}{u} \left(\Delta\varphi - \frac{1}{2} \sum_i \sum_k \left(2\delta_i^k \frac{u_i}{u} - \frac{u_k}{u}\right) \varphi_k \right) \\
&= \frac{\Delta\varphi}{u} - \frac{1}{u} \sum_i \frac{u_i}{u} \varphi_i + \frac{1}{u} \frac{n}{2} \sum_k \frac{u_k}{u} \varphi_k \\
&= \frac{\Delta\varphi}{u} + \frac{n-2}{2} \sum_i \frac{u_i}{u^2} \varphi_i \\
&= \frac{\Delta\varphi}{u} + \frac{n-2}{2} \frac{\langle \nabla u, \nabla \varphi \rangle}{u^2}.
\end{aligned}$$

2.2 b

In this case $\tilde{\Delta}\varphi = 1/u \cdot \Delta\varphi$, so since u is positive we see $\Delta\varphi = 0$ iff $\tilde{\Delta}\varphi = 0$.

3 Problem 2

A function u is superharmonic if $\Delta u \leq 0$. Show that if M is complete, non-compact, and has finite volume, then any positive superharmonic function must be constant. *Note:* This actually holds even when M has at most quadratic volume growth.

3.1 Solution

Let $v = \log u$. Then $\nabla v = u^{-1} \nabla u$. Thus, $\Delta v = \langle \nabla(u^{-1}), \nabla u \rangle + u^{-1} \Delta u = -u^{-2} \langle \nabla u, \nabla u \rangle + u^{-1} \Delta u = u^{-1} \Delta u - u^{-2} |\nabla u|^2$. Thus, we have $\Delta v + |\nabla v|^2 = u^{-1} \Delta u \leq 0$.

Now let φ_k be a good cutoff of $B_k \subset B_{2k}$, where B is ball around any prefixed point. Then by above

$$\int \varphi_k^2 (\Delta v + |\nabla v|^2) \leq 0.$$

However, we have (since φ_k has compact support)

$$0 = \int \operatorname{div}(\varphi_k^2 \nabla v) = \int \langle \nabla \varphi_k^2, \nabla v \rangle + \int \varphi_k^2 \Delta v = 2 \int \varphi_k \langle \nabla \varphi_k, \nabla v \rangle + \int \varphi_k^2 \Delta v.$$

Thus, we see

$$\int -2\varphi_k \langle \nabla \varphi_k, \nabla v \rangle + \varphi_k^2 \langle \nabla v, \nabla v \rangle \leq 0,$$

so

$$\begin{aligned}
\int \varphi_k^2 |\nabla v|^2 &\leq 2 \int \varphi_k \langle \nabla \varphi_k, \nabla v \rangle \\
&\leq 2 \int \varphi_k |\nabla \varphi_k| |\nabla v| \\
&\leq 2 \sqrt{\int \varphi_k^2 |\nabla v|^2} \sqrt{\int |\nabla \varphi_k|^2}.
\end{aligned}$$

As a result, $\sqrt{\int \varphi_k^2 |\nabla v|^2} \leq 2 \sqrt{\int |\nabla \varphi_k|^2}$, so

$$\int \varphi_k^2 |\nabla v|^2 \leq 4 \int |\nabla \varphi_k|^2.$$

Now take any $R > 0$, then for $k > R$, we have (by the book $|\nabla \varphi_k| \leq 1$)

$$k^2 \int_{B_R} |\nabla v|^2 \leq \int \varphi_k^2 |\nabla v|^2 \leq 2 \int |\nabla \varphi_k|^2 \leq 2 \text{Vol}(B_{2k} - B_k) \leq 2 \text{Vol}(B_{2k}) = o(k^2),$$

and taking the limit as $k \rightarrow \infty$ we see $\int_{B_R} |\nabla v|^2 = 0$ for all $R > 0$, hence $\nabla v \equiv 0$, or v is constant (note that this proof works when $\text{Vol}(B_k) = o(k^2)$).

4 Problem 3

Use the divergence theorem to prove that if Ω is compact and u, v are Dirichlet eigenfunctions for Δ on Ω with distinct eigenvalues, then $\int_{\Omega} uv = 0$.

4.1 Solution

By lecture we have $\int u \Delta v = \int v \Delta u$. Suppose $\Delta u = -\lambda u$ and $\Delta v = -\mu v$, then $-\lambda \int uv = \int v \Delta u = \int u \Delta v = -\mu \int uv$, hence because $\lambda \neq \mu$ we see $\int uv = 0$.

5 Problem 4

Compute the Hermite polynomials on \mathbf{R} of degree at most four. Prove, without computing any explicit integrals, that Hermite polynomials of different degrees must be orthogonal in the L^2 space with the gaussian weight $e^{-x^2/4}$.

5.1 Solution

We have $\mathcal{L}u = \partial_x^2 u - x \partial_x u / 2$. Then $\mathcal{L}x^m = m(m-1)x^{m-2} - m/2 \cdot x^m$.

Degree 0: all constants.

Degree 1: $\mathcal{L}x = -1/2x$, so are multiples of x . Note that anything else does not work because after normalizing it we can assume its leading coefficient is 1, so after subtracting it by the special solution we get an eigenfunction of degree 0 with eigenvalue $1/2$, which is impossible by book.

Degree 2: $\mathcal{L}(x^2 - 2) = 2 - x^2$, so are multiples of $x^2 - 2$. Note that anything else does not work because after normalizing it we can assume its leading coefficient is 1, so after subtracting it by the special solution we get an eigenfunction of degree ≤ 1 with eigenvalue 1, which is impossible by book.

Degree 3: $\mathcal{L}(x^3 - 6x) = 9x - 3/2 \cdot x^3$, so are multiples of $x^3 - 6x$. Note that anything else does not work because after normalizing it we can assume its leading coefficient is 1, so after subtracting it by the special solution we get an eigenfunction of degree ≤ 2 with eigenvalue $3/2$, which is impossible by book.

Degree 4: $\mathcal{L}(x^4 - 12x^2 + 12) = -2(x^4 - 12x^2 + 12)$, so are multiples of $x^4 - 12x^2 + 12$. Note that anything else does not work because after normalizing it we can assume its leading coefficient is 1, so after subtracting it by the special solution we get an eigenfunction of degree ≤ 3 with eigenvalue 2, which is impossible by book.

We now prove the second part. Let u, v be Hermite polynomials with different degree. Let $\mathcal{L}u = -\lambda u, \mathcal{L}v = -\mu v$, then $\lambda = \deg u/2 \neq \deg v/2 = \mu$. By divergence theorem

$$\int_{B_R(0)} \operatorname{div} \left(u \nabla v e^{-\frac{x^2}{4}} \right) = \int_{\partial B_R(0)} \operatorname{Flux} \left(u \nabla v e^{-\frac{x^2}{4}} \right) dS \rightarrow 0$$

as $R \rightarrow \infty$ since $e^{-x^2/4}$ is Schwarz. Thus,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \operatorname{div} \left(u \nabla v e^{-\frac{x^2}{4}} \right) \\ &= \int_{\mathbb{R}} \langle \nabla u, \nabla v \rangle e^{-\frac{x^2}{4}} + \int_{\mathbb{R}} u \Delta v e^{-\frac{x^2}{4}} - \int_{\mathbb{R}} u \langle x/2, \nabla v \rangle e^{-\frac{x^2}{4}} \\ &= \int_{\mathbb{R}} \langle \nabla u, \nabla v \rangle e^{-\frac{x^2}{4}} + \int_{\mathbb{R}} u \mathcal{L}v e^{-\frac{x^2}{4}}, \end{aligned}$$

which shows

$$\int_{\mathbb{R}} u \mathcal{L}v e^{-\frac{x^2}{4}} = - \int_{\mathbb{R}} \langle \nabla u, \nabla v \rangle e^{-\frac{x^2}{4}}.$$

By swapping the role of u and v , we see

$$\int_{\mathbb{R}} v \mathcal{L}u e^{-\frac{x^2}{4}} = - \int_{\mathbb{R}} \langle \nabla u, \nabla v \rangle e^{-\frac{x^2}{4}}.$$

Thus,

$$-\lambda \int_{\mathbb{R}} u v e^{-\frac{x^2}{4}} = \int_{\mathbb{R}} v \mathcal{L}u e^{-\frac{x^2}{4}} = \int_{\mathbb{R}} u \mathcal{L}v e^{-\frac{x^2}{4}} = -\mu \int_{\mathbb{R}} u v e^{-\frac{x^2}{4}},$$

showing

$$\int_{\mathbb{R}} u v e^{-\frac{x^2}{4}} = 0,$$

as desired.

6 Problem 5

A triple (M, g, f) is a gradient steady Ricci soliton if $\operatorname{Hess} f + \operatorname{Ric} = 0$. Show that $S + |\nabla f|^2$ is constant in this case.

6.1 Solution

Pick a geodesic normal frame e_i . It suffices to show $d(S + |\nabla f|^2) = 0$. To do this, it suffices to show $dS(e_k) + e_k \langle \nabla f, \nabla f \rangle = 0$. Then at p ,

$$\begin{aligned} dS(e_k) &= 2 \operatorname{div} \operatorname{Ric}(e_k) \\ &= 2 \sum_i (\nabla_{e_i} \operatorname{Ric})(e_i, e_k) \end{aligned}$$

Since $\nabla_{e_i} e_i = \nabla_{e_i} e_k = 0$ at p ,

$$\begin{aligned} &= -2 \sum_i e_i (\operatorname{Hess}_f(e_i, e_k)) \\ &= -2 \sum_i e_i \langle \nabla_{e_k} \nabla f, e_i \rangle \end{aligned}$$

Since $\nabla_{e_i} e_i = 0$ at p ,

$$= -2 \sum_i \langle \nabla_{e_i} \nabla_{e_k} \nabla f, e_i \rangle$$

Since $[e_i, e_k] = 0$,

$$\begin{aligned} &= -2 \sum_i (\operatorname{R}(e_k, e_i, \nabla f, e_i) + \langle \nabla_{e_k} \nabla_{e_i} \nabla f, e_i \rangle) \\ &= -2 \operatorname{Ric}(e_k, \nabla f) - 2 \sum_i e_k \langle \nabla_{e_i} \nabla f, e_i \rangle \\ &= 2 \operatorname{Hess}_f(e_k, \nabla f) - 2 e_k \sum_i \operatorname{Hess}_f(e_i, e_i) \\ &= 2 \langle \nabla_{e_k} \nabla f, \nabla f \rangle - 2 e_k (\Delta f) \\ &= e_k \langle \nabla f, \nabla f \rangle - 2 e_k (\Delta f) \end{aligned}$$

Taking trace to $\operatorname{Hess}_f + \operatorname{Ric} = 0$ we see $\Delta f + S = 0$, so $d(\Delta f + S)(e_k) = 0$, or $e_k(\Delta f) = -dS(e_k)$. Thus, $dS(e_k) = e_k \langle \nabla f, \nabla f \rangle + 2dS(e_k)$, or $dS(e_k) + e_k \langle \nabla f, \nabla f \rangle = 0$, as desired.