Inner product and Cauchy-Schwarz

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1 Introduction

In this document, we will introduce the concept of inner product, and use it to prove the Cauchy-Schwarz inequality.

2 Inner product

Def: Suppose V is a vector space over F, where $F = \mathbb{R}$ or $\mathbb{C}$

An inner product is a function $(\cdot, \cdot) : V \times V \to F$ satisfying:

(i) For a fixed $v \in V$, the map $u \mapsto (u, v)$ is linear

This means that

$$(au_1 + bu_2, v) = a(u_1, v) + b(u_2, b) \ \forall a, b \in F, u_1, u_2 \in V$$

(ii) (Hermitian) $(u, v) = \overline{(v, u)}$

So if $F = \mathbb{R}$, then this condition becomes $(u, v) = (v, u)$

(iii) $(u, u) \geq 0$ with equality holds if and only if $u = 0$

We define the norm $\|u\| \triangleq \sqrt{(u, u)}$

3 Basic results

(1) Use only (i), (ii) to show that $(u, u) \in \mathbb{R}$, hence (iii) is "well-defined."

(2) Show that

$$(u, v_1 + v_2) = (u, v_1) + (u, v_2)$$

And

$$(u, cv) = \bar{c}(u, v) \ \forall c \in F$$

So if $F = \mathbb{R}$, then $(\cdot, \cdot)$ is bilinear.

(3) Show that $\|cu\| = |c|\|u\|$

4 Cauchy-Schwarz

(4) Now suppose $\|u\| = \|v\| = 1$

Additionally, we assume $F = \mathbb{R}$

Prove that $|(u, v)| \leq 1$

(Hint: expand $\|u - v\|^2 = (u - v, u - v) \geq 0$ and $\|u + v\|^2 \geq 0$)

(5) Still suppose $\|u\| = \|v\| = 1$

But now assume $F = \mathbb{C}$, then the above argument may not work. (why?)

Prove that: however, we still have $|(u, v)| \leq 1$

(Hint: Consider $\|u - \xi v\|^2 \geq 0$, for a proper $|\xi| = 1$)

(6) (Cauchy-Schwarz)

$$|(u, v)| \leq \|u\|\|v\| \ \forall u, v \in V$$

5 Example

(7) (Triangular inequality)

$$\|u + v\| \leq \|u\| + \|v\|$$

(Hint: square both side, expand.)

(8) Pick $V = \mathbb{C}^n, F = \mathbb{C}$

For $u = \{z_i\}_1^n, v = \{w_i\}_1^n$, check that

$$(u, v) \triangleq \sum_1^n z_i \overline{w_i}$$

is an inner product.

(9) $z_i, w_i \in \mathbb{C}$, prove that:

$$\sum_1^n z_k \overline{w_k} \sum_1^n \overline{z_k} w_k \leq \sum_1^n z_k \overline{z_k} \sum_1^n w_k \overline{w_k}$$

Specifically, if $z_i, w_i \in \mathbb{R}$, then we get our standard Cauchy-Schwarz inequality:

$$\left(\sum_1^n a_k b_k \right)^2 \leq \sum_1^n a_k^2 \sum_1^n b_k^2$$

(10) Suppose $f : [0, 1] \to \mathbb{R}$ is an integrable function, prove that:

$$\left(\int_0^1 f(x) dx \right)^2 \leq \int_0^1 f(x)^2 dx$$