Incremental Approximate Maxflow on Undirected Graphs in $m^{o(1)}$ Update Time

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Joint work with


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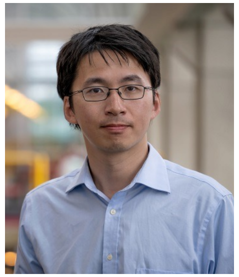
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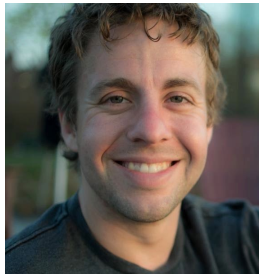
Aaron Sidford
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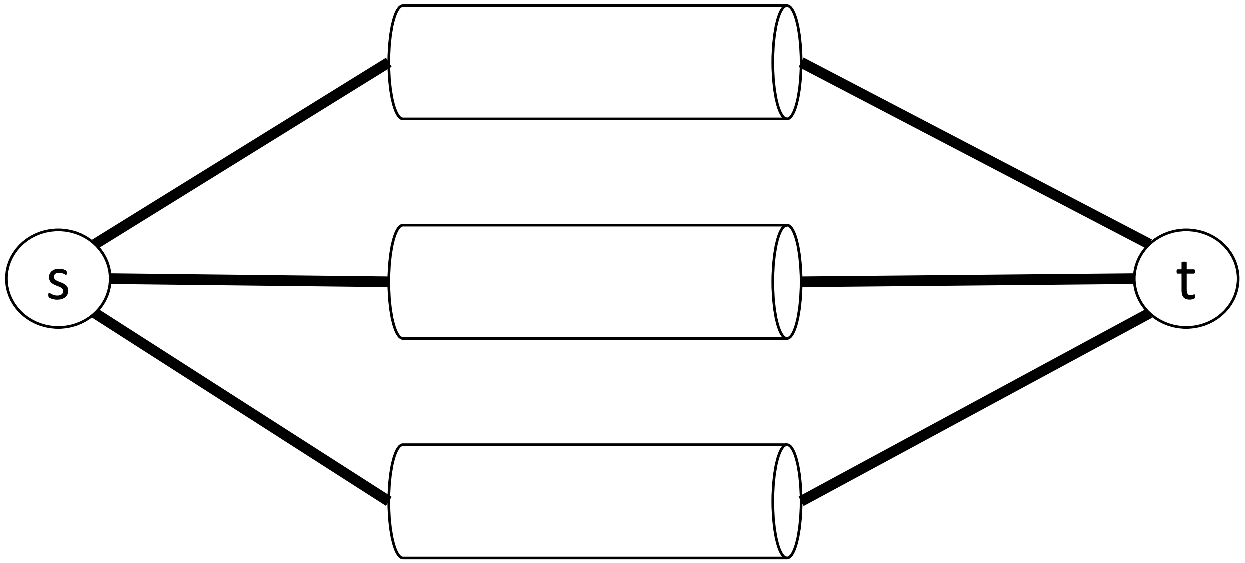
Maxflow

Undirected graph G = (V, E). m edges, n vertices, source s, sink t

edge *capacities* $u_e \geq 0$, integer in $[0, U]$, where $U = m^{O(1)}$



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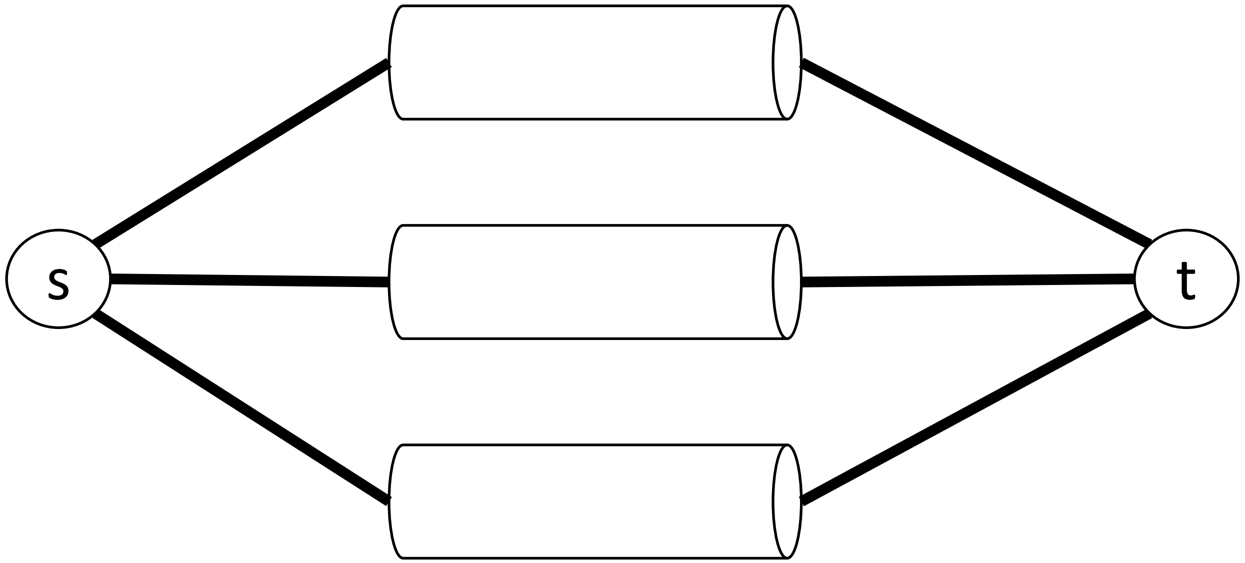
Goal: Route maximum flow from $s \to t$, Subject to capacities u_e



Capacity constraint:
$-\mathrm{u_e} \leq f_e \leq \mathrm{u_e}$

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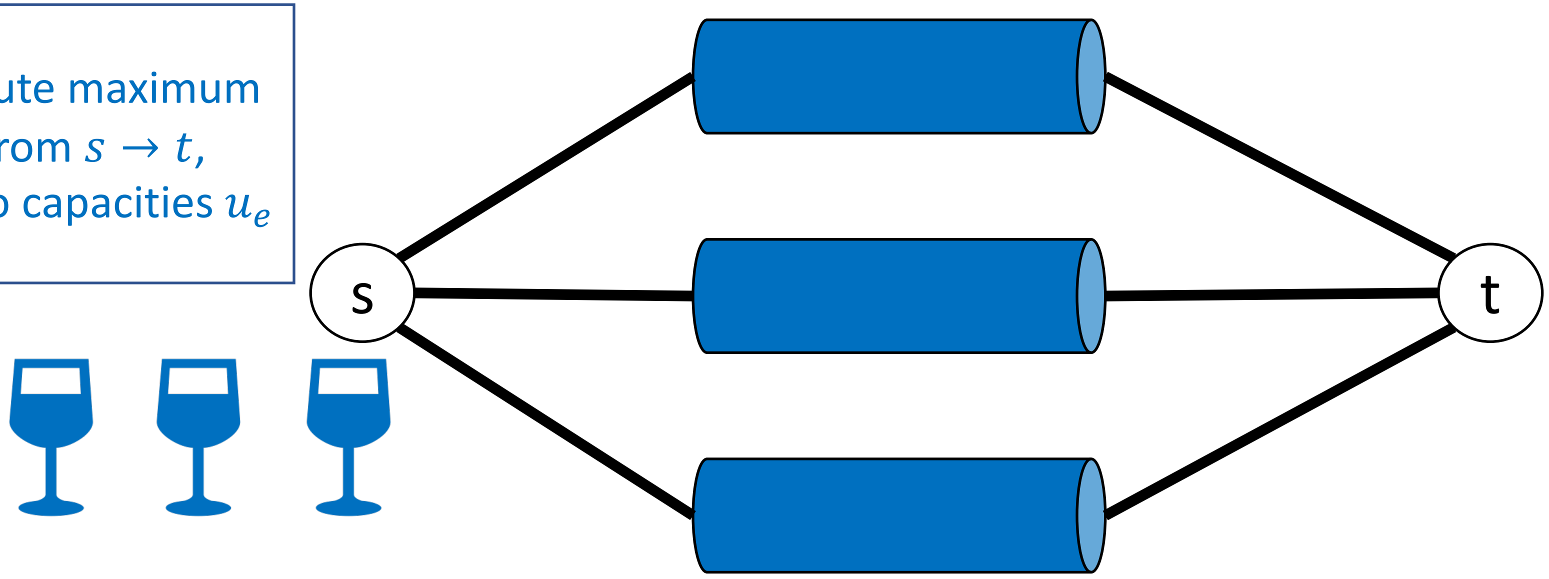

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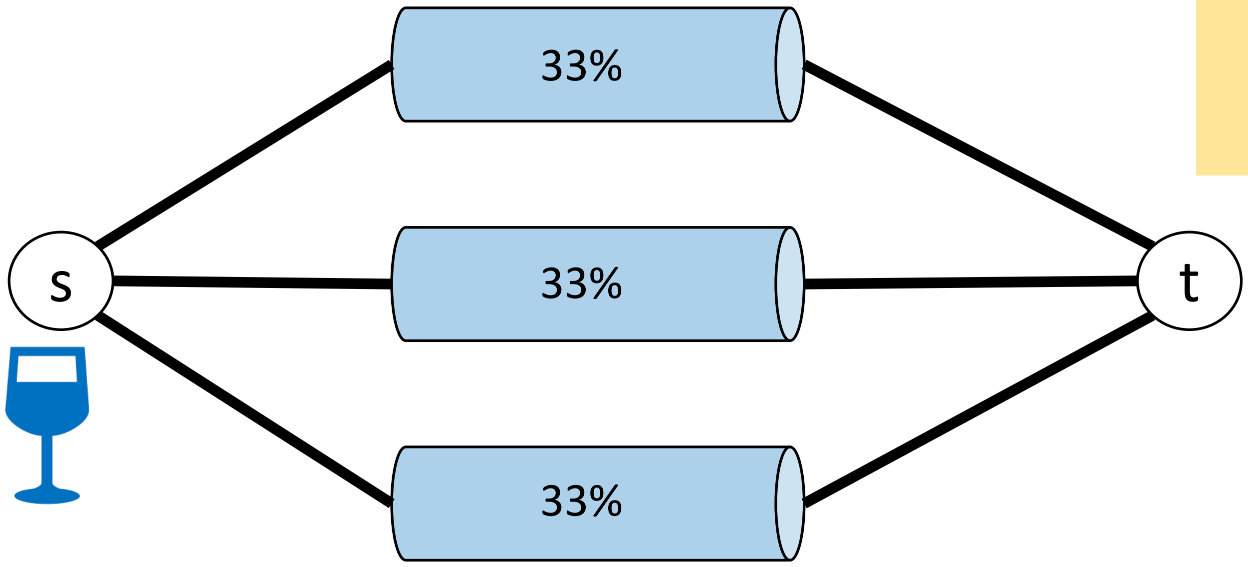
$-\mathrm{u_e} \leq f_e \leq \mathrm{u_e}$

Min Congestion Flow

Congestion of $f = \max_e \frac{|f_e|}{u_e} = \|u^{-1}f\|_\infty$

Goal: Route 1 unit of flow from $s \to t$, minimize congestion

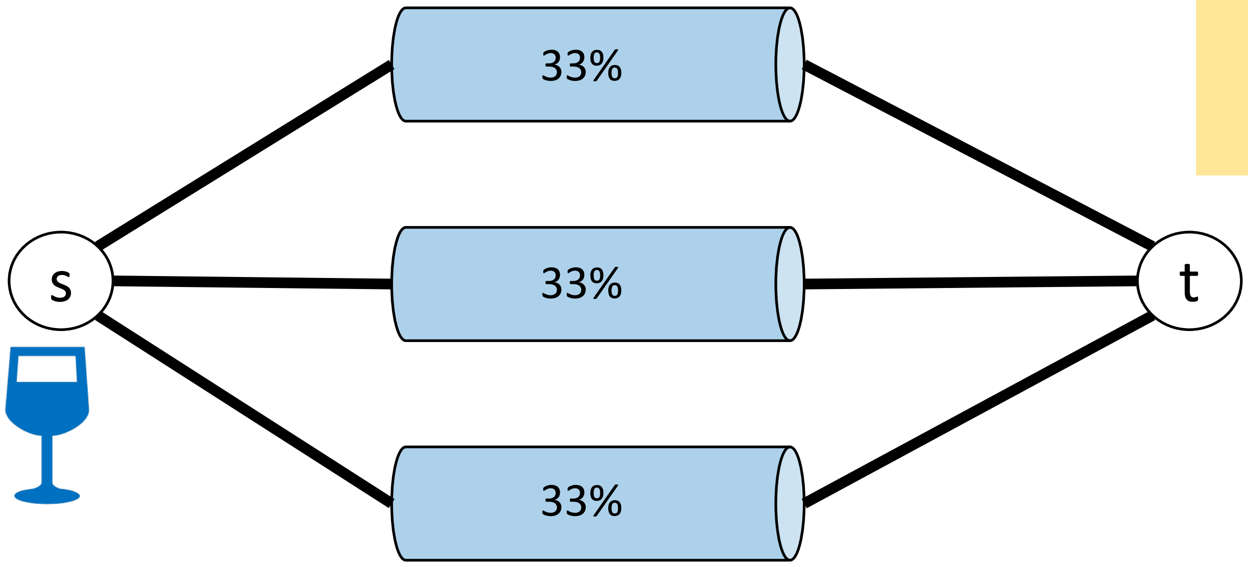


Min congestion
= 1 / Maxflow

Min Congestion Flow

$$\min_{\substack{unit\ s-t \\ flow\ f}} \|u^{-1}f\|_\infty$$

Congestion



Min congestion
= 1 / Maxflow

P-Norm Flow

	$\min\limits_{\substack{unit\ s-t \\ flow\ f}} \|Wf\|_p$	P-norm Energy
$p = \infty$	$\|u^{-1}f\|_\infty$	Min Cong Flow
$p = 2$	$\|Rf\|_2$	Electrical Flow
$p = 1$	$\|Wf\|_1$	Shortest Path
$p = O(\varepsilon^{-1}\log m)$	$\|u^{-1}f\|_p$	$(1+\varepsilon)$-Approx Min Cong Flow

$$\|x\|_\infty \leq \|x\|_p \leq m^{1/p}\|x\|_\infty$$

Result: Incremental Approx Maxflow

Corollary [Brand-C-Kyng-Liu-Peng-Probst Gutenberg-Sachdeva-Sidford]

After each edge insertion,

Maintain a $(1+\varepsilon)$-approx. min congestion flow f ,

in amortized $m^{o(1)}\varepsilon^{-3}$ update time

Randomized w.h.p. against oblivious adversary

Oblivious adversary: sequence of updates fixed beforehand

Subproblem: Incremental Thresholded Lp Flow

Theorem [Brand-C-Kyng-Liu-Peng-Probst Gutenberg-Sachdeva-Sidford]

For $p \geq 2, \delta > 0$, given a threshold F

After each edge insertion, either

- Certify $\min_{B^T f = d} \|Wf\|_p{}^p > F$, or
- Output a flow f s.t. $\|Wf\|_p{}^p \leq F + \delta$

in $\approx p^2 m^{1+o(1)} \log \frac{1}{\delta}$ total time w.h.p. against oblivious adversary

Related works

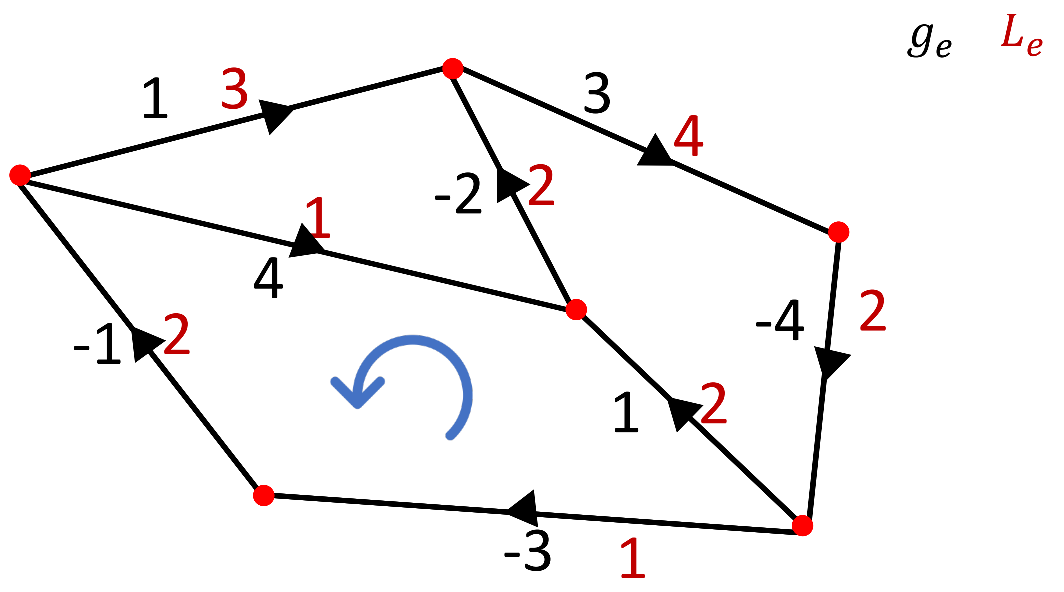
- Insertion/Deletion-only exact maxflow: (directed + unit-capacity)
  $\Omega(n)$ assuming the OMv conjecture [Dah16, HKNS15]
- Incremental + unit capacity + directed: $\sqrt{m}\varepsilon^{-1/2}$ per update [GH23]
  - Incremental augmenting path + lazy recompute
- Incremental + general min cost flow: $\sqrt{n}\varepsilon^{-1}$ per update [BLS23]
  - Continuous potential stable under insertions
  - Dynamic Min Ratio Cycle

Dynamic Algo via Optimization: ours &[BLS23]

- Maintain solution to a convex minimization problem that's
  - Relaxation of maxflow
  - Robust to insertions
- After each update, decrease the objective by updating current solution (implicitly)
- Data structure calls: $m^{1+o(1)}$ iterations of min ratio cycles

Min Ratio Cycle

$$\min_{B^\top \Delta = 0} \frac{g^\top \Delta}{\|L\Delta\|_1}$$

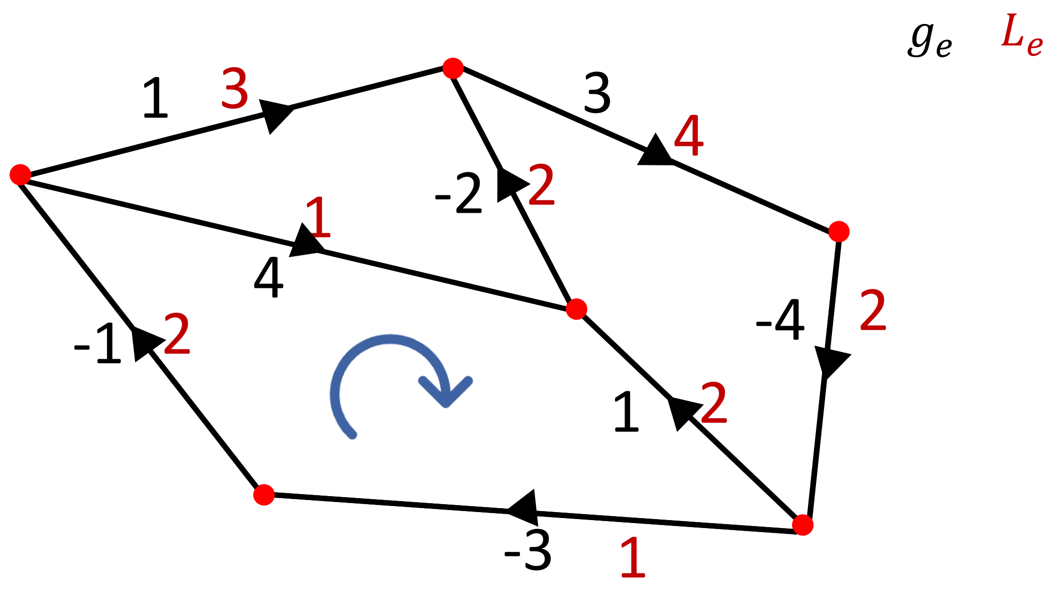


$$\|L\Delta\|_1 = 1 + 2 + 1 + 2 = 6$$

$$g^\top \Delta = -4 + 1 + 3 + 1 = 1$$

Min Ratio Cycle

$$\min_{B^\top \Delta = 0} \frac{g^\top \Delta}{\|L\Delta\|_1}$$



$$\|L\Delta\|_1 = 1 + 2 + 1 + 2 = 6$$

$$g^\top \Delta = 4 - 1 - 3 - 1 = -1$$

Edges and lengths are undirected

Gradient has a direction

Optimal solution is a simple cycle with ratio < 0

Dynamic Min Ratio Cycle w/ Restrictions

[CKLPPS22]

Assuming the min-ratio cycles **change slowly**,

A randomized data-structure that supports in $m^{o(1)}$ amortized time

1. Update g_e, L_e for an edge e
2. Return a $m^{o(1)}$-approximate min-ratio cycle

Differences: Ours vs [BLS23]

- [BLS23]: gradients/lengths changes arbitrarily
  - $\sqrt{n}$ update time for making [CKLPPS22] DS adaptive
- Ours: $m^{o(1)}$ phases, within each phase
  - Edge Lengths only go up
  - One circulation good for the entire phase
  - [CKLPPS22] DS works for free

P norm flow: small error from large error

- At current f, find Δ to minimize $\mathcal{E}(f+\Delta) = {\|f+\Delta\|_p}^p$
- [AKPS19]: "Only the 2nd and the p-th order terms matter"

Incremental for free:

After an edge insertion, f stays feasible

and $\mathcal{E}(f)$ stays the same

$$\mathcal{E}(f+\Delta) - \mathcal{E}(f) - \nabla\mathcal{E}(f)^T\Delta \approx \sum_e (r_e\Delta_e)^2 + |\Delta_e|^p = {\|R\Delta\|_2}^2 + {\|\Delta\|_p}^p$$

- Solve the *residual problem*: if $\min_f \mathcal{E}(f) \le F$

$$\min_{circulation\ \Delta} \mathcal{R}(\Delta) = \nabla\mathcal{E}(f)^T\Delta + {\|R\Delta\|_2}^2 + {\|\Delta\|_p}^p \le \frac{F - \mathcal{E}(f)}{100p} < 0$$

- $m^{o(1)}$-approx. leads to $\approx pm^{o(1)} \log\frac{1}{\delta}$ iterations to find $\mathcal{E}(f) \le F + \delta$

Approx. Residual Problem

- Solving the residual problem to $m^{o(1)}$-approx. equivalent to

  $$\text{find circulation } \Delta \text{ s.t. } g^T\Delta = 1, \|R\Delta\|_2, \|\Delta\|_p \le m^{o(1)}$$
  $$\text{given: } \exists\ \Delta^* \text{ s.t. } \|R\Delta^*\|_2, \|\Delta^*\|_p \le 1$$

- Via MWU, reduce to $m^{1+o(1)}$ iterations of

  $$\text{find circulaiton } \Delta \text{ s.t. } g^T\Delta = 1, \|L\Delta\|_1 \le m^{o(1)}\|\ell\|_1$$
  $$\text{where } \|L\Delta^*\|_1 \le \|\ell\|_1$$

- The edge lengths are slowly increasing due to MWU

Can use Dynamic Min Ratio Cycle from [CKLPPS22]

Incremental Approx. Residual Problem

- $m^{1+o(1)}$ iterations of Min Ratio Cycle

  $$\text{find circulation } \Delta \text{ s.t. } g^T\Delta = 1, \|L\Delta\|_1 \leq m^{o(1)}\|\ell\|_1$$

  $$\text{where } \|L\Delta^*\|_1 \leq \|\ell\|_1$$

- If fail, e.g. $\|L\Delta\|_1 > m^{o(1)}\|\ell\|_1$, Δ^* does not exist

- $\min_{circulation\ \Delta} \mathcal{R}(\Delta)$ is large and $\min_{B^T f = d} \|f\|_p{}^p > F$

Recap

- Incremental approx. maxflow via incremental p-norm flow
- Iterative Refinement: $m^{o(1)} \log \frac{1}{\delta}$ iters of approx. residual problem
- MWU: $m^{o(1)}$-approx. residual solution -> $m^{1+o(1)}$ iters of Min Ratio Cycles
- Lengths only increase, can use the restricted dynamic MRC from [CKLLPS22]
- When the output cycle has large ratio, $\min_{B^T f = d} \|f\|_p{}^p > F$

Future direction

- Adaptive Incremental maxflow?
  - Deterministic Dynamic Min Ratio Cycle in $m^{o(1)}$ update time [CKLMP, arXiv:2311.18295]
- Decremental approx. maxflow?

- Adaptive Incremental maxflow?
    - I think we can [CKLMP, arXiv:2311.18295]
- Decremental approx. maxflow?

Thanks!!

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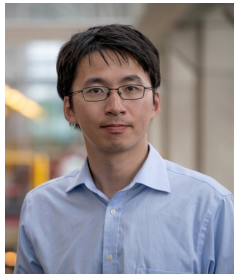

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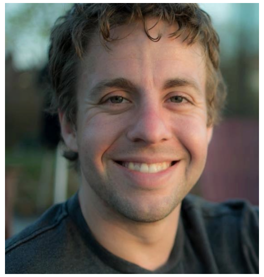

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