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Condensed Matter Theory - Lecture Notes
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Theory & Notes

1. CUENTITAS

Given a physical system, a density operator for it is a positive semi-definite, self-adjoint operator of trace one acting on the system's Hilbert space, denoted by $\mathbb{H}$. The set of all density operators has the structure of a vector space $\mathcal{C}(\mathbb{H})$,

$$\mathcal{C}(\mathbb{H}) = \{\rho \in \mathrm{GL}(N, \mathbb{C}) \mid \rho^\dagger = \rho, \ \rho \geq 0, \ \mathrm{Tr} \, \rho = 1\},$$

where $\mathrm{GL}(N, \mathbb{C})$ is the general linear group over the complex number field, whose elements are squared matrices of $N \times N$-dimension. The following statements can then be proved:

1) $\mathcal{C}(\mathbb{H})$ is a topological space. This is, this space can be imbued with a topology $\mathcal{T}$ which satisfies a set of axioms.
   - In effect, the desired topology may be chosen to be the trivial topology $\mathcal{T} = \{\emptyset, \mathcal{C}(\mathbb{H})\}$,
   - or it may be chosen out to be the discrete topology, ie. any collection of τ-sets, subsets of $\mathcal{C}(\mathbb{H})$, so that that $\mathcal{T} = \bigcup \tau$ adheres to the topological space's axioms.
   - Another interesting election is to define a metric on this space, allowing for the construction of the metric topology. More on this later.
2) $(\mathcal{C}(\mathbb{H}), \mathcal{T})$ is a Hausdorff space, allowing for the distinction of elements via disjoint neighbourhoods,
3) $(\mathcal{C}(\mathbb{H}), \mathcal{T})$ is a topological manifold.
4) $(\mathcal{C}(\mathbb{H}), \mathcal{T})$ is a differentiable manifold,
5) and is a Riemannian non-convex manifold

Density operators can either describe pure or mixed states, which are deffined as follows
- Pure states can be written as an outer product of a vector state with itself, this is

$$\rho \text{ is a pure state if } \exists \, |\psi\rangle \in \mathbb{H} \mid \rho \propto |\psi\rangle \langle\psi| \,.$$

In other words, ρ is a rank-one orthogonal projection. Equivalently, a density matrix is a pure state if there exists a unit vector in the Hilbert space such that ρ is the orthogonal projection onto the span of ψ.

Note as well that

$$|\psi\rangle \langle\psi| \in \mathbb{H} \otimes \mathbb{H}^\star, \text{ but } \mathbb{H} \otimes \mathbb{H}^\star \sim \mathrm{End}(\mathbb{H})$$

ie. the tensor space $\mathbb{H} \otimes \mathbb{H}^\star$ is canonically isomorphic to the vector space of endormorphisms in $\mathbb{H}$, ie. to the space of linear operators from $\mathbb{H}$ to $\mathbb{H}$. It's important to note that this isomorphism is only strictly valid in finite-dimensional Hilbert spaces, wherein for infinite-dimensional Hilbert spaces, the isomorphism holds as well provided the density operators are redefined as being trace-class.
- Mixed states do not adhere to the previous properties.

Let $\mathcal{B}$ be the set of all operators which are endomorphisms on $\mathcal{C}(\mathbb{H})$, ie.

$$\mathcal{B} = \{\mathbf{O} \mid \mathbf{O} : \mathcal{C}(\mathbb{H}) \to \mathcal{C}(\mathbb{H})\}.$$

Note that, by definition, $\mathcal{C}(\mathbb{H}) \subset \mathcal{B}$. Consider an N-partite physical system, then its associated Hilbert space will have $\mathcal{O}(2^N)$ dimension and its associated density operator space will have $\mathcal{O}(2^{2N})$ dimension. Then, all linear operators acting on $\mathcal{C}(\mathbb{H})$ can be classified as k-body operators, with $k \leq N$. This is, in essence, operators whose action is non-trivial only for a total of k particles. Therefore, the N-partite Hilbert space can be written as

$$\mathbb{H} = \bigotimes_{j=1}^{N} \mathfrak{H}_j,$$

where $\mathfrak{H}_j$ is the j-th subsytem's Hilbert space. This definition thus allows for systems with different particles species (eg. fermions, bosons, spins etc.). Then,

$$\mathcal{B}_1(\mathbb{H}) = \{\hat{\mathbf{O}}|\hat{\mathbf{O}} : \mathfrak{H}_j \to \mathfrak{H}_j, \ \forall j \leq N\}$$

is the space of all one-body operators. Then the space of k-body operators can be recursively defined in terms of this set,

$$\mathcal{B}_k(\mathbb{H}) = \{\otimes_{i=1}^k \mathbf{O}_i | \mathbf{O}_i \in \mathcal{B}_1(\mathbb{H})\}, \text{ where } \mathcal{B}(\mathbb{H}) = \bigsqcup_{i=1}^N \mathcal{B}_i(\mathbb{H}).$$

If $\mathbb{H}$ is a Hilbert space and $A \in \mathcal{B}$ is a non-negative self-adjoint operator on $\mathbb{H}$, then it can be shown that A has a well-defined, but possible infinite, trace. Now, if $\mathbf{A}$ is a bounded operator, then $\mathbf{A}^\dagger \mathbf{A}$ is self-adjoint and non-negative. An operator $\mathbf{A}$ is said to be Hilbert-Schmidt if $\text{Tr } \mathbf{A}^\dagger \mathbf{A} < \infty$. Naturally, the space of all Hilbert-Schmidt operators form a vector space, labelled by $\mathfrak{Hs}(\mathbb{H})$. Then, the Hilbert Schmidt inner product can be defined as

$$\langle \cdot, \cdot \rangle_{\text{HS}} : \ \mathfrak{Hs}(\mathbb{H}) \times \mathfrak{Hs}(\mathbb{H}) \to \mathbb{C}, \text{ where} \qquad \begin{aligned} \langle \mathbf{A}, \mathbf{B} \rangle_{\text{HS}} &= \text{Tr } \mathbf{A}^\dagger \mathbf{B} \\ ||\mathbf{A}||_{\text{HS}} &= \sqrt{\text{Tr } \mathbf{A}^\dagger \mathbf{A}}. \end{aligned}$$

If the Hilbert space is finite-dimensional, the trace is well defined and if the Hilbert space is infinite-dimensional, then the trace can be proven to be absolutely convergent and independent of the orthonormal basis choice[1].

This inner product implies that $(\mathfrak{Hs}(\mathbb{H}), \langle \cdot, \cdot \rangle_{\text{HS}})$ is a
- inner product space since the norm is the square root of the inner product of a vector and itself ie.

$$||\mathbf{A}||_{\text{HS}} = \langle \mathbf{A}, \mathbf{A} \rangle_{\text{HS}} = \sqrt{\text{Tr } \mathbf{A}^\dagger \mathbf{A}}$$

- and is a normed vector space since the norm is always well defined over $\mathfrak{hs}(\mathbb{H})$.

Acá va un comentario "importante": no tiene sentido que dos vectores estén infinitamente lejos, no? entónces tengo que definir esto producto interno y métrico solo en HS(H) y no sobre B(H)

Now, every inner product space is a metric space. In effect, since the function

$$\mathbf{A} \to \sqrt{\text{Tr } \mathbf{A}^\dagger \mathbf{A}} \qquad \text{then} \qquad \mathbf{A}, \mathbf{B} \overset{d}{\to} \sqrt{\text{Tr } \mathbf{A}^\dagger \mathbf{B}}$$
$$\text{is a well-defined norm} \qquad\qquad \text{is a well-defined distance}$$

$$d_{\text{HS}}(\cdot, \cdot) : \ \mathfrak{Hs}(\mathbb{H}) \times \mathfrak{Hs}(\mathbb{H}) \to \mathbb{R}$$
$$d_{\text{HS}}(\mathbf{A}, \mathbf{B}) = \sqrt{\text{Tr } \mathbf{A}^\dagger \mathbf{B}}$$

With this metric thus defined, then $(\mathfrak{Hs}(\mathbb{H}), d_{\text{HS}})$ is a metric space. Every metric space can be modified, via the completions of its metric, in such a way that $(\mathfrak{Hs}(\mathbb{H})^\star, d_{\text{HS}}^\star)$ is a complete metric space, in the sense of the convergence of Cauchy series, where $\mathfrak{Hs}(\mathbb{H}) \subset \mathfrak{Hs}(\mathbb{H})^\star$. In this particular case, given that the metric over $\mathfrak{Hs}(\mathbb{H})$ is always a finite number -having removed those elements with infinite trace-, then it is already complete $\mathfrak{Hs}(\mathbb{H}) \sim \mathfrak{Hs}(\mathbb{H})^\star$. Therefore, $\mathfrak{Hs}(\mathbb{H})$ is a Hilbert space with respect to the Hilbert-Schmidt inner product $\langle \cdot, \cdot \rangle_{\text{HS}}$

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[1]In effect, given a non-negative, self-adjoint operator, its trace is always invariant under orthogonal change of basis. Should the trace be a finite number, then it is called a trace class. Any given operator $\mathbf{A} \in \mathcal{B}$ is trace-class if the non-negative self-adjoint operator $\sqrt{\mathbf{A}^\dagger \mathbf{A}}$ is trace class as well. Now, given two Hilbert-Schmidt operators $\mathbf{A}, \mathbf{B} \in \mathfrak{Hs}(\mathbb{H})$, then the new operator $\mathbf{A}^\dagger \mathbf{B}$ is a trace-class operator, meaning that the sum

$$\text{Tr } \mathbf{A}^\dagger \mathbf{B} = \sum_{\lambda \in \Lambda} \langle \mathbf{e}_\lambda, \mathbf{A}^\dagger \mathbf{B} \mathbf{e}_\lambda \rangle$$

is absolutely convergent and the value of the sum is independent of the choice of orthonormal basis $\{\mathbf{e}_\lambda\}_{\lambda \in \Lambda}$.

(or with respect to the Hilbert-Schmidt distance d_{HS}).

Thus defined, the Hilbert-Schmidt inner product is complex-valued, thus not immediately suited for our calculations.

Theorem 1. *Consider the modified Hilbert-Schmidt product given by*

$$\langle \cdot, \cdot \rangle_{\text{HS}}^{\rho_0} : \; \mathfrak{Hs}(\mathbb{H}) \times \mathfrak{Hs}(\mathbb{H}) \to \mathbb{R}$$
$$\langle \mathbf{A}, \mathbf{B} \rangle_{\text{HS}}^{\rho_0} = \tfrac{1}{2}\text{Tr}\, \rho_0 \{\mathbf{A}^\dagger, \mathbf{B}\} \qquad \text{where } \rho_0 \in \mathcal{C}(\mathbb{H}).$$

We claim this is a valid inner product over the space of all linear trace-class endomorphisms on $\mathbb{H}$, $\mathfrak{Hs}(\mathbb{H})$.

Proof. In order to prove this is a well-defined inner product over the space of trace-class operators, we must prove that it linear in it second argument and sesquilinear in its first argument, hermitian and positive-defined.
    In effect,

1) the linearity and sesquilinearity is self evident.

2) Is it hermitian? Yes

(1)
$$\begin{aligned} \langle \mathbf{A}, \mathbf{B} \rangle_{\text{HS}}^{\rho_0} &= \frac{1}{2}\text{Tr}\, \rho_0 \{\mathbf{A}^\dagger, \mathbf{B}\} = \frac{1}{2}\text{Tr}\, \rho_0 \left(\mathbf{A}^\dagger\mathbf{B} + \mathbf{B}\mathbf{A}^\dagger \right) \\ &= \frac{1}{2}\text{Tr}\, \rho_0 \left(\mathbf{B}^\dagger\mathbf{A} + \mathbf{A}\mathbf{B}^\dagger \right)^\dagger = \frac{1}{2}\text{Tr}\, \rho_0 \{\mathbf{B}^\dagger, \mathbf{A}\}^\dagger \\ &= (\langle \mathbf{B}, \mathbf{A} \rangle_{\text{HS}}^{\rho_0})^* \end{aligned}$$

3) Is it positive-defined? Yes, in effect,

(2)
$$\langle \mathbf{A}, \mathbf{A} \rangle_{\text{HS}}^{\rho_0} = \frac{1}{2}\text{Tr}\, \rho_0 \{\mathbf{A}^\dagger, \mathbf{B}\} = \text{Tr}\, \sqrt{\rho}\frac{\mathbf{A}\mathbf{A}^\dagger + \mathbf{A}^\dagger\mathbf{A}}{2}\sqrt{\rho}$$

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In our context, since the calculations are to be computed via the (modified) Hilbert-Schmidt inner product and metric, only trace-class operators are allowed. Therefore, some redefinitions are needed.

(3)
$$\mathfrak{Hs}_1(\mathbb{H}) = \{\hat{\mathbf{O}} | \hat{\mathbf{O}} \in \mathcal{B}_1 \wedge ||\mathbf{O}||_{\text{HS}}^{\rho_0} < \infty, \; \forall j \leq N\} \quad \begin{array}{l}\text{is the space of all}\\ \text{one-body operators,}\end{array} \qquad \text{and where}$$

$$\mathfrak{Hs}_k(\mathbb{H}) = \{\otimes_{i=1}^{k}\hat{\mathbf{O}}_i | \hat{\mathbf{O}}_i \in \mathfrak{Hs}_1(\mathbb{H}) \; \forall i, k \leq N\} \qquad \begin{array}{l}\text{is the space of all}\\ k\text{-body operators}\end{array} \qquad \mathfrak{Hs}(\mathbb{H}) = \bigsqcup_{k=1}^{N} \mathfrak{Hs}_k(\mathbb{H})$$

Consider now an N-body quantum system, where correlations, entanglements, and interactions are present. Different particle species are allowed. The One-body and Two-body Max-Ent frameworks are defined as follows

Näive one-body Max-Ent

In the Näive one-body Max-Ent framework, there are N sets of one-body operators, each one corresponding to one the N subsystems, which are assumed to be the local basis. These operators must be local operators, acting non-trivially in only one Hilbert subspace, and must be trace-class. The framework thus allows for interactions between different particle species since the basis may have different dimension, this is

$$\{\mathbf{O}_i^{(1)}\}_{i=1}^{n_1} \in \mathfrak{Hs}_1(\mathbb{H}^{(1)})$$

$$\vdots$$

$$\{\mathbf{O}_i^{(\ell)}\}_{i=1}^{n_\ell} \in \mathfrak{Hs}_1(\mathbb{H}^{(\ell)})$$

$$\vdots$$

$$\{\mathbf{O}_i^{(N)}\}_{i=1}^{n_N} \in \mathfrak{Hs}_1(\mathbb{H}^{(N)})$$

The operators are then redefined so that they act on the global Hilbert space $\mathbb{H}^{\otimes N}$

$$\mathfrak{b}_1 = \{\mathbf{O}_i^{(1)} \otimes \otimes_{k=2}^N \mathbb{1}^{(k)}\}_{i=1}^{n_1} \subset \mathfrak{Hs}(\mathbb{H}^{\otimes N})$$

$$\vdots$$

$$\mathfrak{b}_\ell = \{\otimes_{k=1}^{\ell} \mathbb{1}^{(k)} \otimes \mathbf{O}_i^{(\ell)} \otimes_{k'=\ell+1}^{N} \mathbb{1}^{(k')}\}_{i=1}^{n_\ell} \subset \mathfrak{Hs}(\mathbb{H}^{\otimes N})$$

$$\vdots$$

$$\mathfrak{b}_N = \{\otimes_{k=1}^{N-1} \mathbb{1}^{(k)} \otimes \mathbf{O}_i^{(N)}\}_{i=1}^{n_N} \subset \mathfrak{Hs}(\mathbb{H}^{\otimes N})$$

Then, the one-body Max-Ent basis $\mathfrak{B}_{ME_1}$ is defined as the union of these sets

$$\mathfrak{B}_{ME_1} = \bigsqcup_{k=1}^{N} \mathfrak{b}_k,$$

whose dimension is given by the sum of the $\mathfrak{b}$-basis dimensions ie. $\dim(\mathfrak{B}_{ME_1}) = \sum_{k=1}^{N} \dim(\mathfrak{b}_k) \sim \mathcal{N}$. Then, the one-body Max-Ent states are given by

(4) $$\boldsymbol{\mathcal{S}}_{ME,1}(\mathfrak{B}_{ME_1}) = \{\rho \in \mathcal{C}(\mathbb{H}) \mid \exists \{\lambda_k\}_{k=1}^{\dim \mathfrak{B}_{ME_1}} \subset \mathbb{R} \text{ such that } \rho \propto \exp\left(-\sum_i \lambda_i \mathbf{O}_i\right)\}.$$

Näive two-body Max-Ent

Here, similarly to the näive one-body Max-Ent framework, we have N sets of local one-body operators at our disposal, which must be trace-class and must only non-trivially act in only one Hilbert subspace. If the one-body local operators are

(5)

$$\{\mathbf{O}_i^{(1)}\}_{i=1}^{n_1} \in \mathfrak{Hs}_1(\mathbb{H}^{(1)}) \qquad\qquad \mathfrak{b}_1 = \{\mathbf{O}_i^{(1)} \otimes \otimes_{k=2}^{N} \mathbb{1}^{(k)}\}_{i=1}^{n_1} \subset \mathfrak{Hs}(\mathbb{H}^{\otimes N})$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$\{\mathbf{O}_i^{(\ell)}\}_{i=1}^{n_\ell} \in \mathfrak{Hs}_1(\mathbb{H}^{(\ell)}) \quad , \quad \begin{array}{c}\text{then the global}\\\text{one-body operators are}\end{array} \quad \mathfrak{b}_\ell = \{\otimes_{k=1}^{\ell} \mathbb{1}^{(k)} \otimes \mathbf{O}_i^{(\ell)} \otimes_{k'=\ell+1}^{N} \mathbb{1}^{(k')}\}_{i=1}^{n_\ell} \subset \mathfrak{Hs}(\mathbb{H}^{\otimes N})$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$\{\mathbf{O}_i^{(N)}\}_{i=1}^{n_N} \in \mathfrak{Hs}_1(\mathbb{H}^{(N)}) \qquad\qquad \mathfrak{b}_N = \{\otimes_{k=1}^{N-1} \mathbb{1}^{(k)} \otimes \mathbf{O}_i^{(N)}\}_{i=1}^{n_N} \subset \mathfrak{Hs}(\mathbb{H}^{\otimes N})$$

However, unlike the previous case, we now allow for two-body operators to be included. These new sets of two-body operators may be defined as follows

$$\mathfrak{c}_{11} = \{\mathbf{O}_i^{(1)}\mathbf{O}_j^{(1)} \otimes_{k=2}^{N} \mathbb{1}^{(k)}\}_{i,j=1}^{\Gamma(n_1,n_1)} \mid \mathbf{O}_i^{(1)} \in \mathfrak{b}_1$$

$$\mathfrak{c}_{12} = \{\mathbf{O}_i^{(1)} \otimes \mathbf{Q}_j^{(2)} \otimes_{k=3}^{N} \mathbb{1}^{(k)}\}_{i,j=1}^{\Gamma(n_1,n_2)} \mid \mathbf{O}_i^{(1)} \in \mathfrak{b}_1, \mathbf{Q}_j^{(2)} \in \mathfrak{b}_2$$

$$\vdots$$

$$\mathfrak{c}_{\ell\ell'} = \{\mathbf{O}_i^{(\ell)} \otimes \mathbf{Q}_j^{(\ell)} \otimes_{\substack{k=1\\k\neq\ell\ell'}}^{N} \mathbb{1}^{(k)}\}_{i,j=1}^{\Gamma(n_\ell,n_{\ell'})} \mid \mathbf{O}_i^{(1)} \in \mathfrak{b}_\ell, \mathbf{Q}_j^{(2)} \in \mathfrak{b}_{\ell'},$$

$$\vdots$$

$$\mathfrak{c}_{NN} = \{\otimes_{k=1}^{N-1} \mathbb{1}^{(k)} \otimes \mathbf{O}_i^{(N)}\mathbf{O}_j^{(N)}\}_{i,j=1}^{\Gamma(n_N,n_N)} \mid \mathbf{O}_i^{(N)} \in \mathfrak{b}_N$$

where $\Gamma(n_a, n_b)$ counts all the possible, non-repeating, order notwithstanding, pair combinations of elements from a n_a-cardinality set with elements from a n_b-cardinality set. In other words,

(6)
$$\Gamma(n_a, n_b) = n_a + n_b + \frac{n_a(n_b - 1)}{2}$$

Then, the two-body Max-Ent basis is

$$\mathfrak{B}_{ME_2} = \bigsqcup_{k=1}^{N} \mathfrak{b}_k \cup \bigsqcup_{k,k'=1}^{N} \mathfrak{c}_{kk'},$$

from which the two-body Max-Ent states are given by

(7)

$$\boldsymbol{\mathcal{S}}_{ME,2}(\mathfrak{B}_{ME_2}) = \{\rho \in \mathcal{C}(\mathbb{H}) \mid \exists \{\lambda_k\}_{k=1}^{\ell \leq N}, \{\gamma_{mn}\}_{\substack{m,n=1\\m<n}}^{\ell \leq N} \subset \mathbb{R} \text{ such that } \rho \propto \exp\left(-\sum_i \lambda_i \mathbf{O}_i - \gamma_{ij}\mathbf{O}_i\mathbf{O}_j\right)\}.$$

Note that $\dim \mathfrak{B}_{ME_2} = \sum_{k=1}^{N} \dim(\mathfrak{b}_k) + \sum_{i,j=1}^{N} \dim(\mathfrak{c}_{ij}) \sim \mathcal{O}(N^2)$

Both of these techniques require substantially fewer parameters than the exact dynamics, which requires $\mathcal{O}(2^{2N})$ complex-valued entries (or alternatively $\mathcal{O}(2^{2N+1})$ real-valued parameters).

Consider a closed quantum many-body system described by a Hamiltonian $\mathbf{H}$ and with its initial state, ρ_0, given by

$$\rho_0 = e^{-\mathbf{K}}$$

where $\rho_0 \in \mathcal{C}(\mathbb{H}^{\otimes N}))$

and with $\mathbf{K} \in \mathfrak{Hs}(\mathbb{H}^{\otimes N}) \Longleftrightarrow \mathbf{K} = -\log \rho$

The system's time evolution of course governed by the Schrödinger equation.

$$i\frac{d\rho(t)}{dt} = [\mathbf{H}, \rho]$$

For the time being, consider ρ_0 as a one-body Max-Ent state, with respect to some general Max-Ent 1 basis composed of a collection of one-body local operators

(8) $\qquad \rho \in \mathcal{S}_{ME,1}(\mathfrak{B}_{ME_1})$ which in turn implies that the $\mathbf{K}$-operator can be uniquely decomposed, upto phase factors, as $\qquad \mathbf{K} = \sum_{\mu}^{\ell} \phi^{\mu}(t)\mathcal{O}_{\mu},$

where we have chosen the Schrodinger picture for the operators. Now we claim the following

Theorem 2. *Since $\rho = e^{-\mathbf{K}}$ is a well-defined density operator $\rho_0 \in \mathcal{C}(\mathbb{H}^{\otimes N}))$, the $\mathbf{K}$-operator's time evolution is governed by a Schrödinger equation as well, this is*

$$i\frac{d\mathbf{K}}{dt} = [\mathbf{K}, \rho].$$

Proof. If $\rho = e^{-\mathbf{K}}$ then, by definition

(9)
$$\rho = \mathbb{1} - \mathbf{K} + \frac{1}{2}\mathbf{K}^2 - \frac{1}{3!}\mathbf{K}^3 + \cdots$$
$$d\rho = 0 - d\mathbf{K} + \frac{1}{2}\Big(d\mathbf{K}\mathbf{K} + \mathbf{K}d\mathbf{K}\Big) - \frac{1}{3!}\Big((d\mathbf{K})\mathbf{K}^2 + \mathbf{K}(d\mathbf{K})\mathbf{K} + \mathbf{K}^2(d\mathbf{K})\Big) + \cdots$$

which, if we are willing to assume that $[\mathbf{K}, d\mathbf{K}] = 0$, yields

(10)
$$d\rho = 0 - d\mathbf{K} + \frac{1}{2}\Big(d\mathbf{K}\mathbf{K} + \mathbf{K}d\mathbf{K}\Big) - \frac{1}{3!}\Big((d\mathbf{K})\mathbf{K}^2 + \mathbf{K}(d\mathbf{K})\mathbf{K} + \mathbf{K}^2(d\mathbf{K})\Big) + \cdots$$
$$= -d\mathbf{K} + \mathbf{K}d\mathbf{K} - \frac{1}{2}\mathbf{K}^2 d\mathbf{K} + \cdots$$
$$= -\Big(\mathbb{1} - \mathbf{K} + \frac{1}{2}\mathbf{K}^2 + \cdots\Big)d\mathbf{K} = -e^{-\mathbf{K}}d\mathbf{K}$$
$$\Rightarrow \frac{d\rho}{dt} = -e^{-\mathbf{K}}\frac{d\mathbf{K}}{dt},$$

and given that ρ's time evolution is governed by the Schrödinger equation, this yields

(11)
$$i\frac{d\rho}{dt} = [\mathbf{H}, \rho]$$
$$ie^{-\mathbf{K}}\frac{d\mathbf{K}}{dt} = [\mathbf{H}, e^{-\mathbf{K}}]$$

[Comentario FTBP: Acá algunas cosas me hacen ruido. Primero, la identidad $\frac{d\rho}{dt} = -e^{-\mathbf{K}}\frac{d\mathbf{K}}{dt}$ se mantiene sí y solo sí asumo que dK y K conmutan, lo cual en general no es el caso. Por otro lado, si intento hacer la cuenta por el lado de la derivada del logaritmo tengo que $d\log\rho/dt = \rho^{-1}d\rho/dt$]

$$\frac{d}{dt} \log \rho(t) = \lim_{\Delta t \to 0} \frac{\log[\rho + \rho' \Delta t] - \log \rho}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\log[\rho \rho^{-1} + \rho' \rho^{-1} \Delta t]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\log[\mathbb{1} + \rho' \rho^{-1} \Delta t]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \log\left(\left[\mathbb{1} + \rho' \rho^{-1} \Delta t\right]\right)^{\frac{1}{\Delta t}}$$

$$= \lim_{\sigma \to 0} \log\left(\left[\mathbb{1} + \sigma\right]\right)^{\rho' \rho^{-1} \sigma^{-1}} \quad \text{where } \sigma = \rho' \rho^{-1} \Delta t$$

$$= \rho' \rho^{-1} \lim_{\sigma \to 0} \log\left(\left[\mathbb{1} + \sigma\right]\right)^{\sigma^{-1}}$$

$$= \rho' \rho^{-1} \lim_{\sigma \to 0} \log e$$

$$= \rho' \rho^{-1}$$

[Comentario FTBP: pero lo que no me gusta de acá es que es que usé que $\log AB^{-1} = \log A - \log B$, que vale solo si conmutan y si $\rho' \rho^{-1}$ conmutan dentro del límite. Me hace ruido]

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