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Notes on Complex Differential Geometry and Algebraic Topology - Lecture Notes

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Theory & Notes

CONTENTS

1. FIBRE BUNDLES

Bundles.texBundles.tex

A manifold is a topological space which is locally isomorphic to $\mathbb{R}^n$, but not necessarily so globally. By introducing a coordinate chart, a local Euclidean structure is endowed on the manifold. A fibre bundle is a topological space which is locally isomorphic to a direct product of two topological spaces.

1.1. Tangent bundles.

A **tangent bundle** TM over an m-dimensional manifold M is a collection of all the tangent spaces of M, namely

$$TM := \bigcup_{p\in M} T_pM$$

The M-manifold over which TM is defined is the **base space**. Let $\{U_i\}^i$ be an open covering of M.

If $x^\mu = \phi_i(p)$ is the coordinate on U_i, an element of $TU_i := \bigcup_{p\in U_i} T_pM$ is specified by a point $p \in M$ and a vector $\mathbf{V} = V^\mu(p)\partial_\mu|_p \in T_pM$.

Given that the open covering U_i are homeomorphic to an open subset $\phi(U_i) \subset \mathbb{R}^m$ and each $T_pM \approx_h \mathbb{R}^m$, it follows that TU_i is identified with a direct product $\mathbb{R}^m \times \mathbb{R}^m$. Explicitly, the mapping reads

$$(p, \mathbf{V}) \in TU_i :\ (p, \mathbf{V}) \mapsto (x^\mu(p), V^\mu(p)). \tag{1.1.0.1}$$

This implies that TU_i is a $2m$-differentiable manifold. Moreover, TU_i can be decomsoped as a direct product $U_i \times \mathbb{R}^i$, i.e. the information contained in the point $u \in TU_i$ can be systematically mapped into a point $p \in M$ and a vector $\mathbf{V} \in T_pM$.

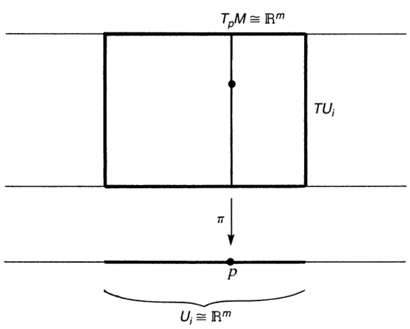

FIGURE 1. Diagram showing a local piece of $TU_i \simeq \mathbb{R}^m \times \mathbb{R}^m$ of the tangent bundle TM. The projection π projects a vector $\mathbf{V} \in T_pM$ to a point $p \in U_i \subset M$.

Thus, the idea of a **bundle projection**, not to be confused with the **natural projection**, arises.

Definition 1. *Given a manifold M with tangent bundle TM, the* **bundle projection** *π at the point $u \in T_pM$ is defined as a map*

$$\pi : TU_i \to U_i,$$

s.t. for any point $u \in TU_i$, $\pi(u)$ is a point $p \in U_i$ at which the vector is defined.

This definition must be contrasted with the notion of **natural projection**,

Definition 2. *Let X and Y be two topological spaces. Then, the* **natural projection** *mapping* $\mathrm{proj}_1 : X \times Y \to X$ *is defined s.t.* $\mathrm{proj}_1((x, y)) = x \in X$.

Remark. *Since both of these mappings are projections, information is lost. In particular in the case of the bundle projection, information about the vector itself is lost. Furthermore, note that $\pi^{-1}(p) = T_pM$. Moreover, the projection π can be globally defined on M, since the definition $\pi(u) = p$ does not depend on a special coordinate chosen, allowing for $\pi : TM \to M$ to be defined globally with no reference to local charts.*

Hence, T_pM is called the **fibre** of M at the point p.

It is obvious that if $M = \mathbb{R}^m$, the tangent bundle itself is expressed as $\mathbb{R}^m \times \mathbb{R}^m$. Naturally, this will not be always the case for more complex structures, since the tangent bundle measures the topological non-triviality of the manifold M. In effect, consider a topology $\tau = \{U_i\}^i$ of charts on M, s.t. $U_i \cap U_j \neq \emptyset$. In particular, consider two charts U_i, U_j and let $y^\mu = \psi(p)$ be the coordinates on U_j. Consider a vector $\mathbf{V} \in T_pM$ s.t. $p \in U_i \cap U_j$. Then, $\mathbf{V}$ has two coordinate presentations,

$$\mathbf{V} = V^\mu \frac{\partial}{\partial x^\mu}\Big|_p = \tilde{V}^\mu \frac{\partial}{\partial y^\mu}\Big|_p, \text{ related as } \tilde{V}^\mu = \frac{\partial y^\mu}{\partial x^\nu}(p) V^\nu, \text{ with} \frac{\partial y^\mu}{\partial x^\nu}(p) \in \mathrm{GL}(m, \mathbb{R})$$

For $\{x^\mu\}$ and $\{y^\mu\}$ to be good coordinate systems, the matrix $G^\mu_\nu \equiv \partial y^\mu / \partial x^\nu(p)$ must be non-singular. Hence, the fibre coordinates are simply related via a linear transformation, an element of the general linear group. The group $\mathrm{GL}(m, \mathbb{R})$ is called the **structure group** of TM. This group then describes precisely how fibres of a tangent bundle are interwoven together to form a tangent bundle, which consequently may have a topologically complicated structure.

Furthermore, let $X \in \chi(M)$ be a vector field on M, which assigns a vector $\mathbf{X}|_p \in T_pM$ at each point $p \in M$. In other words, X is a smooth map $X : M \to TM$. This map is well defined since a point p must be mapped to a point $u \in TM$ s.t. $\pi(u) = p$. Hence, one naturally arrives to the following definition

Definition 3. *Let M be a manifold with fibre bundle TM, a* **section** *or* **cross section** *of TM is a smooth map $s : M \to TM$ s.t. $\pi \circ s = \mathrm{id}_M$.*

If a section $s_i : U_i \to TU_i$ is only defined on a chart U_i, it is called a **local section**.

1.1.1. *Fibre Bundles.*

Definitions and Immediate Results. The tangent bundle of the previous section is an example of a more general framework, a fibre bundle, which is given in terms of several objects

Definition 4. *A* **differentiable fibre bundle** (E, π, M, F, g) *consists of the following elements*

- **Definition 5.** *A differentiable manifold E called the* **total space***.*
- **Definition 6.** *A differentiable manifold M called the* **base space***.*
- **Definition 7.** *A differentiable manifold F called the* **fibre***, or* **typical fibre***.*
- **Definition 8.** *A surjection $\pi : E \to M$ called the* **projection***. Its inverse image $\pi^{-1}(p) = F_p \simeq F$ is called the fibre at p.*
- **Definition 9.** *A Lie group G called the* **structure group***, which acts on F on the left.*
- **Definition 10.** *A set of open coverings $\{U_i\}^i$ on M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ s.t.*
$$\pi \circ \phi_i(p, f) = p.$$
The map ϕ is called the **local trivialization***, since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$.*
- **Definition 11.** *If the map $\phi_i(p, f)$ is relabelled as $\phi_{i,p}(f)$, then the map $\phi_{i,p} : F \to F_p$ is a diffeomorphism. On $U_i \cap U_j \neq \emptyset$, an additional requirement is made, namely that*
$$t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$$
is an element of G. Then, the maps ϕ_i, ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ as
$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f).$$
These maps t_{ij} are called the **transition functions***.*

Several remarks must be made about the preceding definitions.

Remark.

- Let U_i be a chart on the base space M, i.e. an open covering. Remember from definition 7 and definition 8, that this means that
$$\forall p \in M, \text{ and hence } \forall p \in U_i, \ \exists \pi : E \to M \text{ s.t. } \pi^{-1}(p) = F_p \simeq F.$$
More explicitly, for all points p in this open covering, there is a globally-defined projection π, which maps (subsets U_i of) the total space E to the base space M s.t. its preimage $\pi^{-1}(U_i)$ yields a fibre F_p at point p, which is diffeomorphic to the fibre space F. In particular, these charts -according to definition 10- naturally come equipped with a diffeomorphism ϕ, defined as follows
$$\begin{array}{l} \phi_i : U_i \times F \to \pi^{-1}(U_i), \\ \phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F \end{array} \quad \text{Then, } \forall p \in U_i \subset M, \exists \phi_i : \pi^{-1}(U_i) \to U_i \times F,$$
In other words, $\forall p \in U_i, \ \pi^{-1}(p) = F_p \simeq F \to \phi_i : F \to U_i \times F.$

More explicitly, since the projection is map from the fibre space $F_p \simeq F$ to the base space M yielding the point p, its preimage is a mapping from the base space to the fibre at the point p. Then, the diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ can be thought of as a mapping from the direct product of the fibre space with the open set, $U_i \times F$, to the fibre itself. In tecnical terms, $\pi^{-1}(U_i)$ is a **direct-product diffeomorphism** to $U_i \times F$ via ϕ, i.e.

$$\pi^{-1}(U_i) \overset{\phi_i}{\simeq}_d U_i \times F, \text{ s.t. } \phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F.$$

Then the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\phi_i^{-1}"] \arrow[d, "\pi"'] & U_i \times F \arrow[dl, "proj_1"] \\ U_i & \end{tikzcd}$$

If $U_i \cap U_j \neq \emptyset$, there are two maps ϕ_i and ϕ_j on $U_i \cap U_j$, and consider a point u s.t. $\pi(u) = p \in U_i \cap U_j$. Then, there are two possible elements in F to which u can be assigned, one via the mapping ϕ_i^{-1} and the other one via the mapping ϕ_j^{-1}, as follows

(1.1.1.1)
$$\begin{gathered}\phi_i^{-1}(u) = (p, f_i), \quad \phi_j^{-1}(u) = (p, f_j) \\ \implies \exists t_{ij} : U_i \cap U_j \to G, \text{ which relates } f_i \text{ and } f_j \text{ as } \begin{array}{c} f_i = t_{ij}(p) f_j \\ \Updownarrow \\ \phi_j(p, f) = \phi_i\Big(p, t_{ij}(p) f\Big) \end{array}\end{gathered}$$

Some requirements are imposed on these transition functions, namely

(1.1.1.2)
$$\begin{aligned} &t_{ii}(p) = \mathfrak{id}_M, \quad p \in U_i \\ &t_{ij}(p) = t_{ji}(p)^{-1}, \quad p \in U_i \cap U_j \\ &t_{ij}(p) \cdot t_{jk}(p) = t_{ik}(p), \quad p \in U_i \cap U_j \cap U_k. \end{aligned}$$

Unless these conditions are met, local pieces of a fibre bundle cannot be glued together consistently. If all transition functions are identity maps $\mathfrak{id}_M$, the fibre bundle is the **trivial fibre bundle**, which is simply a direct product $M \times F$.

Remark.

- Given a fibre bundle $E \xrightarrow{\pi} M$, the possible set of transition functions is not unique. In effect, consider a covering $\{U_i\}^i$ of M with $\{\phi\}^i$ and $\{\tilde{\phi}_i\}^i$ be two sets of local trivializations giving rise to the same fibre bundle, with transition functions

(1.1.1.3)
$$t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p}, \quad \tilde{t}_{ij}(p) = \tilde{\phi}_{i,p}^{-1} \circ \tilde{\phi}_{j,p}.$$

Let $g_i(p) : F \to F$ at each point $p \in M$ defined by $g_i(p) = \phi_{i,p}^{-1} \circ \tilde{\phi}_{i,p}$, which must be a homeomorphism belonging to G. This requirement must be fulfilled if $\{\phi\}^i$ and $\{\tilde{\phi}_i\}^i$ describe the same fibre bundle.

(1.1.1.4)
$$\begin{aligned} \tilde{t}_{ij}(p) &= \tilde{\phi}_{i,p}^{-1} \circ \tilde{\phi}_{j,p} \\ &= \tilde{\phi}_{i,p}^{-1} \circ \phi_{i,p} \circ \phi_{i,p}^{-1} \circ \phi_{i,p} \circ \phi_{i,p}^{-1} \circ \tilde{\phi}_{j,p} \\ &= g_i(p)^{-1} \circ t_{ij}(p) \circ g_j(p). \end{aligned}$$

In physics, the t_{ij} transformations are the gauge transformations required to paste local charts together, while the g_i correspond to the gauge degrees of freedom within a chart U_i. The most general form of the transition functions is

(1.1.1.5) $$t_{ij}(p) = g_i(p)^{-1} \circ g_j(p).$$

Remark.

- Let $E \xrightarrow{\pi} M$ be a fibre bundle. A ſection o a croſſ ſection $s : M \to E$ is a smooth map which satisfies $\pi \circ s = \mathfrak{id}_M$. It follows that $s(p) = s|_p$ is an element of the fibre at p, $F_p = \pi^{-1}(p)$. The set of sections on M is denoted by $\Gamma(M, F)$. If $U \subset M$. the local ſection is only defined on U. For example, $\Gamma(M, TM)$ is identified with the set of vector fields $\chi(M)$. Note, however, that not all fibre bundles admit global sections.

Example. Consider the following example of a fibre bundle:

Let $E \xrightarrow{\pi} S^1$ be a fibre bundle with a typical fibre $F = \mathbb{R}_{[-1,1]}$. Moreover, let $U_1 = \mathbb{R}_{(0,2\pi)}$ and $U_2 = \mathbb{R}_{(-\pi,\pi)}$ be two open coverings of S^1. Let $A = \mathbb{R}_{(0,\pi)}$ and let $B = \mathbb{R}_{(\pi,2\pi)}$ be the intersection $U - 1 \cap U_2$. The local trivializations (ϕ_1, ϕ_2) are given by

$$\phi_1^{-1}(u) = (\theta, t), \quad \phi_2^{-1}(u) = (\theta, t),$$

for $\theta \in A$, $t \in F$. The transition function $t_{12}(\theta), \theta \in A$, is the identity map $t_{12}(\theta) : t \mapsto t$. On B, however, there are two choices

- for instance,
$$\phi_1^{-1}(u) = (\theta, t), \quad \phi_2^{-1}(u) = (\theta, t),$$
- or
$$\phi_1^{-1}(u) = (\theta, t), \quad \phi_2^{-1}(u) = (\theta, -t),$$

The first case has a trivial transition map, i.e. it is the identity map. Hence, the two pieces of the local bundles are glued together to form a cylinder. In the second case, the transition map is $t_{12}(\theta) : t \mapsto -t, \theta \in B$, giving rise to the Möbius strip.

The cylinder has a trivial structure group $G = \{e\}$, where e is the identity map on F, while the Möbius strip has $G = \{e, g\} \simeq \mathbb{Z}_2$, where $g : t \mapsto -t$. Technically, however, G is not a Lie group. Then, the cylinder corresponds to the trivial bundle $S^1 \times F$, while the Möbius strip is not.

Reconstruction of fibre bundles. A question naturally arises: what is the minimal information required to construct a fibre bundle?

Theorem 1. *Consider a fibre bundle in which the following elements are known:*

- *a base space M,*
- *a topology $\{U_i\}$ of open coverings on M,*
- *the transition functions $t_{ij}(p)$,*
- *the typical fibre F,*
- *the structure group G.*

Then, a fibre bundle (E, π, M, F, G) can be reconstructed.

Proof. In other words, one needs only to define a projection π, a total space E and the local trivializations $\{\phi_i\}^i$, for the fibre bundle to be uniquely characterized. In effect, consider

$$X \equiv \cup_i U_i \times F.$$

Furthermore, let $\sim$ be a globally-defined equivalence relation defined on $\{U_i \times F\}^i$, defined as follows

$$\begin{array}{l} (p, f) \in U_i \times F, \\ (q, f') \in U_j \times F \end{array}, \quad (p, f) \sim (q, f') \text{ and } p = q \wedge f' = t_{ij}(p) f.$$

Sets of equivalence classes can then be defined,

Let $p \in M$ be a point,
Let $f \in F$ be an element of the fibre space $[(p,f)] = \{(q,f') \in U_i \times F \text{ s.t. } (p,f) \sim (q,f')\} = \left\{ (q,f') \in U_i \times F \text{ s.t. } \begin{array}{c} p = q \\ f' = t_{ij}(p) f \end{array} \right\}$

$$\Rightarrow \frac{X}{\sim} = \{[(p,f)] \text{ s.t. } (p,f) \in U_i \times F\}$$

i.e. the set of equivalence classes on $U_i \times F$. A fibre bundle E is then defined as the quotient space

$$E = \frac{X}{\sim}.$$

Two mappings of vital importance then readily arise,

A projection and the local trivializations

$$\begin{aligned} \pi : U_i \times F &\to U_i \\ [(p,f)] &\mapsto p, \end{aligned} \qquad \begin{aligned} \phi_i : U_i \times F &\to \pi^{-1}(U_i) \\ (p,f) &\mapsto [(p,f)]\,, \end{aligned}$$

with their inverse maps

$$\begin{aligned} \pi^{-1} : U_i &\to U_i \times F \\ p &\mapsto [(p,f)]\,, \end{aligned} \qquad \begin{aligned} \phi_i^{-1} : \pi^{-1}(U_i) &\to U_i \times F \\ [(p,f)] &\mapsto (p,f), \end{aligned}$$

with $(p,f) \in U_i \times F$, and $[(p,f)] \in E$. One notes that $U_i \times F = \pi^{-1}(U_i)$.

Moreover, note that $\begin{aligned} \pi \circ \phi_i : U_i \times F &\to U_i \\ (p,f) \mapsto [(p,f)] &\mapsto p \end{aligned}$ is the $proj_1 : U_i \times F \to U_i$ map, see definition 10.

Expanding briefly on the last remark, remember that the inverse map ϕ_i^{-1} maps $\pi^{-1}(U_i)$ to the direct product $U_i \times F$. Hence, the following diagram commutes,

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\phi_i^{-1}"] \arrow[d, "\pi"'] & U_i \times F \arrow[dl, "proj_1"] \\ U_i & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \pi^{-1}(U_i) \arrow[d, "\pi"'] & U_i \times F \arrow[l, "\phi"'] \arrow[dl, "proj_1"] \\ U_i & \end{tikzcd}$$

making $\pi \circ \phi$ the cartesian projection mapping.

Moreover, one can define a global mapping, similar in nature to the local trivializations ϕ_i but instead defined on the entirety of X and not only on $U_i \times F$,

$$\mathfrak{q} : X \to E \simeq \frac{X}{\sim}, \quad \text{the canonical map},$$

$$\mathfrak{q}^{-1} : E \simeq \frac{X}{\sim} \to X$$

s.t. it sends points to their equivalence classes. It is a surjective map, i.e.

$$(p,f),(q,f') \in X, \quad \mathfrak{q}(p,f) \sim \mathfrak{q}(q,f) \text{ if and only if } \mathfrak{q}(p,f) = \mathfrak{q}(q,f'),$$

and consequently $q((p,f)) = \mathfrak{q}^{-1}(\mathfrak{q}(p,f))$ for all $(p,f) \in U_i \times F$.

In particular, this shows that the set of equivalence classes $\frac{X}{\sim}$ is exactly the st of fibers of the local trivialization maps ϕ_i, or its globally defined counterpart: the canonical map. Moreover, if X is a topological space, then $\frac{X}{\sim}$ is imbued with the quotient topology induced by q, hence making $\mathfrak{q} : X \to \frac{X}{\sim}$ a quotient map. Upto a homeomorphism, this construction is representative of all quotient spaces.

This proves that the given data reconstructs a fibre bundle $E \xrightarrow{\pi} M$. $\textit{quod erat dem}\blacksquare$

This procedure can be employed to construct new fibre bundles from and old one. In effect, (E, π, M, F, G) be a fibre bundle, s.t. associated to it there is a new bundle with the same base space M, transition fucntions $t_{ij}(p)$, structure group G but different typical fibre F' on which G acts.

Bundle Maps. Let $E \xrightarrow{\pi} M, E' \xrightarrow{\pi'} M'$ be two fibre bundles. A smooth map $\bar{f} : E' \to E$ is called a $\mathfrak{bundle\ map}$ if it maps each fibre F'_p of E' onto exactly the same fibre F_q of E. If this holds, then $\bar{f}$ naturally induces a smooth map $f : M' \to M$ s.t. $f(p) = q$. Hence, the following diagrams commute

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow[f]{} & M \end{array} \iff \begin{array}{ccc} u & \xrightarrow{\bar{f}} & \bar{f}(u) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ p & \xrightarrow[f]{} & q \end{array}$$

Remark.

Note, however, that not all smooth maps $\bar{f} : E' \to E$ are bundle maps, since the definition of bundle maps requires it to be $\mathfrak{fibre\text{-}preserving}$. More explicitly, let $\bar{f} : E' \to E$ be a mapping from E' to E, s.t. it induces a map $f : M' \to M$ from M' to M, then $\bar{f}$ is a bundle map if and only if $\pi \circ \bar{f} = \pi' \circ f$. In other words, a general smooth map $\bar{f}$ could map $u, v \in F'_p$ of E' to different fibres $\bar{f}(u), \bar{f}(v)$ of E so that their projections differ $\pi(\bar{f}(u)) \neq \pi(\bar{f}(v))$.

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