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Group Theory - Lecture Notes
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Theory & Notes

Definition 1. *A vector space L over a field $\mathbb{F}$, with an operation*

$$L \times L \to L$$
$$(x, y) \to [x, y],$$

called the bracket or commutator of x and y, is called a Lie algebra over $\mathbb{F}$ if the following axioms are satisfied

- *The bracket operation is bilinear,*
- $[x, x] = 0,\ \forall x \in L$*,*
- $[x, [y, z]] = [y, [z, x]] = [z, [x, y]] = 0,\ \forall x, y, z \in L$

,

where the first two axioms imply the bracket's anticommutativity.

Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are isomorphic if there exists a vector space isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$, satisfying $\phi([x, y]) = [\phi(x), \phi(y)],\ \forall x, y \in \mathfrak{g}$. Similarly, a subspace $\mathfrak{k} \subset \mathfrak{g}$ is called a subalgebra if $[x, y] \in \mathfrak{k},\ \forall x, y \in \mathfrak{k}$. Note that any Lie subalgebra is a Lie algebra in its own right, relative to the inherited operations. Note as well, that any non-zero element $x \in \mathfrak{g}$ defines a one-dimensional subalgebra $\mathbb{F}x$, with trivial multiplications.

If V is a finite n-dimensional vector space over $\mathbb{F}$, End(V) is the set of endomorphisms over V, being a n^2-dimensional vector space. Note that End(V) is a ring relative to the usual product operation. Imbued with the Lie bracket operation, End(V) is a Lie algebra over $\mathbb{F}$, denoted by $\mathfrak{gl}(V) \simeq \text{End}(V)$ and is the general linear algebra. If a basis for V is fixed, thereby identifying $\mathfrak{gl}(n, \mathbb{F}) \simeq (V)$, explicit calculations may be performed. In particular if the standard matrix basis e_{ij} is chosen, such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}, , \tag{1}$$

noticing that all coefficients are either 0 or ± 1.

All (simple) Lie algebras fall into four families, $\mathfrak{A}_\ell, \mathfrak{B}_\ell, \mathfrak{C}_\ell, \mathfrak{D}_\ell$ ($\ell > 1$) are the classical algebras, for they correspond to certain classical linear group. In particular, for the last three families, let char $\mathbb{F} \neq 2$.

- $\mathfrak{A}_\ell$: let $\dim V = \ell + 1$. Denote by $\mathfrak{sl}(V)$, or $\mathfrak{sl}(\ell + 1, \mathbb{F})$, the set of endomorphisms on V with zero trace. Since $\text{Tr}(xy) = \text{Tr}(yx)$ and $\text{Tr}(x + y) = \text{Tr}(x) + \text{Tr}(y)$, thus $\mathfrak{sl}(V) \subseteq \mathfrak{gl}(V)$. This algebra is the special linear algebra and is intimately connected to the special linear group $SL(V)$ of endomorphisms of unit determinant.

  What about its dimension? since $\mathfrak{sl}(V)$ is a proper subalgebra of $\mathfrak{gl}(V)$, its dimension is at most $(\ell + 1)^2 - 1$. On the other hand, the number of linearly independent matrices of zero-trace can be readily found. For all $e_{ij},\ i \neq j$ and all $h_i = e_{ii} - e_{i+1,i+1}$, with $1 \leq i \leq \ell$, yielding a total of $\ell + (\ell+1)^2 - (\ell+1)$ matrices, which is the standard basis for $\mathfrak{sl}(V)$.

- $\mathfrak{C}_\ell$: let $\dim V = 2\ell$, with basis $(v_1, \cdots, v_{2\ell})$. Let the non-degenerate skew-symmetric form $f : V \to V$ defined by the matrix

$$s = \begin{pmatrix} 0 & \text{id}_\ell \\ -\text{id}_\ell & 0 \end{pmatrix}.$$

  Let $\mathfrak{sp}(V)$, or $\mathfrak{sp}(2\ell, \mathbb{F})$, the symplectic algebra, which by definition consists of all endomorphism $x : V \to V$ satisfying $f(x(v), w) = -f(v, x(w))$. This algebra is closed under the bracket operation. In matrix terms, the previous condition may be rewritten as

$$x \in \mathfrak{sl}(V) \mid x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}, m,n,p,q \in \mathfrak{gl}(\ell,\mathbb{F}) \wedge sx = -x^{\mathrm{T}}s \leftrightarrow \begin{matrix} n^{\mathrm{T}} = n \\ p^{\mathrm{T}} = p \\ m^{\mathrm{T}} = -q \end{matrix} .$$

This last condition fixes that $\operatorname{Tr} x = 0$. A basis for $\mathfrak{sl}(2\ell,\mathbb{F})$ can now be fixed.

Consider all diagonal matrices $e_{ii} - e_{\ell+1,\ell+1}$, and adding to these all $e_{ij} - e_{\ell+j,\ell+j}$, with $1 \leq i \neq j \leq \ell$, which are $\ell^2 - \ell$ in number. For n, consider the matrices $e_{i,\ell+1}$ and $e_{i,\ell+j} + e_{j,\ell+i}$, for $(1 \leq i < j \leq \ell)$, a total of $\ell + \frac{1}{2}\ell(\ell-1)$, and similarly for positions in p. Adding up yields $\dim \mathfrak{sl}(2\ell,\mathbb{F}) = 2\ell^2 + \ell$.

- $\mathfrak{B}_\ell$: let $\dim V = 2\ell + 1$ be odd, and let f be the non-degenerate symmetric bilinear form on V whose matrix is

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{id}_\ell \\ 0 & \mathrm{id}_\ell & 0 \end{pmatrix}.$$

The orthogonal algebra $\mathfrak{o}(V)$ or $\mathfrak{o}(2\ell+1,\mathbb{F})$, consists on all endomorphisms of V satisfying $f(x(v),w) = -f(v,x(w))$, the same requirement as for $\mathfrak{C}_\ell$. If x is partitioned in the same way as s, say

$$x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix} \Rightarrow sx = -x^{\mathrm{T}}s \leftrightarrow$$

Definition 2. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and let V be a vector space over said field. A representation of $\mathfrak{g}$ on V is a linear map*

$$\rho : \mathfrak{g} \to \mathrm{End}(V), \text{ such that } \rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

Example

Let $\mathfrak{g}$ be the 3-dimensional subspace of $\mathrm{End}(\mathbb{Q})$ spanned by

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let ρ be the linear map from into said space, spanned by the previous matrices, such that

$$\rho(x) = A_1, \rho(y) = A_2, \rho(h) = A_3.$$

Then, ρ is an $\mathfrak{g}$-representation. Furthermore, the representation's kernel is 0, so $\mathfrak{g}$ is isomorphic to its image. Said representations are called faithful.

Definition 3. *Let $\mathfrak{g}$ be a Lie algebra. Define a map*

$$\mathrm{ad}\ : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) \text{ such that } \mathrm{ad}(x)(y) = [x,y].$$

This is a Lie algebra representation, with the ad*-map being called the adjoint representation.*

Definition 4. *Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{k} \subseteq \mathfrak{g}$ is a subalgebra if $[x,y] \in \mathfrak{k},\ \forall x,y \in \mathfrak{k}$*

Definition 5. *Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{I}$ of $\mathfrak{g}$ is called an ideal if $[x,y] \in \mathfrak{I}, \forall x \in \mathfrak{g}, y \in \mathfrak{I}$.*

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{I}$ be an ideal on $\mathfrak{g}$ such that there is a subalgebra $\mathfrak{k} \subseteq \mathfrak{k}$ with the property that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{I}$, ie. a direct sum of vector spaces. Then, $\mathfrak{g}$ is called the semidirect product of $\mathfrak{k}$ and $\mathfrak{I}$, denoted by $\mathfrak{g} = \mathfrak{k} \triangleright \mathfrak{I}$. A special case is the situation where $\mathfrak{k}$ is also an ideal of $\mathfrak{g}$. Then, $\mathfrak{g}$ is called the direct sum of $\mathfrak{k}$ and $\mathfrak{I}$, in which case $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{I}$.

Definition 6. *Let* $\mathfrak{g}$ *be a Lie algebra. Then, the subspace*

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0, \ \forall y \in \mathfrak{g}\},$$

is called the centre of $\mathfrak{g}$.

The centre of a Lie algebra $\mathfrak{g}$ is an ideal in $\mathfrak{g}$. In particular, if $Z(\mathfrak{g}) \simeq \mathfrak{g}$, then $\mathfrak{g}$ is called abelian or commutative.

Definition 7. *Let* $\mathfrak{g}$ *be a Lie algebra and let* K *be a subspace of* $\mathfrak{g}$. *Then*

$$Z_{\mathfrak{g}}(K) = \{x \in \mathfrak{g} \mid [x, y] = 0, \ \forall y \in K\},$$

is called the centraliser of K *in* $\mathfrak{g}$.

If K is an ideal of $\mathfrak{g}$, then $Z_{\mathfrak{g}}(K)$ will also be an ideal of $\mathfrak{g}$. This follows from the Jacobi identity.

Definition 8. *Let* K *be a subspace of the Lie algebra* $\mathfrak{g}$. *Then,*

$$N_{\mathfrak{g}}(K) = \{x \in \mathfrak{g} \mid [x, y] \in K, \ \forall y \in K\},$$

is called the normaliser of K *in* $\mathfrak{g}$.

If K_1 and K_2 are $\mathfrak{g}$-subspaces, then $[K_1, K_2]$ denotes the subspace spanned by all $[x_1, x_2], \ x_1 \in K_1, x_2 \in K_2$.

Definition 9. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra. Then, set* $\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}]$ *with* $\mathfrak{g}_1 = \mathfrak{g}$ *and let* s *be the smallest integer such that* $\mathfrak{g}_s = \mathfrak{g}_{s+1}$. *The series*

$$\mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_s,$$

is the derived series of $\mathfrak{g}$.

A Lie algebra is called solvable if the final term of its derived series is 0. If $\mathfrak{I}$ and $\mathfrak{J}$ are solvable ideals of $\mathfrak{g}$, then it can be proved that $\mathfrak{I} + \mathfrak{J}$ is also a solvable ideal of $\mathfrak{g}$. It follows then that if $\mathfrak{g}$ is a finite-dimensional Lie algebra, then it has maximal solvable ideal. This is called the solvable radical of $\mathfrak{g}$, $R(\mathfrak{g})$.

Definition 10. *Let* $\mathfrak{g}$ *be a finite-dimensional algebra. Set* $\mathfrak{g}^1 = \mathfrak{g}$, *and* $\mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k]$ *and let* t *be the smallest integer such that* $\mathfrak{g}^t = \mathfrak{g}^{t+1}$. *The series*

$$\mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots \supset \mathfrak{g}^t,$$

is called the lower central series of $\mathfrak{g}$.

A finite-dimensional Lie algebra $\mathfrak{g}$ is called nilpotent if $\mathfrak{g}^t = 0$. If $\mathfrak{I}$ and $\mathfrak{J}$ are nilpotent ideals of $\mathfrak{g}$, then it can be proved that so is $\mathfrak{I} + \mathfrak{J}$. It follows that a finite-dimensional Lie algebra $\mathfrak{L}$ has a largest nilpotent ideal. It is called the nilradical and it is denoted by $NR(\mathfrak{g})$.

Definition 11. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra. Set* $Z_1 = Z(\mathfrak{g})$ *and define* Z_{k+1} *recursively by the relation* $Z_{k+1}/Z_k = Z(\mathfrak{g}/Z_k)$ *and let* u *be the smallest number such that* $Z_u = Z_{u+1}$. *Then, the series*

$$Z_1 \supset Z_2 \supset \cdots \supset Z_u,$$

is called the upper central series of $\mathfrak{g}$.

Definition 12. *A Lie algebra* $\mathfrak{g}$ *is called semisimple if* $R(\mathfrak{g}) = 0$.

Definition 13. *Cartan's criterion for semi-simplicity: Let* $\mathfrak{g}$ *be a Lie algebra, with basis* $\{x_1, \cdots, x_n\}$ *and let* $d = \det\{\mathbf{K}\}(x_i, x_j)$. *If* $d \neq 0$, *then* $\mathfrak{g}$ *is semisimple. If* $\mathfrak{g}$ *is defined over a field of characteristic 0, then this is in turn implies* $d \neq 0$.

Definition 14. *A Lie algebra $\mathfrak{g}$ is called simple if $\dim \mathfrak{g} > 1$ and it has no ideals except 0 and $\mathfrak{g}$.*

Let $\mathfrak{g}$ be a Lie algebra. Then, $R(\mathfrak{g})$ can be only 0 or $\mathfrak{g}$. Suppose that $R(\mathfrak{g}) = \mathfrak{g}$, then $\mathfrak{g}$ is solvable and hence $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ not equal to $\mathfrak{g}$ itself. It follows then that $[\mathfrak{g}, \mathfrak{g}] = 0$ so that $\mathfrak{g}$ is Abelian. But then every subspace of $\mathfrak{g}$ is an ideal, contradicting the fact that $\mathfrak{g}$ is simple. The conclusion is that $R(\mathfrak{g}) = 0$ and $\mathfrak{g}$ is semi-simple.