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Group Theory - Lecture Notes
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Theory & Notes

Definition 1. *A vector space L over a field $\mathbb{F}$, with an operation*

$$L \times L \to L$$
$$(x, y) \to [x, y],$$

called the bracket or commutator of x and y, is called a Lie algebra over $\mathbb{F}$ if the following axioms are satisfied

- *The bracket operation is bilinear,*
- $[x, x] = 0, \ \forall x \in L$,
- $[x, [y, z]] = [y, [z, x]] = [z, [x, y]] = 0, \ \forall x, y, z \in L$

,

where the first two axioms imply the bracket's anticommutativity.

Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are isomorphic if there exists a vector space isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$, satisfying $\phi([x, y]) = [\phi(x), \phi(y)], \ \forall x, y \in \mathfrak{g}$. Similarly, a subspace $\mathfrak{k} \subset \mathfrak{g}$ is called a subalgebra if $[x, y] \in \mathfrak{k}, \ \forall x, y \in \mathfrak{k}$. Note that any Lie subalgebra is a Lie algebra in its own right, relative to the inherited operations. Note as well, that any non-zero element $x \in \mathfrak{g}$ defines a one-dimensional subalgebra $\mathbb{F}x$, with trivial multiplications.

If V is a finite n-dimensional vector space over $\mathbb{F}$, End(V) is the set of endomorphisms over V, being a n^2-dimensional vector space. Note that End(V) is a ring relative to the usual product operation. Imbued with the Lie bracket operation, End(V) is a Lie algebra over $\mathbb{F}$, denoted by $\mathfrak{gl}(V) \simeq \text{End}(V)$ and is the general linear algebra. If a basis for V is fixed, thereby identifying $\mathfrak{gl}(n, \mathbb{F}) \simeq (V)$, explicit calculations may be performed. In particular if the standard matrix basis e_{ij} is chosen, such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that

(1) $$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}, ,$$

noticing that all coefficients are either 0 or ± 1.

All (simple) Lie algebras fall into four families, $\mathfrak{A}_\ell, \mathfrak{B}_\ell, \mathfrak{C}_\ell, \mathfrak{D}_\ell$ ($\ell > 1$) are the classical algebras, for they correspond to certain classical linear group. In particular, for the last three families, let char $\mathbb{F} \neq 2$.

- $\mathfrak{A}_\ell$: let $\dim V = \ell + 1$. Denote by $\mathfrak{sl}(V)$, or $\mathfrak{sl}(\ell + 1, \mathbb{F})$, the set of endomorphisms on V with zero trace. Since $\text{Tr}(xy) = \text{Tr}(yx)$ and $\text{Tr}(x + y) = \text{Tr}(x) + \text{Tr}(y)$, thus $\mathfrak{sl}(V) \subseteq \mathfrak{gl}(V)$. This algebra is the special linear algebra and is intimately connected to the special linear group $SL(V)$ of endomorphisms of unit determinant.

  What about its dimension? since $\mathfrak{sl}(V)$ is a proper subalgebra of $\mathfrak{gl}(V)$, its dimension is at most $(\ell + 1)^2 - 1$. On the other hand, the number of linearly independent matrices of zero-trace can be readily found. For all $e_{ij}, \ i \neq j$ and all $h_i = e_{ii} - e_{i+1,i+1}$, with $1 \leq i \leq \ell$, yielding a total of $\ell + (\ell + 1)^2 - (\ell + 1)$ matrices, which is the standard basis for $\mathfrak{sl}(V)$.

- $\mathfrak{C}_\ell$: let $\dim V = 2\ell$, with basis $(v_1, \cdots, v_{2\ell})$. Let the non-degenerate skew-symmetric form $f : V \to V$ defined by the matrix

$$s = \begin{pmatrix} 0 & \text{id}_\ell \\ -\text{id}_\ell & 0 \end{pmatrix}.$$

  Let $\mathfrak{sp}(V)$, or $\mathfrak{sp}(2\ell, \mathbb{F})$, the symplectic algebra, which by definition consists of all endomorphism $x : V \to V$ satisfying $f(x(v), w) = -f(v, x(w))$. This algebra is closed under the bracket operation. In matrix terms, the previous condition may be rewritten as

$$x \in \mathfrak{sl}(V) \mid x = \left(\begin{array}{cc} m & n \\ p & q \end{array} \right), m, n, p, q \in \mathfrak{gl}(\ell, \mathbb{F}) \wedge sx = -x^{\mathrm{T}} s \leftrightarrow \begin{array}{c} n^{\mathrm{T}} = n \\ p^{\mathrm{T}} = p \\ m^{\mathrm{T}} = -q \end{array}.$$

This last condition fixes that $\operatorname{Tr} x = 0$. A basis for $\mathfrak{sl}(2\ell, \mathbb{F})$ can now be fixed.

Consider all diagonal matrices $e_{ii} - e_{\ell+1,\ell+1}$, and adding to these all $e_{ij} - e_{\ell+j,\ell+j}$, with $1 \leq i \neq j \leq \ell$, which are $\ell^2 - \ell$ in number. For n, consider the matrices $e_{i,\ell+1}$ and $e_{i,\ell+j} + e_{j,\ell+i}$, for $(1 \leq i < j \leq \ell)$, a total of $\ell + \frac{1}{2}\ell(\ell-1)$, and similarly for positions in p. Adding up yields $\dim \mathfrak{sl}(2\ell, \mathbb{F}) = 2\ell^2 + \ell$.

- $\mathfrak{B}_\ell$: let $\dim V = 2\ell + 1$ be odd, and let f be the non-degenerate symmetric bilinear form on V whose matrix is

$$s = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & \mathrm{id}_\ell \\ 0 & \mathrm{id}_\ell & 0 \end{array} \right).$$

The orthogonal algebra $\mathfrak{o}(V)$ or $\mathfrak{o}(2\ell+1, \mathbb{F})$, consists on all endomorphisms of V satisfying $f(x(v), w) = -f(v, x(w))$, the same requirement as for $\mathfrak{C}_\ell$. If x is partitioned in the same way as s, say

$$x = \left(\begin{array}{ccc} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{array} \right) \Rightarrow sx = -x^{\mathrm{T}} s \leftrightarrow$$