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Notes on Complex Differential Geometry and Algebraic Topology - Lecture Notes

March 5, 2023

Theory & Notes

CONTENTS

1. COMPLEX GEOMETRY

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1.1. **Complex Structure on a vector space.** Let $\mathcal{V}$ be a $2m$-dimensional real-valued vector space. A *complex structure* on $\mathcal{V}$ is an automorphism $J : \mathcal{V} \to \mathcal{V}$ such that $J^2 = -\mathfrak{id}_{\mathcal{V}}$. With this structure, $\mathcal{V}$ is naturally brought into an m-dimensional complex-valued vector space by letting

$$(\alpha + i\beta)v = \alpha v + \beta J v, \qquad \begin{array}{c} v \in \mathcal{V} \\ \alpha, \beta \in \mathbb{R}. \end{array} \tag{1.1.0.1}$$

In other words, an m-dimensional complex-valued vector space can be thought of as a $2m$-dimensional real-value vector space endowed with the complex structure $J = i_{\mathcal{V}}$. Hence, this vector space $\mathcal{V}$ -equipped with the complex structure J- has an *adapted basis*

$$(v_1, \cdots, v_m, Jv_1, \cdots, Jv_m), \qquad \text{s.t.} \qquad J = \begin{pmatrix} 0 & \mathfrak{id}_{\mathcal{V}} \\ -\mathfrak{id}_{\mathcal{V}} & 0 \end{pmatrix}. \tag{1.1.0.2}$$

An automorphism $\rho : \mathcal{V} \to \mathcal{V}$ preserves a complex structure J on $\mathcal{V}$ if and only if it commutes with J. Hence, these automorphisms form the commutant $\{J\}' \subset \text{GL}(2m, \mathbb{R})$ of J. It turns out that there is an explicit mapping ϕ for the $\{J\}$-commutant of complex-valued m-dimensional matrices and the $\{J\}'$-commutant of real-valued $2m$-dimensional matrices. In effect, note that the commutant $\{J\}' \subset \text{GL}(2m, \mathbb{R})$ of J is the image of the group $\text{GL}(m, \mathbb{C})$ under the monomorphism ϕ, whose action is given as follows

$$\phi : \text{GL}(m, \mathbb{C}) \to \text{GL}(2m, \mathbb{R}), \text{ s.t. } \begin{array}{c} M \in \text{GL}(m, \mathbb{C}), \\ M \overset{\phi}{\mapsto} \begin{pmatrix} \Re M & -\Im M \\ \Im M & \Re M \end{pmatrix} \in \text{GL}(2m, \mathbb{R}). \end{array} \tag{1.1.0.3}$$

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By invoking **??**, one notices that there is an explicit one-to-one correspondence between the complex structures on a $2m$-dimensional real-valued vector space $\mathcal{V}$ and the elements of the quotient $\frac{\text{GL}(2m,\mathbb{R})}{\text{GL}(m,\mathbb{C})}$[1].

Remark. First consider the topological space $X = \text{GL}(2m, \mathbb{R})$. Then consider a subspace of it, $A = \text{GL}(m, \mathbb{C})$. The homomorphism $\phi : \text{GL}(m, \mathbb{C}) \to \text{GL}(2m, \mathbb{R})$, which in reality is an isomorphism, induces an equivalence relationship $\sim_\phi$ *s.t.*

$$A \sim_\phi B \leftrightarrow A = B, \quad A, B \in \text{GL}(2m, \mathbb{R}).$$

Now, the quotient space $\frac{\text{GL}(2m,\mathbb{R})}{\text{GL}(m,\mathbb{C})}$ is, by definition, $\frac{\text{GL}(2m,\mathbb{R})}{\sim_\phi}$, given by

$$\frac{\text{GL}(2m, \mathbb{R})}{\sim_\phi} = \{\text{GL}(2m, \mathbb{R}) - \text{GL}(m, \mathbb{C})\} \cup \{0_{\mathcal{V}}\}. \tag{1.1.0.4}$$

[1]Here, a short summary of the main topological properties of the real-valued and complex-valued general linear groups is presented.

1) The real-valued $\text{GL}(m, \mathbb{R})$ is non-compact. Its maximal compact subgroup is the orthogonal group $O(m)$, while the maximal compact subgroup of $\text{GL}^+(m, \mathbb{R})$ is the special orthogonal group $SO(m)$. As for $SO(m)$, the group $\text{GL}^+(m, \mathbb{R})$ is not simply connected if $m \neq 1$, but rather has a fundamental group

$$\pi_1(SO(m)) = \begin{cases} \mathbb{Z} \text{ for } m = 2 \\ \mathbb{Z}_2 \text{ for } m > 2 \end{cases}.$$

2) The complex-valued $\text{GL}(m, \mathbb{C})$ is a connected space. This follows, in part, since the multiplicative group of complex numbers $\mathbb{C} - \{0\}$ is connected as well. The complex-valued general linear group is not compact however, rather its maximal compact subgroup, $U(m)$, is a compact (group/space). As for $U(m)$, the group manifold $\text{GL}(m, \mathbb{C})$ is not simply connected but has a fundamental group $\pi \simeq \mathbb{Z}$.

A complex structure J on $\mathcal{V}$ generates a complex structure on the dual space to $\mathcal{V}$, $\mathcal{V}^*$, as follows

$$\langle v, J\omega\rangle = \langle Jv, \omega\rangle, \qquad \begin{array}{c} v \in \mathcal{V} \\ \omega \in \mathcal{V}^* \end{array}. \tag{1.1.0.5}$$

Definition 1. A scalar product $h : \mathcal{V} \to \mathbb{C}$ on a real-valued vector space $\mathcal{V}$ equipped with a complex structure J is called *Hermitian* if it is J-invariant, i.e.

$$h(Jv, Jv') = h(v, v'), \quad v, v' \in \mathcal{V}. \tag{1.1.0.6}$$

From this, it follows immediately that $h(Jv, v) = 0$, $\forall v \in \mathcal{V}$. Moreover, $\mathcal{V}$ admits an adapted basis, which is orthonormal with respect to this h-scalar product. Furthermore, one can also define a skew-symmetric bilinear form, which reads

$$\Omega(v, v') \equiv h(Jv, v'), \tag{1.1.0.7}$$

on $\mathcal{V}$, which is J-invariant as well.

One may be interested in defining the so-called *complexification* of a real-valued vector space $\mathcal{V}$ by considering of a morphism between the rings $\mathbb{R}$ and $\mathbb{C}$, as follows,

Definition 2. Let $\mathcal{V}$ be a real-valued vector space. The *complexification* of $\mathcal{V}$ is defined as the tensor product of $\mathcal{V}$ with $\mathbb{C}$, thought of as a two-dimensional real veector space, as follows

$$\mathcal{V}^{\mathbb{C}} = \mathbb{C} \otimes \mathcal{V}, \text{ s.t. } \begin{array}{c} \alpha(v \otimes \beta) = v \otimes (\alpha\beta), \quad v \in \mathcal{V}, \alpha, \beta \in \mathbb{C}. \\ \mathcal{V}^{\mathbb{C}} \simeq \mathcal{V} \oplus i\mathcal{V} \to v = v_1 \otimes 1 + v_2 \otimes i, \quad v_1, v_2 \in \mathcal{V}. \end{array}$$

Alternatively, one may use the direct sum as the definition of the complexification $\mathcal{V}^{\mathbb{C}}$ of $\mathcal{V}$ in such a way that , i.e.

$$\mathcal{V}^{\mathbb{C}} \equiv \mathcal{V} \oplus \mathcal{V},$$

where $\mathcal{V}^{\mathbb{C}}$ is imbued with a linear complex structure operator J s.t. $J(v, w) \equiv (-w, v)$. This linear complex structure, thus, encodes the operation "multiplication by i" in matrix form.

Remark. From its definition, the complexification $\mathcal{V}^{\mathbb{C}}$, the following properties and results hold

- Given a real linear transformation $f : \mathcal{V} \to \mathcal{W}$, between two real vector spaces, there is a natural complex linear transformation $f^{\mathbb{C}}$, the complexification of f, $f^{\mathbb{C}} : \mathcal{V}^{\mathbb{C}} \to \mathcal{W}^{\mathbb{C}}$ given by

$$f^{\mathbb{C}}(v \otimes z) = f(v) \otimes z,$$

s.t.
1) $(\mathfrak{id}_{\mathcal{V}})^{\mathbb{C}} = \mathfrak{id}_{\mathcal{V}^{\mathbb{C}}}$,
2) $(f \circ g) = f^{\mathbb{C}} \circ g^{\mathbb{C}}$,
3) $(f + g) = f^{\mathbb{C}} + g^{\mathbb{C}}$,
4) $(af)^{\mathbb{C}} = af^{\mathbb{C}}, \quad \in \mathbb{R}$.

The map $f^{\mathbb{C}}$ commutes with conjugation, and maps the real subspace of $\mathcal{V}^{\mathbb{C}}$ with the real subspace of $\mathcal{W}^{\mathbb{C}}$. Conversely, a complex linear map $g : \mathcal{V}^{\mathbb{C}} \to \mathcal{W}^{\mathbb{C}}$ is the complexification of a real linear map if and only if it commutes with conjugation. Hence, it follows that

$$\mathrm{Hom}_{\mathbb{R}}(\mathcal{V}, \mathcal{W})^{\mathbb{C}} \simeq \mathrm{Hom}_{\mathbb{R}}(\mathcal{V}^{\mathbb{C}}, \mathcal{W}^{\mathbb{C}}),$$

where $\mathrm{Hom}_{\mathbb{R}}(\mathcal{V}, \mathcal{W})$ is the space of all real-valued linear maps from $\mathcal{V}$ to $\mathcal{W}$.

- The dual $\mathcal{V}^*$ of a real-valued vector space $\mathcal{V}$ is the space $\mathcal{V}^*$ of all real-valued linear maps from $\mathcal{V}$ to $\mathbb{R}$. The complexification of $\mathcal{V}^*$ can be naturally be thought of as $\text{Hom}_{\mathbb{R}}(\mathcal{V}, \mathbb{C})$. This is,

$$(\mathcal{V}^*)^{\mathbb{C}} = \mathcal{V}^* \otimes \mathbb{C} \simeq \text{Hom}_{\mathbb{R}}(\mathcal{V}, \mathbb{C}).$$

The isomorphism is given by $(\omega_1 \otimes 1 + \omega_2 \otimes i) \leftrightarrow \omega_1 + i\omega_2, \quad \omega_1, \omega_2 \in \mathcal{V}^*$.
- *Given a real linear map* $\phi : \mathcal{V} \to \mathbb{C}$, *it may be extended by linearity to yield a complex linear map* $\phi : \mathcal{V}^{\mathbb{C}} \to \mathbb{C}$ *by letting* $\phi(v \otimes z) = z\phi(v)$.
- *Moreover, the previous extension results in an natural isomorphism between the two following structures*

$$(\mathcal{V}^*)^{\mathbb{C}} \simeq (\mathcal{V}^{\mathbb{C}})^*.$$

Then, the complexification $\mathcal{V}$ is a $2m$-dimensional complex space. This gives rise to the following theorem

Theorem 1 *A complex structure J on $\mathcal{V}$ is naturally extended to $\mathcal{V}^{\mathbb{C}}$ by letting $J \circ i = i \circ J$, allowing $\mathcal{V}^{\mathbb{C}}$ to be split into a direct sum of two components*

(1.1.0.8)
$$\mathcal{V}^{\mathbb{C}} = \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}, \text{ where } \begin{array}{rl} \mathcal{V}^{1,0} \text{ is the complex} \\ \text{holomorphic subspace :} & \mathcal{V}^{1,0} = \{v + iJv,\ v \in \mathcal{V}\} \\ \mathcal{V}^{0,1} \text{ is the complex} \\ \text{antiholomorphic subspace :} & \mathcal{V}^{0,1} = \{v - iJv,\ v \in \mathcal{V}\}. \end{array}$$

These are the eigenspaces of J characterized by the eigenvalues i and $-i$ respectively. Complex conjugation on $\mathcal{V}^{\mathbb{C}}$ induces an $\mathbb{R}$-isomorphism $\mathcal{V}^{1,0} \simeq \mathcal{V}^{0,1}$.

Proof. Since $\mathcal{V}^{1,0} \cap \mathcal{V}^{0,1} = \emptyset$, the canonical map $\mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1} \to \mathcal{V}^{\mathbb{C}}$ is injective. . Furthermore, the previous decomposition is an isomorphism due to the existence of the inverse map

$$v \mapsto \frac{1}{2}(v + iJv) \oplus \frac{1}{2}(v - iJv).$$

The second assertion follows from decomposing any vector $v \in \mathcal{V}^{\mathbb{C}}$ as $v = x + iy$, with $x, y \in \mathcal{V}$. Then,

$$\overline{v - iJv} = (x - iy + iJx + Jy) = \bar{v} + iJ\bar{v}.$$

Hence, complex conjugation interchanges the two factors.

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Remark. There is an alternative definition for these (anti)-holomporhic subspaces, as follows

(1.1.0.9)
$$\mathcal{V}^{\mathbb{C}} = \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}, \text{ where } \begin{array}{c} \mathcal{V}^{1,0} = \{J(v) = iv,\ v \in \mathcal{V}\} \\ \mathcal{V}^{0,1} = \{J(v) = -iv,\ v \in \mathcal{V}\}. \end{array}$$

This definition thus induces the existence of two (almost) complex structures on $\mathcal{V}^{\mathbb{C}}$. One is given by J and the other one is given by i, which coincide on $\mathcal{V}^{1,0}$ but differ by a sign on $\mathcal{V}^{0,1}$. Naturally, both $\mathcal{V}^{1,0}$ and $\mathcal{V}^{0,1}$ are complex subspaces of $\mathcal{V}^{\mathbb{C}}$ with respect to these complex structures. If $\mathcal{V}^{\mathbb{C}}$ is taken to be complex vector space with respect to i, the $\mathbb{C}$-linear extension of J is the additional structure that gives rise to the direct sum decomposition. If $\mathcal{V}^{1,0}$ and $\mathcal{V}^{0,1}$ are considered with the complex structure i, then

$$\text{the composition } \mathcal{V} \subset \mathcal{V}^{\mathbb{C}} \to \mathcal{V}^{1,0} \text{ is complex linear,}$$
$$\text{the composition } \mathcal{V} \subset \mathcal{V}^{\mathbb{C}} \to \mathcal{V}^{0,1} \text{ is complex antilinear.}$$

In the previous definition, the one being considered in theorem 1, J is regarded as an overall complex structure, taking on different eigenvalues for different subcomponents of $\mathcal{V}^{\mathbb{C}}$.

Furthermore, there is the *antilinear complex conjugate morphism*

$$v = v_1 + iv_2 \mapsto \bar{v} = v_1 - iv_2, \quad \boldsymbol{\omega} \mapsto \bar{\boldsymbol{\omega}}, \quad \begin{matrix} \boldsymbol{\omega} \in \mathcal{V}^{r,s} \\ \bar{\boldsymbol{\omega}} \in \mathcal{V}^{s,r} \end{matrix} \; r,s = 0,1, \quad \text{and s.t. } \bar{J}v = J\bar{v}. \tag{1.1.0.10}$$

From the previous remarks, it is clear that the complexification $(\mathcal{V}^*)^{\mathbb{C}}$ of the dual $\mathcal{V}^*$ of $\mathcal{V}$ is the complex dual of $\mathcal{V}^{\mathbb{C}}$. Hence, a similar decomposition to the one obtained in theorem 1 holds for the complexification of the dual space $(\mathcal{V}^*)^{\mathbb{C}}$, as follows

Theorem 2 *Let $\mathcal{V}$ be a real vector space endowed with an (almost) complex structure J. Then, the dual space $\mathcal{V}^* = \mathrm{Hom}_{\mathbb{R}}(\mathcal{V}, \mathbb{R})$ has a natural (almost) complex structurer given by $J(f)v = f(J(v))$. This induces a decomposition of the complexification of the dual space $(\mathcal{V}^{\mathbb{C}})^* = (\mathcal{V}^*)^{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(\mathcal{V}, \mathbb{C})$. Then,*

$$(\mathcal{V}^{\mathbb{C}})^* = (\mathcal{V}^{1,0})^* \oplus (\mathcal{V}^{0,1})^*, \text{ where } \begin{matrix} (\mathcal{V}^{1,0})^* \text{ is the subspace } : (\mathcal{V}^{1,0})^* = \{\omega - iJ\omega, \; \omega \in \mathcal{V}^*\} \\ (\mathcal{V}^{0,1})^* \text{ is the subspace } : (\mathcal{V}^{0,1})^* = \{\omega + iJ\omega, \; \omega \in \mathcal{V}^*\}. \end{matrix} \tag{1.1.0.11}$$

are the annihilators[2] *of $\mathcal{V}^{1,0}$ and $\mathcal{V}^{0,1}$ respectively. They are the eigenspaces of the complex structure J on $(\mathcal{V}^*)^{\mathbb{C}}$ characterized by the eigenvalues i and $-i$, respectively.*

Remark. The previous annihilators may be defined alternatively as

$$(\mathcal{V}^{\mathbb{C}})^* = (\mathcal{V}^{1,0})^* \oplus (\mathcal{V}^{0,1})^*, \text{ where } \begin{matrix} (\mathcal{V}^{1,0})^* \text{ is the subspace } : (\mathcal{V}^{1,0})^* = \{f(J(v)) = iJ(f(v)), \; f(\cdot) \in \mathcal{V}^*\} \\ (\mathcal{V}^{0,1})^* \text{ is the subspace } : (\mathcal{V}^{0,1})^* = \{f(J(v)) = -iJ(f(v)), \; f(\cdot) \in \mathcal{V}^*\}. \end{matrix}$$

noting that $(\mathcal{V}^{0,1})^* = \mathrm{Hom}_{\mathbb{C}}\Big((V, J), \mathbb{C}\Big)$.

Lemma 1. *Hence, for a real vector space $\mathcal{V}$ of dimension n, the natural decomposition of its exterior algebra is of the form*

$$\Big(\bigwedge \mathcal{V}\Big)^* = \bigoplus_{k=0}^{d} \bigwedge^{k} \mathcal{V}. \tag{1.1.0.12}$$

Similarly, $(\wedge \mathcal{V}^{\mathbb{C}})^*$ denotes the exterior algebra of the complex vector space $\mathcal{V}^{\mathbb{C}}$, which decomposes as

$$\Big(\bigwedge \mathcal{V}^{\mathbb{C}}\Big)^* = \bigoplus_{k=0}^{d} \bigwedge^{k} \mathcal{V}^{\mathbb{C}}. \tag{1.1.0.13}$$

Furthermore, $(\bigwedge \mathcal{V}^{\mathbb{C}})^* = \bigwedge \mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}$ and $(\wedge \mathcal{V})^*$ is the real subspace of $(\mathcal{V}\mathcal{V}^{\mathbb{C}})^*$ that is left invariant under complex conjugation.

If $\mathcal{V}$ is endowed with an almost complex structure J, then its real dimension d is even, and $\mathcal{V}^{\mathbb{C}}$ naturally decomposes as $\mathcal{V}^{\mathbb{C}} = \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}$, each complex vector space having dimension n.

From the preceding discussion on the complexification and the dual complexifaction of a real vector space, one can it is natural to define the complexifications of higher-order exterior products as

Definition 3. *Let $\mathcal{V}$ be a real-valued vector space imbued with a complex structure J. If $\bigwedge^{p,q} \mathcal{V}$ is the space of (p,q)-real valued tensors, then its complexification can be defined as follows*

$$\bigwedge^{p,q} \mathcal{V} \equiv \bigwedge^{p} \mathcal{V}^{1,0} \otimes_{\mathbb{C}} \bigwedge^{q} \mathcal{V}^{0,1}. \tag{1.1.0.14}$$

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[2]The annihilator of a vector subspace $\mathcal{S}$ of a vector space $\mathcal{V}$ is the set $S^0 \subset \mathcal{V}^*$ of linear functionals s.t. $f(s) = 0, \quad s \in \mathcal{S}$.

Using the preceding remarks and results, a Hermitian scalar product h on $\mathcal{V}$ can be uniquely extended to a symmetric complex J-invariant bilinear fo on $\mathcal{V}^{\mathbb{C}}$ fulliling the following conditions

- $h(\bar{v}, \bar{v}') = \overline{h(v, v')}, \quad v, v' \in \mathcal{V}^{\mathbb{C}}$.
- $h(v, \bar{v}) > 0, \quad v \in \mathcal{V}^{\mathbb{C}} - \{\mathbf{0}\}$.
- $h(v, v') = 0$ if v, v' are simultaneously holomorphic or antiholomorphic.

This complex bilinear form, thus, induces a non-degenerate Hermitian form on $\mathcal{V}^{\mathbb{C}}$, as follows

$$\langle v | v' \rangle_h \equiv h(v, \bar{v}'),$$

s.t. the holomorphic and antiholomorphic spaces are mutually orthogonal. Accordingly, the skew-symmetric form Ω, introduced in equation (1.1.0.7), can be extendd to $\mathcal{V}^C$ so that

$$\begin{gathered} \Omega(\bar{v}, \bar{v}') = \overline{\Omega(v, v')}, \quad v, v' \in \mathcal{V}^{\mathbb{C}} \\ \Omega(v, v') = 0, \quad v, v' \in \mathcal{V}^{r,s}, \quad r, s = 0, 1. \end{gathered} \tag{1.1.0.15}$$

1.2. **Almost-complex manifolds.** Let Z be a $2m$-dimensional smooth real-valued manifold, with coordinate basis $(z^i)_{i=1}^{2m}$.

Definition 4. *An almost-complex structure on Z is defined as a* ***vertical bundle automorphism*** $J : Z \to Z$ *on the tangent bundle TZ s.t.*

$$J \circ J = -\mathfrak{id}_{TZ}.$$

The following statements immediately follow from the previous definition.

Clearly if J is an almost-complex structure, then $J \in \mathrm{GL}(Z)$. Moreover, if Z is the real vector space underlying a complex vector space then $v \mapsto i \cdot v$ defines an almost complex structure J on Z. The converse holds true as well.

Lemma 2. *If J is an almost complex structure on a real vector space Z, then Z admits in a natural way the structure of a complex vector space.*

Proof. In effect, the $\mathbb{C}$-module structure on J is defined as $(a+ib)\cdot v = a\cdot v + b\cdot Jv$ with $a, b \in \mathbb{R}$. The $\mathbb{R}$-linearity of J and the assumption $J^2 = -\mathfrak{id}$ yields that

$$(a+ib)(c+id)\cdot v = (a+ib)\Big((c+id)\cdot v\Big), \quad i(i\cdot v) = -v.$$

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Hence, almost complex structures and complex structures are equivalent notions for vector spaces. Moreover, an almost complex structure can only exist on an even dimensional real vector space.

Lemma 3. *Any almost complex structure J on Z induces a natural orientation on Z.*

Hence, the almost-complex structure J on Z can be represented by a tangent-valued form on Z, as follows

$$J = J^i_k dz^k \otimes \partial_i, \quad \text{s.t.} \quad J^i_k J^k_j = -\delta^i_j. \tag{1.2.0.1}$$

This tangent-valued form defines an automorphism J on the cotangent bundle T^*Z of Z s.t.

$$\langle v, J\omega\langle = \langle Jv, \omega\langle, \quad \begin{matrix} v \in T_z Z \\ \omega \in T^*_z Z \end{matrix}, \quad z \in Z.$$

Furthermore, an almost-complex structure provides Z with an orientation, associated with the **adapted fibre bases** for TZ. The pair (Z, J) is then called an *almost-complex manifold*. A differrmorphism $f : Z \to Z'$ preserves an almost complex structure J on Z if and only if the tangent morphism Tf commutes with J.

References