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Condensed Matter Theory - Lecture Notes
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**Theory & Notes**

## 1. Elements of Matrix Analysis

Consider a linear operator, an endomorphism on $\mathbb{C}^{n\times n}$, $\mathbf{A} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$. Its spectrum is defined as the set of eigenvalues

$$\sigma(\mathbf{A}) = \{\lambda \in \mathbb{C} \mid \ker(A - \lambda \mathbb{1}_n) \neq 0\},$$

which is a finite, non-empty $\mathbb{C}$-subset. Then, the following statements hold

- $\lambda \in \sigma(\mathbf{A}) \leftrightarrow \exists x \in \mathbb{C}^n \mid x \neq 0 \wedge \mathbf{A}x = \lambda x$.
- $\forall \mu \in \mathbb{C}, \sigma(\mathbf{A} + \mu \mathbb{1}) = \sigma(\mathbf{A}) + \mu = \{\lambda + \mu \mid \sigma(\mathbf{A})\}$.
- $\mathbf{A} \in \mathrm{GL}(n, \mathbb{C}) \leftrightarrow 0 \notin \sigma(\mathbf{A})$. Moreover, $\lambda \notin \sigma(\mathbf{A}) \leftrightarrow \mathbf{A} - \lambda \mathbb{1}_n \in \mathrm{GL}(n, \mathbb{C})$.
- If $P_{\mathbf{A}}(x) \in \mathbb{C}[x]$ is $\mathbf{A}$'s characteristic polynomial, then $\lambda \in \sigma(\mathbf{A}) \leftrightarrow P_{\mathbf{A}}(\lambda) = 0$ ie. $\sigma(\mathbf{A})$ is $P_{\mathbf{A}}(x)$'s zeros-set.
- Since $\mathrm{gr}(P_{\mathbf{A}}) = n$, then $0 < |\sigma(\mathbf{A})| \leq n$.
- $\sigma(\mathbf{A}^\dagger) = \sigma(\mathbf{A})^*$ In effect, if
$$\mathbf{A} - \lambda \mathbb{1}_n \notin \mathrm{GL}(n, \mathbb{C}) \to (\mathbf{A} - \lambda \mathbb{1})^\dagger = \mathbf{A}^\dagger - \lambda^* \mathbb{1}_n \notin \mathrm{GL}(n, \mathbb{C}).$$
- If $\mathbf{A} \in \mathrm{GL}(n, \mathbb{C}) \Rightarrow \sigma(\mathbf{A}^{-1}) = \sigma(\mathbf{A})^{-1} = \{\lambda^{-1} : \lambda \in \sigma(\mathbf{A})\}$.

Now, let $\mathbf{A} \in \mathbb{C}^{n\times n}$, then

1) the numerical radius is defined as
$$w(\mathbf{A}) = \max_{x \in \mathbb{C}^n : ||x|| = 1} |\langle \mathbf{A}x, x \rangle|.$$
2) The spectral radius is defined as
$$\rho(\mathbf{A}) = \max_{\lambda \in \sigma(\mathbf{A})} |\lambda|.$$
3) The spectral norm of $\mathbf{A}$ is its operator norm, said norm being induced by the euclidean norm on $\mathbb{C}^n$, this is
$$||\mathbf{A}||_{\text{sp}} = \max_{x \in \mathbb{C}^n : ||x|| = 1} ||\mathbf{A}x|| = \min_{x \in \mathbb{C}^n, C \geq 0} ||\mathbf{A}x|| \leq C||x||.$$
4) The 2-norm or the Frobenius-norm of $\mathbf{A}$ is its euclidean norm, induced by thinking of $\mathbf{A}$ as a $2n$-dimensional vector,
$$||\mathbf{A}||_2^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 = \mathrm{tr}\big(\mathbf{A}^\dagger \mathbf{A}\big).$$

Given an operator $\mathbf{A}$, from its norm-one eigenvectors, it is clear that

$$\rho(\mathbf{A}) \leq w(\mathbf{A}) \leq ||\mathbf{A}||_{\text{sp}}$$

## 2. Essentials of Information Entropy and Related Measures

Rossignoli, Kowalski, Curado (2013)

**Shannon Entropy**. Consider a probability distribution given by

(1) $$\mathfrak{p} = \{p_i\}_{i=1}^{n} \text{ such that } \begin{array}{l} \bullet\ p_i \geq 0 \\ \bullet\ \sum_{i=1}^{n} p_i = 1 \end{array}$$

where $p_i$ indicates the probability of a certain event $i$ in a random experiment. The Shannon entropy is a mesaure of the lack of information associated with the probability distribution and is defined as

(2) $$S(\mathfrak{p}) = -\sum_{i=1}^{n} p_i \log p_i,$$

where the most common choice for the logarithm base is $a = 2$, with the unit of information being the bit. In this case, if $\mathfrak{p} = \left(\frac{1}{2}, \frac{1}{2}\right)$ ie. for an experiment with just two possible and equally likely outcomes. Said quantity is a measure of the lack of information associated with the discrete probability distribution $\mathfrak{p}$, quantifying the uncertainty about the possible outcome of the random experiment. It can also be considered as the average information gained once the outcome is known, as well as a measure of the disorder associated with $\mathfrak{p}$. It satisfies that $S(\mathfrak{p}) \geq 0$, where the lower bound occurs if and only if there is no uncertainty, ie. there just a single event occurring with probability 1, and all others with zero probability, this is

(3) $$S(\mathfrak{p}) = 0 \text{ if and only if } p_i = \delta_{ij}$$

## 3. Theory of Open Quantum Systems

**Probability measures on a Hilbert space**. Consider a fixed orthonormal basis $\{\phi_n\} \subset \mathbb{H}$, then any other $\psi \in \mathbb{H}$ may be decomposed as

$$\psi = \sum_n z_n \phi_n.$$

The probability density functional $P = P[\psi]$ may be regarded as a function $P = P[z_n, z_n^*]$ on the $\mathbb{C}$-variables $z_n, z_n^*$. Alternatively, it can be regarded as function $P = P[\mathbf{a}_n, \mathbf{b}_n]$, wherein

$$z_n = \mathbf{a}_n + i\mathbf{b}_n.$$

An appropriate expression for the volume element in Hilbert space can be found as the usual Euclidean volume element in a real space, with coordinate atlas given by $(\mathbf{a}_n, \mathbf{b}_n)$, that is

$$D\psi D\psi^* = \prod_n d\mathbf{a}_n d\mathbf{b}_n, \text{ where } \begin{array}{c} d\mathbf{a}_n = \frac{1}{2}(dz_n + dz_n^*) \\ d\mathbf{b}_n = \frac{1}{2i}(dz_n - dz_n^*) \end{array} \Rightarrow D\psi D\psi^* = \prod_n \frac{i}{2} dz_n dz_n^*.$$

Then, a functional integration on the Hilbert space can be written as

$$\int_A D\psi D\psi^* P[\psi] = \int_A \prod_n \frac{i}{2} dz_n dz_n^*.$$

This functional volume element on the Hilbert space is invariant under linear unitary transformations

$$\psi \to U\psi \Rightarrow D\psi D\psi^* \to D\psi' D\psi'^*.$$

In effect, the unitary transformation $U \in U(N)$ may be decomposed into its real and imaginary parts,

$$U = \mathfrak{R}(U) + I\mathfrak{I}(U).$$

The unitary of $U$ leads to the following relations

$$\mathfrak{R}(U)\mathfrak{R}(U)^{\mathrm{T}} + \mathfrak{I}(U)\mathfrak{I}(U)^{\mathrm{T}} = \mathbb{1}_N, \tag{4}$$

$$\mathfrak{I}(U)\mathfrak{R}(U)^{\mathrm{T}} - \mathfrak{R}(U)\mathfrak{I}(U)^{\mathrm{T}} = 0. \tag{5}$$

In the chosen representation, the $U$-matrix describes a unitary transformation $z_n \to z_n'$, from the coefficients $z_n$ in the $\psi_n$-decomposition to $z_n'$-coefficients in the $\psi'$-basis decomposition. The corresponding transformation of the real coefficients $\mathbf{a}_n, \mathbf{b}_n$ defined by $z_n = \mathbf{a} + i\mathbf{b}_n$ is provided by the real matrix

$$\tilde{U} = \begin{pmatrix} \mathfrak{R}(U) & -\mathfrak{I}(U) \\ \mathfrak{I}(U) & \mathfrak{R}(U) \end{pmatrix},$$

which is an orthogonal matrix satisfying $|\det \tilde{U}| = 1$. Thus, as it was expected, the unitary transformation $U$ on the Hilbert space $\mathbb{H}$ induces an orthogonal transformation $\tilde{U}$ on the $\mathbb{R}$-variables, $\mathbf{a}_n, \mathbf{b}_n$, which were introduced to define a volume element in a Hilbert space. The transformation formula for multidimensional integrals conclude that

$$\prod_n d\mathbf{a}_n' d\mathbf{b}_n' = |\det \tilde{U}| \prod_n d\mathbf{a}_n d\mathbf{b}_n = \prod_n d\mathbf{a}_n d\mathbf{b}_n,$$

thus proving the unitary invariance of the volume element.