# Trigonometric and elliptic Selberg integrals

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**Beta function** $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \quad (\mathrm{Re}\,\alpha>0,\ \mathrm{Re}\,\beta>0)$

**Gamma function** $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx \quad (\mathrm{Re}\,\alpha>0) \quad \Longrightarrow \quad \boxed{B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$

This relation is extended to the following cases of multiple integrals.

**Dixon–Anderson integral (Type I) [Dixon 1905, Anderson 1991]**

$$\int_{z_n=x_{n-1}}^{x_n}\cdots\int_{z_2=x_1}^{x_2}\int_{z_1=x_0}^{x_1}\prod_{i=1}^{n}\prod_{j=0}^{n}|z_i-x_j|^{s_j-1}\prod_{1\le k<l\le n}(z_l-z_k)\,dz_1dz_2\cdots dz_n=\frac{\Gamma(s_0)\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(s_0+s_1+\cdots+s_n)}\prod_{0\le i<j\le n}(x_j-x_i)^{s_i+s_j-1}$$

**Selberg integral (Type II) [Selberg, 1942]**

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{n}z_i^{\alpha-1}(1-z_i)^{\beta-1}\prod_{1\le i<j\le n}|z_i-z_j|^{2\tau}\,dz_1dz_2\cdots dz_n=\prod_{j=1}^{n}\frac{\Gamma(\alpha+(j-1)\tau)\Gamma(\beta+(j-1)\tau)\Gamma(j\tau+1)}{\Gamma(\alpha+\beta+(n+j-2)\tau)\Gamma(\tau+1)}$$

Each integral can be written as a product of gamma functions. When $n=1$, both integrals coincide with the ordinary Beta function. Chapter 4 of Peter's book [Log-Gases and Random Matrices] is devoted to how to prove the above relation of Selberg integral.

various derivations of Selberg's formula are presented.

Set
$$S_n(\alpha,\beta,\tau) := \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} z_i^{\alpha-1}(1-z_i)^{\beta-1} \prod_{1\le i<j\le n} |z_i - z_j|^{2\tau}\, dz_1 \cdots dz_n.$$

Then the following two are typical methods:

**Anderson's derivation**

A method for calculating a multiple integral of $2n+1$ dimension in two ways [Fubini's theorem] to obtain a relation of $S_n(\alpha,\beta,\tau)$ with respect to $n$ (two-term relation between $S_n$ and $S_{n+1}$). The Dixon–Anderson integral (Type I) is used in the process.

**Aomoto's derivation** ($\leftarrow$ this talk)

**1. Difference equation** Derive the difference equations for the parameters that $S_n(\alpha,\beta,\tau)$ satisfies (In this case two-term relation is obtained).

**2. Asymptotic behavior** As a boundary condition, determine the asymptotic behavior as parameters go to infinity. (The steepest descent method (saddle point method), etc, of integration is used.)

Please imagine $n=1$ case. During a calculus class we learned about two derivations of the relation between Beta function and Gamma function. One is to use a method for calculating a double integral in two ways. This is similar to Anderson's derivation. The other is to derive the difference equation using Integration by Parts. This is corresponding to Aomoto's derivation.

## q-Analog of Selberg integral

The $q$-analog ($q$-deformation) of Dixon–Anderson integral and Selberg integral are given by

**$q$-Dixon–Anderson integral (Type I) [Evans 1992] (See [I.–Forrester, J. Math. Anal. Appl, 2015])**

$$\int_{z_n=x_{n-1}}^{x_n}\cdots\int_{z_2=x_1}^{x_2}\int_{z_1=x_0}^{x_1}\prod_{i=1}^{n}\prod_{j=0}^{n}\frac{(qz_i/x_j;q)_\infty}{(q^{s_j}z_i/x_j;q)_\infty}\prod_{1\le i<j\le n}(z_j-z_i)\,d_qz_1d_qz_2\cdots d_qz_n$$
$$=\frac{\Gamma_q(s_0)\Gamma_q(s_1)\cdots\Gamma_q(s_n)}{\Gamma_q(s_0+s_1+\cdots+s_n)}\prod_{0\le i<j\le n}(x_j-x_i)\frac{(qx_i/x_j;q)_\infty(qx_j/x_i)_\infty}{(q^{s_j}x_i/x_j;q)_\infty(q^{s_i}x_j/x_i;q)_\infty}.$$

**$q$-Selberg integral (Type II) [Askey, et al, 1980–90's, Aomoto 1998] (See [I.–Forrester, Trans. AMS, 2017])**

$$\int_{z_1=0}^{1}\int_{z_2=0}^{q^\tau z_1}\cdots\int_{z_n=0}^{q^\tau z_{n-1}}\prod_{i=1}^{n}z_i^\alpha\frac{(qz_i;q)_\infty}{(q^\beta z_i;q)_\infty}\prod_{1\le j<k\le n}z_j^{2\tau-1}\frac{(q^{1-\tau}z_k/z_j;q)_\infty}{(q^\tau z_k/z_j;q)_\infty}(z_j-z_k)\frac{d_qz_n}{z_n}\cdots\frac{d_qz_1}{z_1}$$
$$=q^{\alpha\tau\binom{n}{2}+2\tau^2\binom{n}{3}}\prod_{j=1}^{n}\frac{\Gamma_q(\alpha+(j-1)\tau)\Gamma_q(\beta+(j-1)\tau)\Gamma_q(j\tau)}{\Gamma_q(\alpha+\beta+(n+j-2)\tau)\Gamma_q(\tau)}$$

When $q\to 1$, the above $q$-integrals degenerate to the ordinary integrals, respectively. I will explain how to prove the q-Selberg integral formula using $q$-version of Aomoto's derivation when $n=1$. Even in this simplest case, if you settled this, you can easily analogize the situation to general $n$ case.
On the next slide I will explain the case of $q$-beta function.

# Notation

Set $\mathbb{C}^* = \mathbb{C} - \{0\}$.

## Definition ($q$-shifted factorials)

Fix $q \in \mathbb{C}^*$ as $|q| < 1$. We use the symbols $(x;q)_\infty$, $(x;q)_n$ as

$$(x;q)_\infty = \prod_{i=0}^{\infty}(1-q^i x) = (1-x)(1-qx)(1-q^2x)\cdots.$$

$$(x;q)_n = \frac{(x;q)_\infty}{(q^n x;q)_\infty} = \begin{cases} (1-x)(1-qx)\cdots(1-q^{n-1}x) & (n=1,2,\cdots) \\ 1 & (n=0) \\ \dfrac{1}{(1-q^{-n}x)(1-q^{-(n-1)}x)\cdots(1-q^{-1}x)} & (n=-1,-2,\cdots) \end{cases}.$$

We also use the symbol

$$(x_1, x_2, \ldots, x_m; q)_\infty = (x_1;q)_\infty (x_2;q)_\infty \cdots (x_m;q)_\infty$$

for abbriviation.

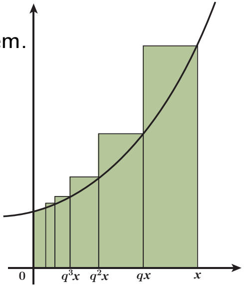
**$q$-binomial theorem** $\displaystyle\sum_{\nu=0}^{\infty} \frac{(q^\alpha;q)_\nu}{(q;q)_\nu} x^\nu = \frac{(q^\alpha x;q)_\infty}{(x;q)_\infty} \quad \xrightarrow[q\to 1]{} \quad \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu}{\nu!} x^\nu = \frac{1}{(1-x)^\alpha} \quad |x|<1$

where $(\alpha)_\nu := \alpha(\alpha+1)\cdots(\alpha+\nu-1)$

**Jackson integral**

$$\int_0^x f(z)d_qz := (1-q)\sum_{\nu=0}^{\infty} q^\nu z\, f(q^\nu z), \text{ which is equivalent to } \int_0^x f(z)\frac{d_qz}{z} = (1-q)\sum_{\nu=0}^{\infty} f(q^\nu z).$$

$$\boxed{\frac{(q^\alpha z;q)_\infty}{(q^\beta z;q)_\infty} \underset{q\to1}{\longrightarrow} (1-z)^{\beta-\alpha}}$$ by $q$-binomial theorem.




**Andrews' $q$-Beta function**

$$B_q(\alpha,\beta) := \int_0^1 z^\alpha \frac{(qz;q)_\infty}{(q^\beta z;q)_\infty}\frac{d_qz}{z} \underset{q\to1}{\longrightarrow} \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz$$

**Askey's $q$-Gamma function** $\quad \Gamma_q(\alpha) := \dfrac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha} \underset{q\to1}{\longrightarrow} \Gamma(\alpha)$

satisfies

$$\Gamma_q(\alpha+1) = \frac{1-q^\alpha}{1-q}\Gamma_q(\alpha) \left(\text{In particular, if } n\in\mathbb{N}, \text{ then } \Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n} \underset{q\to1}{\longrightarrow} n!\right)$$

$$\boxed{\textbf{Theorem} \quad B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}} \quad \textbf{LHS} = (1-q)\sum_{\nu=0}^{\infty} q^{\nu\alpha}\frac{(q^{1+\nu};q)_\infty}{(q^{\beta+\nu};q)_\infty}, \quad \textbf{RHS} = (1-q)\frac{(q^{\alpha+\beta},q;q)_\infty}{(q^\alpha,q^\beta;q)_\infty}$$

Let us prove this formula using Aomoto's derivation.

# Mellin's method (1907) for hypergeometric function ${}_2F_1$

The idea behind the $q$-version of Aomoto's derivation is based on Mellin's derivation, which is a method for deriving the differential equation that the hypergeometric series ${}_2F_1$ sarisfies.

**Gauss's hypergeometric series**

$$\boxed{{}_2F_1(\alpha,\beta;\gamma;x) = \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu(\beta)_\nu}{(\gamma)_\nu\ \nu!} x^\nu \quad |x|<1}$$

If we set $y = {}_2F_1(\alpha,\beta;\gamma;x)$, then $y$ satisfies

**Gauss's hypergeometric differential equation**

$$x(1-x)\frac{d^2y}{dx^2} + (\gamma-(\alpha+\beta+1)x)\frac{dy}{dx} - \alpha\beta y = 0.$$

Let us confirm this fact following Mellin's derivation.

If we set $\Phi(z) := x^z \dfrac{\Gamma(\alpha+z)\Gamma(\beta+z)}{\Gamma(\gamma+z)\Gamma(1+z)}$, then $\Phi(\nu)=0$ if $\nu=-1,-2,-3,\cdots$.

Moreover $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ implies $(\alpha)_\nu = \dfrac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)}$. Then $\boxed{{}_2F_1(\alpha,\beta;\gamma;x) = \dfrac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\sum_{\nu=-\infty}^{\infty} \Phi(\nu)}$

This series can be formally regarded as a bilateral series (i.e. a sum over $\mathbb{Z}$).

For any point $\xi \in \mathbb{C}$ and any meromorphic function $\varphi(z)$ on $\mathbb{C}$, we set

$$\boxed{\langle \varphi, \xi \rangle := \sum_{\nu=-\infty}^{\infty} \varphi(\xi+\nu)\Phi(\xi+\nu)} \quad \text{c.f.} \quad {}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\langle 1, 0 \rangle.$$

If we set $(\nabla\varphi)(z) := \varphi(z) - \dfrac{\Phi(z+1)}{\Phi(z)}\varphi(z+1) = \varphi(z) - x\dfrac{(\alpha+z)(\beta+z)}{(\gamma+z)(1+z)}\varphi(z+1)$,

then $\boxed{\textbf{Key Lemma } \langle \nabla\varphi, \xi \rangle = 0}$ holds.

$$\left(\text{Key Lemma is equivalent to } \sum_{\nu=-\infty}^{\infty} \varphi(\xi+\nu)\Phi(\xi+\nu) = \sum_{\nu=-\infty}^{\infty} \varphi(\xi+\nu+1)\Phi(\xi+\nu+1).\right)$$

In particular, if we take $\varphi(z) = (\gamma-1+z)z$, then $\nabla\varphi(z) = (\gamma-1+z)z - x(\alpha+z)(\beta+z)$.

Taking account of $x\dfrac{d}{dx}\Phi(z) = z\Phi(z)$, we see the fact $x\dfrac{d}{dx}\langle f, \xi \rangle = \langle z\, f, \xi \rangle$. Therefore

$$\boxed{\left(\gamma-1+x\frac{d}{dx}\right)x\frac{d}{dx}\langle 1, \xi \rangle - x\left(\alpha+x\frac{d}{dx}\right)\left(\beta+x\frac{d}{dx}\right)\langle 1, \xi \rangle = 0}$$

This equation coincides with Gauss's hypergeometric differential equation.

# Extended Jackson integral ($q$-version of Aomoto's derivation)

$$\boxed{\int_0^{\xi\infty} f(z)\frac{d_qz}{z} := (1-q)\sum_{\nu=-\infty}^{\infty} f(q^\nu\xi)}$$

If the right-hand side converges, we call it the extended Jackson integral.

$$B_q(\alpha,\beta;\xi) := \int_0^{\xi\infty}\Phi(z)\frac{d_qz}{z},$$

where $\Phi(z) := z^\alpha\dfrac{(qz;q)_\infty}{(q^\beta z;q)_\infty}$

In particular, if $\xi=1$, then $B_q(\alpha,\beta;1) := \displaystyle\int_0^1\Phi(z)\frac{d_qz}{z} = B_q(\alpha,\beta)$.

We set $\nabla\varphi(z) := \varphi(z) - \dfrac{\Phi(qz)}{\Phi(z)}\varphi(qz) = \varphi(z) - q^\alpha\dfrac{1-q^\beta z}{1-qz}\varphi(qz)$

and $\langle\varphi,\xi\rangle := \displaystyle\int_0^{\xi\infty}\varphi(z)\Phi(z)\frac{d_qz}{z}$. In particular, if $\varphi(z)=1$, then $\langle 1,\xi\rangle = B_q(\alpha,\beta;\xi)$.

$\boxed{\textbf{Key Lemma } \langle\nabla\varphi,\xi\rangle = 0}$ holds. $\left(\text{Key Lemma is equivalent to } \displaystyle\int_0^{\xi\infty}\varphi(z)\Phi(z)\frac{d_qz}{z} = \int_0^{\xi\infty}\varphi(qz)\Phi(qz)\frac{d_qz}{z}.\right)$

$$\varphi(z)=1-z \implies \nabla\varphi(z) = (1-z) - q^\alpha\frac{1-q^\beta z}{1-qz}(1-qz) = (1-z) - q^\alpha(1-q^\beta z) = (1-q^\alpha) - (1-q^{\alpha+\beta})z.$$

From Key Lemma, we have $(1-q^\alpha)\langle 1,\xi\rangle - (1-q^{\alpha+\beta})\langle z,\xi\rangle = 0$.

Since $B_q(\alpha+1,\beta;\xi) = \displaystyle\int_0^{\xi\infty} z\Phi(z)\frac{d_qz}{z} = \langle z,\xi\rangle$, we obtain $\boxed{B_q(\alpha,\beta;\xi) = \dfrac{1-q^{\alpha+\beta}}{1-q^\alpha}B_q(\alpha+1,\beta;\xi)}$.

**$q$-difference equation**

# Boundary condition (Asymptotic behavior)

$\boxed{B_q(\alpha,\beta;\xi) = \dfrac{1-q^{\alpha+\beta}}{1-q^{\alpha}} B_q(\alpha+1,\beta;\xi)}$ This relation is independent of the choice of $\xi$. In particular, if we put $\xi = 1$, then we have

$$B_q(\alpha,\beta) = \frac{1-q^{\alpha+\beta}}{1-q^{\alpha}} B_q(\alpha+1,\beta).$$

Repeated use of this $q$-difference equation

$$\begin{aligned} B_q(\alpha,\beta) &= \frac{1-q^{\alpha+\beta}}{1-q^{\alpha}} B_q(\alpha+1,\beta) = \frac{(1-q^{\alpha+\beta})(1-q^{\alpha+\beta+1})}{(1-q^{\alpha})(1-q^{\alpha+1})} B_q(\alpha+2,\beta) = \cdots = \frac{(q^{\alpha+\beta};q)_N}{(q^{\alpha};q)_N} B_q(\alpha+N,\beta) \\ &= \frac{(q^{\alpha+\beta};q)_\infty}{(q^{\alpha};q)_\infty} \lim_{N\to\infty} B_q(\alpha+N,\beta). \end{aligned}$$

On the other hand, by definition,

$$B_q(\alpha,\beta) = \int_0^1 \Phi(z)\frac{d_qz}{z} = (1-q)\sum_{\nu=0}^{\infty}\Phi(q^{\nu}) = (1-q)\left(\frac{(q;q)_\infty}{(q^{\beta};q)_\infty} + q^{\alpha}\frac{(q^2;q)_\infty}{(q^{\beta+1};q)_\infty} + q^{2\alpha}\frac{(q^3;q)_\infty}{(q^{\beta+2};q)_\infty} + \cdots\right),$$

$$\lim_{N\to\infty} B_q(\alpha+N,\beta) = (1-q)\frac{(q;q)_\infty}{(q^{\beta};q)_\infty}.$$

Therefore

$$B_q(\alpha,\beta) = \frac{(q^{\alpha+\beta};q)_\infty}{(q^{\alpha};q)_\infty} \lim_{N\to\infty} B_q(\alpha+N,\beta) = \frac{(q^{\alpha+\beta};q)_\infty}{(q^{\alpha};q)_\infty} \times (1-q)\frac{(q;q)_\infty}{(q^{\beta};q)_\infty} = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}. \quad \textbf{Q.E.D.}$$

# $q$-Integrals associated with root systems

$q$-Dixon–Anderson integral and $q$-Selberg integral can be extended to the following two contour integrals.

## $BC_n$ Type I integral ($2n+2$ parameters) [Gustafson,1994]

$$\frac{1}{(2\pi\sqrt{-1})^n}\int\cdots\int_{\mathbb{T}^n}\prod_{i=1}^{n}\frac{(z_i^2,z_i^{-2};q)_\infty}{\prod_{k=1}^{2n+2}(a_kz_i,a_kz_i^{-1};q)_\infty}\prod_{1\le i<j\le n}(z_iz_j,z_iz_j^{-1},z_i^{-1}z_j,z_i^{-1}z_j^{-1};q)_\infty\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$=\frac{2^n n!}{(q;q)_\infty^n}\frac{(a_1a_2\cdots a_{2n+2};q)_\infty}{\prod_{1\le i<j\le 2n+2}(a_ia_j;q)_\infty},$$

where $\mathbb{T}^n$ is the $n$-fold direct product of the unit circle

## $BC_n$ Type II integral ($4+1$ parameters) [Gustafson,1994]

$$\frac{1}{(2\pi\sqrt{-1})^n}\int\cdots\int_{\mathbb{T}^n}\prod_{i=1}^{n}\frac{(z_i^2,z_i^{-2};q)_\infty}{\prod_{k=1}^{4}(a_kz_i,a_kz_i^{-1};q)_\infty}\prod_{1\le i<j\le n}\frac{(z_iz_j,z_iz_j^{-1},z_i^{-1}z_j,z_i^{-1}z_j^{-1};q)_\infty}{(t\,z_iz_j,t\,z_iz_j^{-1},t\,z_i^{-1}z_j,t\,z_i^{-1}z_j^{-1};q)_\infty}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$=\frac{2^n n!}{(q;q)_\infty^n}\prod_{k=1}^{n}\frac{(t;q)_\infty}{(t^k;q)_\infty}\frac{(a_1a_2a_3a_4t^{2n-k-1};q)_\infty}{\prod_{1\le i<j\le 4}(a_ia_jt^{k-1};q)_\infty}$$

Using residue calculation the above contour integrals can be convert to Jackson integral representations. The integrand of type II defines the orthogonal inner product of the Macdonald–Koornwinder polynomials.

$n=1\Rightarrow$ Askey–Wilson integral $$\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{(z^2;q)_\infty(z^{-2};q)_\infty}{\prod_{k=1}^{4}(a_kz;q)_\infty(a_kz^{-1};q)_\infty}\frac{dz}{z}=\frac{2}{(q;q)_\infty}\frac{(a_1a_2a_3a_4;q)_\infty}{\prod_{1\le i<j\le 4}(a_ia_j;q)_\infty}.$$

# Examples (other root systems)

## $G_2$ Type I integral (4 parameters) [Gustafson,1994]

Suppose that $a_k \in \mathbb{C}^*$ $(1 \le k \le 4)$ satisfy $|a_k| < 1$. Then we have

$$\frac{(q;q)_\infty^2}{12\,(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\prod_{i=1}^{3}\frac{(x_i, x_i^{-1};q)_\infty}{\prod_{k=1}^{4}(a_k x_i, a_k x_i^{-1};q)_\infty}\prod_{1\le j<k\le 3}(x_j x_k^{-1}, x_k x_j^{-1};q)_\infty \frac{dx_1}{x_1}\frac{dx_2}{x_2} \quad (\text{where} \quad x_3 = x_1^{-1}x_2^{-1})$$

$$= \frac{(a_1^2a_2^2a_3^2a_4^2;q)_\infty}{(a_1a_2a_3a_4;q)_\infty}\prod_{i=1}^{4}\frac{(a_i;q)_\infty}{(a_i^2;q)_\infty}\prod_{1\le i<j\le 4}\frac{1}{(a_ia_j;q)_\infty}\prod_{1\le i<j<k\le 4}\frac{1}{(a_ia_ja_k;q)_\infty}.$$

## $G_2$ Type II integral $(1+1$ parameters) [Habsieger,1986]

Suppose that $a, t \in \mathbb{C}^*$ satisfy $|a| < 1$ and $|t| < 1$. Then we have

$$\frac{(q;q)_\infty^2}{12\,(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\prod_{i=1}^{3}\frac{(x_i, x_i^{-1};q)_\infty}{(a\,x_i, a\,x_i^{-1};q)_\infty}\prod_{1\le j<k\le 3}\frac{(x_jx_k^{-1}, x_kx_j^{-1};q)_\infty}{(t\,x_jx_k^{-1}, t\,x_kx_j^{-1};q)_\infty}\frac{dx_1}{x_1}\frac{dx_2}{x_2} \quad (\text{where} \quad x_3 = x_1^{-1}x_2^{-1})$$

$$= (qat^2, qa^2t^3;q)_\infty\frac{(a;q)_\infty}{(a^2;q)_\infty}\frac{(at;q)_\infty}{(a^3t^3;q)_\infty}\frac{(t, qt;q)_\infty}{(t^2, qt^3;q)_\infty}$$

Regarding $F_4$, there also exist Type I and Type II integrals, and the results are similar to those for $G_2$ case.

We fix $p, q \in \mathbb{C}^*$ as $|p| < 1$ and $|q| < 1$. **The Ruijsenaars elliptic gamma function** is defined as

$$\Gamma(x;p,q) = \frac{(pqx^{-1};p,q)_\infty}{(x;p,q)_\infty}, \quad \text{where} \quad (x;p,q)_\infty = \prod_{\mu,\nu=0}^{\infty}(1-p^\mu q^\nu x).$$

We also use the notation $\Gamma(x_1,\dots,x_m;p,q) = \Gamma(x_1;p,q)\cdots\Gamma(x_m;p,q)$. Note that $\Gamma(x;p,q)$ satisfies

$$\boxed{\Gamma(qx;p,q) = \theta(x;p)\Gamma(x;p,q) \quad \text{and} \quad \Gamma(px;p,q) = \theta(x;q)\Gamma(x;p,q),}$$

where $\theta(x;p) = (x,p/x;p)_\infty$ is a theta function satisfying $\theta(px;p) = -\theta(x;p)/u$, and also satisfies

$$\Gamma(pqx^{-1};p,q) = \frac{1}{\Gamma(x;p,q)}, \quad \frac{1}{\Gamma(x,x^{-1};p,q)} = -x^{-1}\theta(x;p)\theta(x;q).$$

**Remark.** $\boxed{\textbf{Elliptic} \xrightarrow[p\to 0]{} \textbf{Trigonometric } (q\textbf{-analog}) \xrightarrow[q\to 1]{} \textbf{Rational (Classical)}}$

$$\Gamma(x;p,q) = \frac{(pqx^{-1};p,q)_\infty}{(x;p,q)_\infty} \xrightarrow[p\to 0]{} \frac{1}{(x;q)_\infty}$$

$$\Gamma(px;p,q) = \frac{(qx^{-1};p,q)_\infty}{(px;p,q)_\infty} \xrightarrow[p\to 0]{} (qx^{-1};q)_\infty$$

## Elliptic analog of $BC_n$ Type I and Type II integrals

Under the balancing condition $a_1 \cdots a_{2n+4} = pq$,

$$\frac{1}{(2\pi\sqrt{-1})^n} \int \cdots \int_{\mathbb{T}^n} \prod_{i=1}^{n} \frac{\prod_{m=1}^{2n+4} \Gamma(a_m z_i, a_m z_i^{-1}; p, q)}{\Gamma(z_i^2, z_i^{-2}; p, q)} \prod_{1 \le j < k \le n} \frac{1}{\Gamma(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}$$
$$= \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{1 \le j < k \le 2n+4} \Gamma(a_j a_k; p, q),$$

where $a_1, \ldots, a_{2n+4}$ are complex parameters with $|a_m| < 1$ $(m = 1, \ldots, 2n+4)$. [Rains 2010]

Under the balancing condition $a_1 \cdots a_6 t^{2n-2} = pq$,

$$\frac{1}{(2\pi\sqrt{-1})^n} \int \cdots \int_{\mathbb{T}^n} \prod_{i=1}^{n} \frac{\prod_{m=1}^{6} \Gamma(a_m z_i, a_m z_i^{-1}; p, q)}{\Gamma(z_i^2, z_i^{-2}; p, q)} \prod_{1 \le j < k \le n} \frac{\Gamma(t z_j z_k, t z_j z_k^{-1}, t z_j^{-1} z_k, t z_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}$$
$$= \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{i=1}^{n} \left( \frac{\Gamma(t^i; p, q)}{\Gamma(t; p, q)} \prod_{1 \le j < k \le 6} \Gamma(t^{i-1} a_j a_k; p, q) \right),$$

where $a_1, \ldots, a_6, t$ are complex parameters with $|a_m| < 1$ $(m = 1, \ldots, 6)$, $|t| < 1$.
[van Diejen–Spiridonov 2001, Rains 2010 (Anderson method) I.–Noumi 2017 (Aomoto method)]

**Askey–Wilson integral** $$\frac{(q;q)_\infty}{2(2\pi\sqrt{-1})}\int_{\mathbb{T}}\frac{(z^2;q)_\infty(z^{-2};q)_\infty}{\prod_{k=1}^4(a_kz;q)_\infty(a_kz^{-1};q)_\infty}\frac{dz}{z}=\frac{(a_1a_2a_3a_4;q)_\infty}{\prod_{1\le i<j\le 4}(a_ia_j;q)_\infty}.$$

$$\uparrow (a_5\to 0)$$

**Nassrallah–Rahman integral** $$\frac{(q;q)_\infty}{2(2\pi\sqrt{-1})}\int_{\mathbb{T}}\frac{(qa_6^{-1}z,qa_6^{-1}z^{-1},z^2,z^{-2};q)_\infty}{\prod_{k=1}^5(a_kz;q)_\infty(a_kz^{-1};q)_\infty}\frac{dz}{z}=\frac{\prod_{m=1}^5(qa_6^{-1}a_m^{-1};q)_\infty}{\prod_{1\le i<j\le 5}(a_ia_j;q)_\infty}.$$

where $|a_k|<1$ $(k=1,\dots,6)$, under the balancing condition $a_1a_2\cdots a_6=q$.

$$\uparrow (a_6\to pa_6,\quad \text{then}\quad p\to 0)$$

**Spiridonov $BC_1$ integral** $$\frac{(p;p)_\infty(q;q)_\infty}{2(2\pi\sqrt{-1})}\int_{\mathbb{T}}\frac{\prod_{k=1}^6\Gamma(a_kx,a_kx^{-1};p,q)}{\Gamma(x^2,x^{-2};p,q)}\frac{dx}{x}=\prod_{1\le i<j\le 6}\Gamma(a_ia_j;p,q),$$

where $|a_k|<1$ $(k=1,\dots,6)$, under the balancing condition $a_1a_2\cdots a_6=pq$.

# Elliptic analog of $G_2$ Type I integrals

**Elliptic $G_2$ type I integral (5 parameters+Balancing condition) [I.–Noumi 2020]**

Suppose that $x_1x_2x_3 = 1$ and $a_k \in \mathbb{C}^*$ $(1 \le k \le 5)$ satisfy $|a_k| < 1$. Under the balancing condition $(a_1a_2a_3a_4a_5)^2 = pq$, we have

$$\frac{(p;p)_\infty^2(q;q)_\infty^2}{12\,(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\frac{\prod_{i=1}^{3}\prod_{k=1}^{5}\Gamma(a_kx_i, a_kx_i^{-1};p,q)}{\prod_{1\le i<j\le 3}\Gamma(x_ix_j, x_i^{-1}x_j, x_ix_j^{-1}, x_i^{-1}x_j^{-1};p,q)}\frac{dx_1}{x_1}\frac{dx_2}{x_2}$$

$$= \prod_{i=1}^{5}\frac{\Gamma(a_i^2;p,q)}{\Gamma(a_i;p,q)}\prod_{1\le i<j\le 5}\Gamma(a_ia_j;p,q)\prod_{\substack{1\le i<j\\<k\le 5}}\Gamma(a_ia_ja_k;p,q)\prod_{\substack{1\le i<j\\<k<l\le 5}}\Gamma(a_ia_ja_ka_l;p,q).$$

This includes Gustafson's $G_2$ q-integral (type I) as a limiting case;
first replace $a_5$ with $\sqrt{p}a_5$, and then take the limit $p \to 0$.