

Trigonometric and elliptic Selberg integrals

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7 Aug 2025 (Happy Birthday, Peter!)

Selberg integral

Beta function $B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad (\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0)$

Gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad (\operatorname{Re} \alpha > 0) \implies B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

This relation is extended to the following cases of multiple integrals.

Dixon–Anderson integral (Type I) [Dixon 1905, Anderson 1991]

$$\int_{z_n=x_{n-1}}^{x_n} \cdots \int_{z_2=x_1}^{x_2} \int_{z_1=x_0}^{x_1} \prod_{i=1}^n \prod_{j=0}^n |z_i - x_j|^{s_j-1} \prod_{1 \leq k < l \leq n} (z_l - z_k) dz_1 dz_2 \cdots dz_n = \frac{\Gamma(s_0)\Gamma(s_1) \cdots \Gamma(s_n)}{\Gamma(s_0 + s_1 + \cdots + s_n)} \prod_{0 \leq i < j \leq n} (x_j - x_i)^{s_i+s_j-1}$$

Selberg integral (Type II) [Selberg, 1942]

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n z_i^{\alpha-1} (1-z_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |z_i - z_j|^{2\tau} dz_1 dz_2 \cdots dz_n = \prod_{j=1}^n \frac{\Gamma(\alpha + (j-1)\tau)\Gamma(\beta + (j-1)\tau)\Gamma(j\tau + 1)}{\Gamma(\alpha + \beta + (n+j-2)\tau)\Gamma(\tau + 1)}$$

Each integral can be written as a product of gamma functions. When $n = 1$, both integrals coincide with the ordinary Beta function. Chapter 4 of Peter's book [Log-Gases and Random Matrices] is devoted to how to prove the above relation of Selberg integral.

various derivations of Selberg's formula are presented.

Set

$$S_n(\alpha, \beta, \tau) := \int_0^1 \cdots \int_0^1 \prod_{i=1}^n z_i^{\alpha-1} (1 - z_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |z_i - z_j|^{2\tau} dz_1 \cdots dz_n.$$

Then the following two are typical methods:

Anderson's derivation

A method for calculating a multiple integral of $2n + 1$ dimension in two ways [Fubini's theorem] to obtain a relation of $S_n(\alpha, \beta, \tau)$ with respect to n (two-term relation between S_n and S_{n+1}). The Dixon–Anderson integral (Type I) is used in the process.

Aomoto's derivation

(← this talk)

- 1. Difference equation** Derive the difference equations for the parameters that $S_n(\alpha, \beta, \tau)$ satisfies (In this case two-term relation is obtained).
- 2. Asymptotic behavior** As a boundary condition, determine the asymptotic behavior as parameters go to infinity. (The steepest descent method (saddle point method), etc, of integration is used.)

Please imagine $n = 1$ case. During a calculus class we learned about two derivations of the relation between Beta function and Gamma function. One is to use a method for calculating a double integral in two ways. This is similar to Anderson's derivation. The other is to derive the difference equation using Integration by Parts. This is corresponding to Aomoto's derivation.

q -Analog of Selberg integral

The q -analog (q -deformation) of Dixon–Anderson integral and Selberg integral are given by

q -Dixon–Anderson integral (Type I) [Evans 1992] (See [I.–Forrester, J. Math. Anal. Appl, 2015])

$$\int_{z_n=x_{n-1}}^{x_n} \cdots \int_{z_2=x_1}^{x_2} \int_{z_1=x_0}^{x_1} \prod_{i=1}^n \prod_{j=0}^n \frac{(qz_i/x_j; q)_\infty}{(q^{s_j} z_i/x_j; q)_\infty} \prod_{1 \leq i < j \leq n} (z_j - z_i) d_q z_1 d_q z_2 \cdots d_q z_n \\ = \frac{\Gamma_q(s_0) \Gamma_q(s_1) \cdots \Gamma_q(s_n)}{\Gamma_q(s_0 + s_1 + \cdots + s_n)} \prod_{0 \leq i < j \leq n} (x_j - x_i) \frac{(qx_i/x_j, qx_j/x_i; q)_\infty}{(q^{s_j} x_i/x_j, q^{s_i} x_j/x_i; q)_\infty}.$$

q -Selberg integral (Type II) [Askey, et al, 1980–90's, Aomoto 1998] (See [I.–Forrester, Trans. AMS, 2017])

$$\int_{z_1=0}^1 \int_{z_2=0}^{q^\tau z_1} \cdots \int_{z_n=0}^{q^\tau z_{n-1}} \prod_{i=1}^n z_i^\alpha \frac{(qz_i; q)_\infty}{(q^\beta z_i; q)_\infty} \prod_{1 \leq j < k \leq n} z_j^{2\tau-1} \frac{(q^{1-\tau} z_k/z_j; q)_\infty}{(q^\tau z_k/z_j; q)_\infty} (z_j - z_k) \frac{d_q z_n}{z_n} \cdots \frac{d_q z_1}{z_1} \\ = q^{\alpha\tau \binom{n}{2} + 2\tau^2 \binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha + (j-1)\tau) \Gamma_q(\beta + (j-1)\tau) \Gamma_q(j\tau)}{\Gamma_q(\alpha + \beta + (n+j-2)\tau) \Gamma_q(\tau)}$$

When $q \rightarrow 1$, the above q -integrals degenerate to the ordinary integrals, respectively. I will explain how to prove the q -Selberg integral formula using q -version of Aomoto's derivation when $n = 1$. Even in this simplest case, if you settled this, you can easily analogize the situation to general n case.

On the next slide I will explain the case of q -beta function.

Set $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Definition (q -shifted factorials)

Fix $q \in \mathbb{C}^*$ as $|q| < 1$. We use the symbols $(x; q)_\infty$, $(x; q)_n$ as

$$(x; q)_\infty = \prod_{i=0}^{\infty} (1 - q^i x) = (1 - x)(1 - qx)(1 - q^2 x) \cdots$$

$$(x; q)_n = \frac{(x; q)_\infty}{(q^n x; q)_\infty} = \begin{cases} (1 - x)(1 - qx) \cdots (1 - q^{n-1} x) & (n = 1, 2, \dots) \\ 1 & (n = 0) \\ \frac{1}{(1 - q^{-n} x)(1 - q^{-(n-1)} x) \cdots (1 - q^{-1} x)} & (n = -1, -2, \dots) \end{cases}.$$

We also use the symbol

$$(x_1, x_2, \dots, x_m; q)_\infty = (x_1; q)_\infty (x_2; q)_\infty \cdots (x_m; q)_\infty$$

for abbreviation.

q -binomial theorem

$$\sum_{\nu=0}^{\infty} \frac{(q^\alpha; q)_\nu}{(q; q)_\nu} x^\nu = \frac{(q^\alpha x; q)_\infty}{(x; q)_\infty} \xrightarrow{q \rightarrow 1} \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu}{\nu!} x^\nu = \frac{1}{(1 - x)^\alpha} \quad |x| < 1$$

where $(\alpha)_\nu := \alpha(\alpha + 1) \cdots (\alpha + \nu - 1)$

q -Beta function and q -gamma function

Jackson integral

$$\int_0^x f(z) d_q z := (1-q) \sum_{\nu=0}^{\infty} q^{\nu} z f(q^{\nu} z), \text{ which is equivalent to } \int_0^x f(z) \frac{d_q z}{z} = (1-q) \sum_{\nu=0}^{\infty} f(q^{\nu} z).$$

$$\boxed{\frac{(q^{\alpha} z; q)_{\infty}}{(q^{\beta} z; q)_{\infty}} \xrightarrow{q \rightarrow 1} (1-z)^{\beta-\alpha}} \text{ by } q\text{-binomial theorem.}$$

Andrews' q -Beta function

$$B_q(\alpha, \beta) := \int_0^1 z^{\alpha} \frac{(qz; q)_{\infty}}{(q^{\beta} z; q)_{\infty}} \frac{d_q z}{z} \xrightarrow{q \rightarrow 1} \int_0^1 z^{\alpha-1} (1-z)^{\beta-1} dz$$

Askey's q -Gamma function

$$\Gamma_q(\alpha) := \frac{(q; q)_{\infty}}{(q^{\alpha}; q)_{\infty}} (1-q)^{1-\alpha} \xrightarrow{q \rightarrow 1} \Gamma(\alpha)$$

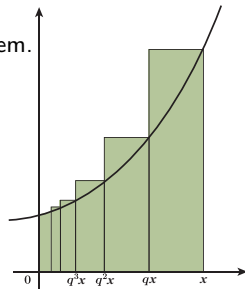
satisfies

$$\Gamma_q(\alpha + 1) = \frac{1-q^{\alpha}}{1-q} \Gamma_q(\alpha) \left(\text{In particular, if } n \in \mathbb{N}, \text{ then } \Gamma_q(n+1) = \frac{(q; q)_n}{(1-q)^n} \xrightarrow{q \rightarrow 1} n! \right)$$

$$\text{Theorem } B_q(\alpha, \beta) = \frac{\Gamma_q(\alpha) \Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)}$$

$$\text{LHS} = (1-q) \sum_{\nu=0}^{\infty} q^{\nu \alpha} \frac{(q^{1+\nu}; q)_{\infty}}{(q^{\beta+\nu}; q)_{\infty}}, \quad \text{RHS} = (1-q) \frac{(q^{\alpha+\beta}, q; q)_{\infty}}{(q^{\alpha}, q^{\beta}; q)_{\infty}}$$

Let us prove this formula using Aomoto's derivation.



Mellin's method (1907) for hypergeometric function ${}_2F_1$

The idea behind the q -version of Aomoto's derivation is based on Mellin's derivation, which is a method for deriving the differential equation that the hypergeometric series ${}_2F_1$ satisfies.

Gauss's hypergeometric series

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{\nu=0}^{\infty} \frac{(\alpha)_{\nu}(\beta)_{\nu}}{(\gamma)_{\nu} \nu!} x^{\nu} \quad |x| < 1$$

If we set $y = {}_2F_1(\alpha, \beta; \gamma; x)$, then y satisfies

Gauss's hypergeometric differential equation

$$x(1-x)\frac{d^2y}{dx^2} + (\gamma - (\alpha + \beta + 1)x)\frac{dy}{dx} - \alpha\beta y = 0.$$

Let us confirm this fact following Mellin's derivation.

If we set $\Phi(z) := x^z \frac{\Gamma(\alpha + z)\Gamma(\beta + z)}{\Gamma(\gamma + z)\Gamma(1 + z)}$, then $\Phi(\nu) = 0$ if $\nu = -1, -2, -3, \dots$.

Moreover $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ implies $(\alpha)_{\nu} = \frac{\Gamma(\alpha + \nu)}{\Gamma(\alpha)}$. Then

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{\nu=-\infty}^{\infty} \Phi(\nu)$$

This series can be formally regarded as a bilateral series (i.e. a sum over \mathbb{Z}).

For any point $\xi \in \mathbb{C}$ and any meromorphic function $\varphi(z)$ on \mathbb{C} , we set

$$\langle \varphi, \xi \rangle := \sum_{\nu=-\infty}^{\infty} \varphi(\xi + \nu) \Phi(\xi + \nu) \quad \text{c.f.} \quad {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \langle 1, 0 \rangle.$$

If we set $(\nabla \varphi)(z) := \varphi(z) - \frac{\Phi(z+1)}{\Phi(z)} \varphi(z+1) = \varphi(z) - x \frac{(\alpha+z)(\beta+z)}{(\gamma+z)(1+z)} \varphi(z+1),$

then **Key Lemma** $\langle \nabla \varphi, \xi \rangle = 0$ holds.

$$\left(\text{Key Lemma is equivalent to } \sum_{\nu=-\infty}^{\infty} \varphi(\xi + \nu) \Phi(\xi + \nu) = \sum_{\nu=-\infty}^{\infty} \varphi(\xi + \nu + 1) \Phi(\xi + \nu + 1). \right)$$

In particular, if we take $\varphi(z) = (\gamma - 1 + z)z$, then $\nabla \varphi(z) = (\gamma - 1 + z)z - x(\alpha + z)(\beta + z).$

Taking account of $x \frac{d}{dx} \Phi(z) = z \Phi(z)$, we see the fact $x \frac{d}{dx} \langle f, \xi \rangle = \langle z f, \xi \rangle$. Therefore

$$\left(\gamma - 1 + x \frac{d}{dx} \right) x \frac{d}{dx} \langle 1, \xi \rangle - x \left(\alpha + x \frac{d}{dx} \right) \left(\beta + x \frac{d}{dx} \right) \langle 1, \xi \rangle = 0$$

This equation coincides with Gauss's hypergeometric differential equation.

Extended Jackson integral (q -version of Aomoto's derivation)

$$\int_0^{\xi\infty} f(z) \frac{d_q z}{z} := (1-q) \sum_{\nu=-\infty}^{\infty} f(q^\nu \xi)$$

If the right-hand side converges, we call it the extended Jackson integral.

$$B_q(\alpha, \beta; \xi) := \int_0^{\xi\infty} \Phi(z) \frac{d_q z}{z},$$

$$\text{where } \Phi(z) := z^\alpha \frac{(qz; q)_\infty}{(q^\beta z; q)_\infty}$$

In particular, if $\xi = 1$, then $B_q(\alpha, \beta; 1) := \int_0^1 \Phi(z) \frac{d_q z}{z} = B_q(\alpha, \beta)$.

$$\text{We set } \nabla \varphi(z) := \varphi(z) - \frac{\Phi(qz)}{\Phi(z)} \varphi(qz) = \varphi(z) - q^\alpha \frac{1 - q^\beta z}{1 - qz} \varphi(qz)$$

$$\text{and } \langle \varphi, \xi \rangle := \int_0^{\xi\infty} \varphi(z) \Phi(z) \frac{d_q z}{z}. \text{ In particular, if } \varphi(z) = 1, \text{ then } \langle 1, \xi \rangle = B_q(\alpha, \beta; \xi).$$

$$\text{Key Lemma } \langle \nabla \varphi, \xi \rangle = 0 \text{ holds. (Key Lemma is equivalent to } \int_0^{\xi\infty} \varphi(z) \Phi(z) \frac{d_q z}{z} = \int_0^{\xi\infty} \varphi(qz) \Phi(qz) \frac{d_q z}{z}.)$$

$$\varphi(z) = 1 - z \implies \nabla \varphi(z) = (1 - z) - q^\alpha \frac{1 - q^\beta z}{1 - qz} (1 - qz) = (1 - z) - q^\alpha (1 - q^\beta z) = (1 - q^\alpha) - (1 - q^{\alpha+\beta})z.$$

From Key Lemma, we have $(1 - q^\alpha) \langle 1, \xi \rangle - (1 - q^{\alpha+\beta}) \langle z, \xi \rangle = 0$.

Since $B_q(\alpha + 1, \beta; \xi) = \int_0^{\xi\infty} z \Phi(z) \frac{d_q z}{z} = \langle z, \xi \rangle$, we obtain

$$B_q(\alpha, \beta; \xi) = \frac{1 - q^{\alpha+\beta}}{1 - q^\alpha} B_q(\alpha + 1, \beta; \xi).$$

q -difference equation

Boundary condition (Asymptotic behavior)

$$B_q(\alpha, \beta; \xi) = \frac{1 - q^{\alpha+\beta}}{1 - q^\alpha} B_q(\alpha + 1, \beta; \xi)$$

This relation is independent of the choice of ξ . In particular, if we put

$\xi = 1$, then we have

$$B_q(\alpha, \beta) = \frac{1 - q^{\alpha+\beta}}{1 - q^\alpha} B_q(\alpha + 1, \beta).$$

Repeated use of this q -difference equation

$$\begin{aligned} B_q(\alpha, \beta) &= \frac{1 - q^{\alpha+\beta}}{1 - q^\alpha} B_q(\alpha + 1, \beta) = \frac{(1 - q^{\alpha+\beta})(1 - q^{\alpha+\beta+1})}{(1 - q^\alpha)(1 - q^{\alpha+1})} B_q(\alpha + 2, \beta) = \cdots = \frac{(q^{\alpha+\beta}; q)_N}{(q^\alpha; q)_N} B_q(\alpha + N, \beta) \\ &= \frac{(q^{\alpha+\beta}; q)_\infty}{(q^\alpha; q)_\infty} \lim_{N \rightarrow \infty} B_q(\alpha + N, \beta). \end{aligned}$$

On the other hand, by definition,

$$\begin{aligned} B_q(\alpha, \beta) &= \int_0^1 \Phi(z) \frac{d_q z}{z} = (1 - q) \sum_{\nu=0}^{\infty} \Phi(q^\nu) = (1 - q) \left(\frac{(q; q)_\infty}{(q^\beta; q)_\infty} + q^\alpha \frac{(q^2; q)_\infty}{(q^{\beta+1}; q)_\infty} + q^{2\alpha} \frac{(q^3; q)_\infty}{(q^{\beta+2}; q)_\infty} + \cdots \right), \\ \lim_{N \rightarrow \infty} B_q(\alpha + N, \beta) &= (1 - q) \frac{(q; q)_\infty}{(q^\beta; q)_\infty}. \end{aligned}$$

Therefore

$$B_q(\alpha, \beta) = \frac{(q^{\alpha+\beta}; q)_\infty}{(q^\alpha; q)_\infty} \lim_{N \rightarrow \infty} B_q(\alpha + N, \beta) = \frac{(q^{\alpha+\beta}; q)_\infty}{(q^\alpha; q)_\infty} \times (1 - q) \frac{(q; q)_\infty}{(q^\beta; q)_\infty} = \frac{\Gamma_q(\alpha) \Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)}. \quad \text{Q.E.D.}$$

q -Integrals associated with root systems

q -Dixon–Anderson integral and q -Selberg integral can be extended to the following two contour integrals.

BC_n Type I integral ($2n + 2$ parameters) [Gustafson, 1994]

$$\begin{aligned} & \frac{1}{(2\pi\sqrt{-1})^n} \int \cdots \int_{\mathbb{T}^n} \prod_{i=1}^n \frac{(z_i^2, z_i^{-2}; q)_\infty}{\prod_{k=1}^{2n+2} (a_k z_i, a_k z_i^{-1}; q)_\infty} \prod_{1 \leq i < j \leq n} (z_i z_j, z_i z_j^{-1}, z_i^{-1} z_j, z_i^{-1} z_j^{-1}; q)_\infty \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \\ &= \frac{2^n n!}{(q; q)_\infty^n} \frac{(a_1 a_2 \cdots a_{2n+2}; q)_\infty}{\prod_{1 \leq i < j \leq 2n+2} (a_i a_j; q)_\infty}, \quad \text{where } \mathbb{T}^n \text{ is the } n\text{-fold direct product of the unit circle} \end{aligned}$$

BC_n Type II integral ($4 + 1$ parameters) [Gustafson, 1994]

$$\begin{aligned} & \frac{1}{(2\pi\sqrt{-1})^n} \int \cdots \int_{\mathbb{T}^n} \prod_{i=1}^n \frac{(z_i^2, z_i^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z_i, a_k z_i^{-1}; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(z_i z_j, z_i z_j^{-1}, z_i^{-1} z_j, z_i^{-1} z_j^{-1}; q)_\infty}{(t z_i z_j, t z_i z_j^{-1}, t z_i^{-1} z_j, t z_i^{-1} z_j^{-1}; q)_\infty} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \\ &= \frac{2^n n!}{(q; q)_\infty^n} \prod_{k=1}^n \frac{(t; q)_\infty}{(t^k; q)_\infty} \frac{(a_1 a_2 a_3 a_4 t^{2n-k-1}; q)_\infty}{\prod_{1 \leq i < j \leq 4} (a_i a_j t^{k-1}; q)_\infty} \end{aligned}$$

Using residue calculation the above contour integrals can be convert to Jackson integral representations. The integrand of type II defines the orthogonal inner product of the Macdonald–Koornwinder polynomials.

$$n = 1 \Rightarrow \text{Askey–Wilson integral} \quad \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2; q)_\infty (z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z; q)_\infty (a_k z^{-1}; q)_\infty} \frac{dz}{z} = \frac{2}{(q; q)_\infty} \frac{(a_1 a_2 a_3 a_4; q)_\infty}{\prod_{1 \leq i < j \leq 4} (a_i a_j; q)_\infty}.$$

Examples (other root systems)

G_2 Type I integral (4 parameters) [Gustafson,1994]

Suppose that $a_k \in \mathbb{C}^*$ ($1 \leq k \leq 4$) satisfy $|a_k| < 1$. Then we have

$$\begin{aligned} & \frac{(q; q)_\infty^2}{12(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \prod_{i=1}^3 \frac{(x_i, x_i^{-1}; q)_\infty}{\prod_{k=1}^4 (a_k x_i, a_k x_i^{-1}; q)_\infty} \prod_{1 \leq j < k \leq 3} (x_j x_k^{-1}, x_k x_j^{-1}; q)_\infty \frac{dx_1}{x_1} \frac{dx_2}{x_2} \quad (\text{where } x_3 = x_1^{-1} x_2^{-1}) \\ &= \frac{(a_1^2 a_2^2 a_3^2 a_4^2; q)_\infty}{(a_1 a_2 a_3 a_4; q)_\infty} \prod_{i=1}^4 \frac{(a_i; q)_\infty}{(a_i^2; q)_\infty} \prod_{1 \leq i < j \leq 4} \frac{1}{(a_i a_j; q)_\infty} \prod_{1 \leq i < j < k \leq 4} \frac{1}{(a_i a_j a_k; q)_\infty}. \end{aligned}$$

G_2 Type II integral (1 + 1 parameters) [Habsieger,1986]

Suppose that $a, t \in \mathbb{C}^*$ satisfy $|a| < 1$ and $|t| < 1$. Then we have

$$\begin{aligned} & \frac{(q; q)_\infty^2}{12(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \prod_{i=1}^3 \frac{(x_i, x_i^{-1}; q)_\infty}{(a x_i, a x_i^{-1}; q)_\infty} \prod_{1 \leq j < k \leq 3} \frac{(x_j x_k^{-1}, x_k x_j^{-1}; q)_\infty}{(t x_j x_k^{-1}, t x_k x_j^{-1}; q)_\infty} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \quad (\text{where } x_3 = x_1^{-1} x_2^{-1}) \\ &= (q a t^2, q a^2 t^3; q)_\infty \frac{(a; q)_\infty}{(a^2; q)_\infty} \frac{(a t; q)_\infty}{(a^3 t^3; q)_\infty} \frac{(t, q t; q)_\infty}{(t^2, q t^3; q)_\infty} \end{aligned}$$

Regarding F_4 , there also exist Type I and Type II integrals, and the results are similar to those for G_2 case.

Ruijsenaars elliptic gamma function

We fix $p, q \in \mathbb{C}^*$ as $|p| < 1$ and $|q| < 1$. **The Ruijsenaars elliptic gamma function** is defined as

$$\Gamma(x; p, q) = \frac{(pqx^{-1}; p, q)_\infty}{(x; p, q)_\infty}, \quad \text{where} \quad (x; p, q)_\infty = \prod_{\mu, \nu=0}^{\infty} (1 - p^\mu q^\nu x).$$

We also use the notation $\Gamma(x_1, \dots, x_m; p, q) = \Gamma(x_1; p, q) \cdots \Gamma(x_m; p, q)$. Note that $\Gamma(x; p, q)$ satisfies

$$\Gamma(qx; p, q) = \theta(x; p)\Gamma(x; p, q) \quad \text{and} \quad \Gamma(px; p, q) = \theta(x; q)\Gamma(x; p, q),$$

where $\theta(x; p) = (x, p/x; p)_\infty$ is a theta function satisfying $\theta(px; p) = -\theta(x; p)/u$, and also satisfies

$$\Gamma(pqx^{-1}; p, q) = \frac{1}{\Gamma(x; p, q)}, \quad \frac{1}{\Gamma(x, x^{-1}; p, q)} = -x^{-1}\theta(x; p)\theta(x; q).$$

Remark.

$$\begin{array}{ccccc} \text{Elliptic} & \xrightarrow{p \rightarrow 0} & \text{Trigonometric (q-analog)} & \xrightarrow{q \rightarrow 1} & \text{Rational (Classical)} \end{array}$$

$$\Gamma(x; p, q) = \frac{(pqx^{-1}; p, q)_\infty}{(x; p, q)_\infty} \xrightarrow{p \rightarrow 0} \frac{1}{(x; q)_\infty}$$

$$\Gamma(px; p, q) = \frac{(qx^{-1}; p, q)_\infty}{(px; p, q)_\infty} \xrightarrow{p \rightarrow 0} (qx^{-1}; q)_\infty$$

Elliptic analog of BC_n Type I and Type II integrals

Under the balancing condition $a_1 \cdots a_{2n+4} = pq$,

$$\begin{aligned} & \frac{1}{(2\pi\sqrt{-1})^n} \int \cdots \int_{\mathbb{T}^n} \prod_{i=1}^n \frac{\prod_{m=1}^{2n+4} \Gamma(a_m z_i, a_m z_i^{-1}; p, q)}{\Gamma(z_i^2, z_i^{-2}; p, q)} \prod_{1 \leq j < k \leq n} \frac{1}{\Gamma(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n} \\ &= \frac{2^n n!}{(p; p)_\infty^n (q; q)_\infty^n} \prod_{1 \leq j < k \leq 2n+4} \Gamma(a_j a_k; p, q), \end{aligned}$$

where a_1, \dots, a_{2n+4} are complex parameters with $|a_m| < 1$ ($m = 1, \dots, 2n+4$). [Rains 2010]

Under the balancing condition $a_1 \cdots a_6 t^{2n-2} = pq$,

$$\begin{aligned} & \frac{1}{(2\pi\sqrt{-1})^n} \int \cdots \int_{\mathbb{T}^n} \prod_{i=1}^n \frac{\prod_{m=1}^6 \Gamma(a_m z_i, a_m z_i^{-1}; p, q)}{\Gamma(z_i^2, z_i^{-2}; p, q)} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t z_j z_k, t z_j z_k^{-1}, t z_j^{-1} z_k, t z_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n} \\ &= \frac{2^n n!}{(p; p)_\infty^n (q; q)_\infty^n} \prod_{i=1}^n \left(\frac{\Gamma(t^i; p, q)}{\Gamma(t; p, q)} \prod_{1 \leq j < k \leq 6} \Gamma(t^{i-1} a_j a_k; p, q) \right), \end{aligned}$$

where a_1, \dots, a_6, t are complex parameters with $|a_m| < 1$ ($m = 1, \dots, 6$), $|t| < 1$.

[van Diejen–Spiridonov 2001, Rains 2010 (Anderson method) I.–Noumi 2017 (Aomoto method)]

Askey–Wilson integral

$$\frac{(q; q)_\infty}{2(2\pi\sqrt{-1})} \int_{\mathbb{T}} \frac{(z^2; q)_\infty (z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z; q)_\infty (a_k z^{-1}; q)_\infty} \frac{dz}{z} = \frac{(a_1 a_2 a_3 a_4; q)_\infty}{\prod_{1 \leq i < j \leq 4} (a_i a_j; q)_\infty}.$$

$$\uparrow (a_5 \rightarrow 0)$$

Nassrallah–Rahman integral

$$\frac{(q; q)_\infty}{2(2\pi\sqrt{-1})} \int_{\mathbb{T}} \frac{(qa_6^{-1}z, qa_6^{-1}z^{-1}, z^2, z^{-2}; q)_\infty}{\prod_{k=1}^5 (a_k z; q)_\infty (a_k z^{-1}; q)_\infty} \frac{dz}{z} = \frac{\prod_{m=1}^5 (qa_6^{-1}a_m^{-1}; q)_\infty}{\prod_{1 \leq i < j \leq 5} (a_i a_j; q)_\infty}.$$

where $|a_k| < 1$ ($k = 1, \dots, 6$), under the balancing condition $a_1 a_2 \cdots a_6 = q$.

$$\uparrow (a_6 \rightarrow pa_6, \text{ then } p \rightarrow 0)$$

Spiridonov BC_1 integral

$$\frac{(p; p)_\infty (q; q)_\infty}{2(2\pi\sqrt{-1})} \int_{\mathbb{T}} \frac{\prod_{k=1}^6 \Gamma(a_k x, a_k x^{-1}; p, q)}{\Gamma(x^2, x^{-2}; p, q)} \frac{dx}{x} = \prod_{1 \leq i < j \leq 6} \Gamma(a_i a_j; p, q),$$

where $|a_k| < 1$ ($k = 1, \dots, 6$), under the balancing condition $a_1 a_2 \cdots a_6 = pq$.

Elliptic G_2 type I integral (5 parameters+Balancing condition) [I.-Noumi 2020]

Suppose that $x_1 x_2 x_3 = 1$ and $a_k \in \mathbb{C}^*$ ($1 \leq k \leq 5$) satisfy $|a_k| < 1$. Under the balancing condition $(a_1 a_2 a_3 a_4 a_5)^2 = pq$, we have

$$\begin{aligned} & \frac{(p; p)_\infty^2 (q; q)_\infty^2}{12 (2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \frac{\prod_{i=1}^3 \prod_{k=1}^5 \Gamma(a_k x_i, a_k x_i^{-1}; p, q)}{\prod_{1 \leq i < j \leq 3} \Gamma(x_i x_j, x_i^{-1} x_j, x_i x_j^{-1}, x_i^{-1} x_j^{-1}; p, q)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \\ &= \prod_{i=1}^5 \frac{\Gamma(a_i^2; p, q)}{\Gamma(a_i; p, q)} \prod_{1 \leq i < j \leq 5} \Gamma(a_i a_j; p, q) \prod_{\substack{1 \leq i < j \\ < k \leq 5}} \Gamma(a_i a_j a_k; p, q) \prod_{\substack{1 \leq i < j \\ < k < l \leq 5}} \Gamma(a_i a_j a_k a_l; p, q). \end{aligned}$$

This includes Gustafson's G_2 q -integral (type I) as a limiting case; first replace a_5 with $\sqrt{p}a_5$, and then take the limit $p \rightarrow 0$.