Edge statistics for random band matrices

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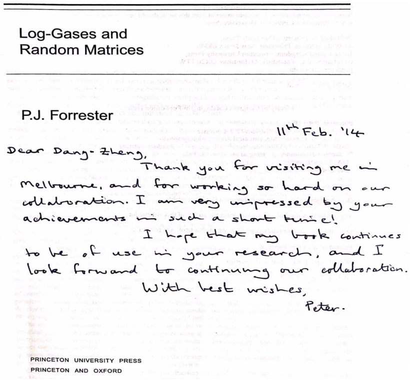
Log-gases in Caeli Australi
Recent Developments in and around Random Matrix Theory
4-15 August, 2025
Joint work with Guangyi Zou, arXiv:2401.00492v2.



Log-Gases and
Random Matrices

P.J. Forrester

11th Feb. '14

Dear Dang-Zheng,
Thank you for visiting me in Melbourne, and for working so hard on our collaboration. I am very impressed by your achievements in such a short time!
I hope that my book continues to be of use in your research, and I look forward to continuing our collaboration.
With best wishes,
Peter.

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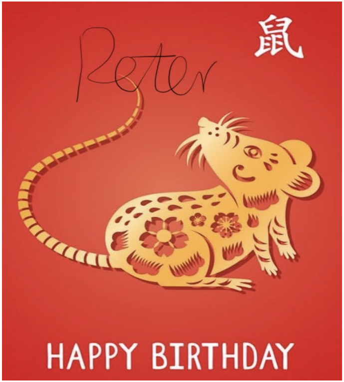


Fri, Feb 14, 2014@

Dear Peter,

I am now at USTC. Thanks very much for inviting me to visit Melbourne. Hope that I can visit Melbourne again or you visit China.

This trip has been both enjoyable and productive! Your profound knowlege of RMT is impressive! ...

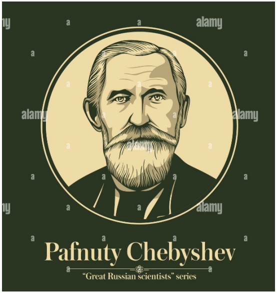
Table of Contents

The method of moments I

Application of the method of moments in probability and statistics. Moments in mathematics, 1987, 125–142. Diaconis:

It is widely acknowledged that the first proof of the central limit theorem up to modern standards of rigor was given by Chebyshev in 1887.

Pafnutii Lvovich Chebyshev (1821–1894), 1887 original Russian version, French translation *"Sur deux théorèmes relatifs aux probabilités"*, *Acta mathematica 1890*



The method of moments II

♡ **Moment Convergence Theorem**
If $\lim_{n\to\infty} \mathbb{E}[X_n^k] = \mathbb{E}[X^k]$ for $k \geq 1$ and X is uniquely determined by its moments (Carleman's condition: $\sum_{m=1}^{\infty} \left(\mathbb{E}[|X|^{2m}]\right)^{-1/(2m)} = \infty$) then $X_n \xrightarrow{D} X$ as $n \to \infty$.

♠ **CLT** $S_n = \sum_{i=1}^{n} X_i$ sum of iid variables with mean zero, variance 1 and finite higher moments, then $\frac{1}{\sqrt{n}} S_n \to N(0,1)$.

$$S_n^r = \sum_{k=1}^{r} \sum_{\substack{r_1+\cdots+r_k=r \\ r_j \geq 1}} \frac{r!}{r_1! \cdots r_k!} \sum_{1 \leq i_1 \neq \cdots \neq i_k \leq n} X_{i_1}^{r_1} \cdots X_{i_k}^{r_k}$$

$$\mathbb{E}\left[S_n^r\right] = \sum_{k=1}^{r} \sum_{\substack{r_1+\cdots+r_k=r \\ r_j \geq 1}} \frac{r!}{r_1! \cdots r_k!} \sum_{1 \leq i_1 \neq \cdots \neq i_k \leq n} \mathbb{E}\left[X_{i_1}^{r_1} \cdots X_{i_k}^{r_k}\right]$$

Only when all $r_j = 2$, the r-th moment of standard normal distribution! See Billingsley, Probability and Measure Sect. 30

Wigner's 1955 seminal paper

ANNALS OF MATHEMATICS
Vol. 62, No. 3, November, 1955
Printed in U.S.A.

CHARACTERISTIC VECTORS OF BORDERED MATRICES WITH INFINITE DIMENSIONS

BY EUGENE P. WIGNER

(Received April 18, 1955)

Introduction

The statistical properties of the characteristic values of a matrix the elements of which show a normal (Gaussian) distribution are well known (cf. [6] Chapter XI) and have been derived, rather recently, in a particularly elegant fashion.[1] The present problem arose from the consideration of the properties of the wave functions of quantum mechanical systems which are assumed to be so complicated that statistical considerations can be applied to them. Since

All the remaining work will deal with real symmetric matrices of very high dimensionality. The first and last problems concern infinite bordered matrices; the second one a finite matrix the consideration of which served as an intermediate step toward the solution of the last one. We mean by a bordered matrix the sum of a diagonal matrix $\mathbf{k}$ and a border matrix $\mathbf{v}$. The diagonal elements of $\mathbf{k}$ are all the integers $\cdots, -2, -1, 0, 1, 2, \cdots$. The border matrix $\mathbf{v}$ has non vanishing elements only up to a distance N from the diagonal, the absolute value of all the non vanishing elements is the same

$$
\begin{aligned}
|v_{mn}| &= v \qquad &&\text{for } |m-n| \leqq N, \ (-\infty < m, n < \infty) \\
&= 0 &&\text{for } |m-n| > N.
\end{aligned} \tag{1}
$$

Since the matrix $H = \mathbf{k} + \mathbf{v}$ is symmetric, $v_{mn} = v_{nm}$. Subject to this condition, however, the signs of the v_{ij} are random, i.e. we consider ensembles of matrices with all possible signs of v_{mn} subject to the conditions of symmetry. In the

Semicircle law for Wigner matrices

♠ Wigner matrices $H = (H_{uv})_{u,v=1}^N$ with iid entries (mean zero, variance 1 and all higher moments)
Wigner, two papers, Ann. Math., 1955,1958

♠

$$\mathrm{Tr}(H^m) = \sum_{p=(u_0,u_1,\ldots,u_{m-1},u_m=u_0)} \prod_{j=0}^{m-1} H_{u_j u_{j+1}}$$
$$= \sum_{1\le u\le v\le N} \prod_{u\le v} \left(H_{uv}^{\#\{(u_j,u_{j+1})=(u,v)\}} \cdot H_{vu}^{\#\{(u_j,u_{j+1})=(v,u)\}} \right)$$

♠ Calculate all finite moments

$$\int x^{2k}\rho_{\mathrm{sc}}(x)dx = \frac{1}{k+1}\binom{2k}{k}.$$

Limits for largest eigenvalues

Wigner matrices with IID matrix entries

♡ **Bai-Yin law** [Annals of Probability, 1993] $\lambda_{max}(H) \to 2$ a.s.

$$\mathbb{E}[\lambda_{max}^{2m}] \le \mathbb{E}\big[\mathrm{Tr}(H^{2m})\big] = \mathbb{E}\big[\sum_j \lambda_j^{2m}\big] \le N\mathbb{E}[\lambda_{max}^{2m}], \ m \gg \log N$$

♠ **Airy kernel,** $2/3$ **exponent** P.J. Forrester, The spectrum edge of random matrix ensembles, Nuclear Physics B402 (1993), 709—728
Tracy-Widom law [Tracy-Widom, CMP 1994]

♡ TW distribution for Wigner matrices [Soshnikov, CMP 1999]
$N^{2/3}\big(\lambda_{max}(H) - 2\big) \to$ TW law.
Heuristically,

$$\mathbb{E}\big[\mathrm{Tr}\big(\frac{1}{2}H\big)^m\big] \sim \sum_{\text{edge}} \mathbb{E}[(\frac{1}{2}\lambda_i)^{tN^{\frac{2}{3}}}] \sim \sum_{\text{edge}} \mathbb{E}[e^{\frac{t\theta_i}{2}}]$$

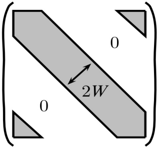
where $m = [tN^{\frac{2}{3}}]$ and $\lambda_i := 2 + N^{-\frac{2}{3}}\theta_i$.

Chebyshev polynomial method

♡ Chebyshev polynomials of second kind $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$. $U_n(\frac{1}{2}x)$ orthogonal w.r.t. $\rho_{\rm sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}$. Hard edge limit

$$\lim_{M\to\infty} \frac{1}{n+1} U_n\big(1+\frac{y}{2M^2}\big) = \frac{\sin\big(t\sqrt{-y}\big)}{t\sqrt{-y}}, \quad n = \lfloor tM \rfloor.$$

♡ Random band matrices with random signs and with bandwith W



$$\mathbb{E}[\mathrm{Tr}\, U_n(\frac{H}{2})] = \big(1+o(\frac{n}{W})\big) \sum_{u_i} \mathbb{E}[\prod_{i=0}^{n-1} H_{u_i u_{i+1}}],$$

where the sum is over non-backtracking random walks $u_i \neq u_{i+2}$; Sodin 2010. **Cf. Bai-Yin'93, Feldheim-Sodin'10, Erdős-Knowles'11, also Benoit Collins talk**

Sodin's 5/6-theorem

Random band matrices with random signs, lattice dimension $d = 1$
Sasha Sodin, The spectral edge of some random band matrices, Annals of Mathematics, 172 (2010), 2223–2251

Phase transition for edge eigenvalue statistics:

i) (Delocalization) $W \gg N^{5/6}$, Tracy-Widom distributions
ii) (Localization) $W \ll N^{5/6}$, edge statistics seems to be Poissonian
iii) (Crossover) Not known in the critical regime $W \sim N^{5/6}$

Table of Contents

Anderson's 1958 paper

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey

(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

Anderson Model

$$H = -\Delta + \lambda V,$$

where Δ is graph Laplacian on $\Lambda := ([-L/2, L/2) \cap \mathbb{Z})^d$ and $V = (V_x)_{x\in\Lambda}$ has IID random variables, $\lambda > 0$ measures the disorder strength. The support of spectrum $[-2d, 2d] + \lambda\,\mathrm{supp}(V_0)$. Not mean-field models, with spatial and geometric structure. Anderson Metal-Insulator Transition

Aizenman-Warzel Book

Random Operators Disorder Effects on Quantum Spectra and Dynamics, GSM 168, 2015; Chapter 1, page 4

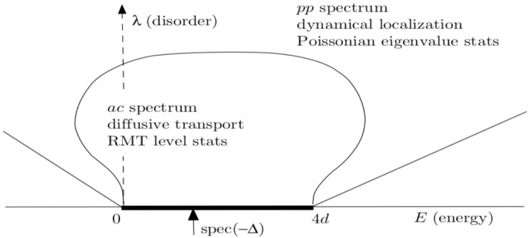


Figure 1.2. The predicted shape of the phase diagram of the Anderson model (1.2) in dimensions $d > 2$ for site potentials given by bounded iid random variables with a distribution similar to (1.5).

In his seminal work P. W. Anderson posited **[27]** that under random potential there would be a transition in the transport properties of the model which heavily depend on the dimension d of the underlying lattice, the strength $\lambda \in \mathbb{R}$ of the disorder, and the energy. The term **mobility edge** was coined for the boundary of the regime at which conduction starts.

Definition of RBM

Random band matrices are more realistic interpolating models between Wigner matrices and Anderson models.**Geometric structure, as typical transitions occur between nearby states.**
Lattice $\Lambda := ([-L/2, L/2) \cap \mathbb{Z})^d$, bandwidth parameter W, define a self-adjoint matrix $H = (H_{xy})_{x,y\in\Lambda}$

(i) $H_{xy} = \sigma_{xy} A_{xy}$, where $\mathbb{E}[|A_{xy}|^2] = 1$ and $\{A_{xy}\}$ are i.i.d. random variables.

(ii) Variance profile function $f(x)$

$$\sigma_{xy}^2 = \frac{1}{M} f(\frac{x-y}{W}), \quad M \sim W^d$$

RBM and Anderson model are expected to have same qualitative properties when $\lambda \sim W^{-1}$
Cf. Spencer, Random banded and sparse matrices, 2011; Bourgade, Random band matrices, Proceedings of ICM 2018

Unimodular random band matrices

$\Lambda := ([-L/2, L/2) \cap \mathbb{Z})^d$, W: bandwidth parameter.
Define RBM model $H = (H_{xy})_{x,y\in\Lambda}$

(**Unimodularity**) $H_{xy} = \sigma_{xy} A_{xy}$, with $|A_{xy}| = 1$ be a i.i.d. uniform variables on $\{+1, -1\}$ for real case ($\beta = 1$) and $\{e^{i\theta:\theta\in[0,2\pi)}\}$ for complex case ($\beta = 2$).

(**Periodic condition**)

$$\sigma_{xy}^2 = \frac{1}{M} \sum_{n\in\mathbb{Z}^d} f(\frac{x - y + nL}{W}).$$

(**Gaussian profile**) Density function $f = Ce^{-\frac{1}{2}|x|^2}$

L.-Zou '24, I

L., Guangyi Zou, arXiv:2401.00492v2

Theorem (Supercritical regime, RMT statistics)

For the unimodular RBM with Gaussian profile and in the Hermitian case, as $L \to \infty$ if

$$W \gg \begin{cases} L^{1-\frac{d}{6}}, & d < 4, \\ L^{\frac{1}{3}+\epsilon}, & d \geq 4, \end{cases}$$

where $\epsilon > 0$ is arbitrary, then the GUE Edge Universality holds true!

Remark: This really characterizes the transition exponent and all the physical dimensions. When $d \leq 4$, GUE Edge Universality is SHARP! Not sharp when $d \geq 5$.

L.-Zou '24, II

Theorem (Subcritical regime, Poisson)

In the Hermitian case, $d = 1, 2, 3$ and $1 \ll W \ll L^{1-\frac{d}{6}}$, limiting edge point process exists (Poission).

Theorem (Critical regime, crossover)

In the Hermitian case, $d = 1, 2, 3$, Critical phenomena in the the critical regime.

In the critical regime, transition density for Brownian Motions on tori, related to Jacobi theta functions

$$\theta(x, \tau) = \sum_{n \in \mathbb{Z}^d} \frac{1}{(\sqrt{2\pi\tau})^d} e^{-\frac{1}{2\tau}|x+n|^2}, \quad x \in \mathbb{Z}^d$$

appears and connects Gaussians and Uniformness.

Recent progress in bulk case

Bulk Universality or bulk Delocalization:

- Erdös and Knowles eigenvector weak delocalization $W \gg L^{\frac{6}{d+6}}$.
- Erdös, Knowles, Yau and Yin $W \gg L^{\frac{d+2}{2d+2}}$
- He and Marcozzi $W \sim L^{\frac{d+1}{2d+1}}, d = 1$.
- Xu, Yang, Yau, and Yin eigenvalues for $W \gg L^{95/(d+95)}, d \geq 7$.
- Yang, Yau, and Yin eigenvector for $W \gg L^{\epsilon}, d \geq 8$.
- Yau, Yin, Delocalization of One-Dimensional Random Band Matrices, arXiv:2501.01718
- Dubova, Yang, Yau, Yin, Delocalization of two-dimensional random band matrices, arXiv:2503.07606
- Erdos, Riabov, the Zigzag Strategy for Random Band Matrices, arXiv:2506.06441
- Dubova, Yang, Yau, Yin, Delocalization of Non-Mean-Field Random Matrices in Dimensions $d \geq 3$, arXiv:2507.20274

Table of Contents

Singularity for Feynman graph integrals

Subcritical regime $W \ll L^{1-\frac{d}{6}}$, encountering graph integrals for Γ with degree less than 4, similar to

$$I = \int \cdots \int_{\alpha_e \in [0,1]} \mathcal{U}_\Gamma(\{\alpha_e\})^{-\frac{d}{2}} \prod_{e \in E(\Gamma)} d\alpha_e.$$

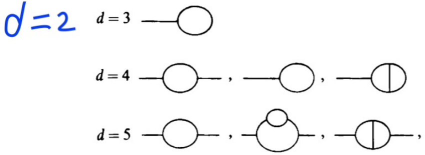
where (Symanzik) graph polynomials

$$\mathcal{U}_\Gamma(\{\alpha_e\}) = \sum_{\text{spanning tree } T} \prod_{e \notin T} \alpha_e.$$

Ultraviolet divergence might happen! Just consider a simpler integral, as $\epsilon \to 0$

$$\int_\epsilon^1 t^{-\frac{d}{2}} dt = \begin{cases} \textit{finite}, & d = 1, \\ \log(\epsilon^{-1}), & d = 2, \\ \infty, & d > 2. \end{cases}$$

Connection with ϕ^3 theory (Collins, **Renormalization**)



and tadpole graphs to 4 loops

Fig. 5.7.1. All the graphs with overall divergences in ϕ^3 theory at those space-time dimensions where it is super-renormalizable.

118 *Renormalization*

If $d = 6$, only the one-, two-, and three-point functions are divergent, with degree of divergence 4, 2, and 0, respectively. The permissible counterterms are just terms of the form of those in $\mathscr{L}_0 + \mathscr{L}_b$, so the theory is renormalizable if $d = 6$. Moreover, there is a divergence in every order of g (except for tree graphs, of course).

If $d = 3$, 4, or 5, then only a finite set of graphs, illustrated in Fig. 5.7.1, have overall divergences, and renormalization is needed only for the mass and for the tadpole coupling. Again we have renormalizability.

Method of polynomial moments

When $d \geq 2$, to deal with regular diagrams in both $\beta = 1$ and $\beta = 2$ cases.

Definition (Modified Chebyshev polynomials)

Modified Chebyshev polynomial $\mathcal{P}_n(z)$ of degree n defined by the four-term recursion

$$\mathcal{P}_n(z) = z\mathcal{P}_{n-1}(z) - \mathcal{P}_{n-2}(z) + a_4\mathcal{P}_{n-4}(z), \quad n = 1, 2, \ldots,$$

with $\mathcal{P}_0(z) = 1$ and $\mathcal{P}_{-n}(z) = 0$ for any integer $n > 0$. a_4 depends on W and goes to zero.

In the real case, $d \geq 2$, Tadpole diagram **Renormalization Polynomials**

$$\widetilde{\mathcal{P}}_n^{(R)}(H) = H\widetilde{\mathcal{P}}_{n-1}^{(R)}(H) - \widetilde{\mathcal{P}}_{n-2}^{(R)}(H) + a_4\widetilde{\mathcal{P}}_{n-4}^{(R)}(H) - \sum_{l=3}^{3R} a_{2l}\widetilde{\mathcal{P}}_{n-2l}^{(R)}(H), \quad n \geq 3,$$

Thanks for your attention!