MTH 231 Quiz 1

Name: Answer Key

Recitation Time: ______

Instructions:

- Answer the following questions with complete, orderly sentences (when applicable.)
- Answers without work shown will not receive credit.
- Read each question completely before attempting to solve.
- Write your name and recitation time in the blanks provided above.

1. (5 points) Determine whether $(\neg q \land (p \to q)) \to \neg p$ is a tautology and justify your answer.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \land (p \to q)$	$(\neg q \land (p \to q)) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Because $(\neg q \land (p \to q)) \to \neg p$ is true for all possible true/false combinations of p and q, it is true always and therefore $(\neg q \land (p \to q)) \to \neg p$ is a tautology.

True or False:

2. (8 points) Determine if the following are true or false, and justify your answer. Let the domain be all real numbers.

(a) $\exists x(x^2 = 2)$

True. Let $x = \sqrt{2}$, which is a real number. We have that $(\sqrt{2})^2 = 2$, so there exists a real number x such that $x^2 = 2$.

(b) $\exists x(x^2 = -1)$

False. If $x^2 = -1$, then $x = \pm\sqrt{-1}$. But $\sqrt{-1} = i$ is not a real number, so there does not exist a real number x that satisfies $x^2 = -1$.

(c) $\forall x(x^2 + 2 \geq 1)$

True. We can write $x^2 + 2 \geq 1$ as $x^2 \geq -1$. the square of any real number is never negative, so this is true for all real numbers.

(d) $\forall x(x^2 \neq x)$

False. We know that 0 is a real number, and $0^2 = 0$. Therefore $x^2 \neq x$ is not true for all real numbers.

3. (5 points) Express the following statement using predicates, quantifiers, and logical operators.

"Everyone needs coffee or sleep during finals week."

Let $C(x)$ denote "x needs coffee during finals week" and $S(x)$ denote "x needs sleep during finals week."

then "Everyone needs coffee or sleep during finals week" can be written as "For all x, x needs coffee or x needs sleep during finals week", or

$$\boxed{\forall x(C(x) \vee S(x)).}$$

4. (7 points) Negate the following statement and translate the negated expression to an English sentence.

$$\forall x \exists y(P(x) \to Q(x, y)).$$

Let:

$P(x) =$ "x is from Washington"
$Q(x, y) =$ "x has visited y",

where the domain of x is all OSU students and the domain of y is all US states.

Negation: $\neg(\forall x \exists y(P(x) \to Q(x,y))) \equiv \exists x \neg(\exists y(P(x) \to Q(x,y)))$ by DeMorgan's Laws

$\equiv \exists x \forall y \neg(P(x) \to Q(x,y))$ by De Morgan's Laws

$\equiv \exists x \forall y(P(x) \wedge \neg Q(x,y))$ Log. Eq. "$\neg(p \to q) \equiv p \wedge \neg q$", see table on back

the negation of $\forall x \exists y(P(x) \to Q(x,y))$ is the statement:

$$\boxed{\exists x \forall y(P(x) \wedge \neg Q(x,y)),}$$

which can be written as:

"There exists an OSU Student such that for all U.S. States, the student is from Washington and has not visited any U.S. state", or

$\boxed{\text{"There exists an OSU student from Washington who has not visited any U.S. State."}}$

(Note that this is very likely false, since being from Washington implies that they have at least been to Washington.)

5. (Extra Credit: 5 points) Hat Problem: Level 1.

There is a box containing three blue hats and two red hats. Three people sitting in a circle are blindfolded and a hat – either a red hat or a blue hat – is placed on each of their heads. The three people are not allowed to communicate with each other. The first person's blindfold is removed and after looking at the hats on the other two people he is asked to state the color of his own hat (which he cannot see.) The first person replies that he does not know the color of his hat. The second person has her blindfold removed and also states that she does not know the color of her hat. The third person, before her blindfold is removed, correctly states the color of her own hat. What is the color of the third person's hat and how does she know the color?

The third person's hat is blue.

Since there are three blue hats and two red hats, the following possibilities could occur:

- Three blue hats
- Two blue hats, one red hat
- Two red hats, one blue hat.

The first person can only be certain of their own hat color if they see two red hats – this would guarantee that they had a blue hat. Since they did not know their hat color, they did not see two red hats. Similarly, the second person could not have seen two red hats. But if the third person had been wearing a red hat, the second person would have known her hat was blue, since both she and the third person cannot have a red hat together. Since the second person was not able to make this claim, the third person knows her hat must be blue.

Let R_1 = First Person has red hat, (R_2 and R_3 defined similarly). Here $\neg R_1 \equiv B_1$ (blue hat).

(1) $\neg(R_2 \wedge R_3) \equiv \neg R_2 \vee \neg R_3$ (or $B_2 \vee B_3$)

(2) $R_3 \rightarrow \neg R_2$ (or $R_3 \rightarrow B_2$)

Since we could not conclude $\neg R_2$ (or B_2), then we must have $\neg R_3$. (i.e., ~~the~~ looking at the contrapositive $R_2 \rightarrow \neg R_3$, the only way to not have a false implication when R_2 could be either true or false is to have $\neg R_3$ be true.)

Therefore B_3 is true and the third person has a blue hat.

Useful Equivalences

$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity Laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's Laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation Laws
$p \rightarrow q \equiv \neg p \vee q$ $\neg(p \rightarrow q) \equiv p \wedge \neg q$ (proved in class)	Logical Equivalences Involving Conditionals