Answer Key

MTH 231

Selected Review Problems

(pp. 111-113)

Let C1: $p \to q \equiv \neg p \vee q$

C2: $\neg(p \to q) \equiv p \wedge \neg q$

1. Let p be the proposition "I will do every exercise in this book" and q be the proposition "I will get an A in this course". Express each of these as a combination of p and q.

   (a) I will get an A in this course only if I do every exercise in this book.

   $q \to p$

   (b) I will get an A in this course and I will do every exercise in this book.

   $q \wedge p$

   (c) Either I will not get an A in this course or I will not do every exercise in this book.

   $\neg q \vee \neg p$

   (d) For me to get an A in this course, it is necessary and sufficient that I do every exercise in this book.

   $q \leftrightarrow p$

2. Find the truth table of the compound proposition $(p \vee q) \to (p \wedge \neg r)$.

p	q	r	$\neg r$	$p \vee q$	$p \wedge \neg r$	$(p \vee q) \to (p \wedge \neg r)$
T	T	T	F	T	F	F
T	T	F	T	T	T	T
T	F	T	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	F	F
F	T	F	T	T	F	F
F	F	T	F	F	F	T
F	F	F	T	F	F	T

3. Show that these compound propositions are tautologies.

   (a) $(\neg q \wedge (p \to q)) \to \neg p$

p	q	$\neg q$	$p \to q$	$\neg q \wedge (p \to q)$	$\neg p$	$(\neg q \wedge (p \to q)) \to \neg p$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

or

$$
\begin{aligned}
(\neg q \wedge (p \to q)) \to \neg p &\equiv \neg[\neg q \wedge (p \to q)] \vee \neg p && \text{by C1} \\
&\equiv \neg(\neg q) \vee \neg(p \to q) \vee \neg p && \text{DeMorgan} \\
&\equiv q \vee \neg p \vee \neg(p \to q) && \text{Double Neg.} \\
&\equiv (q \vee \neg p) \vee (p \wedge \neg q) && \text{C2} \\
&\equiv [(q \vee \neg p) \vee p] \wedge [(q \vee \neg p) \vee \neg q] && \text{Distributive Laws} \\
&\equiv (q \vee T) \wedge (T \vee \neg p) && \text{Negation Laws} \\
&\equiv T \wedge T && \text{Domination Laws} \\
&\equiv T && \text{Idempotent Laws}
\end{aligned}
$$

emoji tears!!! →

   (b) $((p \vee q) \wedge \neg p) \to q$

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$[(p \vee q) \wedge \neg p] \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

or

$$
\begin{aligned}
[(p \vee q) \wedge \neg p] \to q &\equiv \neg[(p \vee q) \wedge \neg p] \vee q && \text{C1} \\
&\equiv [\neg(p \vee q) \vee \neg(\neg p)] \vee q && \text{DeMorgan's Law} \\
&\equiv \neg(p \vee q) \vee (p \vee q) && \text{Double Negation} \\
&\equiv T && \text{Negation}
\end{aligned}
$$

4. Find a compound proposition involving the propositional variables p, q, r, and s that is true when exactly three of these propositional variables are true and is false otherwise.

$(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s)$

5. Show that these statements are inconsistent: "If Miranda does not take a course in discrete mathematics, then she will not graduate." "If Miranda does not graduate, then she is not qualified for the job." "If Miranda reads this book, she is qualified for the job." "Miranda does not take a course in discrete mathematics, but she reads this book."

Let d = "Miranda takes a course in discrete mathematics"
g = "Miranda will graduate"
q = "Miranda is qualified for the job"
b = "Miranda reads this book"

(1) $\neg d \rightarrow \neg g$
(2) $\neg g \rightarrow \neg q$
(3) $b \rightarrow q$
(4) $\neg d \wedge b$

We assume all the statements are true.

- If (4) is true, both $\neg d$ and b are true, so $\boxed{d = F}$ and $\boxed{b = T}$.
- By (3), $b \rightarrow q$ with $b = T$, so $\boxed{q = T}$ also since we cannot have $T \rightarrow F$.
- Looking at $q \rightarrow g$, the contrapositive of (2), we see that since $q = T$ we must have $\boxed{g = T}$ also.
- But ~~since~~ the contrapositive of (1), the statement $g \rightarrow d$, implies that if $g = T$, then $\boxed{d = T}$. This is a contradiction since $d = F$.

Therefore the statements are inconsistent.

6. Hearken back to the times of knights and knaves where knaves always lie and knights always tell the truth. Suppose you meet three people, Anita, Boris, and Carmen. What are Anita, Boris, and Carmen if Anita says, "I am a knave and Boris is a knight" and Boris says, "Exactly one of the three of us is a knight."

Anita is lying since neither a knight nor a knave would ever say they were a knave. Then Anita is a knave, and since her statement is true about her being a knave, it must be false about Boris being a knight. So Boris is also a knave. Then Boris is lying when he says exactly one of them is a knight, and since Carmen is the only person who could have been a knight, Carmen cannot be a knight and is therefore a knave. We conclude Anita, Boris, and Carmen are all knaves.

Let (1) $\neg A_{knight} \wedge B_{knight}$
(2) $(A_{knight} \wedge \neg B_{knight} \wedge \neg C_{knight}) \vee (\neg A_{knight} \wedge B_{knight} \wedge \neg C_{knight}) \vee (\neg A_{knight} \wedge \neg B_{knight} \wedge C_{knight})$

where A_{knight} denotes "Anita is a knight", etc. If $A_{knight} = T$, then (1) is true. But this means that $\neg A_{knight}$ is true, which is a contradiction. So $A_{knight} = F$, and (1) is false. We have that the negation of (1) is equivalent to $A_{knight} \vee \neg B_{knight}$, and since $A_{knight} = F$, $\neg B_{knight}$ must be true so $B_{knight} = F$. Then (2) is false. If (2) is false, each piece is false. The first two pieces are false because $A_{knight} = F$ and $B_{knight} = F$, but for the third piece to be false we need C_{knight} to be false.

Conclusion: $A_{knight} = F$, $B_{knight} = F$, $C_{knight} = F$.

7. Let $P(x)$ be the statement "Student x knows calculus" and let $Q(y)$ be the statement "Class y contains a student who knows calculus." Express each of these as quantifications of $P(x)$ and $Q(y)$.

   (a) Some students know calculus.

   $\exists x\, P(x)$

   (b) Not every student knows calculus.

   $\exists x\, \neg P(x)$

   (c) Every class has a student in it who knows calculus.

   $\forall y\, Q(y)$

   (d) Every student in every class knows calculus.

   $\forall x \forall y\, P(x)$ or $\forall x \forall y\, [P(x) \wedge Q(y)]$

   (e) There is at least one class with no students who know calculus.

   or $\exists y\, \neg Q(y)$

8. Let $P(m,n)$ be the statement "m divides n", where the domain for both variables consists of all possible integers. (By "m divides n" we mean that $n = km$ for some integer k.) Determine the truth value of each of these statements.

   (a) $P(4,5)$ F, $5 \neq 4k$ for any integer k

   (b) $P(2,4)$ T, $4 = 2(k)$ when $k = 2$

   (c) $\forall m \forall n P(m,n)$ F, counterexample in part (a)

   (d) $\exists m \forall n P(m,n)$ T, let $m=1$. $\forall n\, P(1,n)$ is true since $n = 1(k)$ if $k=n$ for all integers n.
   ($m=-1$ also works)

   (e) $\exists n \forall m P(m,n)$ T, let $n=0$. then $\forall m\, P(m,0)$ is true since $0 = m(k)$ when $k=0$ for all integers m.

   (f) $\forall n P(1,n)$ T, same as (d)

9. Use existential and universal quantifiers to express the statement "No one has more than three grandmothers" using the propositional function $G(x,y)$, which represents "x is the grandmother of y."

Negate:
$\exists y\, \exists x_1 \exists x_2 \exists x_3 \exists x_4 \left(G(x_1,y) \wedge G(x_2,y) \wedge G(x_3,y) \wedge G(x_4,y)\right)$ to get
$\neg\left(\exists y\, \exists x_1 \exists x_2 \exists x_3 \exists x_4 \left(G(x_1,y) \wedge G(x_2,y) \wedge G(x_3,y) \wedge G(x_4,y)\right)\right)$
$\equiv \forall y\, \forall x_1 \forall x_2 \forall x_3 \forall x_4 \left(\neg G(x_1,y) \vee \neg G(x_2,y) \vee \neg G(x_3,y) \vee \neg G(x_4,y)\right)$.

10. Let $P(x)$ and $Q(x)$ be propositional functions. Show that $\exists x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \exists Q(x)$ always have the same truth value.

$$\exists x(P(x) \rightarrow Q(x)) \equiv \exists x(\neg P(x) \vee Q(x)) \text{ and}$$

$$\forall x P(x) \rightarrow \exists x Q(x) \equiv \neg(\forall x P(x)) \vee \exists x Q(x)$$
$$\equiv \exists x \neg P(x) \vee \exists x Q(x).$$

If $Q(a)$ is true for some a, then $\neg P(a) \vee Q(a)$ is true and $\exists x(\neg P(x) \vee Q(x))$. Also $\exists x Q(x)$ is true, so $\exists x \neg P(x) \vee \exists x Q(x)$ is true. Similarly, if $\neg P(a)$ is true for some a, then $\neg P(a) \vee Q(a)$ is true and $\exists x(\neg P(x) \vee Q(x))$ is true. Likewise $\exists x \neg P(x)$ is true and $\exists x \neg P(x) \vee \exists x Q(x)$. thus the two statements are equivalent.

11. Find the negations of these statements.

(a) If it snows today, then I will go skiing tomorrow.

It is snowing today and I will not go skiing tomorrow.

(b) Every person in this class understands mathematical induction.

there is a person in the class that doesn't understand mathematical induction.

(c) Some students in this class do not like discrete mathematics.

Every Student in the class likes discrete mathematics.

(d) In every mathematics class there is some student who falls asleep during lectures. ☹

There exists a mathematics class such that none of the students fall asleep during lecture.

12. Express this statement using quantifiers: "Every student in this class has taken some course in every department in the school of mathematical sciences."

Let $P(x,y,z)$ denote "x has taken course y in deparment z in the school of mathematical sciences," where the domain of x is students in the class, y is the courses offered, and z is the departments in the school of mathematical sciences.

$$\forall x \forall z \exists y\, P(x,y,z)$$

$P(x,y) =$ " x has taken course y "

13. Express this statement using quantifiers: "There is exactly one student in this class who has taken exactly one mathematics class at this school" using the following the uniqueness quantifier. Then express this statement using quantifiers, without using the uniqueness quantifier.

$\exists! x \exists! y\, P(x,y)$

Answers may vary:

Unique class: "there exists a class y_1 such that for all classes y_2, $P(x, y_2)$ if and only if $y_1 = y_2$."

$\exists y_1 \forall y_2 (P(x,y_2) \leftrightarrow y_1 = y_2)$

Unique student: "there exists a student x_1 such that for all students x_2, $\exists y_1 \forall y_2 (P(x_2,y_2) \leftrightarrow y_1 = y_2)$ if and only if $x_1 = x_2$.

Solution: $\exists x_1 \forall x_2 [\exists y_1 \forall y_2 (P(x_2,y_2) \leftrightarrow y_1 = y_2) \leftrightarrow x_1 = x_2]$

14. Use rules of inference to show that if the premises $\forall x(P(x) \rightarrow Q(x))$, $\forall x(Q(x) \rightarrow R(x))$, and $\neg R(a)$, where a is in the domain, are true, then the conclusion $\neg P(a)$ is true.

1. $\forall x (P(x) \rightarrow Q(x))$	Premise 1
2. $P(a) \rightarrow Q(a)$	Universal Instantiation (1)
3. $\forall x (Q(x) \rightarrow R(x))$	Premise 2
4. $Q(a) \rightarrow R(a)$	Universal Instantiation (3)
5. $P(a) \rightarrow R(a)$	Hypothetical Syllogism (2,4)
6. $\neg R(a) \rightarrow \neg P(a)$	Contrapositive (5)
7. $\neg R(a)$	Premise 3
8. $\neg P(a)$	Modus Ponens (6,7)

15. Prove that if x^3 is irrational, then x is irrational.

Proof by Contraposition: Let x be a rational number. Then $x = \frac{a}{b}$ for some integers a and b. Then $x^3 = (\frac{a}{b})^3 = \frac{a^3}{b^3}$. Since a^3 is an integer and b^3 is an integer, $\frac{a^3}{b^3}$ is a rational number. Then x^3 is a rational number. By contraposition, if x^3 is irrational, then x is irrational.

16. Prove that if x is irrational and $x \geq 0$, then $\sqrt{x}$ is irrational.

(this can be proved with contraposition, but since I did #15 that way I am going to do a proof by contradiction for a little variety.)

Let x be irrational with $x \geq 0$. Suppose by way of contradiction that $\sqrt{x}$ is rational. Then $\sqrt{x} = \frac{a}{b}$ for some integers a and b. Then $(\sqrt{x})^2 = (\frac{a}{b})^2$, and $x = \frac{a^2}{b^2}$ where a^2 and b^2 are both integers. But this is a contradiction since x was supposed to be irrational and consequently cannot be written as a ratio of two integers. Therefore $\sqrt{x}$ cannot be rational and is instead irrational.