Example of logical equivalences using THEOREM 2.1.1 of the book and conditional statement equivalence.

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$:	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

And $p \to q \equiv \sim p \vee q$

Example 1:

Show that $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$.

L.H. S = $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q)$ 1. De Morgan's law

$\equiv (\sim p \wedge \sim(\sim q)) \vee (\sim p \wedge \sim q)$ 2. Double negation law

$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q)$ 3. Distributive law

$\equiv (\sim p \wedge (q \vee \sim q))$ 4. Negation law

$\equiv (\sim p \wedge t)$ 5. Identity law

$\equiv \sim p$

Example 2:

Show that $\sim (q \to p) \vee (p \wedge q) \equiv q$.

L.H.S $= \sim (q \to p) \vee (p \wedge q)$ 1. Implication / conditional equivalence

$\equiv \sim (\sim q \vee p) \vee (p \wedge q)$ 2. De Morgan's law

$\equiv (\sim (\sim q) \wedge \sim p) \vee (p \wedge q)$ 3. Double negation law and commutative law $(p \wedge q) = (q \wedge p)$

$\equiv (q \wedge \sim p) \vee (q \wedge p)$ 4. Distributive law

$\equiv \quad (q \wedge (\sim p \vee p))$ 5. Negation law

$\equiv \quad (q \wedge T)$ 6. Identity law

$\equiv \quad q$

Example 3:

Show that $(p \vee \sim q) \wedge (\sim p \vee \sim q) \equiv \sim q$

$(p \vee \sim q) \wedge (\sim p \vee \sim q)$ 1. Commutative law

$\equiv (\sim q \vee p) \wedge (\sim q \vee \sim p)$ 2. Distributive law

$\equiv \sim q \vee (p \wedge \sim p)$ 3. Negation law

$\equiv \sim q \vee c$ 4. Identity law

$\equiv \sim q$

Example 4: (Show that the following statement is a tautology)

L.H. S $= (p \wedge q) \rightarrow (p \vee q)$ 1. Implication equivalence

$\equiv \sim (p \wedge q) \vee (p \vee q)$ 2. De Morgan's law

$\equiv (\sim p \vee \sim q) \vee (p \vee q)$ 3. Associative law

$\equiv (\sim p \vee \sim q \vee p) \vee q$ 4. Commutative law

$\equiv (\sim p \vee p \vee \sim q) \vee q$ 5. Associative law

$\equiv (\sim p \vee p) \vee (\sim q \vee q)$ 6. Negation law

$\equiv t \vee t$ 7. Idempotent law

$\equiv t$, It's a tautology .

Example 5: (Show that the following statement is a tautology)

L.H.S $= (p \wedge q) \rightarrow p$ 1.Implication equivalence

$\equiv \sim (p \wedge q) \vee p$ 2. De Morgan's law

$\equiv (\sim p \vee \sim q) \vee p$ 3. Commutative law

$\equiv (\sim q \vee \sim p) \vee p$ 4. Associative law

$\equiv \sim q \vee (\sim p \vee p)$ 5. Negation law

$\equiv \sim q \vee t$ 6. Universal bound law

$\equiv t$, it's a tautology.

Example 6:

Show that $\sim ((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$.

L.H.S $= \sim ((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$ 1. De Morgan's law

$\equiv (\sim (\sim p \wedge q) \wedge \sim (\sim p \wedge \sim q)) \vee (p \wedge q)$ 2. De Morgan's law

$\equiv (\sim(\sim p) \vee \sim q) \wedge ((\sim(\sim p) \vee \sim (\sim q)) \vee (p \wedge q)$ 3. Double negation law

$\equiv ((p \vee \sim q) \wedge (p \vee q)) \vee (p \wedge q)$ 4. Distributive law

$\equiv ((p \vee (\sim q \wedge q)) \vee (p \wedge q)$ 5. Negation law

$\equiv (p \vee c) \vee (p \wedge q)$ 6. Identity law

$\equiv p \vee (p \wedge q)$ 7. Absorption law

$\equiv p$

Example 7:

Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.

L.H.S $= (p \rightarrow q) \vee (p \rightarrow r)$ 1. Implication Equivalence

$\equiv (\sim p \vee q) \vee (\sim p \vee r)$ 2. Associative law

$\equiv \sim p \vee q \vee \sim p \vee r$ 3. Commutative law

$\equiv \sim p \vee \sim p \vee q \vee r$ 4. Idempotent law

$\equiv \sim p \vee (q \vee r)$ 5. Implication law

$\equiv p \to q \vee r$, they are logically equivalent.

Example 8:

Show that $\sim (p \vee (\neg p \wedge q)) \equiv (\sim p \wedge \sim q) \vee c$.

L.H.S $= \sim (p \vee (\neg p \wedge q))$ 1. De Morgan's law

$\equiv \sim p \wedge \sim (\sim p \wedge q)$ 2. De Morgan's law

$\equiv \sim p \wedge [\sim (\sim p) \vee \sim q]$ 3. Double negation law

$\equiv \sim p \wedge (p \vee \sim q)$ 4. Distributive law

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ 5. Negation law

$\equiv c \vee (\sim p \wedge \sim q)$ 6. Commutative law

$\equiv (\sim p \wedge \sim q) \vee c$ 7. Identity law

$\equiv \sim p \wedge \sim q$

**Try it yourself: Show that $(q \wedge (p \to \sim q)) \to \sim p$ is a tautology.