11.3 Lecture 1 (1.1 Systems of Linear Equations)

Basic object of study: linear equations (ℓ.e.)

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

where

- $a_1, \ldots, a_n$ are coefficients;
- $x_1, \ldots, x_n$ are variables;
- $a_1, \ldots, a_n, b \in \mathbb{R}$ (or $\mathbb{C}$).

Example 11.1 (☆☆☆☆☆ Examples of ℓ.e.)

(1) $3x_1 + 2x_2 + 2022x_3 = \log_2 17 \to \ell$.e.

(2) $3x_1^2 + 2x_1 x_3 + \sin x_4 = 8 \to$ not ℓ.e.

Definition 11.3

- A **system of linear equations** (s.ℓ.e.) is a collection of one or more linear equations.
- A **solution** to a system of linear equations is an n-tuple $(s_1, s_2, \cdots, s_n)$ that makes each equation in the system of linear equations true after you replace x_i with s_i.
- The **solution set** of a system of linear equations is the set of all solutions to a system of linear equations.
- Two systems of linear equations are **equivalent** if they have the same solution set.

Example 11.2 (☆☆☆☆☆ solution of s.ℓ.e.)

$$\begin{cases} 3x_1 + 2x_2 = 3 \\ -x_1 + x_2 = 4. \end{cases} \tag{11.1}$$

is a system of linear equations, and $(s_1, s_2) = (-1, 3)$ is a solution.

Question **(Fundamental questions)** Given a systems of linear equations, we ask

(1) Does a solution to the systems of linear equations exist?

(2) If a solution exists, is it unique?

These two questions (in various disguises) appear throughout the course.

How big is a solution set to a systems of linear equations?

Example 11.3 (☆☆☆☆☆ **Intuition from 2-variable case**) Equations of lines in the plane:

$$\begin{cases} a_1\, x_1 + a_2\, x_2 = b, \\ c_1\, x_1 + c_2\, x_2 = d. \end{cases} \tag{11.2}$$

Two lines in the plane can intersect in 3 ways:

(1) one point

(2) meet at no points → parallel lines

(3) everywhere → the same line

Theorem 11.2 A system of linear equations has either

(1) no solution (inconsistent);

(2) exactly one solution (consistent);

(3) infinitely many solutions (consistent).

Definition 11.4 **(consistent and inconsistent)** A s.ℓ.e. is consistent if it has one or infinite number of solutions; otherwise, a s.ℓ.e. is inconsistent.

Linear algebra also studies matrices, rectangular arrays of numbers. A system of linear equations can be stored compactly as a matrix. For example,

$$\text{s.}\ell\text{.e.}: \begin{cases} \quad x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad\quad 2x_2 - 8x_3 = 8 \\ 5x_1 \qquad\quad - 5x_3 = 10 \end{cases} \tag{11.3}$$

$$\text{augmented matrix}: \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \tag{11.4}$$

To solve a system of linear equations, our strategy is to replace one system with an equivalent system that is easier to solve. To solve, we manipulate the augmented matrix.

Basic operations:

Definition 11.5 (Elementary row operations)

(1) (Scaling) Multiply one equation by a nonzero constant.

(2) (Replacement) Add a constant multiple of one equation to another equation.

(3) (Interchange) Interchange any two equations.

These three operations do not change the solution set.

Example 11.4 (☆☆☆☆☆ Elementary row operations)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ 5x_1 \phantom{{}-2x_2} - 5x_3 = 10 \end{cases} \tag{11.5}$$

Solution

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \begin{array}{l} \cdot(-5) \\ \\ + \end{array}$$ augmented matrix; use (Replacement) in Definition 11.5

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \begin{array}{l} \\ \mid \cdot \frac{1}{2} \\ \mid \cdot \frac{1}{10} \end{array}$$ use (Scaling) in Definition 11.5

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{array}{l} \\ \cdot(-1) \\ + \end{array}$$ use (Replacement) in Definition 11.5

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 3 & -3 \end{bmatrix} \begin{array}{l} \\ \\ \mid \cdot (\frac{1}{3}) \end{array}$$ use (Scaling) in Definition 11.5

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$ lower triangular system $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = -1 \end{cases}$$ start back-substitution

$\Longrightarrow x_2 = 4x_3 + 4 = 4 \cdot (-1) + 4 = 0$ use $x_3 = -1$ and $x_2 - 4x_3 = 4$

$\Longrightarrow x_1 = 2x_2 - x_3 = 2 \cdot (0) - (-1) = 1$ use $x_3 = -1$, $x_2 = 0$, and $x_1 - 2x_2 + x_3 = 0$

$\Longrightarrow x_1 = 1, x_2 = 0, x_3 = -1$ unique solution

Key ideas:

- ℓ.e. and s.ℓ.e.
- fundamental questions: does a s.ℓ.e. have a solution?
- s.ℓ.e. and matrix notation
- basic operations that preserves s.ℓ.e.

11.5 Lecture 2 (1.2 Row Reduction and Echelon Forms)

Definition 11.11 (echelon form (or row echelon form)[a]) A rectangular matrix is in echelon form (or row echelon form) if

(1) all nonzero rows are above any rows of all zeros; [b]

(2) each leading entry (leftmost nonzero entry) is in a column to the right of the leading entry in the row above;[c]

(3) all entries in a column below a leading entry are zeros. [d]

The matrix is in reduced echelon form (or reduced row echelon form) if

(1') each leading entry (in each nonzero row) is 1[e];

(2') each leading 1 is the only non zero entry in its column[f].

[a]列梯形式矩陣

[b]所有非零列（矩陣的列至少有一個非零元素）在所有全零列的上面，即全零列都在矩陣的底部。

[c]非零列的首項 (最左邊的元素)，也稱作主元（leading entry）。每列最左邊的首個非零元素，嚴格地比上面列的首項係數更靠右。也就是上面列的主元比下面列的主元更靠左。

[d]首項係數所在行，在該首項係數下面的元素都是零（前兩條的推論）。主元所在之行，下面元素都是零。

[e]每個 leading entry 都要是 1。

[f]每個 leading entry1 是其所在之行唯一非零元素，亦即，所在之行其他元素皆為零。

Example 11.8 (☆☆☆☆☆ Echelon form and reduced echelon form ())

1. $\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 0 & 0 & 0 & 15 \end{bmatrix}$: (row) echelon form, not reduced (row) echelon form.

   (1) 無全零列。✓

   (2) 第二列的主元6，所在之行，是在第一列的主元1的右邊。第一列的主元1，比第二列的主元6更靠左。✓[a]

   (3) 主元6跟1，所在之行，6跟1下面元素都是 0 。✓

   (1') 主元有非 1 元素。✕

   (2') 主元所在之行，有非 1 元素 (紫色)。✕

2. $\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$: (row) echelon form, not reduced (row) echelon form.

   (1) 無全零列。✓

   (2) 第二列的主元1，所在之行，是在第一列的主元1的右邊。第一列的主元1，比第二列的主元1更靠左。✓

   (3) 主元1跟1，所在之行，1跟1下面元素都是 0 。✓

   (1') 主元皆為 1。✓

   (2') 主元所在之行，有非 0 元素 (紫色)。✕

3. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$: reduced row echelon form

   (1) 無全零列。✓

   (2) 第二 (三) 列的主元1，所在之行，是在第一 (二) 列的主元1的右邊。第一 (二) 列的主元1，比第二 (三) 列的主元1更靠左。✓

   (3) 主元1跟1，所在之行，1跟1下面元素都是 0 。✓

   (1') 主元皆為 1。✓

   (2') 主元所在之行，無非 0 元素 (紫色)。✓

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[a] 將第一列與第二列互換 $\begin{bmatrix} 0 & 6 & -6 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 0 & 0 & 15 \end{bmatrix}$ 就不符合了。或是將第三列移至第一列 $\begin{bmatrix} 0 & 0 & 0 & 15 \\ 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \end{bmatrix}$ 亦然。也就「上面列的主元比下面列的主元更靠左」這件事不成立。或是也可以看「第一列的主元、第二列的主元、第三列的主元是否有由左至右」。

Definition 11.12 (Elementary row operations) Gaussian elimination: procedure to put a matrix into reduced (row) echelon form using 3 rules:

(1) (Replacement) Add a constant multiple of one row to another.

(2) (Interchange) Interchange two rows.

(3) (Scaling) Multiply a row by a nonzero constant.

Example 11.9 (Elementary row operations ())

$$\begin{bmatrix} 0 & 2 & -2 & 1 \\ 1 & 2 & -1 & -3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

begin with leftmost column $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

interchange top row so a nonzero entry is at top of its column

$$\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \\ 1 & -1 & 2 & 3 \end{bmatrix} \begin{array}{l} \cdot(-1) \\ \\ + \end{array}$$

use elementary row operation to create zeros in rest of the column

$$\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \\ 0 & -3 & 3 & 6 \end{bmatrix} \begin{array}{l} \\ | \cdot (3) \\ | \cdot (2) \end{array}$$

once all entries below leading entry are 0, ignore top row.

Repeat procedure on remaining submatrix

$$\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 0 & -6 & 6 & 12 \end{bmatrix} \begin{array}{l} \\ \cdot(1) \\ + \end{array}$$

row 2+row 3

$$\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

Stop when you reach echelon form

$$\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 0 & 0 & 0 & 15 \end{bmatrix} \begin{matrix} \\ |\cdot(\frac{1}{6}) \\ |\cdot(\frac{1}{15}) \end{matrix}$$

multiply each row by $1/a$,

where a is the leading entry

$$\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow + \\ \\ \cdot(3) \end{matrix} \quad \begin{matrix} \\ \leftarrow + \\ \cdot(\frac{-1}{2}) \end{matrix}$$

to get reduced row echelon form, begin with rightmost

leading 1 and create 0's in the columns above it.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow + \\ \cdot(-2) \\ \\ \end{matrix}$$

Then move left

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reduced row echelon form

Definition 11.13 (Row equivalent) Two matrices A and B are **row equivalent** if B can be obtained from A via elementary row operations.

Theorem 11.5 (Uniqueness of reduced row echelon form) Every matrix is row equivalent to a unique reduced row echelon matrix.[a]

[a] 簡約列梯形式的唯一性 (線代啟示錄)

Example 11.10 (☆☆☆☆☆ Row equivalent matrices ())

In Example 11.9, $\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 0 & 0 & 0 & 15 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are row equivalent.

Remark (**Leading entries in row echelon forms**) Although row echelon forms are not unique, leading entries are in the same spot. In Example 11.9,

- $\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 0 & 0 & 0 & 15 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Definition 11.14 (Pivot position and Pivot column)

- **(Pivot position)** A pivot position in a matrix A is a location in A that corresponds to a leading 1 in reduced row echelon form of A.[a]
- **(Pivot column)** A pivot column is a column that contains a pivot. [b]

[a] reduced row echelon form 中，pivot position 就是 leading 1 所在的位置。1 這元素就叫做 pivot。
[b] reduced row echelon form 中，Pivot column 就是所有有 pivot 的 column。

Example 11.11 (☆☆☆☆☆ **Pivot position and Pivot column ()**)

Pivot positions

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivot columns

[a]

[a] 使用 tikz 的畫圖技術，尚未知如何同時標示一元素為 pivot position 與 pivot column。

Key ideas:

- (row) echelon form
- reduced (row) echelon form: uniqueness
- row equivalent
- pivot, pivot position, and pivot column

11.6 Lecture 3 (1.2 Row Reduction and Echelon Forms (Continued)

Key idea:

- "Shape" of the echelon matrix tells us the # of solutions
- Gaussian elimination can find these solutions
- Infinitely many solutions $\implies$ parametric solution

Recall from Theorem 11.4 that, a system of linear equations has

(1) no solution, or

(2) exactly one solution, or

(3) infinitely many solutions.

把增廣矩陣化約為 echelon form (不需為 reduced echelon form) 後，就可判斷是哪個 case。

Remark (**Row echelon forms for numbers of solutions to system of linear equations**) The "shape" of the echelon form of the augmented matrix determines which case.

(1) (No solution) Echelon form has row of the form:

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & b \end{array}\right] \text{with } b \neq 0 \Rightarrow\Leftarrow 0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = b \tag{11.6}$$

(2) (Exactly one solution) [a]Echelon form has a pivot (non-zero leading entry) in each row and column except the last column[b]

$$\left[\begin{array}{cccc|c} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{array}\right], \tag{11.7}$$

where

$\blacksquare$: nonzero entry

$*$: any value

(3) (Infinitely many number of solutions) Echelon form has no pivot in last column, and (number of pivots) < (number of column−1)[c]

$$\left[\begin{array}{cccc|c} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{array}\right] \tag{11.8}$$

[a]除了最後一行 (代表方程式的右邊項)，每一行每一列都要有 pivot

[b]最後一列 $\blacksquare \cdot x_n = *$，因為 $\blacksquare \neq 0$，故可解出 x_n。再往上一列，$\blacksquare \cdot x_{n-1} + * \cdot x_n = *$，$x_n$ 已得，因 $\blacksquare \neq 0$，可解出 x_{n-1}。如此再往上一列 $\cdots$，即可解出方程組之解。

[c]如此例，number of pivots= 3 代表有多少個「有效」方程式，而 number of column$-1 = 4 > 3$ 為未知數之數目。

Example 11.12 (☆☆☆☆☆ **Exactly one solution()**) In Example 11.4, we solved

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases} \tag{11.9}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \quad \text{augmented matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{reduced echelon form; lower triangular system} \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = -1 \end{cases} \quad \text{start back-substitution}$$

$$\Longrightarrow x_2 = 4x_3 + 4 = 4 \cdot (-1) + 4 = 0 \quad \text{use } x_3 = -1 \text{ and } x_2 - 4x_3 = 4$$

$$\Longrightarrow x_1 = 2x_2 - x_3 = 2 \cdot (0) - (-1) = 1 \quad \text{use } x_3 = -1,\ x_2 = 0, \text{ and } x_1 - 2x_2 + x_3 = 0$$

$$\Longrightarrow x_1 = 1, x_2 = 0, x_3 = -1 \quad \text{unique solution}$$

Example 11.13 (☆☆☆☆☆ **No solution ()**) Solve

$$\begin{cases} x_1 - 2x_2 + 5x_3 = 9 \\ -x_1 + 3x_2 = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$$

Solution Reduced row echelon form: $\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Longrightarrow$ no solution.

Example 11.14 (☆☆☆☆☆ **Infinitely many solutions ()**) Solve

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 1 \\ 2x_1 - 4x_2 + x_3 = 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 = 4 \end{cases} \tag{11.10}$$

Solution

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \qquad \text{augmented matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{reduced row echelon form}$$

$\Longrightarrow$

#pivot$= 2$

#column$-1 = 5 - 1 = 4$

$\Longrightarrow$ $2 < 4$

$\Longrightarrow$ Infinitely many solutions

Question How to express all solution in Example 11.14?

- variables corresponding to pivot columns=basic variables or leading variables
- variables corresponding to non-pivot columns=free variables

Solution **(conti. Example 11.14)**

- variables corresponding to pivot columns=basic variables or leading variables: x_1, x_3
- variables corresponding to non-pivot columns=free variables: x_2, x_4

$\Longrightarrow$ x_1, x_3 can be expressed in terms of x_2 and x_4

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{reduced row echelon form (from Example 11.14)}$$

$$\Longrightarrow \begin{cases} x_1 - 2x_2 \quad + \quad x_4 = 2 \\ \qquad\qquad x_3 - 2x_4 = 1 \end{cases} \qquad \text{system of 2 linear equations}$$

$$\Longrightarrow \begin{cases} x_1 = 2x_2 - x_4 + 2, \\ x_3 = 2x_4 + 1. \end{cases} \qquad \text{free variables: } x_2,\ x_4$$

$$\Longrightarrow \begin{cases} x_1 = 2s - t + 2, \\ x_2 = s, \\ x_3 = 2t + 1, \\ x_4 = t, \end{cases} \qquad \text{let } x_2 = s,\ x_4 = t, \text{ where } s, t \in \mathbb{R}; \text{ solution in parametric form}$$

11.7 Lecture 4 (1.3 Vector Equations)

僕の言葉は絶対だ (Joke 211104)

- dachshund=Dooooooooog
- caterpillar(live)=ommmmmmm
- caterpillar(dead)=owwwwwww

- 臘腸狗的英文 =Dooooooooog
- 毛毛蟲 (活的) 的英文 =ommmmmmm
- 毛毛蟲 (死的) 的英文 =owwwwwwww

Key idea:

- vector (column vector), vector sum, scalar mulplication
- linear combinations of vector
- span

Definition 11.15 (Column vector)

- A matrix with one column is a column vector or simply a vector.
- Two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ are equal if and only if $u_i = v_i$ for all $i = 1, \ldots, n$.
- The zero vector in $\mathbb{R}^n$ is $\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

[a]

[a] 我們規定向量是用直的表示。

Example 11.18 (☆☆☆☆☆ vectors)

(1) $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(2) $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$

(3) $\mathbb{R}^n = \{ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \big| u_i \in \mathbb{R}, i = 1, \ldots, n \}$ (n-space: all vectors with n entries)

Definition 11.16 (Operation on vectors)

- (vector addition) $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \overset{\text{define}}{:=} \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$
- (scalar multiplication) $c \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \overset{\text{define}}{:=} \begin{bmatrix} c\,u_1 \\ c\,u_n \end{bmatrix}$, where $c \in \mathbb{R}$ is a scalar.

Theorem 11.6 (Algebraic Properties of $\mathbb{R}^n$) For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u} := (-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

Looking ahead: these properties imply that $\mathbb{R}^n$ is vector space.

Remark (**Vectors and Geometry**)

- When $n = 2$ or $n = 3$, we can visualize $\mathbb{R}^2$ or $\mathbb{R}^3$.
- Visualize $\mathbb{R}^2$ as the plane. Identify $\begin{bmatrix} a \\ b \end{bmatrix}$ with the point (a,b) and draw directed arrow from $(0,0)$ to (a,b).
- The sum $\mathbf{u}+\mathbf{v}$ is the fourth vertex of the parallelogram whose other vertices are $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$.
- Scalar multiplication stretches the line through $\mathbf{0}$ and $\mathbf{u}$.
- Geometric idea extends to $\mathbb{R}^n$ (although harder to draw!). A vector $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ corresponds to directed arrow from $(0,\dots,0)$ to $(u_1,\dots,u_n)$.

Definition 11.17 **(Linear combination)** Given vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_p \in \mathbb{R}$, the vector

$$y := c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \cdots + c_p\,\vec{v}_p \tag{11.12}$$

is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$. The $c_i's$ are weights.

Example 11.19 (☆☆☆☆☆ **Linear combination**) Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$. Then $2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 0 \end{bmatrix}$, which is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Example 11.20 (☆☆☆☆☆ **Find linear combination**) Let $\mathbf{a}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Write $\mathbf{b} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Solution Find $c_1, c_2 \in \mathbb{R}$ such that $c_1\begin{bmatrix} 6 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$, i.e.

$$\begin{cases} 6\,c_1 - 3\,c_2 = -3 \\ -c_1 + 4\,c_2 = 11. \end{cases} \tag{11.13}$$

$\Longrightarrow c_1 = 1,\ c_2 = 3.$

Remark (**Vector equations & augmented matrix of linear systems**) [a] A vector equation

$$x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \cdots + x_n\,\mathbf{a}_n = \mathbf{b} \tag{11.14}$$

has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{array}\right] \tag{11.15}$$

[a] vector equation 的 solution set，跟把 vector equation 的向量們還有右邊項形成的增廣矩陣的 solution set，是一致的。

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ of vectors.

Definition 11.18 (**Spanning sets**) [a] If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$, then the subset of $\mathbb{R}^n$ spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$ is the set

$$\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} := \{c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \cdots + c_p\,\vec{v}_p \mid c_1, c_2, \ldots, c_p \in \mathbb{R}\} \tag{11.16}$$

[a] 把給定向量們的線性組合收集起的集合，就稱為 Spanning set。由給定向量們所 span 出來的集合稱為 Spanning set。

Example 11.21 (☆☆☆☆☆ **Spanning sets**)

(a) From Example 11.20, $1\begin{bmatrix}6\\-1\end{bmatrix} + 3\begin{bmatrix}-3\\4\end{bmatrix} = \begin{bmatrix}-3\\11\end{bmatrix} \implies \begin{bmatrix}-3\\11\end{bmatrix} \in \mathrm{Span}\{\begin{bmatrix}-3\\11\end{bmatrix}, \begin{bmatrix}-3\\4\end{bmatrix}\}$

(b) For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in \mathbb{R}^n$, we have $\mathbf{0} = 0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_1 + \cdots + 0\cdot\mathbf{v}_p \implies \mathbf{0} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$

(c) $c\,\mathbf{v}_i = 0\cdot\mathbf{v}_1 + \cdots + c\cdot\mathbf{v}_i + \cdots + 0\cdot\mathbf{v}_p \implies \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ contains all multiples of $\mathbf{v}_i$

Remark (**Geometry and $\mathbb{R}^3$**) Geometrically, if $\mathbf{u}, \mathbf{u} \in \mathbb{R}^3$:

- $\mathrm{Span}\{\mathbf{u}\}$ is the line in $\mathbb{R}^3$ through the origin and $\mathbf{u} \in \mathbb{R}^3$.
- If $\mathbf{v}$ is not a multiple of $\mathbf{u}$, then $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ through $\mathbf{u}$, $\mathbf{v}$.

Remark (**Spanning & finding a solution to a linear system**) [a] Determine if $\mathbf{b} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is equivalent to finding a solution to

$$x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 + \cdots + x_p\,\mathbf{a}_p = \mathbf{b}, \tag{11.17}$$

which is equivalent to finding a solution to

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p & \mathbf{b} \end{array}\right] \tag{11.18}$$

[b]

[a] 要決定 **b** 是否在行向量們所形成的 spanning set, 端看以行向量與 **b** 所形成的增廣矩陣是否有解，有解就是在，無解就是不在。

[b] See for Example 11.20.

11.8 Lecture 5 (1.4 The Matrix Equation $Ax = b$)

Key idea:

- new viewpoint of systems of linear equations $A\mathbf{x} = \mathbf{b}$
- when $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$
- properties of $A\mathbf{x}$

Definition 11.19 (Matrix equation) Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. If $\mathbf{x} \in \mathbb{R}^n$, then the product of A and $\mathbf{x}$, denoted $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries of $\mathbf{x}$ as weights, i.e.

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \cdots + x_n\,\mathbf{a}_n \tag{11.19}$$

Example 11.22 (☆☆☆☆☆ Matrix equations)
Write the following system of linear equations in the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 + 5x_2 + 6x_3 = -3 \\ 7x_1 + 8x_2 + 9x_3 = 6 \end{cases} \tag{11.20}$$

Solution

- (Vector equations) $x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix}$
- (Matrix equations) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix}$

Three ways to view system of linear equations:

Theorem 11.7 (Matrix equations & vector equations & augmented matrix of linear systems) Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$. Then the equation $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ has the same solution set as the vector equations

$$x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \cdots + x_n\,\mathbf{a}_n = \mathbf{b}, \tag{11.21}$$

which has the same solution set as the system of linear equations with augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{array}\right] \tag{11.22}$$

Question [a] Given A,

- When does $A\mathbf{x} = \mathbf{b}$ has a solution for fixed $\mathbf{b}$?
- When does $A\mathbf{x} = \mathbf{b}$ has a solution for all (free) $\mathbf{b}$?

[a] 前面我們已經解過很多 $A\mathbf{x} = \mathbf{b}$ 的解，這些 case 中，當然 $\mathbf{b}$ 都是給定的。現在我們問一個不同的問題，對哪些 $\mathbf{b}$，$A\mathbf{x} = \mathbf{b}$ 有解？答案當然是可能對某些 $\mathbf{b}$ 才有解，如 Example 11.23 所示。

Example 11.23 (★★★★★ $A\mathbf{x} = \mathbf{b}$ **has a solution for some b**)
Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 14 \\ 1 & 3 & 5 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Does $A\mathbf{x} = \mathbf{b}$ has a solution for all possible $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$?

Solution

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 8 & 14 & b_2 \\ 1 & 3 & 5 & b_3 \end{bmatrix} \qquad \text{augmented matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 4 & 8 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 - \frac{1}{4}(b_2 - 2b_1) \end{bmatrix} \qquad \text{row echelon form}$$

$\Longrightarrow$ matrix is consistent if and only if $b_3 - b_1 - \frac{1}{4}(b_2 - 2b_1) = 0$
$\Longleftrightarrow$ $-\frac{1}{2}b_1 - \frac{1}{4}b_2 + b_3 = 0$ [a]

[a] 最後變成兩個獨立的方程式，三個未知數。

Theorem 11.8 () [a] Let A be an $m \times n$ matrix. The following are equivalent[b]:

(a) $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$.

(b) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

(c) Columns of A span $\mathbb{R}^n$.

(d) A[c] has a pivot position in each row.

---

[a] Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ span $\mathbb{R}^n$ means Span$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\} = \mathbb{R}^n$ or every $\mathbf{b} \in \mathbb{R}^m$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$. (c) 是 A 的 Columns span $\mathbb{R}^n$，意思就是，對於任意 $\mathbb{R}^m$ 中的 vector，都可以用 A 的 Columns 的線性組合產生，故這句話就是 (b)。而 (a) 與 (b) 沒什麼差別。(d)

[b] 等價意思是，這些敘述全真或全假。

[c] 注意！不是 $A\mathbf{x} = \mathbf{b}$ 的增廣矩陣，是只有 A。

Theorem 11.9 **(Properties of the Matrix-Vector Product $A\mathbf{x}$)** Let A be an $m \times n$ matrix, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

(b) $A(c\mathbf{u}) = c(A\mathbf{u})$

11.9 Lecture 6 (1.5 Solution Sets of Linear Equations)

Key ideas:

- Homogeneous system of linear equations
- Describe all solutions to a system of linear equations as a span of vectors

僕の言葉は絶対だ **((211118))**

- Q: What happened in 1809?
- A: Abraham Lincoln was born.
- Q: What happened in 1819?
- A: Abraham Lincoln was ten years old.

僕の言葉は絶対だ **((211118) from A Joke-A-Day: 200 Kid-Friendly Jokes For The Classroom)**

- Q: Which is faster, heat or cold?
- A: Heat, because you can catch a cold.

Definition 11.20 (Homogeneous linear systems) A system of linear equations is homogeneous if it has the form $A\mathbf{x} = \mathbf{0}$.

Remark (A homogeneous linear system has the trivial solution 0) A homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely

$$\mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{11.24}$$

normally called trivial solution.

Question Does a homogeneous system of linear equations have only the trivial solution or an infinite numbers of solutions?[a]

[a] 線性方程組解的個數只有三種可能，既然不是零個解，那便是唯一解或是無窮多解。

Theorem 11.10 (non-trivial solution $\Longleftrightarrow$ one free variable) $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution if and only if the system has at least one free variable.

Example 11.26 (☆☆☆☆☆ **Homogeneous system**) Solve the homogeneous system of linear equations:

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 0 \\ -2x_1 - 5x_2 + 4x_3 = 0 \\ -x_1 + 2x_2 + 2x_3 = 0 \end{cases} \tag{11.25}$$

Solution

$$\begin{bmatrix} 1 & 3 & -2 & 0 \\ -2 & -5 & 4 & 0 \\ -1 & 2 & 2 & 0 \end{bmatrix} \quad \text{augmented matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{bmatrix} \quad \text{row operation}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{reduced row echelon form}$$

$$= \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{\#pivot= 2; \#column= 4; free variable: } x_3$$

$$\Rightarrow \begin{cases} x_1 = 2x_3, \\ x_2 = 0, \\ x_3 = x_3 \end{cases} \quad \text{solution in parametric form}$$

$$\Rightarrow \mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \text{solution in column vector form}$$

$$\Rightarrow \mathbf{x} = \text{Span}\left\{ c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, c \in \mathbb{R} \right\} \quad \text{describe all solutions as a span of vector(s)}$$

[a]

[a] 找出沒有 pivot 的 column。第幾個 column 沒有 pivot, free variable 就是第幾個變數。

Definition 11.21 **(Non-homogeneous linear systems)** A system of linear equations is non-homogeneous if it has the form $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \neq 0$.

Question How are the solution sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ related?

Example 11.27 (☆☆☆☆☆ **Non-homogeneous system**) Find all solutions to

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 1 \\ -2x_1 - 5x_2 + 4x_3 = 2 \\ -x_1 + 2x_2 + 2x_3 = 19 \end{cases} \tag{11.26}$$

Solution

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ -2 & -5 & 4 & 2 \\ -1 & 2 & 2 & 19 \end{bmatrix} \quad \cdot(2),\ \cdot(1)$$ augmented matrix

$$\sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 5 & 0 & 20 \end{bmatrix} \quad \cdot(-5)$$ row operation

$$\sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row echelon form

$$= \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ #pivot= 2; #column= 4; free varlabile: x_3

$$\Rightarrow \begin{cases} x_1 = 2x_3 - 11, \\ x_2 = 4, \\ x_3 = x_3 \end{cases}$$ solution in parametric form

$$\Rightarrow \mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -11 \\ 4 \\ 0 \end{bmatrix}$$ solution in column vector form

$$\Rightarrow \mathbf{x} = c\mathbf{v} + \mathbf{p}$$ $\mathbf{v} := \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{p} := \begin{bmatrix} -11 \\ 4 \\ 0 \end{bmatrix}$, $x_3 := c$

Example 11.26 $\Rightarrow$ $c\mathbf{v}$ is the solution to $A\mathbf{x} = \mathbf{0}$

Solution sets are "translation".

Theorem 11.11 (How to find all solutions to a non-homogeneous system) [a] Let $\mathbf{p}$ be any solution to $A\mathbf{x}=\mathbf{b}$. Then

$$\begin{aligned} &\{\mathbf{w} \mid A\mathbf{w}=\mathbf{b}\} && \text{find all solutions to } A\mathbf{x}=\mathbf{b} \\ =&\{\mathbf{p}+\mathbf{v} \mid A\mathbf{v}=\mathbf{0}\} && \text{find one solution to } A\mathbf{x}=\mathbf{b} \\ & && \text{and add it to all solutions of } A\mathbf{x}=\mathbf{0} \end{aligned}$$

[a] 意思是，要找 $A\mathbf{w}=\mathbf{b}$ 所有解時，可先找 $A\mathbf{w}=\mathbf{b}$ 之任一解 $\mathbf{p}$，然後把 $A\mathbf{w}=\mathbf{0}$ 所有解 $\mathbf{v}$ 找出來，最後 $\mathbf{p}+\mathbf{v}$ 就是 $A\mathbf{w}=\mathbf{b}$ 之所有解。

Proof. Key: A is linear. Use the fact: $\mathcal{A}\subseteq\mathcal{B}$ and $\mathcal{B}\subseteq\mathcal{A} \iff \mathcal{A}=\mathcal{B}$.[1]

(1) Prove $\{\mathbf{w} \mid A\mathbf{w}=\mathbf{b}\} \subseteq \{\mathbf{p}+\mathbf{v} \mid A\mathbf{v}=\mathbf{0}\}$[2]

$$\begin{aligned} \mathbf{w} =& \mathbf{p}+(\mathbf{w}-\mathbf{p}) && \mathbf{w} \text{ satisfies } A\mathbf{w}=\mathbf{b}\text{; decompose } \mathbf{w} \text{ as } \mathbf{w}=\mathbf{p}+(\mathbf{w}-\mathbf{p}) \\ \Rightarrow A(\mathbf{w}-\mathbf{p}) =& A\mathbf{w}-A\mathbf{p} && A \text{ is linear} \\ =& \mathbf{b}-\mathbf{b} && A\mathbf{w}=\mathbf{b};\ A\mathbf{p}=\mathbf{b} \\ =& \mathbf{0} && \text{simplify} \\ \Rightarrow A\mathbf{v} =& \mathbf{0} && \text{let } \mathbf{w}-\mathbf{p}:=\mathbf{v} \end{aligned}$$

(2) Prove $\{\mathbf{p}+\mathbf{v} \mid A\mathbf{v}=\mathbf{0}\} \subseteq \{\mathbf{w} \mid A\mathbf{w}=\mathbf{b}\}$[3]

$$\begin{aligned} A(\mathbf{p}+\mathbf{v}) =& A\mathbf{p}+A\mathbf{v} && A \text{ is linear} \\ =& \mathbf{b}+\mathbf{0} && \mathbf{p} \text{ is a solution to } A\mathbf{x}=\mathbf{b};\ A\mathbf{v}=\mathbf{0} \\ =& \mathbf{b} && \Rightarrow (\mathbf{p}+\mathbf{v}) \in \{\mathbf{w} \mid A\mathbf{w}=\mathbf{b}\} \end{aligned}$$

□

[1] 證明兩集合相等常用手法，互相包含，你中有我，我中有你 (The spirit of you is upon me, and the spirit of me is upon you.)。

[2] 將 $\mathbf{w}$ decompose 成 $\mathbf{w}=\mathbf{p}+(\mathbf{w}-\mathbf{p})$，然後只要證明 $A(\mathbf{w}-\mathbf{p})=0$ 即可。看到 $\{\mathbf{p}+\mathbf{v} \mid A\mathbf{v}=\mathbf{0}\}$ 中的 $\mathbf{p}+\mathbf{v}$ 與 $A\mathbf{v}=\mathbf{0}$，所以就是要把 $\mathbf{w}$ 抽出 $\mathbf{v}$，然後算 $A\mathbf{v}=\mathbf{0}$。

[3] 只要計算 $A(\mathbf{p}+\mathbf{v})$ 即可。

11.10 Lecture 7 (Linear Independence)

Key ideas:

- Definition of linear independence
- Connection to homogeneous system of linear equations
- Special cases of linear independence and linear dependence

僕の言葉は絶対だ **((211116) 腦急; A joke in Chinese that cannot be translated)**
人長的帥但字寫的醜。猜三國人物。[a]

[a]顏良文醜。

僕の言葉は絶対だ **((211116) 謎面[a] (from https://kknews.cc/history/kzgppq.html))** 劉邦聞之大笑，劉備聞之大哭。(打一字)

[a]翠。

Definition 11.22 **(Linear independence and linear dependence)** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \in \mathbb{R}^n$

- is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = x_2 = \ldots = x_p = 0$;

- is said to be **linearly dependent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has a non-trivial solution (some $x_i \neq 0$).

僕の言葉は絶対だ **((211118) You don't like happy new year?)** $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2021 \end{bmatrix}$ or $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2022 \end{bmatrix}$?

Example 11.28 (★★★★★ Linear independence and linear dependence)

(a) Show that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2021 \end{bmatrix}$ are linearly independent.

(b) Show that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ are linearly dependent.

Solution

(a) Find $c_1, c_2 \in \mathbb{R}$ such that $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2022 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e.

$$\begin{cases} c_1 - \quad 0\,c_2 = 0 \\ 0\,c_1 + 2022\,c_2 = 0. \end{cases} \tag{11.27}$$

$\implies c_1 = 0,\ c_2 = 0. \implies$ linearly independent

(b) Find $c_1, c_2 \in \mathbb{R}$ such that $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$, i.e.

$$\begin{cases} c_1 - \quad c_2 = 6 \\ 0\,c_1 + 2\,c_2 = 10. \end{cases} \tag{11.28}$$

$\implies c_1 = 11,\ c_2 = 5. \implies$ linearly dependent

Definition 11.23 Given a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$. Columns of A are said to be linearly independent if $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ is linearly independent.

Remark (**Linear independence and homogeneous systems**) Given a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$. The following are equivalent.

(a) Columns of A are said to be linearly independent

(b) $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p & \mathbf{0} \end{bmatrix}$ has only trivial solution.

(c) $A\mathbf{x} = \mathbf{0}$ has only trivial solution.

僕の言葉は絶対だ **((211125) from A Joke-A-Day: 200 Kid-Friendly Jokes For The Classroom)**

(1)
- Q: Why did the teacher have to wear sunglasses?
- A: Because her students were so bright.

(2)
- Q: What do you call a fish with no eye?
- A: A fsh.

(3)
- Q: What always comes at the end of Thanksgiving?
- A: The letter "g."

(4)
- Q: What type of key is the most important at Thanksgiving dinner?
- A: The tur-key.

(5)
- Q: What do you get if you cross a pine tree with an apple?
- A: A pine-apple.

Proposition 11.1 (Properties of a set of vectors) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

(a) If $\mathbf{0} \in S$, then S is linearly dependent.

(b) If $p = 1$, and $\mathbf{v}_1 \neq \mathbf{0}$, then S is linearly independent.

(c) If $p = 2$, $\mathbf{v}_1, \mathbf{v}_2 \neq \mathbf{0}$ and $\mathbf{v}_2 \neq c\mathbf{v}_1$ for any $c \in \mathbb{R}$, then S is linearly independent.

Proof.

(a) Let $\mathbf{v}_1 = \mathbf{0}$

$\Rightarrow x_1 \cdot \mathbf{0} + 0 \cdot \mathbf{v}_2 + \cdots + 0 \cdot \mathbf{v}_p = \mathbf{0}$ has a non-trivial solution $(x_1 \neq 0)$

$\Rightarrow S$ is linearly dependent.

(b) $\mathbf{v}_1 \neq \mathbf{0}$

$\Rightarrow x_1 \mathbf{v}_1 = \mathbf{0}$ has only the trivial solution $x_1 = 0$

$\Rightarrow S$ is linearly independent.

(c) Suppose that S is linearly dependent

$\Rightarrow x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = \mathbf{0}$ has a non-trivial solution with $x_2 \neq 0$.[4]

$\Rightarrow \mathbf{v}_2 = -\frac{x_1}{x_2}\mathbf{v}_1$ contradicts $\mathbf{v}_2 \neq c\mathbf{v}_1$ for any $c \in \mathbb{R}$

$\Rightarrow S$ is linearly independent by contradiction.

□

Theorem 11.12 (Characterization of Linearly Dependent Sets) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ with $\mathbf{v}_1 \neq \mathbf{0}$. Then the following are equivalent.

(a) S is linearly dependent.

(b) There exists some $j > 1$ such that

$$\mathbf{v}_j \in \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}\}.$$

Proof.

- Proof of (a) $\Rightarrow$ (b)

$$\begin{aligned}
\mathbf{0} &= x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p && \text{let } j \text{ be the largest subscript for which } x_j \neq 0. \\
\mathbf{0} &= x_1\mathbf{v}_1 + \cdots + x_j\mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_p && \text{if } j = 1 \Rightarrow x_1\mathbf{v}_1 = 0 \Rightarrow x_1 = 0 \text{ (since } \mathbf{v}_1 \neq 0) \Rightarrow j > 1 \\
\mathbf{0} &= x_1\mathbf{v}_1 + \cdots + x_j\mathbf{v}_j && \text{simplify} \\
\mathbf{v}_j &= -\frac{1}{x_j}(x_1\mathbf{v}_1 + \cdots + x_{j-1}\mathbf{v}_{j-1}) && \text{rearrange; } x_j \neq 0 \\
\Rightarrow \mathbf{v}_j &\in \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}\} && \text{Definition 11.8}
\end{aligned}$$

[4] Without loss of generality (WLOG).

- Proof of (b) $\Rightarrow$ (a)

$$
\begin{aligned}
\mathbf{v}_j &= -\frac{1}{x_j}(x_1\mathbf{v}_1 + \cdots + x_{j-1}\mathbf{v}_{j-1}) && \text{Definition 11.18; } x_j \neq 0\\
\mathbf{0} &= x_1\mathbf{v}_1 + \cdots + x_j\mathbf{v}_j && \text{rearrange}\\
\mathbf{0} &= x_1\mathbf{v}_1 + \cdots + x_j\mathbf{v}_j + 0\,\mathbf{v}_{j+1} + \cdots + 0\,\mathbf{v}_p && \text{add } 0 = 0\,\mathbf{v}_{j+1} + \cdots + 0\,\mathbf{v}_p\\
&\Rightarrow S \text{ is linearly dependent} && \text{Definition 11.22; } x_j \neq 0
\end{aligned}
$$

□

Theorem 11.13 ($p > n$) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vectors in $\mathbb{R}^n$ with $p > n$[a], then S is linearly dependent.

[a] 想成未知數數目 p 比方程式數目 n 多。

Proof.

$$
\begin{aligned}
\mathbf{0} &= x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p && \text{the vector equation (SLE)}\\
&\left[\begin{array}{ccccc|c} * & * & * & * & * & 0\\ * & * & * & * & * & 0\\ * & * & * & * & * & 0 \end{array}\right] && \text{augmented matrix: \#of columns} = p+1, \text{ \#of rows} = n\\
&\Rightarrow \text{cannot have a pivot in each column} && \text{Remark 11.6}\\
&\Rightarrow \text{SLE has a free variable} && \text{Theorem 11.10}\\
&\Rightarrow \text{SLE has a non-trivial solution} && \text{Definition 11.22}\\
&\Rightarrow S \text{ is linearly dependent}
\end{aligned}
$$

□

Example 11.29 (★☆☆☆☆) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}6\\10\end{bmatrix}\right\}$ is linearly dependent since $p = 3 > 2 = n$.

Remark (**When** $p \leq n$) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is a set with $p \leq n$, then S may or may not be linearly independent.

- $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}\right\}$ in $\mathbb{R}^3$ is not linearly independent.
- $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ in $\mathbb{R}^3$ is linearly independent.

11.11 Lecture 8 (Introduction to linear transformations)

Key ideas:

- matrix transformations
- linear transformations
- matrix transformations=linear transformations
- Another point of view of SLE

僕の言葉は絶対だ **((211202))** In what sense does "triangle=integral"?

僕の言葉は絶対だ **((211202) from A Joke-A-Day: 200 Kid-Friendly Jokes For The Classroom)**

(1)
- Q: What happened to the man who stole a calendar from the store?
- A: He got 12 months.

(2)
- Q: Why can't the elephant use the computer?
- A: Because he's afraid of the mouse.

(3)
- Q: What kind of table can you have for dinner?
- A: A vege-table.

(4)
- Q: Why was the computer cold?
- A: Because it left the Windows open!

(5)
- Q: Why do birds fly south for the winter?
- A: Because it' s too far to walk.

(6)
- Q: What has 18 legs and catches flies?
- A: A baseball team.

Definition 11.24 (Transformation) A transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$, i.e. a rule that assigns to each $x \in \mathbb{R}^n$ exactly one $T(x) \in \mathbb{R}^m$.[a]

Terminology:

- $\mathbb{R}^n$: the domain of T
- $\mathbb{R}^m$: the codomain of T
- if $x_0 \in \mathbb{R}^n$, then $T(x_0)$ is the image of x_0
- range of $T := \{T(x) = Ax \mid x \in \mathbb{R}^n\}$

[a] That is, a transformation is nothing but a generalization of functions in some sense.

In Calculus,

- single-variable Calculus $f:\mathbb{R}^1 \to \mathbb{R}^1$, e.g. $f(x)=x^2$, $f(x)=\sin x$
- multi-variable Calculus $g:\mathbb{R}^n \to \mathbb{R}^1$, where $n \geq 2$, e.g. $g(x,y)=x^2+2021xy+2022y^3$

Example 11.30 (☆☆☆☆☆ **Transformation**) Define $T:\mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1^2 + x_2 \\ x_2 \\ x_1^2 x_2 + x_2^3 \end{bmatrix}$$

$\Rightarrow$ each vector in $\mathbb{R}^2$ is mapped (by T) to a vector in $\mathbb{R}^3$. For example,

$$T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1^2 + 2 \\ 2 \\ 1^2 \cdot 2 + 2^3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 10 \end{bmatrix}.$$

Example 11.31 (☆☆☆☆☆ $m \times n$ **matrix** A **is a transformation**[a]) Every $m \times n$ matrix A[b] defines a transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ by[c]

$$T(x) = Ax.$$

For example, let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$. Define $T:\mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 + x_2 \end{bmatrix}$$

[a]就把這種 transformation 叫做 matrix transformation

[b]很明顯，任意矩陣皆可視為 transformation。

[c]$T_{m\times 1} = A_{m\times n} x_{n\times 1}$

Since every $m\times n$ matrix A can be regarded as a transformation, we call it a "matrix transformation".

Definition 11.25 **(Matrix transformation)** If $T:\mathbb{R}^n \to \mathbb{R}^m$ is a transformation that can be given by an $m \times n$ matrix A, i.e. $T(x) = Ax$, then T is a matrix transformation.[a]

[a]簡單說，Matrix transformation 就是長相為 Ax。

Example 11.32 (☆☆☆☆☆ **Matrix transformation**) [a]Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 + x_2 \end{bmatrix}$$

[b] $\Rightarrow$ it is a matrix transformation since

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 + x_2 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$$

[a]要判斷是不是 matrix transformation，就是看能否找出那個矩陣 A。

[b]seems linear in x_1 and x_2!?

To show if a transformation T is a matrix transformation or not, we need to find if there exists A such that $T(x) = Ax$.

Example 11.33 (☆☆☆☆☆ **NOT all transformations are matrix transformation**) Show that $T : \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = x_1 x_2$$

[a] is not a matrix transformation.

[a]$x_1 x_2$ is NOT linear both in x_1 and x_2

Solution If it was a matrix transformation, then we can find a 1×2 matrix $A = \begin{bmatrix} a & b \end{bmatrix}$ such that

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 x_2$$

for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

- $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a = 1 \cdot 0 \Rightarrow a = 0$
- $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b = 0 \cdot 1 \Rightarrow b = 0$

$\Longrightarrow A = \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

On the other hand, $T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 0 \neq 1 \cdot 1 = 1 \implies$ No such A exists!

Linear transformations preserve the operations of vector addition and scalar multiplication.

Definition 11.26 (Linear transformation) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear[a] if

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$.[b]

(ii) $T(cu) = cT(u)$ for all $u \in \mathbb{R}^n$ and $c \in \mathbb{R}$.[c]

[a](211130) Intuitively, consider the scalar function $f(x) = ax + b$, where $a \neq 0$ and b are constants. It is easy to see that this function satisfies the two properties given in Definition 11.26. Indeed, it represents a straight line on the xy plane

[b]addition → linear transformation and linear transformation → addition result in the same consequence

[c]scalar multiplication → linear transformation and linear transformation → scalar multiplication result in the same consequence

Example 11.34 (☆☆☆☆☆ **Linear transformation**) Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(\begin{bmatrix}x_1\\x_2\end{bmatrix}) = \begin{bmatrix}3x_1 + 2x_2\\x_1 - 4x_2\end{bmatrix} \tag{11.29}$$

is linear.

Solution For any $u = \begin{bmatrix}u_1\\u_2\end{bmatrix} \in \mathbb{R}^2$ and $v = \begin{bmatrix}v_1\\v_2\end{bmatrix} \in \mathbb{R}^2$, and $c \in \mathbb{R}$,

(i)

$$\begin{aligned}
T(\begin{bmatrix}u_1\\u_2\end{bmatrix} + \begin{bmatrix}v_1\\v_2\end{bmatrix}) &= T(\begin{bmatrix}u_1+v_1\\u_2+v_2\end{bmatrix}) && \text{elementary matrix operation}\\
&= \begin{bmatrix}3(u_1+v_1) + 2(u_2+v_2)\\(u_1+v_1) - 4(u_2+v_2)\end{bmatrix} && (11.29)\\
&= \begin{bmatrix}3u_1 + 2u_2\\u_1 - 4u_2\end{bmatrix} + \begin{bmatrix}3v_1 + 2v_2\\v_1 - 4v_2\end{bmatrix} && \text{expand and appropriately collect}\\
&= T(\begin{bmatrix}u_1\\u_2\end{bmatrix}) + T(\begin{bmatrix}v_1\\v_2\end{bmatrix}) && (11.29)
\end{aligned}$$

(ii)

$$\begin{aligned} T(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}) &= T(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}) && \text{elementary matrix operation} \\ &= \begin{bmatrix} 3(cu_1)+2(cu_2) \\ (cu_1)-4(cu_2) \end{bmatrix} && (11.29) \\ &= c\begin{bmatrix} 3u_1+2u_2 \\ u_1-4u_2 \end{bmatrix} && \text{expand and appropriately collect} \\ &= cT(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}) && (11.29) \end{aligned}$$

$\Longrightarrow$ T is linear by Definition 11.26.

Example 11.35 (☆☆☆☆☆ **Linear transformation → matrix transformation**) The linear transformation T in Example 11.34 is a matrix transformation since

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 3x_1+2x_2 \\ x_1-4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11.30}$$

Question

(a) All linear transformations are matrix transformations?

(b) All matrix transformations are linear transformations?

Theorem 11.14 (**Linear transformations=Matrix transformations**) Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a transformation. Then T is a linear transformation if and only if T is a matrix transformation.

Proof.

- Proof: T is a matrix transformation $\Longrightarrow$ T is a linear transformation. Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a matrix transformation. Then $\exists$ $m\times n$ matrix A such that $T(x) = Ax$.

  (i) $T(u+v) = A(u+v) = Au + Av = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$.

  (ii) $T(cu) = A(cu) = c(Au) = cT(u)$ for all $u \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

- Proof: T is a linear transformation $\Longrightarrow$ T is a matrix transformation. See Theorem 11.15

□

Remark (**SLE and linear/matrix transformations**)

(a) (Another point-of-view of SLE) linear/matrix transformations give a new point-of-view of SLE.

(b) (Solve $Ax = b$ in language of transformations) Consider the matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$. Given $b \in \mathbb{R}^m$, solving the SLE $Ax = b$ is asking if there is $x \in \mathbb{R}^n$ that maps to b by A.

- SLE $Ax = b$ has a solution $\iff$ b is in

$$\begin{aligned}\text{range of } T :=& \{T(x) = Ax \mid x \in \mathbb{R}^n\} \\ =& \text{span of columns of } A\end{aligned}$$

- SLE $Ax = b$ has a unique solution $\iff$ exactly one vector in $\mathbb{R}^n$ maps to b

11.12 Lecture 9 (The Matrix of a Linear Transformation)

Key ideas:

- Identity matrix and standard basis
- Linear transformations $\Longrightarrow$ Matrix transformations (Theorem 11.15)
- Geometry of linear transformations
- Onto and one-to-one

Definition 11.27 (Identity matrix; Standard basis)

(a) ($n \times n$ identity matrix) $$\mathbf{I_n} = \underbrace{\left.\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}\right\}}_{n \text{ columns}} n \text{ rows}$$

- e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) (Standard basis) $\{e_1, e_2, \dots, e_n\}$, where

$$e_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

is the j^{th} column of I_n.

- e.g. $n = 3$, $\{e_1, e_2, \dots, e_3\} = \{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$
- $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = I_n$

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T is completely determined by what T does to $e_1, e_2, \dots, e_n$. This fact can be seen by the following example.

Example 11.36 (☆☆☆☆☆ **A linear transformation T is determined by T(standard basis) (Example 1.9-1)**) The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$[a]. Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

and

$$T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}.$$

With no additional information, find a formula for the image of an arbitrary x in $\mathbb{R}^2$.

[a] Q: What if e_1 and e_2 are replaced by two linearly independent non-standard basis v_1 and v_2?

Solution Observe that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2. \tag{11.31}$$

$$\begin{aligned} T(x) &= x_1 T(e_1) + x_2 T(e_2) && \text{(11.31); } T \text{ is linear using Definition 11.26} \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} && T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \text{ and } T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} && \text{write in the form } Ax \end{aligned}$$

We prove that every linear transformation is a matrix transformation.

Theorem 11.15 (**Linear transformations $\Longrightarrow$ Matrix transformations**) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique A such that $T(x) = Ax$.

Proof. Idea: Just copy-paste Example 11.36[5].

[5] 證明思路：只要把 Example 11.36 的陣型擺上，然後修改即可。

Write

$$\mathbf{x} = I_n \mathbf{x} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} \mathbf{x} = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n. \tag{11.32}$$

$$\begin{aligned} T(\mathbf{x}) &= x_1 T(\mathbf{e}_1) + \cdots + x_n T(\mathbf{e}_n) && \text{(11.32); } T \text{ is linear using Definition 11.26} \\ &= \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} && \text{write in the form } A\mathbf{x} \\ &= A\mathbf{x} && A := \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \end{aligned}$$

The uniqueness of A is treated in Exercise 33. □

Definition 11.28 (Standard matrix) A in Theorem 11.15 is called the standard matrix for linear transformation T.

Many geometric transformations (e.g. reflection, dilation, rotation, shear) in $\mathbb{R}^2$ are examples of linear transformations.

Example 11.37 (☆☆☆☆☆ Rotation transformation (Example 1.9-3)) Fix angle φ. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through angle φ.

(a) Show that T is a linear transformation.

(b) Find standard matrix of T.

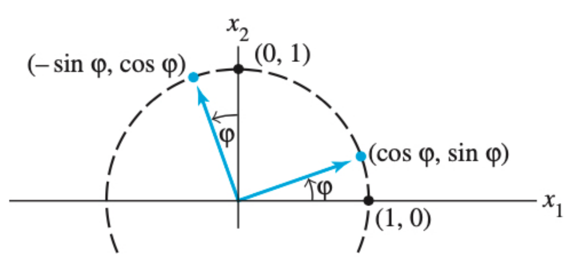

Figure 11.1: A rotation transformation.

Solution Given

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \tag{11.33}$$

- Goal: express $(\bar{x}, \bar{y})$ in terms of (x, y) and φ
- $r := \sqrt{x^2 + y^2} = \sqrt{\bar{x}^2 + \bar{y}^2}$

$$\begin{cases} x = r\cos\theta, & \bar{x} = r\cos(\theta+\varphi) \\ y = r\sin\theta, & \bar{y} = r\sin(\theta+\varphi) \end{cases} \tag{11.34}$$

$\Rightarrow$

$$\begin{aligned} \bar{x} &= r\cos(\theta+\varphi) && (11.34) \\ &= r(\cos\theta\cos\varphi - \sin\theta\sin\varphi) && \text{sum formula for } \cos(a+b) \\ &= x\cos\varphi - y\sin\varphi && (11.34):\ x = r\cos\theta,\ y = r\sin\theta \end{aligned}$$

$$\begin{aligned} \bar{y} &= r\sin(\theta+\varphi) && (11.34) \\ &= r(\sin\theta\cos\varphi + \cos\theta\sin\varphi) && \text{sum formula for } \sin(a+b) \\ &= y\cos\varphi + x\sin\varphi && (11.34):\ x = r\cos\theta,\ y = r\sin\theta \end{aligned}$$

$\Rightarrow$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x\cos\varphi - y\sin\varphi \\ y\cos\varphi + x\sin\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \tag{11.35}$$

$\Rightarrow$ T is linear and the standard matrix $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$.

Definition 11.29 (Onto and one-to-one)

- (Onto) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto (or surjective) if for each $\mathbf{b} \in \mathbb{R}^m$, there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.
  $\Rightarrow$ Onto is asking about existence, i.e. does there exist an $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{b}$.
- (One-to-one) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one (or injective) if for each $\mathbf{b} \in \mathbb{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or no solution. ($\Longleftrightarrow$ for all $\mathbf{x} \neq \mathbf{y}$ in $\mathbb{R}^n$, then $T(\mathbf{x}) \neq T(\mathbf{y})$)
  $\Rightarrow$ One-to-one is asking about uniqueness, i.e. is the only $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{b}$.

When T is a linear transformation, onto and one-to-one encoded in standard matrix.

Theorem 11.16 (Equivalence of onto) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then the following statements are equivalent.

(a) T is onto $\mathbb{R}^m$.

(b) $T(x) = Ax = b$ has a solution for all $b \in \mathbb{R}^m$.

(c) Columns of A span $\mathbb{R}^m$.

(d) A has a pivot in each row.

Proof.

- (a) $\iff$ (b): Definition 11.29
- (b) $\iff$ (c) $\iff$ (d): Theorem 11.8

□

Theorem 11.17 (**Equivalence of one-to-one**) Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then the following statements are equivalent.

(a) T is one-to-one.

(b) $Ax = 0$ has only trivial solution.

(c) Columns of A are linearly independent.

(d) $Ax = 0$ has no free variables.

Proof.

- (a) $\iff$ (b): Use Definition 11.29 and Theorem 11.11
- (b) $\iff$ (c): Definition 11.22
- (b) $\iff$ (d): Theorem 11.10

□

Example 11.38 (★★☆☆☆) The linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 + 4x_3 \\ x_2 - x_3 \end{bmatrix}$$

has standard matrix $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & -1 \end{bmatrix}$

(a) Is T onto?

(b) Is T one-to-one?

Solution

(a) - A has a pivot in each row

$\implies$ T is onto by Theorem 11.16

- $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has a solution for all $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$

  $\Longrightarrow$ T is onto by Theorem 11.16
- $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

  $\Longrightarrow$ for all $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ such x_1, x_2, and x_3 exist

  $\Longrightarrow$ T is onto by Theorem 11.16

(b) A has a free variable $\Longrightarrow$ T is not one-to-one by Theorem 11.17

11.13 Lecture 10 (Applications of Linear Systems)

Key ideas:

- Balancing Chemical Equations
- Traffic flows

僕の言葉は絶対だ **((211116) 謎面[a] (from https://kknews.cc/history/kzgppq.html))** 劉邦聞之大笑，劉備聞之大哭。(打一字)

[a] 翠。

Example 11.39 (☆☆☆☆☆ Balancing chemical equations) Balance the chemical equation:

$$(x_1)C_3H_8 + (x_2)O_2 \longrightarrow (x_3)CO_2 + (x_4)H_2O,$$

where

- C_3H_8: propane (丙烷)
- O_2: oxygen
- CO_2: carbon dioxide
- H_2O: water

Solution $\begin{bmatrix} C \\ H \\ O \end{bmatrix}$

- step 1: C_3H_8: $\begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$, O_2: $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, CO_2: $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, H_2O: $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.
- step 2: $x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$
- step 3: $\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = \frac{1}{4}x_4, \\ x_2 = \frac{5}{4}x_4, \\ x_3 = \frac{3}{4}x_4, \end{cases}$ with x_4 free
- step 4: take $x_4 = 4 \Rightarrow$

$$C_3H_8 + 5\,O_2 \longrightarrow 3\,CO_2 + 4\,H_2O$$

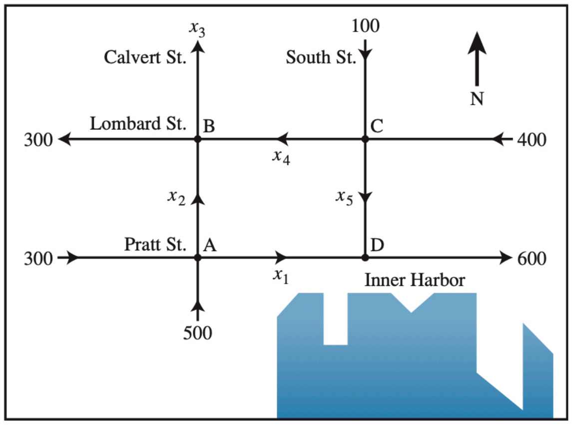

Figure 11.2: Baltimore streets.

Example 11.40 (☆☆☆☆☆ **Network flow; Traffic flows (Example 1.6-2)**) The network in Figure 11.2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

Intersection	Flow in	Flow out
A	$300 + 500$	$x_1 + x_2$
B	$x_2 + x_4$	$300 + x_3$
C	$100 + 400$	$x_4 + x_5$
D	$x_1 + x_5$	600

Solution

(1)

$$\begin{cases} x_1 + x_2 & = 800 \\ \quad x_2 - x_3 + x_4 & = 300 \\ \quad\quad x_4 + x_5 & = 500 \\ x_1 \quad\quad + x_5 & = 600 \end{cases}$$

$$\Longrightarrow$$

$$\begin{cases} x_1 \quad + x_5 & = 600 \\ \quad x_2 \quad - x_5 & = 200 \\ \quad\quad x_3 & = 400 \\ \quad\quad x_4 + x_5 & = 500 \end{cases}$$

$\Longrightarrow$

$$\begin{cases} x_1 = 600 - x_5, \\ x_2 = 200 + x_5, \\ x_3 = 400, \\ x_4 = 500 - x_5, \end{cases}$$

with x_5 free (seem free, actually not free!)

(2) the streets in this problem are one-way $\Rightarrow \begin{cases} x_1 \geq 0 \Longrightarrow x_5 \leq 600, \\ x_4 \geq 0 \Longrightarrow x_5 \leq 500, \end{cases} \Rightarrow x_5 \leq 500 \Rightarrow$

$$\begin{cases} 100 \leq x_1 \leq 600, \\ 200 \leq x_2 \leq 700, \end{cases}$$