

Problem Set 1

Group 2

2026-01-29

1.) Prove $\lim_{x \rightarrow -1} 2x + 1 = -1$

Solution:

Determine $\delta > 0$ so that if $|x - (-1)| < \delta$ then $|2x + 1 - (-1)| < \epsilon$.

$$\begin{aligned}|(2x + 1) - (-1)| &< \epsilon \\ |2x + 2| &< \epsilon \\ 2|x + 1| &< \epsilon \\ 2|x - (-1)| &< \epsilon \\ |x - (-1)| &< \frac{\epsilon}{2}\end{aligned}$$

Given $\epsilon > 0$, Choose $\delta = \frac{\epsilon}{2}$, thus,

$$\begin{aligned}|x + 1| &< \delta \\ |x + 1| &< \frac{\epsilon}{2} \\ 2|x + 1| &< \epsilon \\ |2x + 2| &< \epsilon \\ |2x + 1 + 1| &< \epsilon \\ |2x + 1 - (-1)| &< \epsilon\end{aligned}$$

2.) Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function and graph using R with the point/s identified. $f(x) = x^3 - 4x^2 - 2x - 5$ on $[-10, 10]$.

Solution:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3c^2 - 8c - 2 = \frac{[(10)^3 - 4(10) - 2(10) - 5] - [(-10)^3 - 4(-10) - 2(-10) - 5]}{10 - (-10)}$$

$$3c^2 - 8c - 2 = \frac{575 - (-1385)}{20}$$

$$3c^2 - 8c - 2 = 98$$

$$3c^2 - 8c - 100 = 0$$

$$c = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-100)}}{6}$$

$$c = \frac{8 \pm \sqrt{1264}}{6}$$

$$c = \frac{8 \pm 4\sqrt{79}}{6}$$

$$c \approx -4.59$$

$$c \approx 7.25$$

[1] 7.258796 -4.592130

3.) Find the point c that satisfies the mean value theorem for integrals on the interval $[-1, 1]$. The function is $f(x) = 2e^x$

Solution:

$$\int_{-1}^1 2e^x dx = (2e^c)(1 - (-1))$$

$$2e^x \Big|_{-1}^1 = (2e^c)2$$

$$(2e - 2e^{-1}) = 4e^c$$

$$\frac{2(e - e^{-1})}{4} = e^c$$

$$\frac{(e - e^{-1})}{2} = e^c$$

$$\ln\left(\frac{(e - e^{-1})}{2}\right) = c$$

$$0.16143936157 \approx c$$

4.) Consider the function $f(x) = \cos(x)$

a. Find the fourth Taylor polynomial for f at $x = \pi$.

Solution:

$$f(x) = \cos(x) \rightarrow f(\pi) = -1$$

$$f'(x) = -\sin(x) \rightarrow f'(\pi) = 0$$

$$f''(x) = -\cos(x) \rightarrow f''(\pi) = 1$$

$$f'''(x) = \sin(x) \rightarrow f'''(\pi) = 0$$

$$f^{(4)}(x) = -\cos(x) \rightarrow f^{(4)}(\pi) = -1$$

$$P_4(x) = -1 + \frac{1}{2}(x - \pi)^2 + \frac{1}{24}(x - \pi)^4$$

b. Use the fourth Taylor polynomial to approximate $\cos(0)$.

Substitute $x = 0$:

$$P_4(0) = -1 + \frac{1}{2}(-\pi)^2 + \frac{1}{24}(-\pi)^4$$

$$P_4(0) = -1 + \frac{(-\pi)^2}{2} + \frac{(-\pi)^4}{24}$$

Numerically,

$$P_4(0) \approx -0.124$$

$$\cos(0) \approx -0.124$$

c. Use the fourth Taylor polynomial to bound the error.

5.) If $fl(x)$ is the machine approximated number of a real number x and ϵ is the corresponding relative error, then show that $fl(x) = (1 - \epsilon)x$.

Solution Let $fl(x)$ denote the machine approximation of a real number x . The relative error ϵ is defined as

$$\epsilon = \frac{x - fl(x)}{x}, \quad x \neq 0$$

Multiplying both sides by x gives

$$\epsilon x = x - fl(x)$$

Rearranging,

$$fl(x) = x - \epsilon x$$

Factoring out x , we obtain that

$$fl(x) = x(1 - \epsilon)$$

6.) For the following numbers x and their corresponding approximations x_A , find the number of significant digits in x_A with respect to x and find the relative error.

a. $x = 451.01, x_A = 451.023$

b. $x = -0.04518, x_A = -0.045113$

c. $x = 23.4604, x_A = 23.4213$

7.) Find the condition number for the following functions

a. $f(x) = 2x^2$

$$f'(x) = 4x$$

$$cn = \left| \frac{x(4x)}{2x^2} \right| = 2$$

$$cn = 2$$

b. $f(x) = 2\pi^x$

$$\begin{aligned}
f'(x) &= 2\pi^x \ln(\pi) \\
cn &= \left| \frac{x(2\pi^x \ln(\pi))}{2\pi^x} \right| = x \ln(\pi) \\
cn &= x \ln(\pi)
\end{aligned}$$

c. $f(x) = 2b^x$

$$\begin{aligned}
f'(x) &= 2b^x \ln(b) \\
cn &= \left| \frac{x(2b^x \ln(b))}{2b^x} \right| = x \ln(b) \\
cn &= x \ln(b)
\end{aligned}$$

8.) Determine if the following series converges or diverges. If it converges determine its sum.

$$\Sigma_{i=1}^{\infty} \frac{1}{2^n}$$

Solution:

This is a geometric series with a common ratio

$$r = \frac{1}{2}$$

Since $|r| < 1$, the series converge. The sum of geometric series starting at $n = 1$ is

$$\begin{aligned}
\Sigma_{i=1}^{\infty} ar^n &= \frac{ar}{1-r} \\
\Sigma_{i=1}^{\infty} ar^n &= \frac{1/2}{1-1/2} = 1
\end{aligned}$$