EXCERPTED FROM

STEPHEN WOLFRAM A NEW KIND OF SCIENCE

CHAPTER 5

Two Dimensions and Beyond

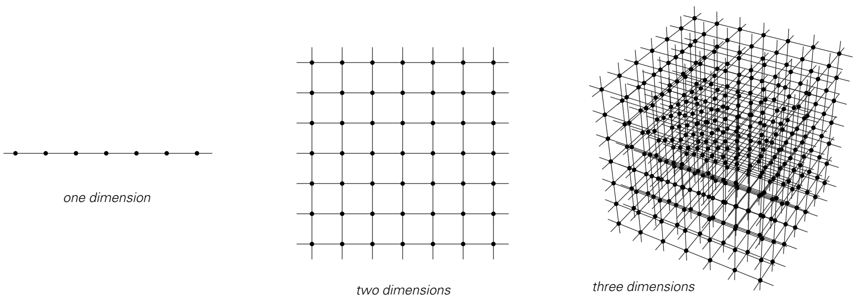
5 Two Dimensions and Beyond

Introduction

The physical world in which we live involves three dimensions of space. Yet so far in this book all the systems we have discussed have effectively been limited to just one dimension.

The purpose of this chapter, therefore, is to see how much of a difference it makes to allow more than one dimension.

At least in simple cases, the basic idea—as illustrated in the pictures below—is to consider systems whose elements do not just lie along a one-dimensional line, but instead are arranged for example on a two-dimensional grid.



Examples of simple arrangements of elements in one, two and three dimensions. In two dimensions, what is shown is a square grid; triangular and hexagonal grids are also possible. In three dimensions, what is shown is a cubic lattice; various other lattices, analogous to those for regular crystals, are also possible—as are arrangements that are not repetitive.

Traditional science tends to suggest that allowing more than one dimension will have very important consequences. Indeed, it turns out that many of the phenomena that have been most studied in traditional science simply do not occur in just one dimension.

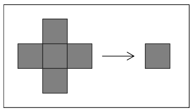
Phenomena that involve geometrical shapes, for example, usually require at least two dimensions, while phenomena that rely on the existence of knotted structures require three dimensions. But what about the phenomenon of complexity? How much does it depend on dimension?

It could be that in going beyond one dimension the character of the behavior that we would see would immediately change. And indeed in the course of this chapter, we will come across many examples of specific effects that depend on having more than one dimension.

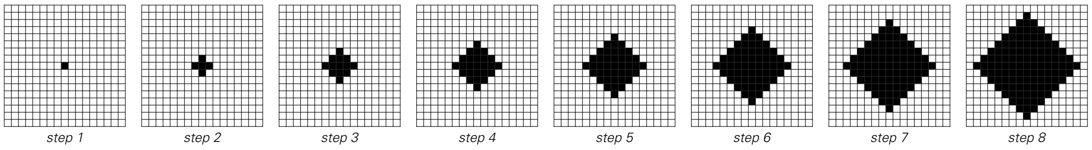
But what we will discover in the end is that at an overall level the behavior we see is not fundamentally much different in two or more dimensions than in one dimension. Indeed, despite what we might expect from traditional science, adding more dimensions does not ultimately seem to have much effect on the occurrence of behavior of any significant complexity.

Cellular Automata

The cellular automata that we have discussed so far in this book are all purely one-dimensional, so that at each step, they involve only a single line of cells. But one can also consider two-dimensional cellular automata that involve a whole grid of cells, with the color of each cell being updated according to a rule that depends on its neighbors in all four directions on the grid, as in the picture below.

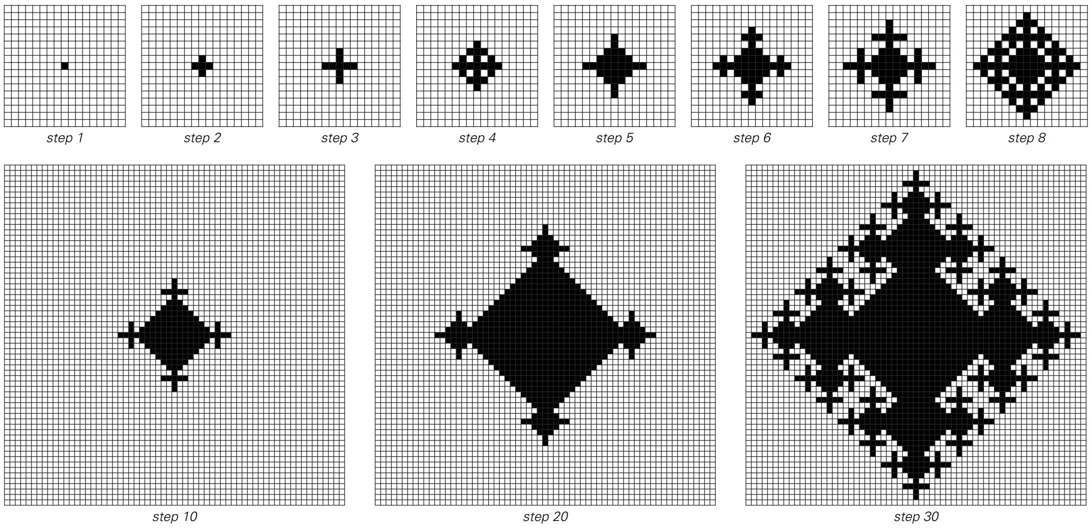

The form of the rule for a typical two-dimensional cellular automaton. In the cases discussed in this section, each cell is either black or white. Usually I consider so-called totalistic rules in which the new color of the center cell depends only on the average of the previous colors of its four neighbors, as well as on its own previous color.

The pictures below show what happens with an especially simple rule in which a particular cell is taken to become black if any of its four neighbors were black on the previous step.



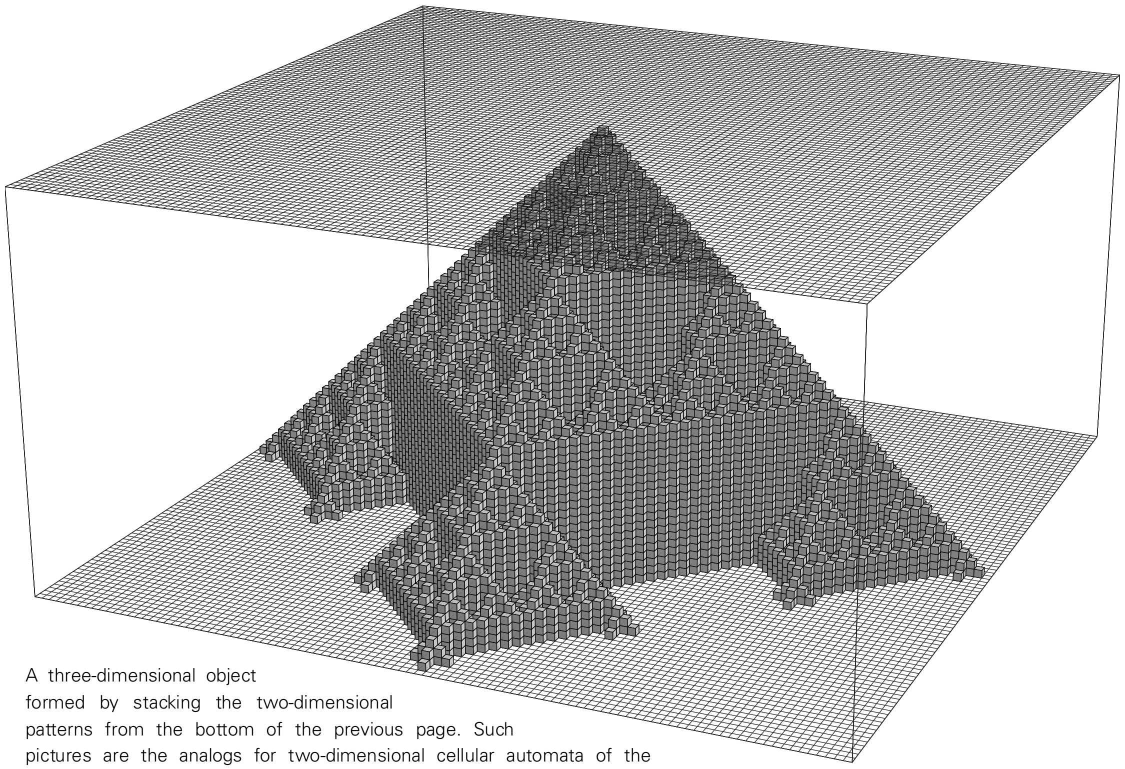
Successive steps in the evolution of a two-dimensional cellular automaton whose rule specifies that a particular cell should become black if any of its neighbors were black on the previous step. (In the numbering scheme described on page 173 this rule is code 1022.)

Starting from a single black cell, this rule just yields a uniformly expanding diamond-shaped region of black cells. But by changing the rule slightly, one can obtain more complicated patterns of growth. The pictures below show what happens, for example, with a rule in which each cell becomes black if just one or all four of its neighbors were black on the previous step, but otherwise stays the same color as it was before.



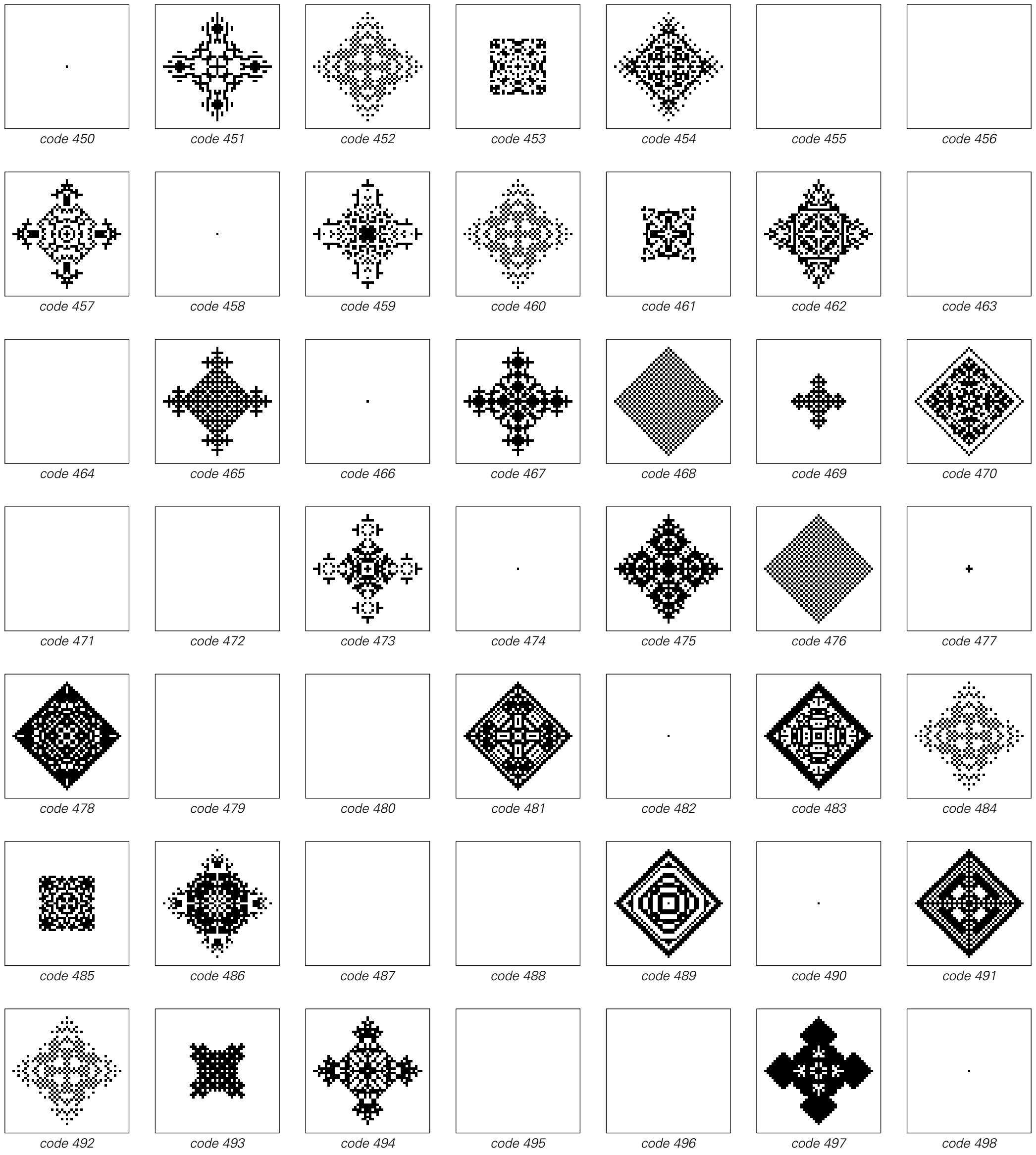
Steps in the evolution of a two-dimensional cellular automaton whose rule specifies that a particular cell should become black if exactly one or all four of its neighbors were black on the previous step, but should otherwise stay the same color. Starting with a single black cell, this rule yields an intricate, if very regular, pattern of growth. (In the numbering scheme on page 173, the rule is code 942.)

The patterns produced in this case no longer have a simple geometrical form, but instead often exhibit an intricate structure somewhat reminiscent of a snowflake. Yet despite this intricacy, the patterns still show great regularity. And indeed, if one takes the patterns from successive steps and stacks them on top of each other to form a three-dimensional object, as in the picture below, then this object has a very regular nested structure.

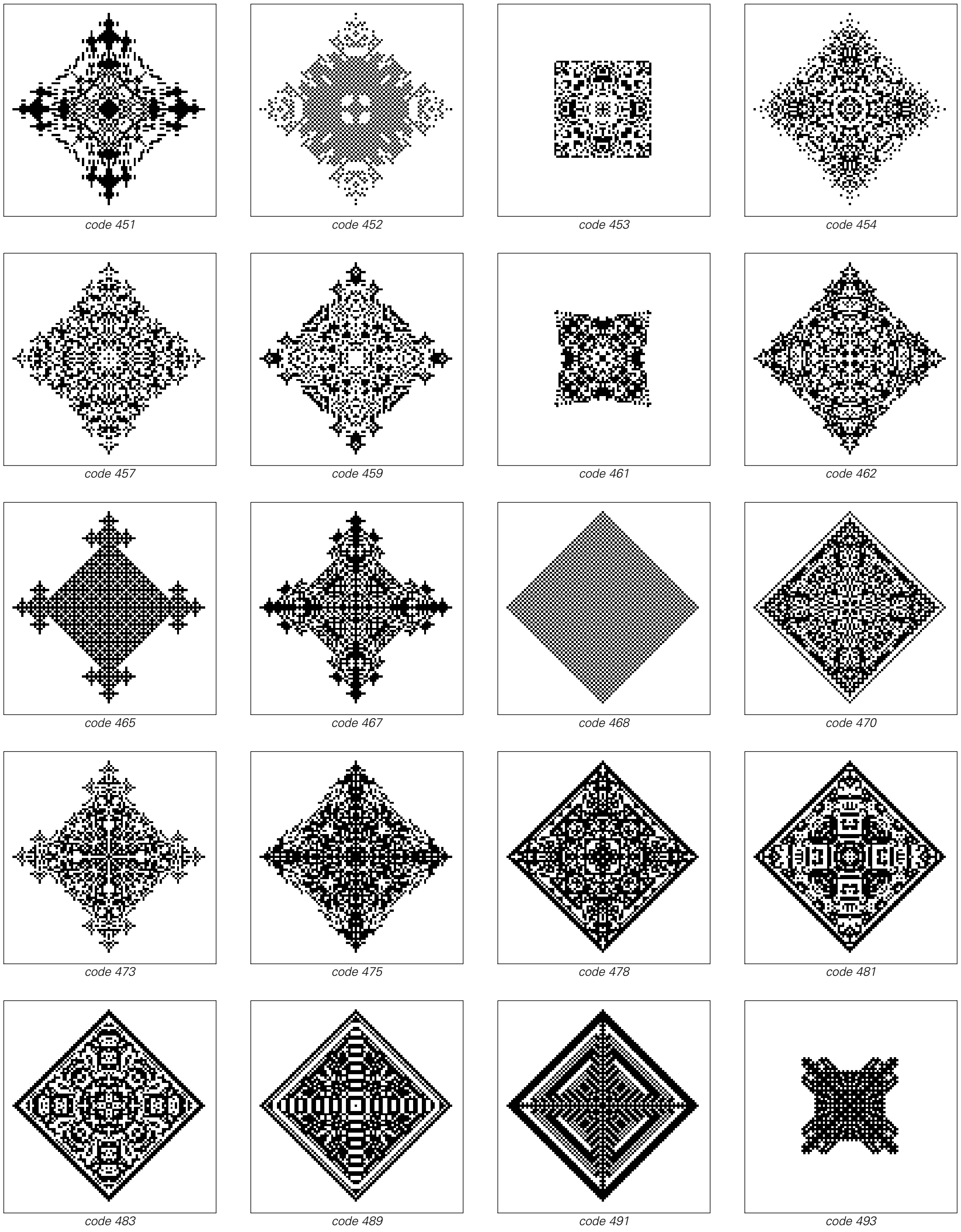


A three-dimensional object formed by stacking the two-dimensional patterns from the bottom of the previous page. Such pictures are the analogs for two-dimensional cellular automata of the two-dimensional pictures that I often generate for one-dimensional cellular automata.

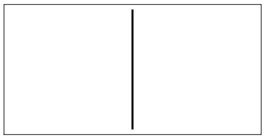
But what about other rules? The facing page and the one that follows show patterns produced by two-dimensional cellular automata with a sequence of different rules. Within each pattern there is often considerable complexity. But this complexity turns out to be very similar to the complexity we have already seen in one-dimensional



Patterns generated by a sequence of two-dimensional cellular automaton rules. The patterns are produced by starting from a single black square and then running for 22 steps. In each case the base 2 digit sequence for the code number specifies the rule as follows. The last digit specifies what color the center cell should be if all its neighbors were white on the previous step, and it too was white. The second-to-last digit specifies what happens if all the neighbors are white, but the center cell itself is black. And each earlier digit then specifies what should happen if progressively more neighbors are black. (Compare page 60.)



Patterns generated by two-dimensional cellular automata from the previous page, but now after twice as many steps.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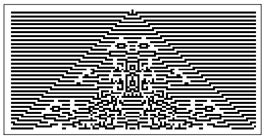
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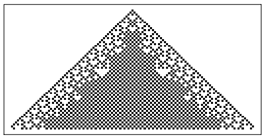
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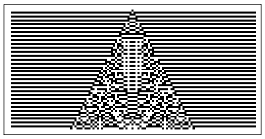
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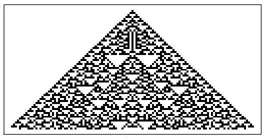
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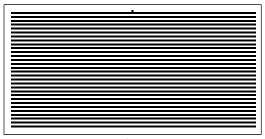
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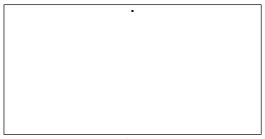
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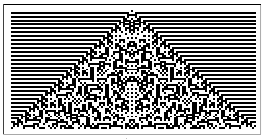
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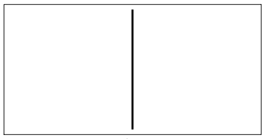
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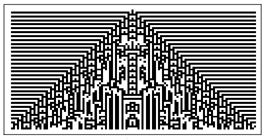
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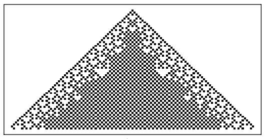
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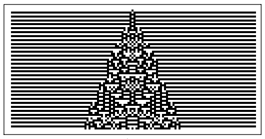
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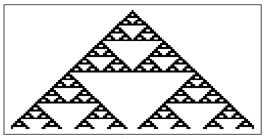
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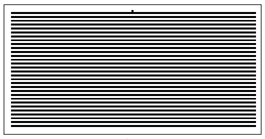
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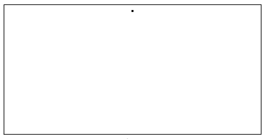
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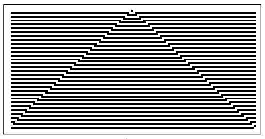
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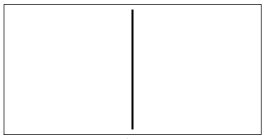
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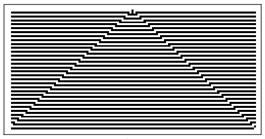
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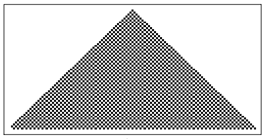
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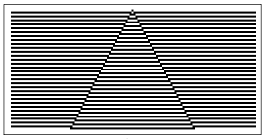
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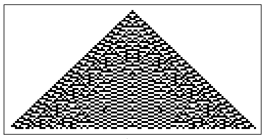
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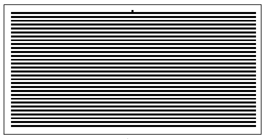
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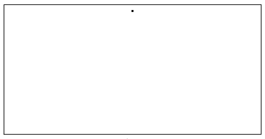
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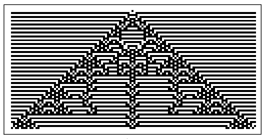
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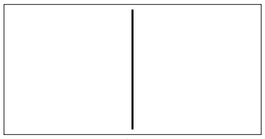
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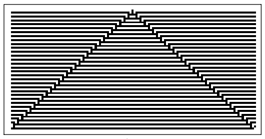
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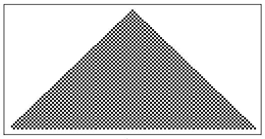
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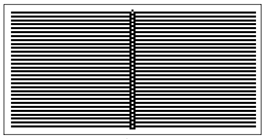
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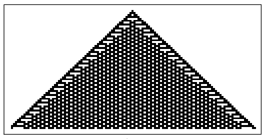
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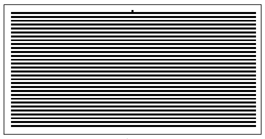
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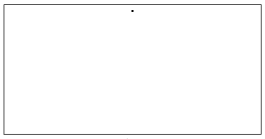
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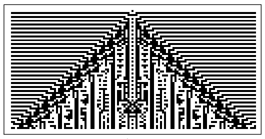
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Evolution of one-dimensional slices through some of the two-dimensional cellular automata from the previous two pages. Each picture shows the colors of cells that lie on the one-dimensional line that goes through the middle of each two-dimensional pattern. The results are strikingly similar to ones we saw in previous chapters in purely one-dimensional cellular automata.

cellular automata. And indeed the previous page shows that if one looks at the evolution of a one-dimensional slice through each two-dimensional pattern the results one gets are strikingly similar to what we have seen in ordinary one-dimensional cellular automata.

But looking at such slices cannot reveal much about the overall shapes of the two-dimensional patterns. And in fact it turns out that for all the two-dimensional cellular automata shown on the last few pages, these shapes are always very regular.

But it is nevertheless possible to find two-dimensional cellular automata that yield less regular shapes. And as a first example, the picture on the facing page shows a rule that produces a pattern whose surface has seemingly random irregularities, at least on a small scale.

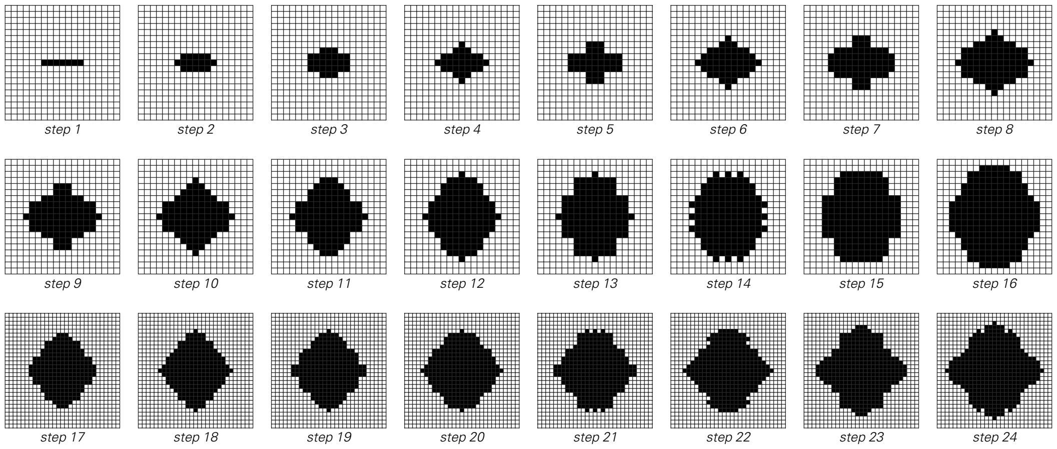
In this particular case, however, it turns out that on a larger scale the surface follows a rather smooth curve. And indeed, as the picture on page 178 shows, it is even possible to find cellular automata that yield overall shapes that closely approximate perfect circles.

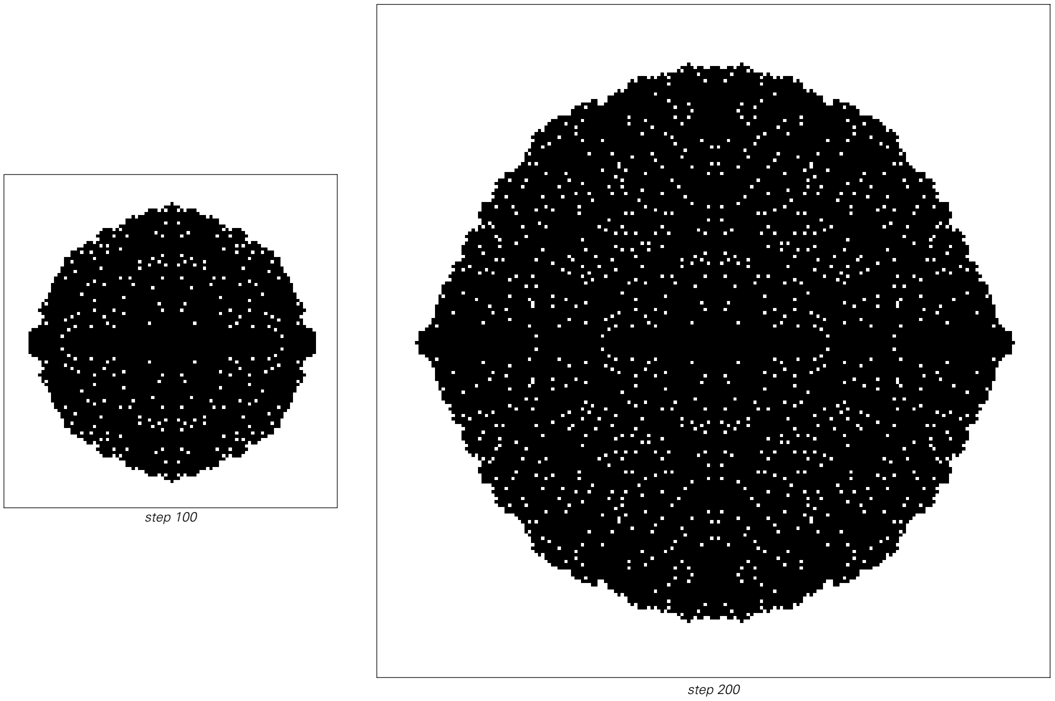
But it is certainly not the case that all two-dimensional cellular automata produce only simple overall shapes. The pictures on pages 179–181 show one rule, for example, that does not. The rule is actually rather simple: it just states that a particular cell should become black whenever exactly three of its eight neighbors—including diagonals—are black, and otherwise it should stay the same color as it was before.

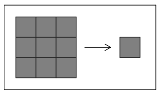
In order to get any kind of growth with this rule one must start with at least three black cells. The picture at the top of page 179 shows what happens with various numbers of black cells. In some cases the patterns produced are fairly simple—and typically stop growing after just a few steps. But in other cases, much more complicated patterns are produced, which often apparently go on growing forever.

The pictures on page 181 show the behavior produced by starting from a row of eleven black cells, and then evolving for several hundred steps. The shapes obtained seem continually to go on changing, with no simple overall form ever being produced.

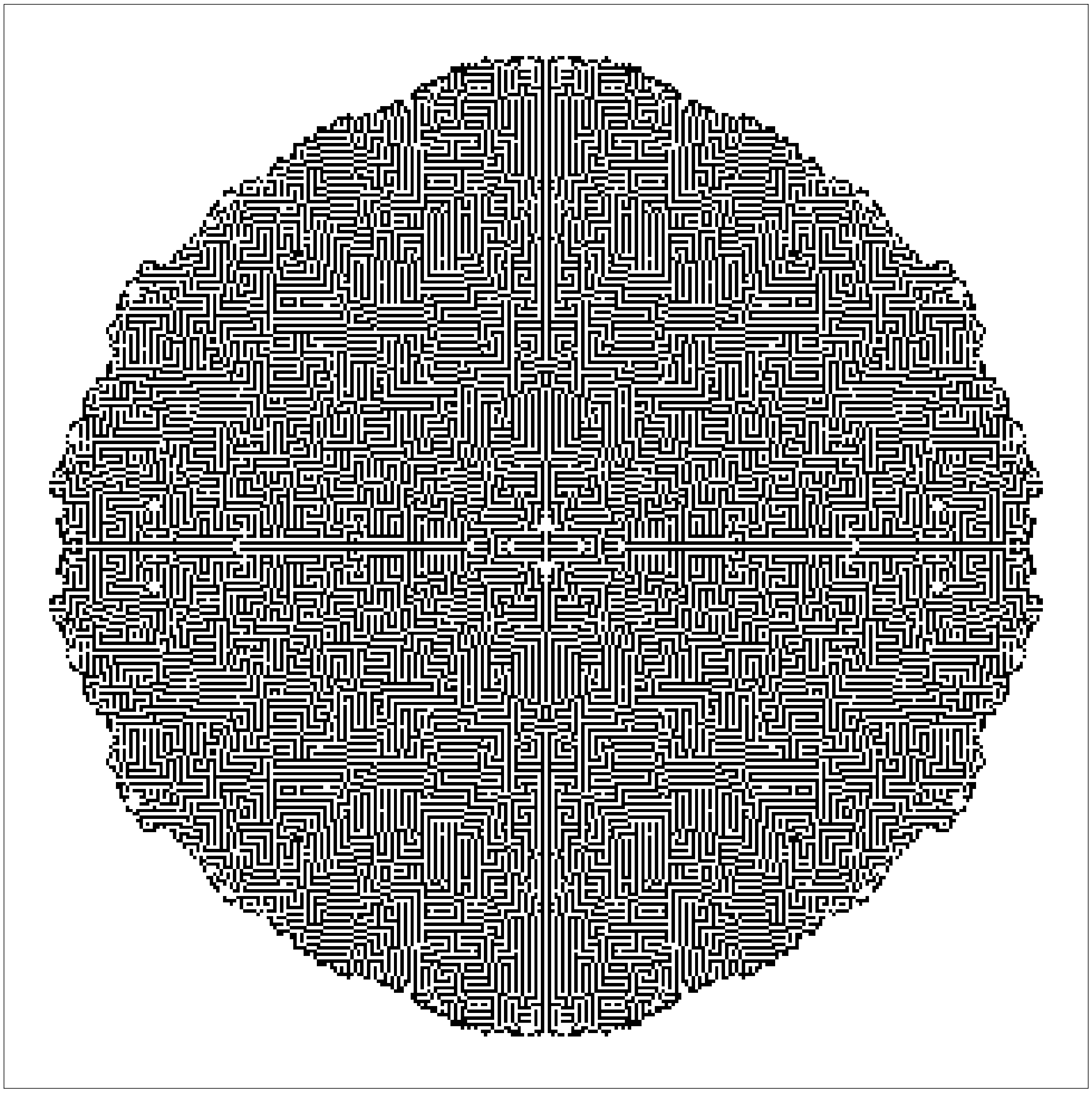
And so it seems that there can be great complexity not only in the detailed arrangement of black and white cells in a two-dimensional cellular automaton pattern, but also in the overall shape of the pattern.



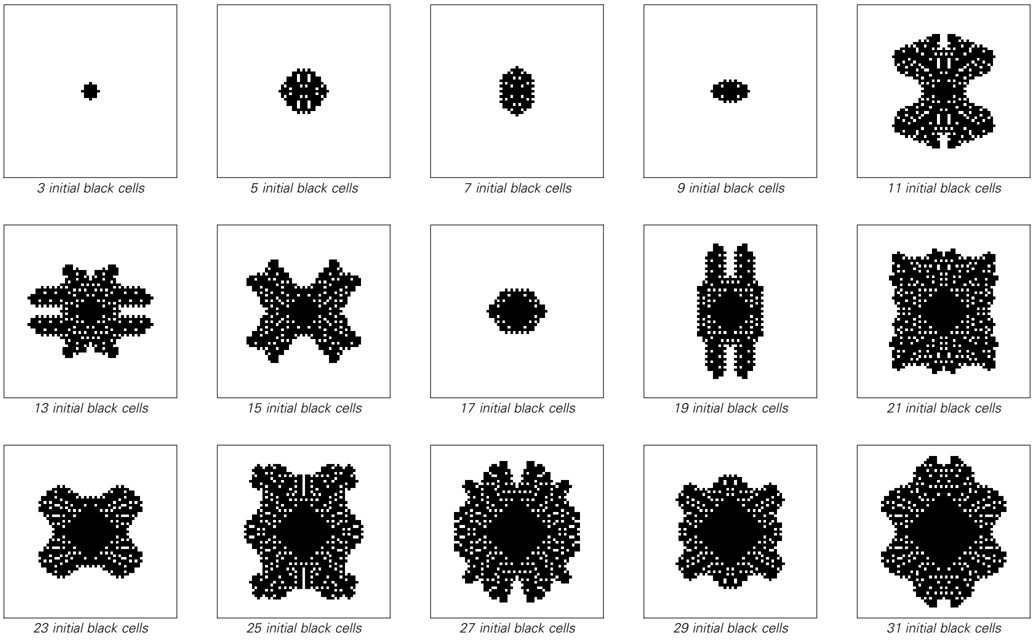




A two-dimensional cellular automaton that yields a pattern with a rough surface. The rule used here includes diagonal neighbors, and so involves a total of 8 neighbors for each cell, as indicated in the icon on the left. The rule specifies that the center cell should become black if either 3 or 5 of its 8 neighbors were black on the step before, and should otherwise stay the same color as it was before. The initial condition in the case shown consists of a row of 7 black cells. In an extension to 8 neighbors of the scheme used in the pictures a few pages back, the rule has code number 175850.



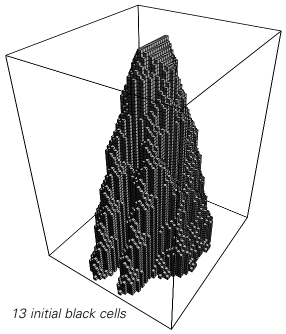
A cellular automaton that yields a pattern whose shape closely approximates a circle. The rule used is of the same kind as on the previous page, but now takes the center cell to become black only if it has exactly 3 black neighbors. If it has 1, 2 or 4 black neighbors then it stays the same color as it was before, and if it has 5 or more black neighbors, then it becomes white on the next step (code number 746). The initial condition consists of a row of 7 black cells, just as in the picture on the previous page. The pattern shown here is the result of 400 steps in the evolution of the system. After t steps, the radius of the approximate circle is about $0.37\,t$.



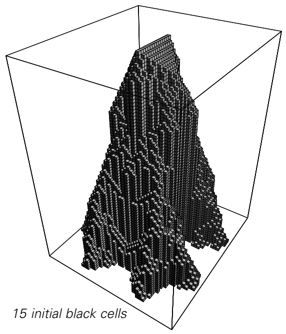
Patterns produced by evolution according to a simple two-dimensional cellular automaton rule starting from rows of black cells of various lengths. The rule used specifies that a particular cell should become black if exactly three out of its eight neighbors (with diagonal neighbors included) are black (code number 174826). The patterns in the picture are obtained by 60 steps of evolution according to this rule. The smaller patterns above have all stopped growing after this number of steps, but many of the other patterns apparently go on growing forever.

So what about three-dimensional cellular automata? It is straightforward to generalize the setup for two-dimensional rules to the three-dimensional case. But particularly on a printed page it is fairly difficult to display the evolution of a three-dimensional cellular automaton in a way that can readily be assimilated.

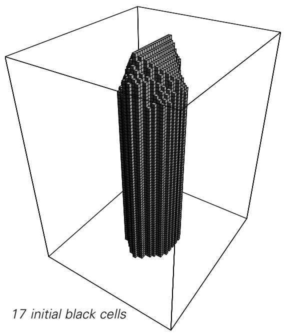
Pages 182 and 183 do however show a few examples of three-dimensional cellular automata. And just as in the two-dimensional case, there are some specific new phenomena that can be seen. But overall it seems that the basic kinds of behavior produced are just the same as in one and two dimensions. And in particular, the basic phenomenon of complexity does not seem to depend in any crucial way on the dimensionality of the system one looks at.



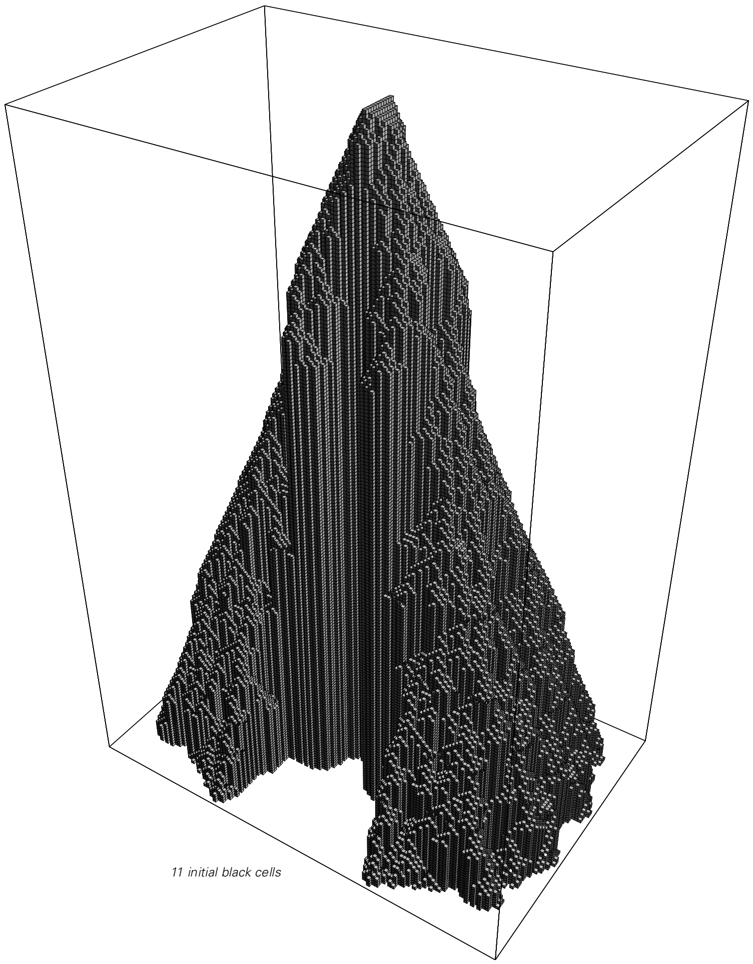
13 initial black cells



15 initial black cells

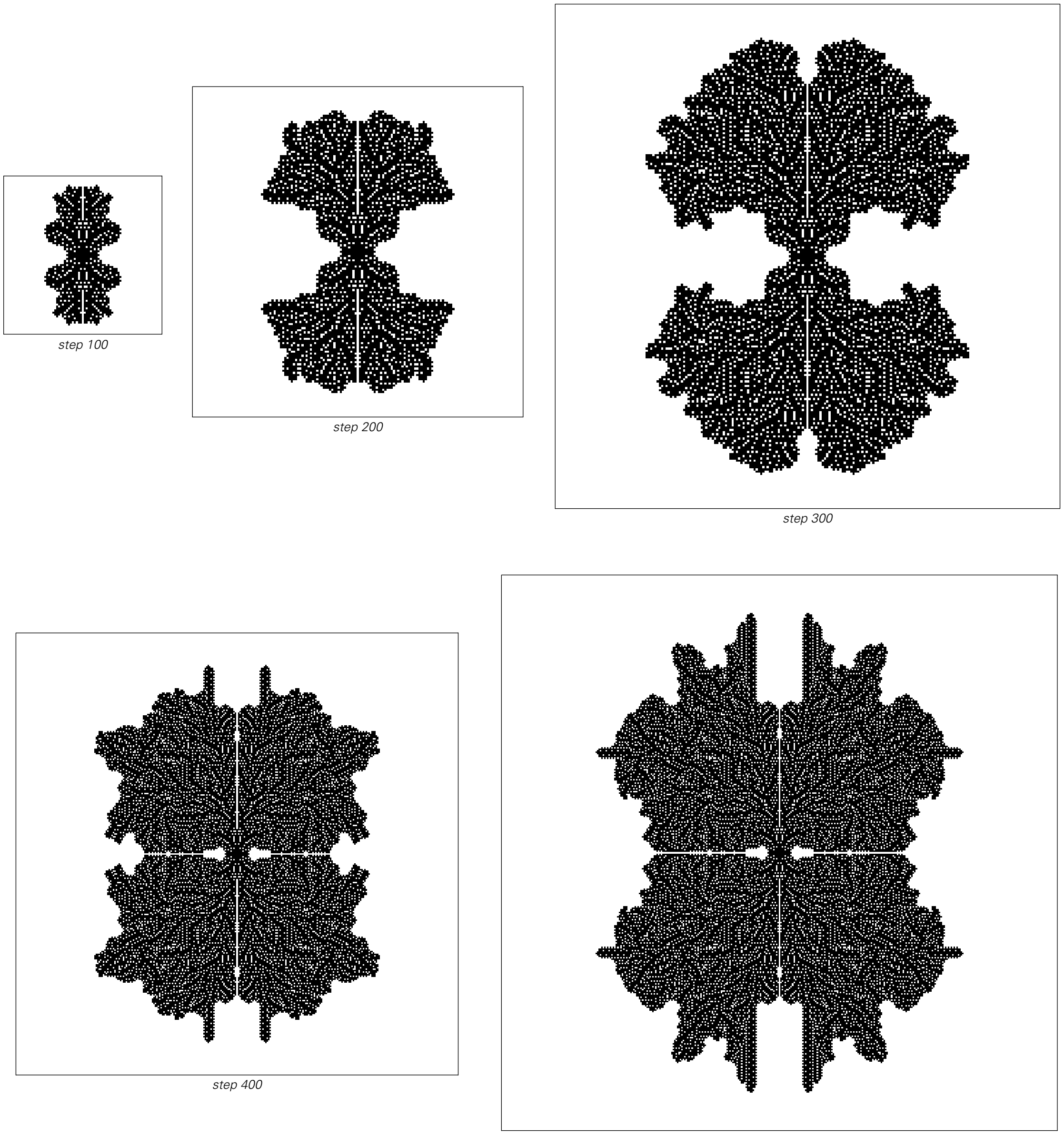


17 initial black cells

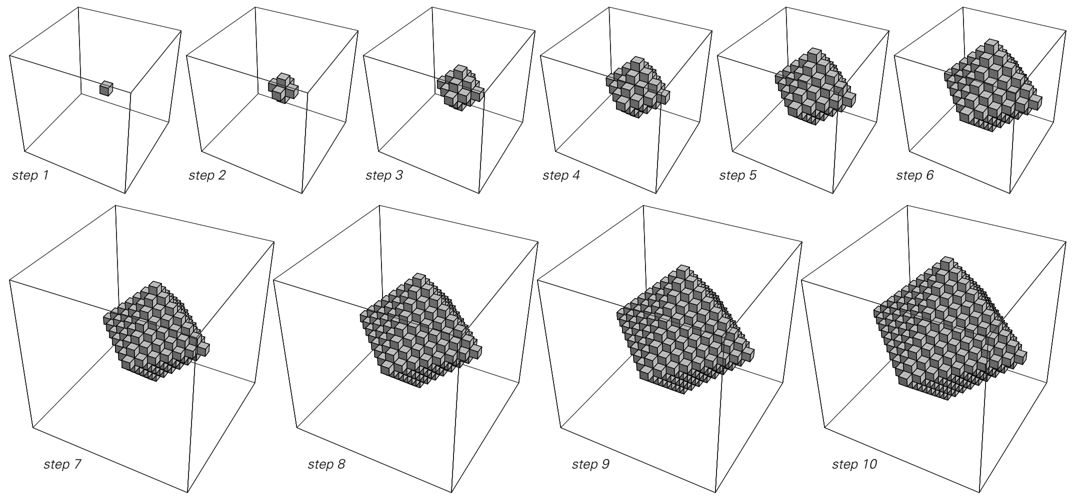


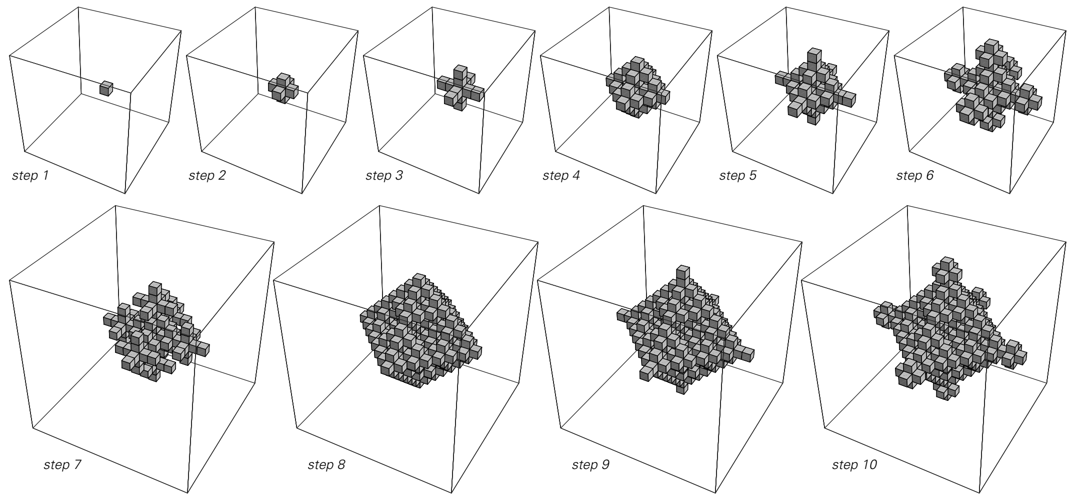
11 initial black cells

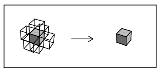
Three-dimensional objects formed by stacking successive two-dimensional patterns produced in the evolution of the cellular automaton from the previous page. The large picture on the right shows 200 steps of evolution.



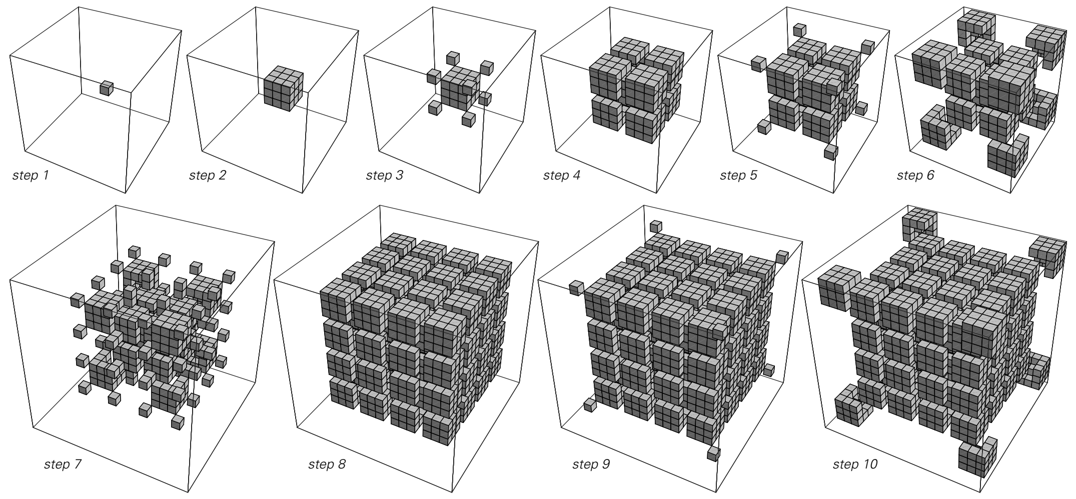
Stages in the evolution of the cellular automaton from the facing page, starting with an initial condition consisting of a row of 11 black cells.

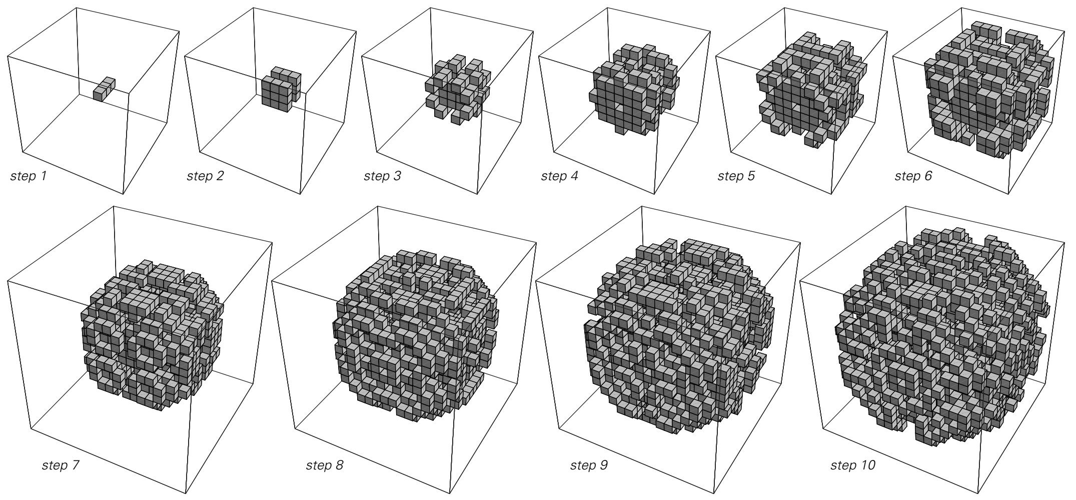


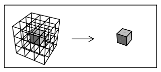




Examples of three-dimensional cellular automata. In the top set of pictures, the rule specifies that a cell should become black whenever any of the six neighbors with which it shares a face were black on the step before. In the bottom pictures, the rule specifies that a cell should become black only when exactly one of its six neighbors was black on the step before. In both cases, the initial condition contains a single black cell. In the top pictures, the limiting shape obtained is a regular octahedron. In the bottom pictures, it is a nested pattern analogous to the two-dimensional one on page 171.



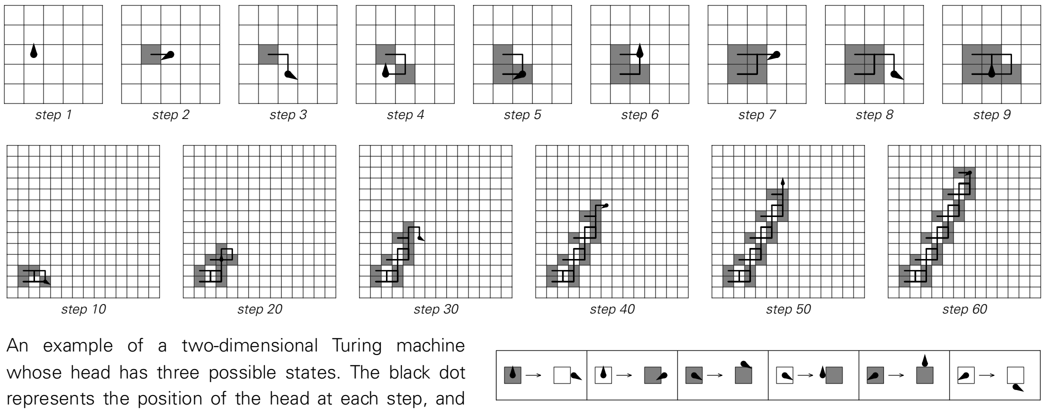




Further examples of three-dimensional cellular automata, but now with rules that depend on all 26 neighbors that share either a face or a corner with a particular cell. In the top pictures, the rule specifies that a cell should become black when exactly one of its 26 neighbors was black on the step before. In the bottom pictures, the rule specifies that a cell should become black only when exactly two of its 26 neighbors were black on the step before. In the top pictures, the initial condition contains a single black cell; in the bottom pictures, it contains a line of three black cells.

Turing Machines

Much as for cellular automata, it is straightforward to generalize Turing machines to two dimensions. The basic idea—shown in the picture below—is to allow the head of the Turing machine to move around on a two-dimensional grid rather than just going backwards and forwards on a one-dimensional tape.



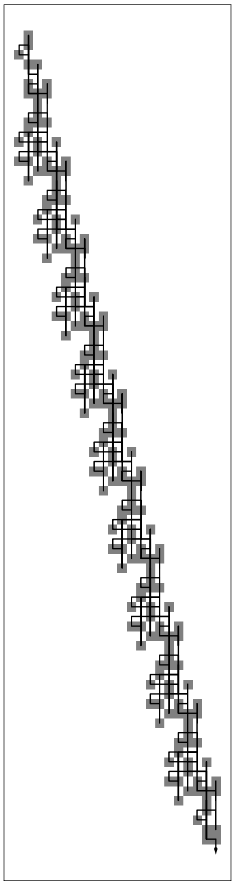
An example of a two-dimensional Turing machine whose head has three possible states. The black dot represents the position of the head at each step, and the three possible orientations of the arrow on this dot correspond to the three possible states of the head. The rule specifies in which of the four possible directions the head should move at each step. Note that the orientation of the arrow representing the state of the head has no direct relationship to directions on the grid—or to which way the head will move at the next step.

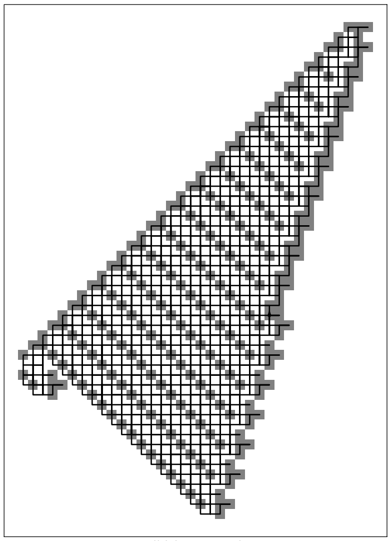
When we looked at one-dimensional Turing machines earlier in this book, we found that it was possible for them to exhibit complex behavior, but that such behavior was rather rare.

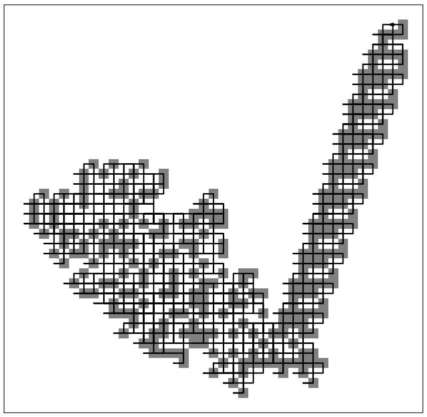
In going to two dimensions we might expect that complex behavior would somehow immediately become more common. But in fact what we find is that the situation is remarkably similar to one dimension.

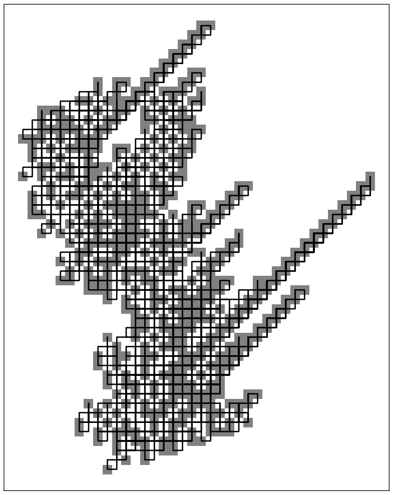
For Turing machines with two or three possible states, only repetitive and nested behavior normally seem to occur. With four states, more complex behavior is possible, but it is still rather rare.

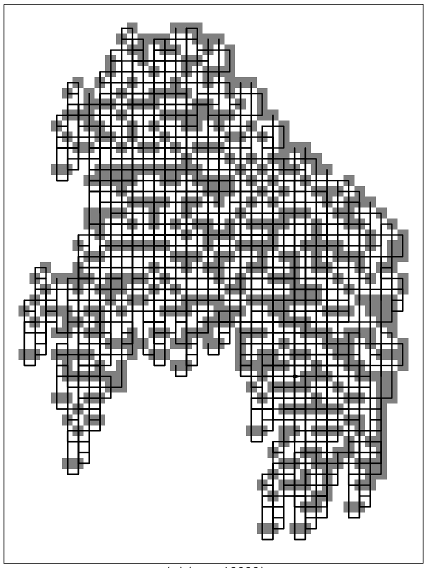
The facing page shows some examples of two-dimensional Turing machines with four states. Simple behavior is overwhelmingly the most common. But out of a million randomly chosen rules, there will typically be a few that show complex behavior. Page 186 shows one example where the behavior seems in many respects completely random.

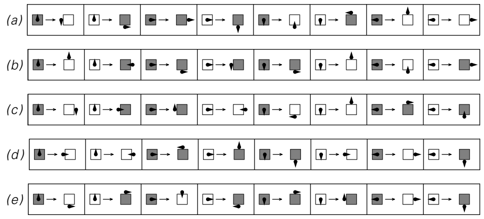

(a) (step 1000)


(b) (step 2500)

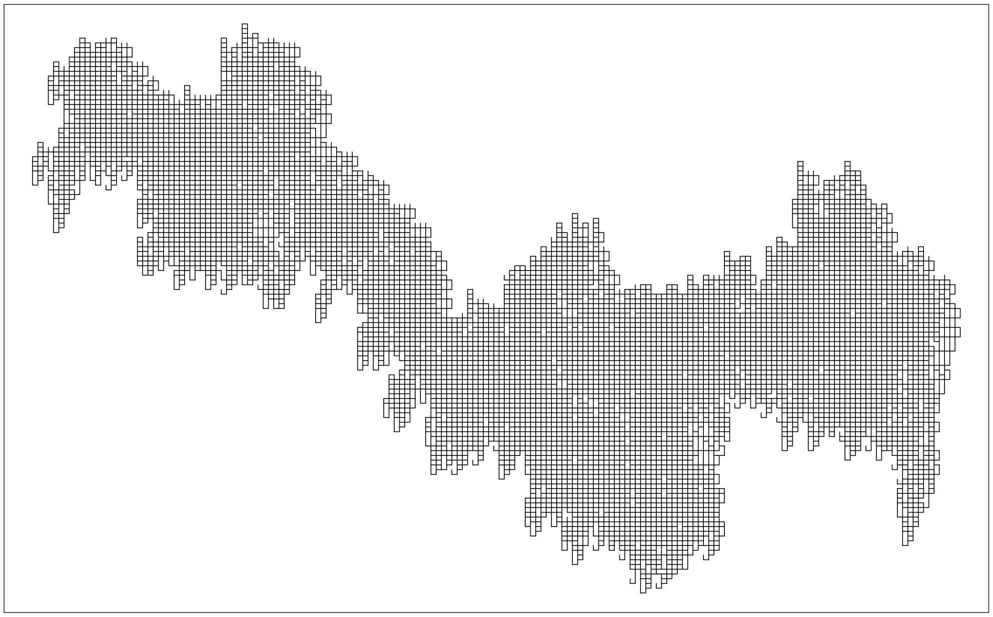

(c) (step 3000)


(d) (step 8000)

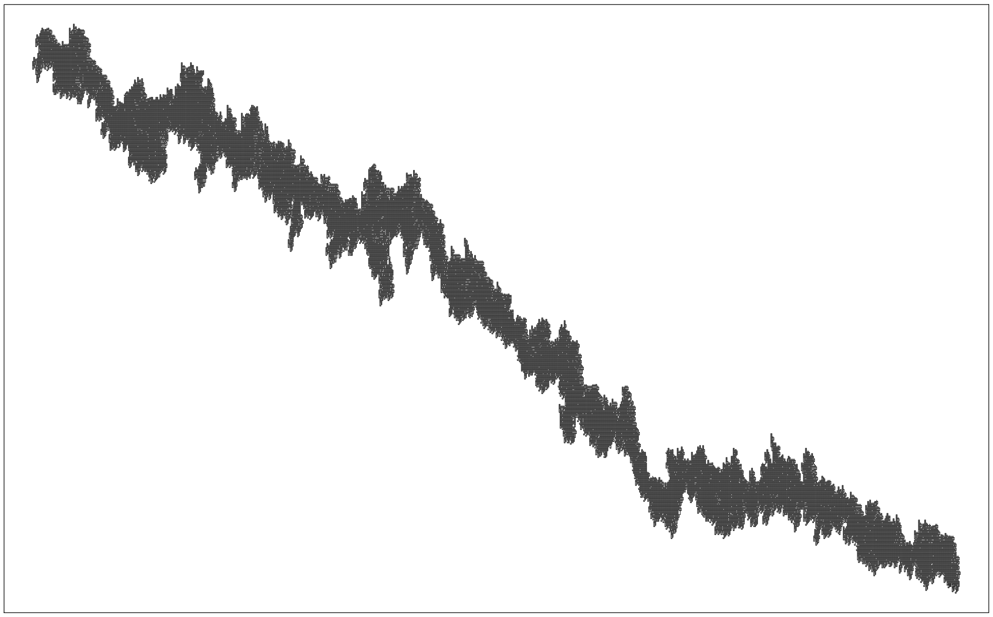

(e) (step 10000)



Examples of patterns produced by two-dimensional Turing machines whose heads have four possible states. In each case, all cells are initially white, and one of the rules given on the left is applied for the specified number of steps. Note that in the later cases shown, the head often visits the same position on the grid many times.



100,000 steps



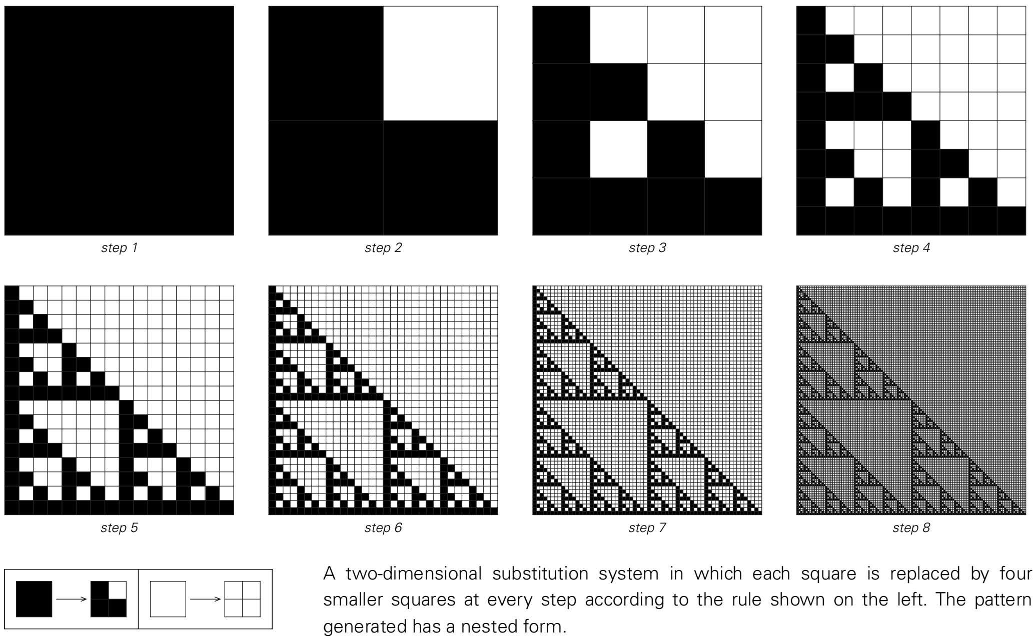
500,000 steps

The path traced out by the head of the two-dimensional Turing machine with rule (e) from the previous page. There are many seemingly random fluctuations in this path, though in general it tends to grow to the right.

Substitution Systems and Fractals

One-dimensional substitution systems of the kind we discussed on page 82 can be thought of as working by progressively subdividing each element they contain into several smaller elements.

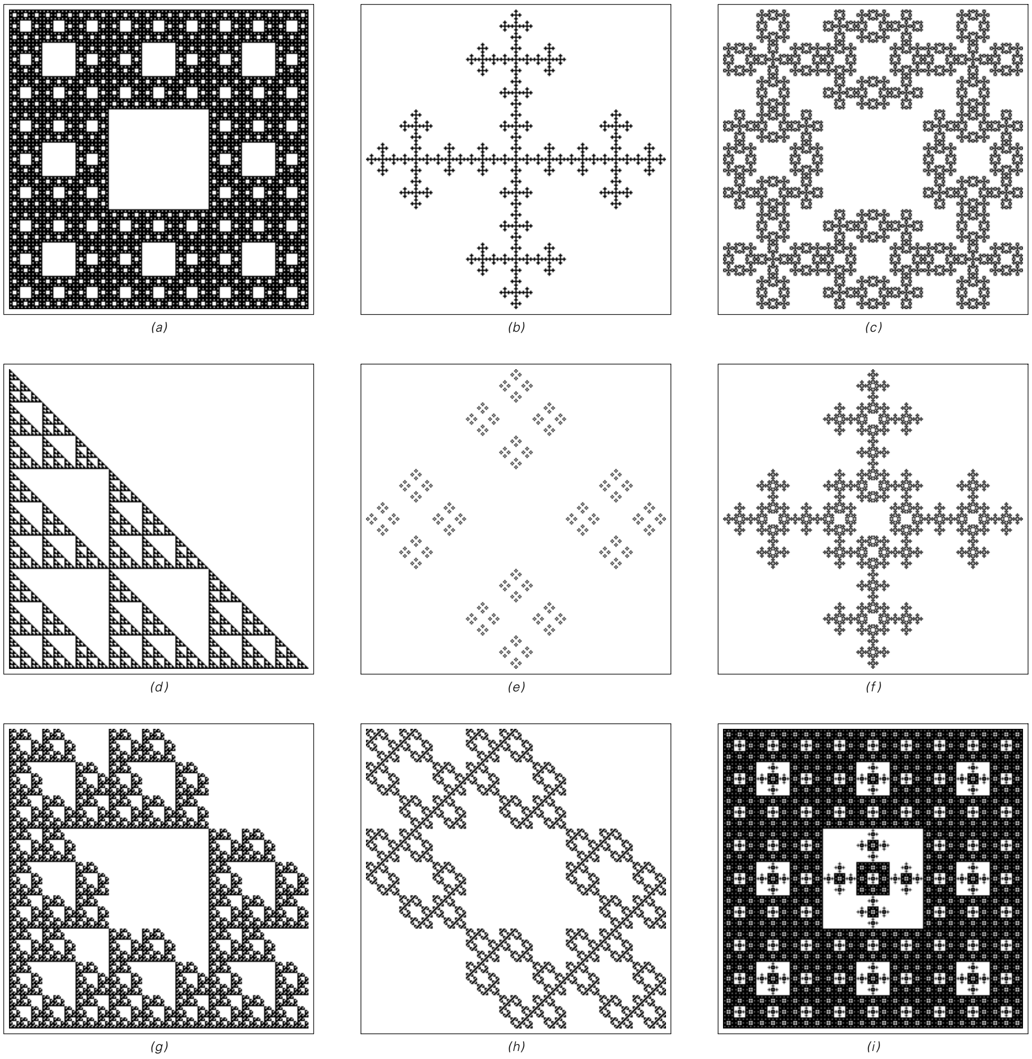
One can construct two-dimensional substitution systems that work in essentially the same way, as shown in the pictures below.



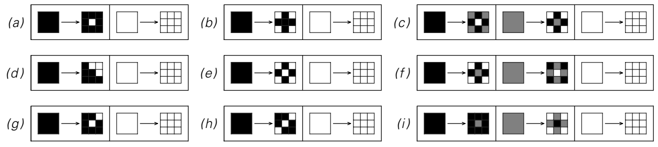
A two-dimensional substitution system in which each square is replaced by four smaller squares at every step according to the rule shown on the left. The pattern generated has a nested form.

The next page gives some more examples of two-dimensional substitution systems. The patterns that are produced are certainly quite intricate. But there is nevertheless great regularity in their overall forms. Indeed, just like patterns produced by one-dimensional substitution systems on page 83, all the patterns shown here ultimately have a simple nested structure.

Why does such nesting occur? The basic reason is that at every step the rules for the substitution system simply replace each black square with several smaller black squares. And on subsequent steps, each of these new black squares is then in turn replaced in exactly the

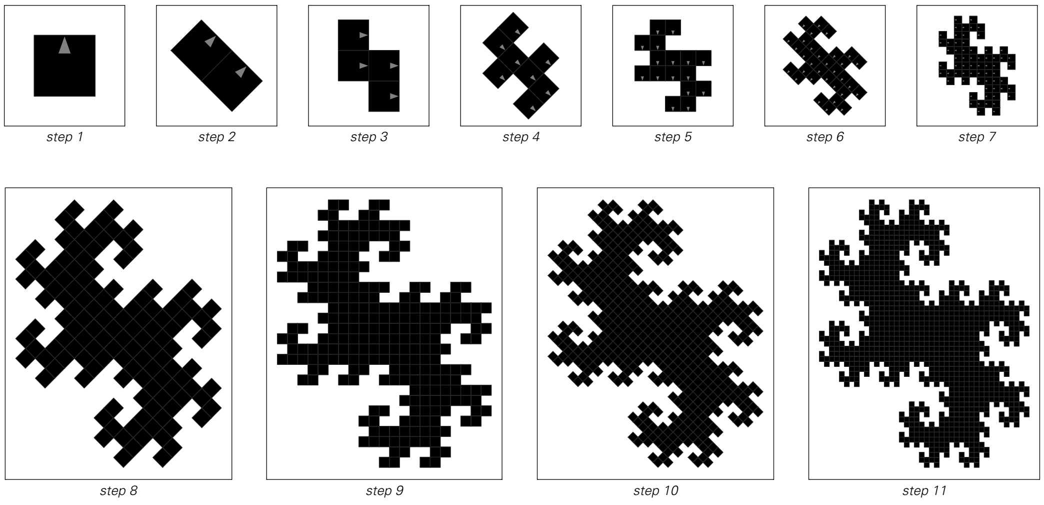


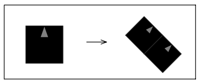
Patterns from various two-dimensional substitution systems. In each case what is shown is the pattern obtained after five steps of evolution according to the rules on the right, starting with a single black square.



same way, so that it ultimately evolves to produce an identical copy of the whole pattern.

But in fact there is nothing about this basic process that depends on the squares being arranged in any kind of rigid grid. And the picture below shows what happens if one just uses a simple geometrical rule to replace each black square by two smaller black squares. The result, once again, is that one gets an intricate but highly regular nested pattern.

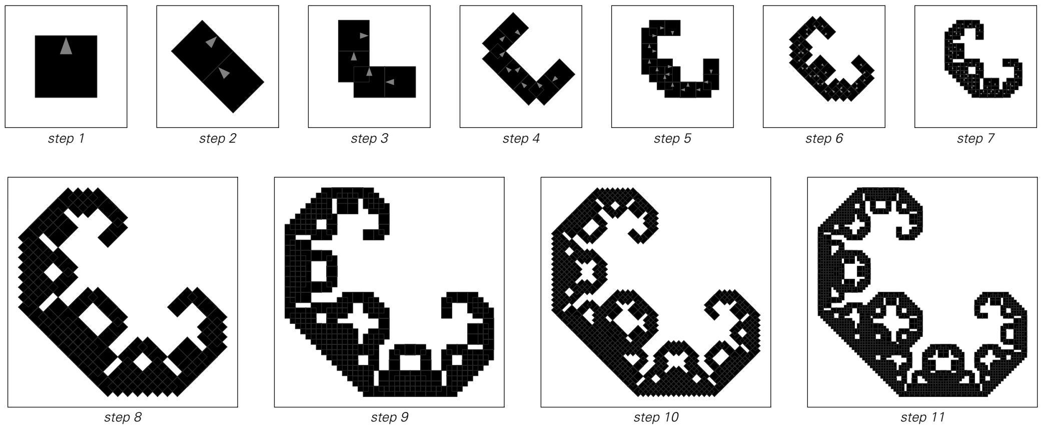




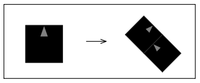
The pattern obtained by starting with a single black square and then at every step replacing each black cell with two smaller black cells according to the simple geometrical rule shown on the left. Note that in applying the rule to a particular square, one must take account of the orientation of that square. The final pattern obtained has an intricate nested structure.

In a substitution system where black squares are arranged on a grid, one can be sure that different squares will never overlap. But if there is just a geometrical rule that is used to replace each black square, then it is possible for the squares produced to overlap, as in the picture on the next page. Yet at least in this example, the overall pattern that is ultimately obtained still has a purely nested structure.

The general idea of building up patterns by repeatedly applying geometrical rules is at the heart of so-called fractal geometry. And the



The pattern obtained by repeatedly applying the simple geometrical rule shown on the right. Even though this basic rule does not involve overlapping squares, the pattern obtained even by step 3 already has squares that overlap. But the overall pattern obtained after a large number of steps still has a nested form.

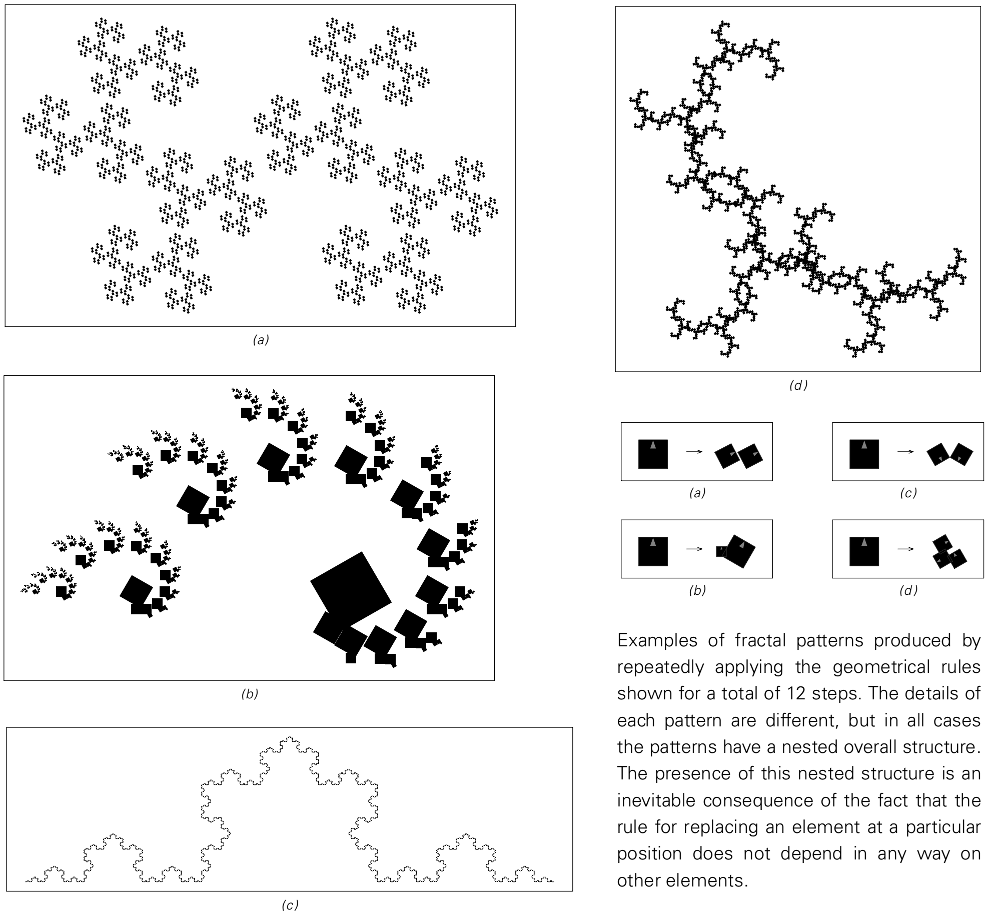


pictures on the facing page show several more examples of fractal patterns produced in this way.

The details of the geometrical rules used are different in each case. But what all the rules have in common is that they involve replacing one black square by two or more smaller black squares. And with this kind of setup, it is ultimately inevitable that all the patterns produced must have a completely regular nested structure.

So what does it take to get patterns with more complicated structure? The basic answer, much as we saw in one-dimensional substitution systems on page 85, is some form of interaction between different elements—so that the replacement for a particular element at a given step can depend not only on the characteristics of that element itself, but also on the characteristics of other neighboring elements.

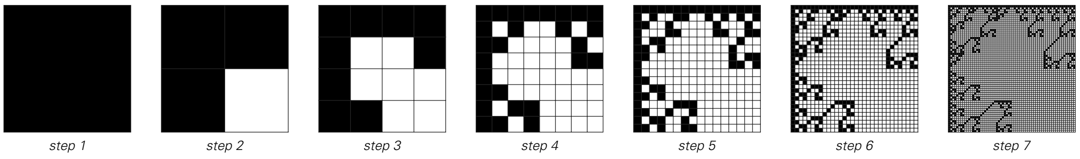
But with geometrical replacement rules of the kind shown on the facing page there is a problem with this. For elements can end up anywhere in the plane, making it difficult to define an obvious notion of neighbors. And the result of this has been that in traditional fractal geometry the idea of interaction between elements is not considered—so that all patterns that are produced have a purely nested form.

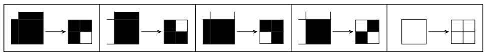


Examples of fractal patterns produced by repeatedly applying the geometrical rules shown for a total of 12 steps. The details of each pattern are different, but in all cases the patterns have a nested overall structure. The presence of this nested structure is an inevitable consequence of the fact that the rule for replacing an element at a particular position does not depend in any way on other elements.

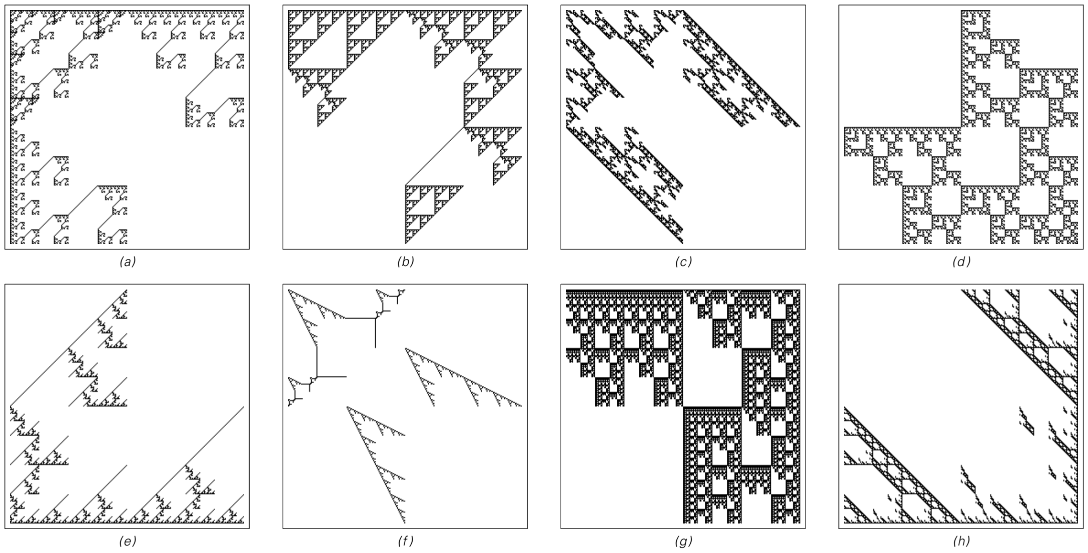
Yet if one sets up elements on a grid it is straightforward to allow the replacements for a given element to depend on its neighbors, as in the picture at the top of the next page. And if one does this, one immediately gets all sorts of fairly complicated patterns that are often not just purely nested—as illustrated in the pictures on the next page.

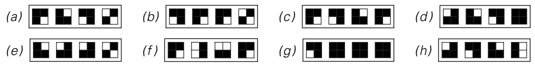
In Chapter 3 we discussed both ordinary one-dimensional substitution systems, in which every element is replaced at each step, and sequential substitution systems, in which just a single block of elements are replaced at each step. And what we did to find which block of elements should be replaced at a given step was to scan the whole sequence of elements from left to right.





A two-dimensional neighbor-dependent substitution system. The grid of cells is assumed to wrap around in both its dimensions.





Patterns generated by 8 steps of evolution in various two-dimensional neighbor-dependent substitution systems.

So how can this be generalized to higher dimensions? On a two-dimensional grid one can certainly imagine snaking backwards and forwards or spiralling outwards to scan all the elements. But as soon as one defines any particular order for elements—however they may be laid out—this in effect reduces one to dealing with a one-dimensional system.

And indeed there seems to be no immediate way to generalize sequential substitution systems to two or more dimensions. In Chapter 9, however, we will see that with more sophisticated ideas it is in fact possible in any number of dimensions to set up substitution systems in which elements are scanned in order—but whatever order is used, the results are in some sense always the same.

Network Systems

One feature of systems like cellular automata is that their elements are always set up in a regular array that remains the same from one step to the next. In substitution systems with geometrical replacement rules there is slightly more freedom, but still the elements are ultimately constrained to lie in a two-dimensional plane.

Indeed, in all the systems that we have discussed so far there is in effect always a fixed underlying geometrical structure which remains unchanged throughout the evolution of the system.

It turns out, however, that it is possible to construct systems in which there is no such invariance in basic structure, and in this section I discuss as an example one version of what I will call network systems.

A network system is fundamentally just a collection of nodes with various connections between these nodes, and rules that specify how these connections should change from one step to the next.

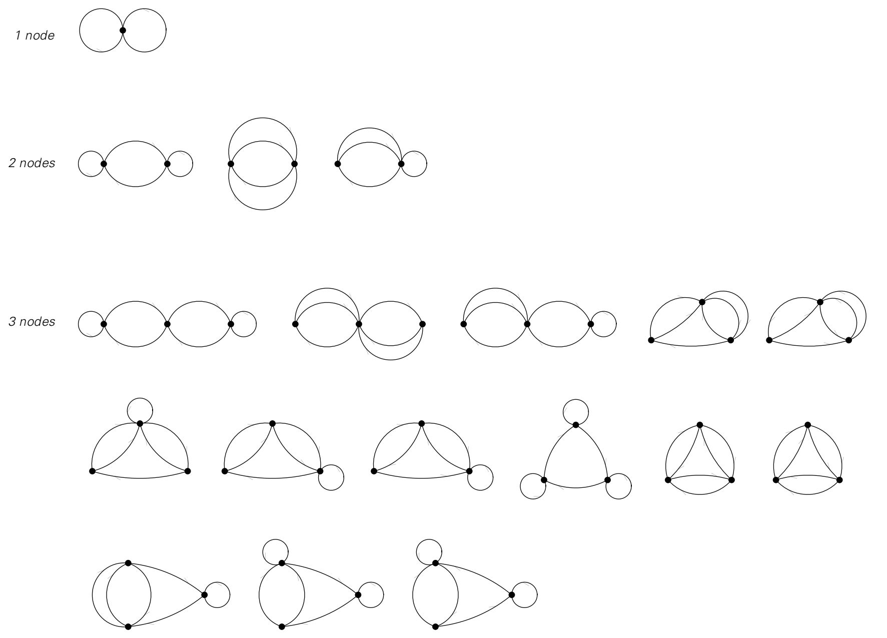
At any particular step in its evolution, a network system can be thought of a little like an electric circuit, with the nodes of the network corresponding to the components in the circuit, and the connections to the wires joining these components together.

And as in an electric circuit, the properties of the system depend only on the way in which the nodes are connected together, and not on any specific layout for the nodes that may happen to be used.

Of course, to make a picture of a network system, one has to choose particular positions for each of its nodes. But the crucial point is that these positions have no fundamental significance: they are introduced solely for the purpose of visual representation.

In constructing network systems one could in general allow each node to have any number of connections coming from it. But at least for the purposes of this section nothing fundamental turns out to be lost if one restricts oneself to the case in which every node has exactly two outgoing connections—each of which can then either go to another node, or can loop back to the original node itself.

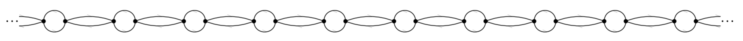
With this setup the very simplest possible network consists of just one node, with both connections from the node looping back, as in the top picture below. With two nodes, there are already three possible patterns of connections, as shown on the second line below. And as the number of nodes increases, the number of possible different networks grows very rapidly.



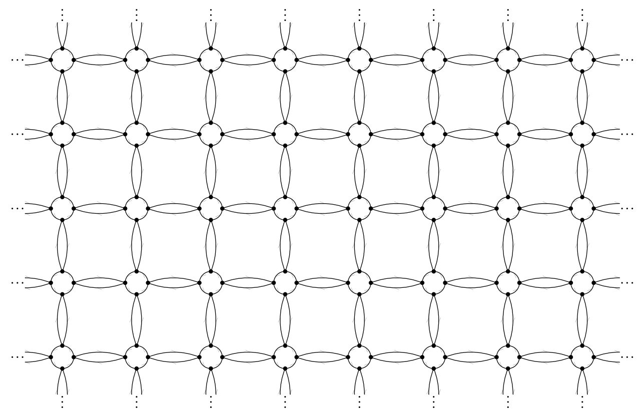
Possible networks formed by having one, two or three nodes, with two connections coming out of each node. The picture shows all inequivalent cases ignoring labels, but excludes networks in which there are nodes which cannot be reached by connections from other nodes.

For most of these networks there is no way of laying out their nodes so as to get a picture that looks like anything much more than a random jumble of wires. But it is nevertheless possible to construct many specific networks that have easily recognizable forms, as shown in the pictures on the facing page.

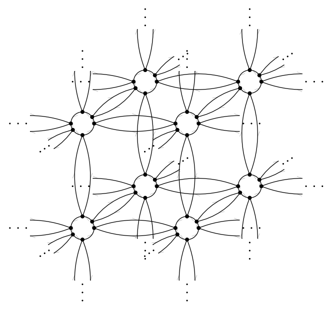
Each of the networks illustrated at the top of the facing page consists at the lowest level of a collection of identical nodes. But the remarkable fact that we see is that just by changing the pattern of



(a) one dimension



(b) two dimensions



(c) three dimensions

Examples of networks that correspond to arrays in one, two and three dimensions. At an underlying level, each network consists just of a collection of nodes with two connections coming from each node. But by setting up appropriate patterns for these connections, one can get networks with very different effective geometrical structures.

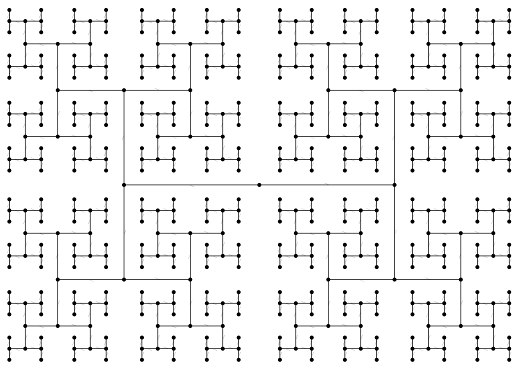
connections between these nodes it is possible to get structures that effectively correspond to arrays with different numbers of dimensions.

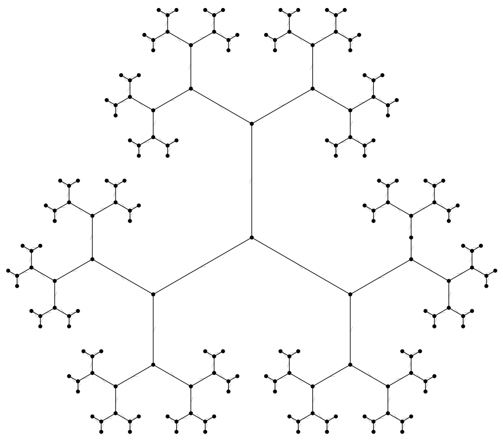
Example (a) shows a network that is effectively one-dimensional. The network consists of pairs of nodes that can be arranged in a sequence in which each pair is connected to one other pair on the left and another pair on the right.

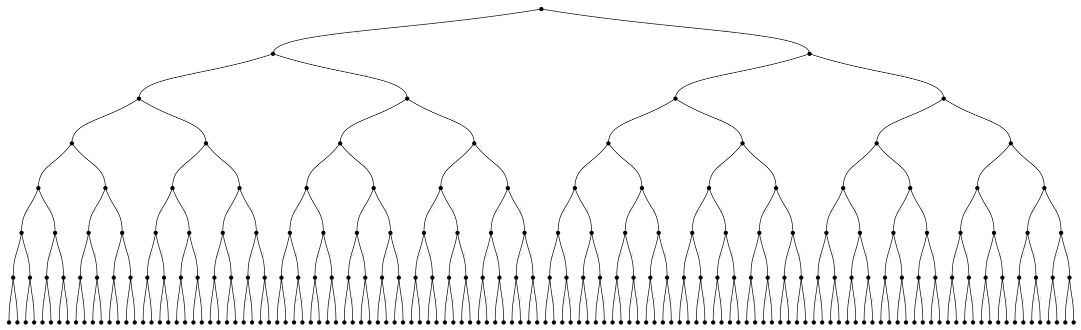
But there is nothing intrinsically one-dimensional about the structure of network systems. And as example (b) demonstrates, it is just a matter of rearranging connections to get a network that looks like a two-dimensional rather than a one-dimensional array. Each individual node in example (b) still has exactly two connections coming out of it, but now the overall pattern of connections is such that every block of nodes is connected to four rather than two neighboring blocks, so that the network effectively forms a two-dimensional square grid.

Example (c) then shows that with appropriate connections, it is also possible to get a three-dimensional array, and indeed using the same principles an array with any number of dimensions can easily be obtained.

The pictures below show examples of networks that form infinite trees rather than arrays. Notice that the first and last networks shown actually have an identical pattern of connections, but they look different here because the nodes are arranged in a different way on the page.

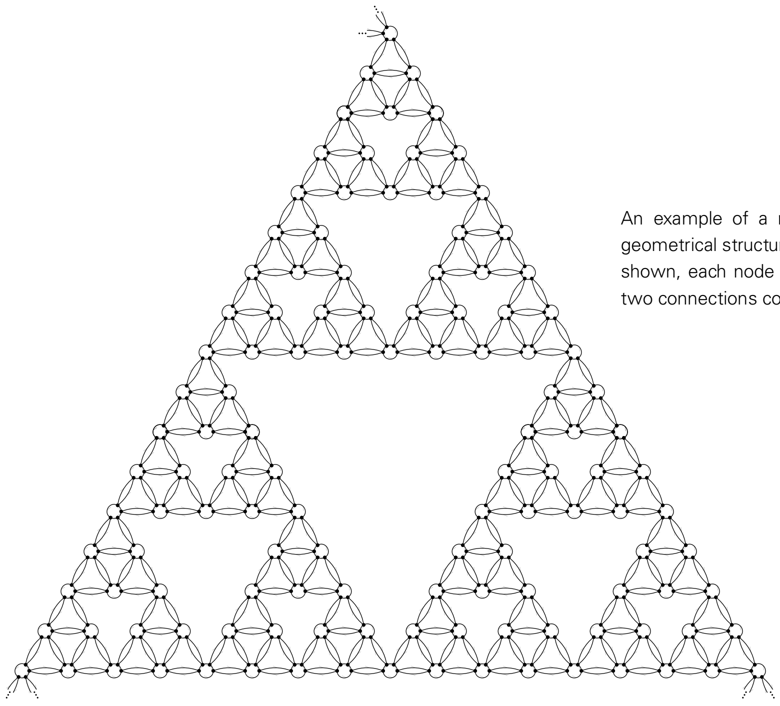

(a)


(b)


(c)

Examples of networks that correspond to infinite trees. Note that networks (a) and (c) are identical, though they look different because the nodes are laid out differently on the page. All the networks shown are truncated at the leaves of each tree.

In general, there is great variety in the possible structures that can be set up in network systems, and as one further example the picture below shows a network that forms a nested pattern.

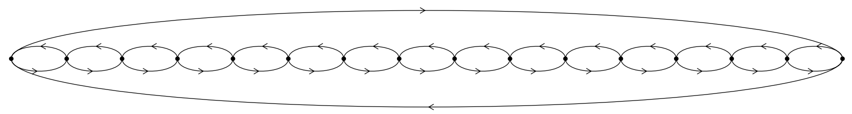


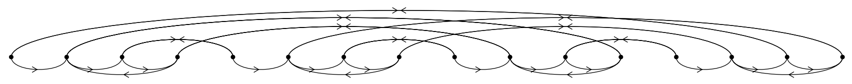
An example of a network that forms a nested geometrical structure. As in all the other networks shown, each node here is identical, and has just two connections coming out of it.

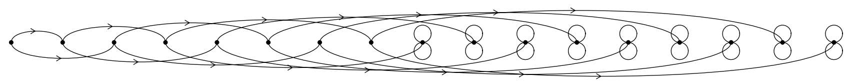
In the pictures above we have seen various examples of individual networks that might exist at a particular step in the evolution of a network system. But now we must consider how such networks are transformed from one step in evolution to the next.

The basic idea is to have rules that specify how the connections coming out of each node should be rerouted on the basis of the local structure of the network around that node.

But to see the effect of any such rules, one must first find a uniform way of displaying the networks that can be produced. The pictures at the top of the next page show one possible approach based on always arranging the nodes in each network in a line across the page. And although this representation can obscure the geometrical structure


(a)


(b)


(c)

Networks from previous pictures laid out in a uniform way. Network (a) corresponds to a one-dimensional array, (b) to a two-dimensional array, and (c) to a tree. In the layout shown here, all the networks have their nodes arranged along a line. Note that in cases (a) and (b) the connections are arranged so that the arrays effectively wrap around; in case (c) the leaves of the tree are taken to have connections that loop back to themselves.

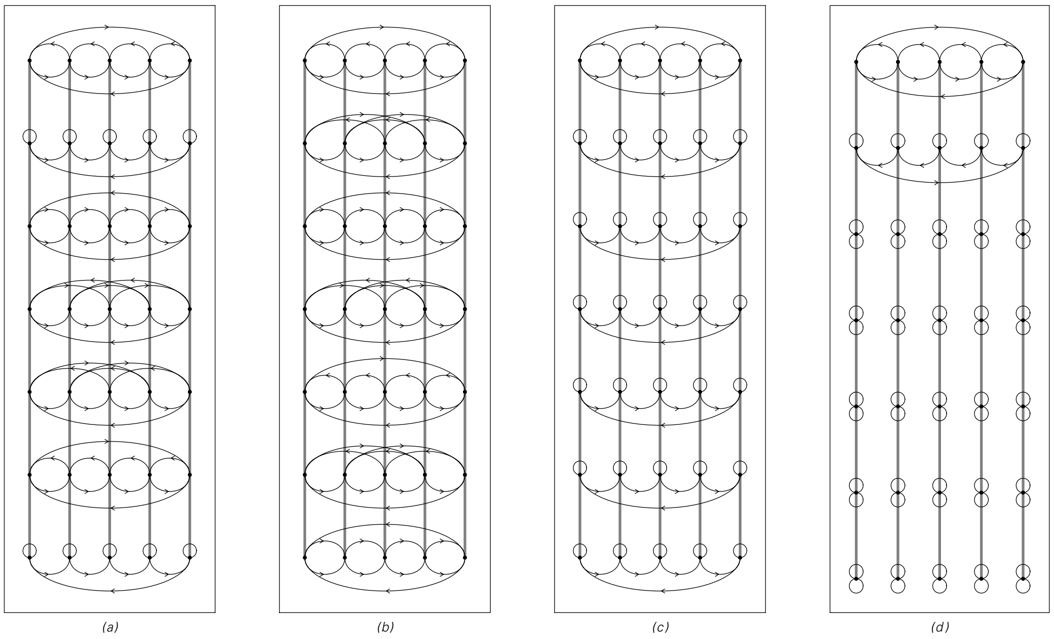
of a particular network, as in the second and third cases above, it more readily allows comparison between different networks.

In setting up rules for network systems, it is convenient to distinguish the two connections that come out of each node. And in the pictures above one connection is therefore always shown going above the line of nodes, while the other is always shown going below.

The pictures on the facing page show examples of evolution obtained with four different choices of underlying rules. In the first case, the rule specifies that the "above" connection from each node should be rerouted so that it leads to the node obtained by following the "below" connection and then the "above" connection from that node. The "below" connection is left unchanged.

The other rules shown are similar in structure, except that in cases (c) and (d), the "above" connection from each node is rerouted so that it simply loops back to the node itself.

In case (d), the result of this is that the network breaks up into several disconnected pieces. And it turns out that none of the rules I consider here can ever reconnect these pieces again. So as a consequence, what I do in the remainder of this section is to track only the piece that includes the first node shown in pictures such as those

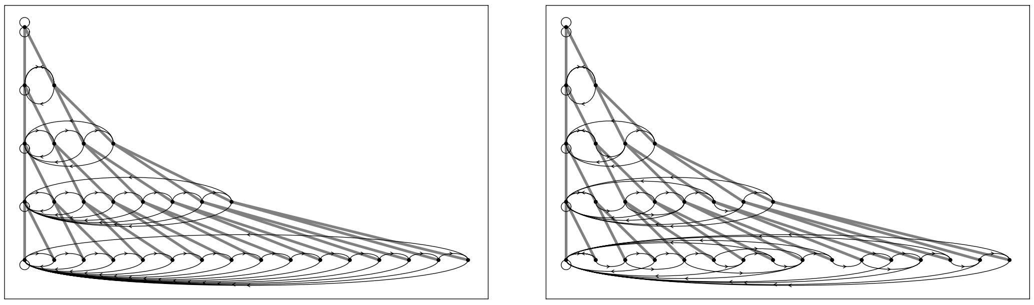


The evolution of network systems with four different choices of underlying rules. Successive steps in the evolution are shown on successive lines down the page. In case (a), the "above" connection of each node is rerouted at each step to lead to the node reached by following first the below connection and then the above connection from that node; the below connection is left unchanged. In case (b), the above connection of each node is rerouted to the node reached by following the above connection and then the above connection again; the below connection is left unchanged. In case (c), the above connection of each node is rerouted so as to loop back to the node itself, while the below connection is left unchanged. And in case (d), the above connection is rerouted so as to loop back, while the below connection is rerouted to lead to the node reached by following the above connection. With the "above" connection labelled as 1 and the "below" connection as 2, these rules correspond to replacing connections *{{1}, {2}}* at each node by (a) *{{2, 1}, {2}}*, (b) *{{1, 1}, {2}}*, (c) *{{}, {2}}*, and (d) *{{}, {1}}*.

above. And in effect, this then means that other nodes are dropped from the network, so that the total size of the network decreases.

By changing the underlying rules, however, the number of nodes in a network can also be made to increase. The basic way this can be done is by breaking a connection coming from a particular node by inserting a new node and then connecting that new node to nodes obtained by following connections from the original node.

The pictures on the next page show examples of behavior produced by two rules that use this mechanism. In both cases, a new node is inserted in the "above" connection from each existing node in the network. In the first case, the connections from the new node are exactly the same as the connections from the existing node, while in the second case, the "above" and "below" connections are reversed.



Evolution of network systems whose rules involve the addition of new nodes. In both cases, the new nodes are inserted in the "above" connection from each node. In case (a), the connections from the new node lead to the same nodes as the connections from the original node. In case (b), the above and below connections for the new node are reversed. In the pictures above, new nodes are placed immediately after the nodes that give rise to them, and gray lines are used to indicate the origin of each node. Note that the initial conditions consist of a network that contains only a single node.

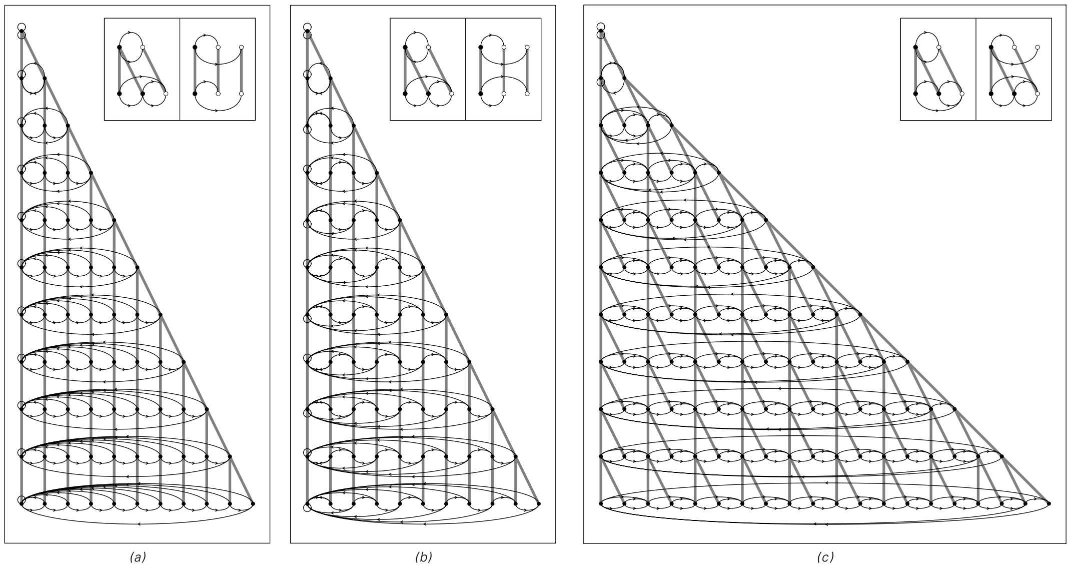
But in both cases the behavior obtained is quite simple. Yet much like neighbor-independent substitution systems these network systems have the property that exactly the same operation is always performed at each node on every step.

In general, however, one can set up network systems that have rules in which different operations are performed at different nodes, depending on the local structure of the network near each node.

One simple scheme for doing this is based on looking at the two connections that come out of each node, and then performing one operation if these two connections lead to the same node, and another if the connections lead to different nodes.

The pictures on the facing page show some examples of what can happen with this scheme. And again it turns out that the behavior is always quite simple—with the network having a structure that inevitably grows in an essentially repetitive way.

But as soon as one allows dependence on slightly longer-range features of the network, much more complicated behavior immediately

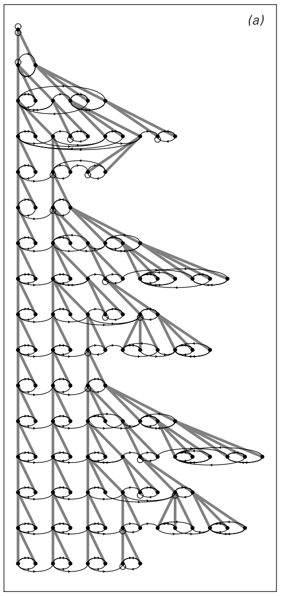


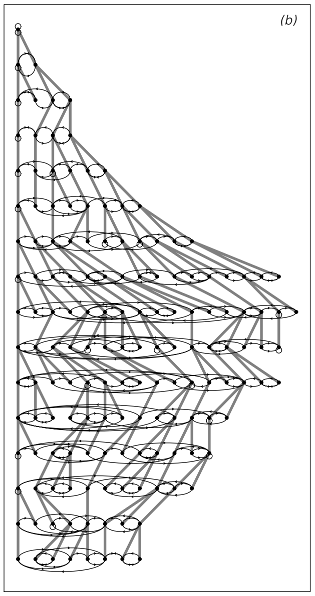
Examples of network systems with rules that cause different operations to be performed at different nodes. Each rule contains two cases, as shown above. The first case specifies what to do if both connections from a particular node lead to the same node; the second case specifies what to do when they lead to different nodes. In the rules shown, the connections from a particular node (indicated by a solid circle) and from new nodes created from this node always go to the nodes indicated by open circles that are reached by following just a single above or below connection from the original node. Even if this restriction is removed, however, more complicated behavior does not appear to be seen.

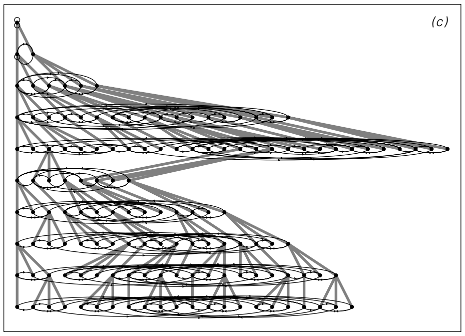
becomes possible. And indeed, the pictures on the next two pages show examples of what can happen if the rules are allowed to depend on the number of distinct nodes reached by following not just one but up to two successive connections from each node.

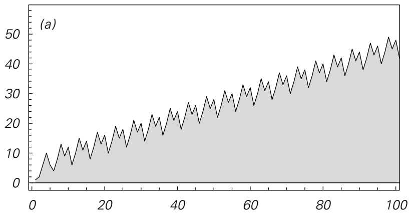
With such rules, the sequence of networks obtained no longer needs to form any kind of simple progression, and indeed one finds that even the total number of nodes at each step can vary in a way that seems in many respects completely random.

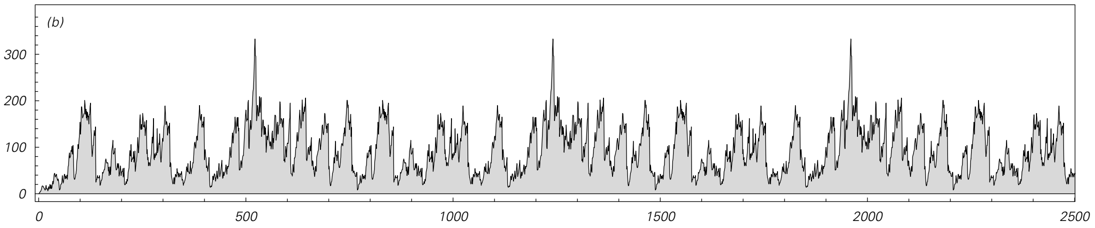
When we discuss issues of fundamental physics in Chapter 9 we will encounter a variety of other types of network systems—and I suspect that some of these systems will in the end turn out to be closely related to the basic structure of space and spacetime in our universe.

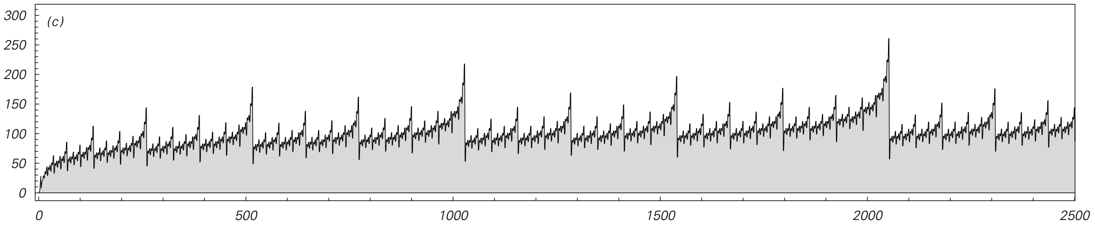










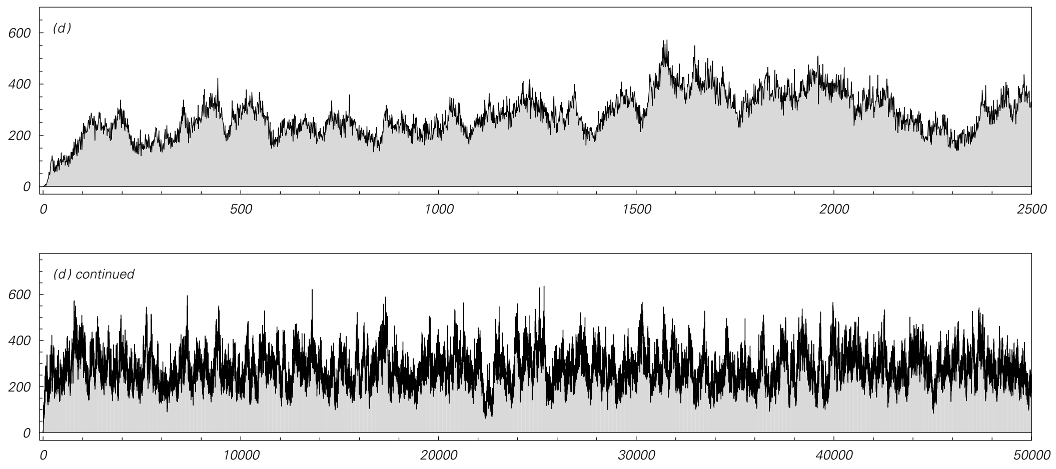


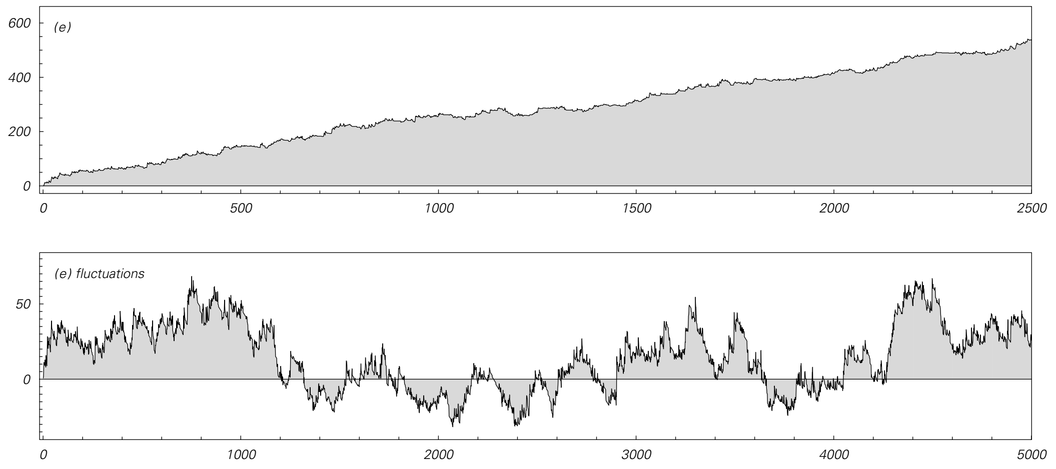
```
(a) {{1, 1}→{{1}, {{2, 1}, {2, 1}}},  {1, 2}→{{{}, {1, 1}}, {{1, 1}, {}}},  {2, 1}→{{{}, {}}, {{1}, {2, 1}}},  {2, 2}→{{{1, 1}, {2, 1}}, {{2}, {2, 1}}},
    {2, 3}→{{{}, {}}, {2}},  {2, 4}→{{{2, 2}, {}}, {}}}

(b) {{1, 1}→{{{}, {1, 1}}, {2}},  {1, 2}→{{2}, {{}, {}}},
        {2, 1}→{{2, 1}, {{}, {1}}},  {2, 2}→{{{2}, {1}}, {}},  {2, 3}→{{1, 2}, {2}},  {2, 4}→{{{1}, {1}}, {2, 1}}}

(c) {{1, 1}→{{{1, 1}, {1}}, {2}},  {1, 2}→{{{1, 2}, {2}}, {{2, 2}, {}}},  {2, 1}→{{{2, 2}, {2}}, {{1}, {}}},  {2, 2}→{{{1}, {1}}, {{2, 1}, {1, 1}}},
    {2, 3}→{{2, 1}, {2}},  {2, 4}→{{{1}, {1, 2}}, {{1, 2}, {}}}}
```

Network systems in which the rule depends on the number of distinct nodes reached by going up to distance two away from each node. The plots show the total number of nodes obtained at each step. In cases (a) and (b), the behavior of the system is eventually repetitive. In case (c), it is nested—the size of the network at step t is related to the number of 1's in the base 2 digit sequence of t.





(d) {{1, 1}→{{{1, 2}, {1, 2}}, {}}, {1, 2}→{{2, 2}, {{1}, {1}}}, {2, 1}→{{1}, {{}, {2}}}, {2, 2}→{{1, 2}, {2, 1}}, {2, 3}→{{{2, 1}, {2}}, {1}}, {2, 4}→{{1}, {1, 1}}}

(e) {{1, 1}→{{}, {{1, 1}, {1, 2}}}, {1, 2}→{{{}, {1}}, {{1, 1}, {1, 2}}}, {2, 1}→{{2}, {}}, {2, 2}→{{{2, 1}, {1}}, {{1, 1}, {2}}}, {2, 3}→{{2, 2}, {2}}, {2, 4}→{{2, 1}, {2}}}

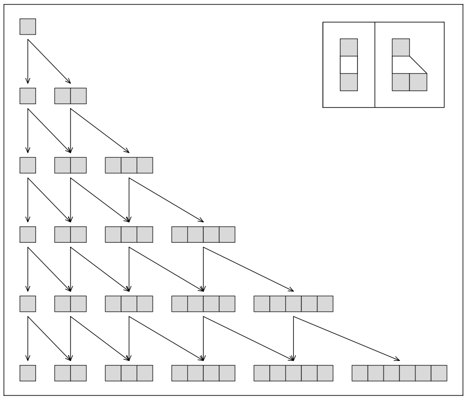
Network systems in which the total number of nodes obtained on successive steps appears to vary in a largely random way forever. About one in 10,000 randomly chosen network systems seem to exhibit the kind of behavior shown here.

Multiway Systems

The network systems that we discussed in the previous section do not have any underlying grid of elements in space. But they still in a sense have a simple one-dimensional arrangement of states in time. And in fact, all the systems that we have considered so far in this book can be thought of as having the same simple structure in time. For all of them are ultimately set up just to evolve progressively from one state to the next.

Multiway systems, however, are defined so that they can have not just a single state, but a whole collection of possible states at any given step.

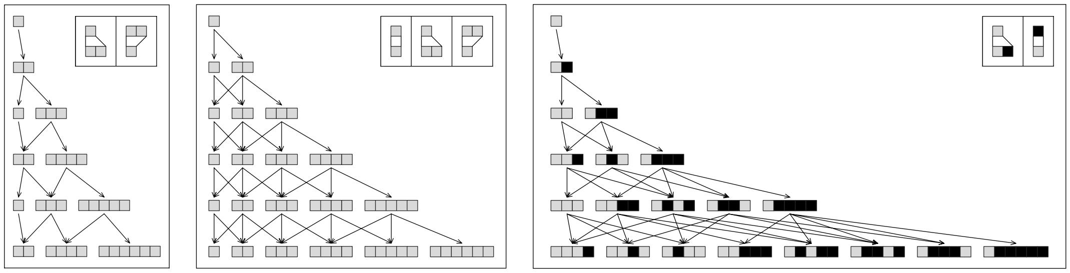
The picture below shows a very simple example of such a system.


A very simple multiway system in which one element in each sequence is replaced at each step by either one or two elements. The main feature of multiway systems is that all the distinct sequences that result are kept.

Each state in the system consists of a sequence of elements, and in the particular case of the picture above, the rule specifies that at each step each of these elements either remains the same or is replaced by a pair of elements. Starting with a single state consisting of one element, the picture then shows that applying these rules immediately gives two possible states: one with a single element, and the other with two.

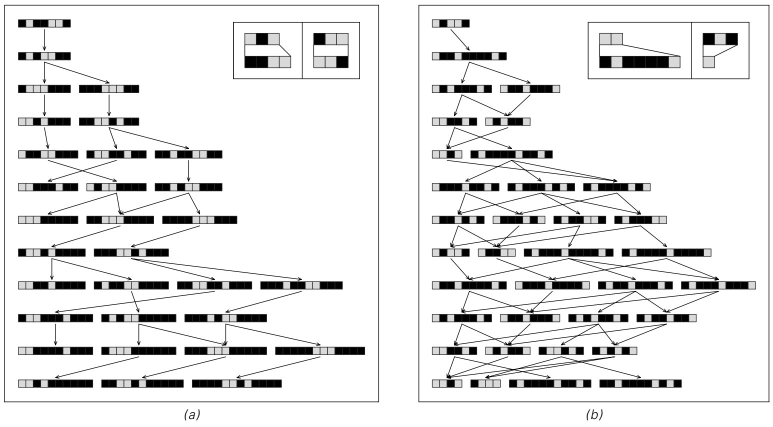
Multiway systems can in general use any sets of rules that define replacements for blocks of elements in sequences. We already saw exactly these kinds of rules when we discussed sequential substitution systems on page 88. But in sequential substitution systems the idea was to do just one replacement at each step. In multiway systems, however, the idea is to do all possible replacements at each step—and then to keep all the possible different sequences that are generated.

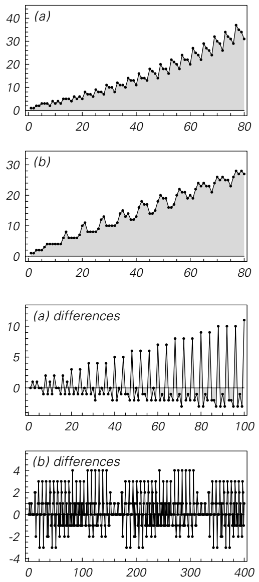
The pictures below show what happens with some very simple rules. In each of these examples the behavior turns out to be rather simple—with for example the number of possible sequences always increasing uniformly from one step to the next.



Examples of simple multiway systems. The number of distinct sequences at step *t* in these three systems is respectively *Ceiling[t/2]*, *t* and *Fibonacci[t + 1]* (which increases approximately like 1.618^t).

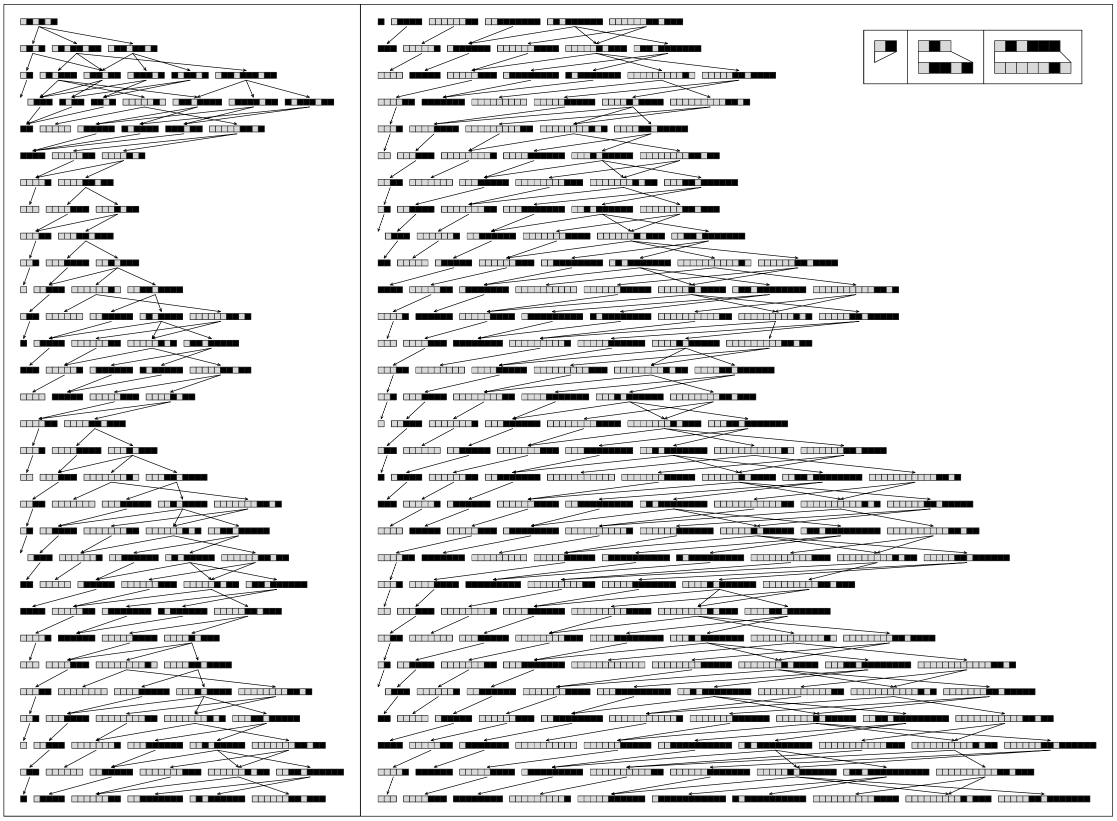
In general, however, this number need not exhibit such uniform growth, and the pictures below show examples where fluctuations occur.

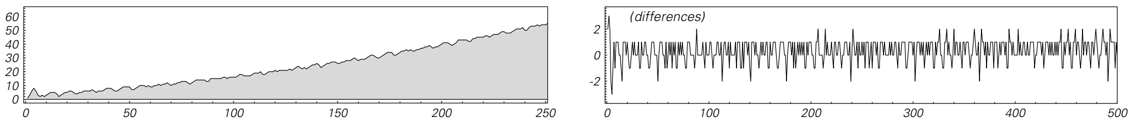




Examples of multiway systems with slightly more complicated behavior. The plots on the right show the total number of possible states obtained at each step, and the differences of these numbers from one step to the next. In both cases, essentially repetitive behavior is seen, every 40 and 161 steps respectively. Note that in case (a), the total number of possible states at step *t* increases roughly like t^2, while in case (b) it increases only like *t*.

But in both these cases it turns out to be not too long before these fluctuations essentially repeat. The picture below shows an example where a larger amount of apparent randomness is seen. Yet even in this case one finds that there ends up again being essential repetition—although now only every 1071 steps.

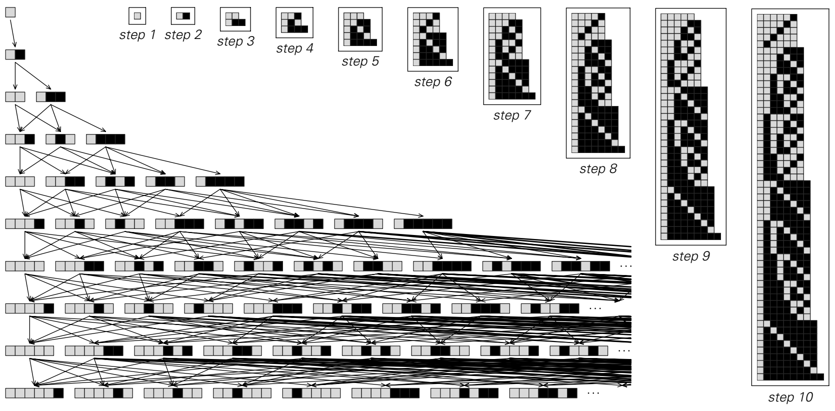


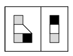


A multiway system with behavior that shows some signs of apparent randomness. The rule for this system involves three possible replacements. Note that the first replacement only removes elements and does not insert new ones. In the pictures sequences containing zero elements therefore sometimes appear. At least with the initial condition used here, despite considerable early apparent randomness, the differences in number of elements do repeat (shifted by 1) every 1071 steps.

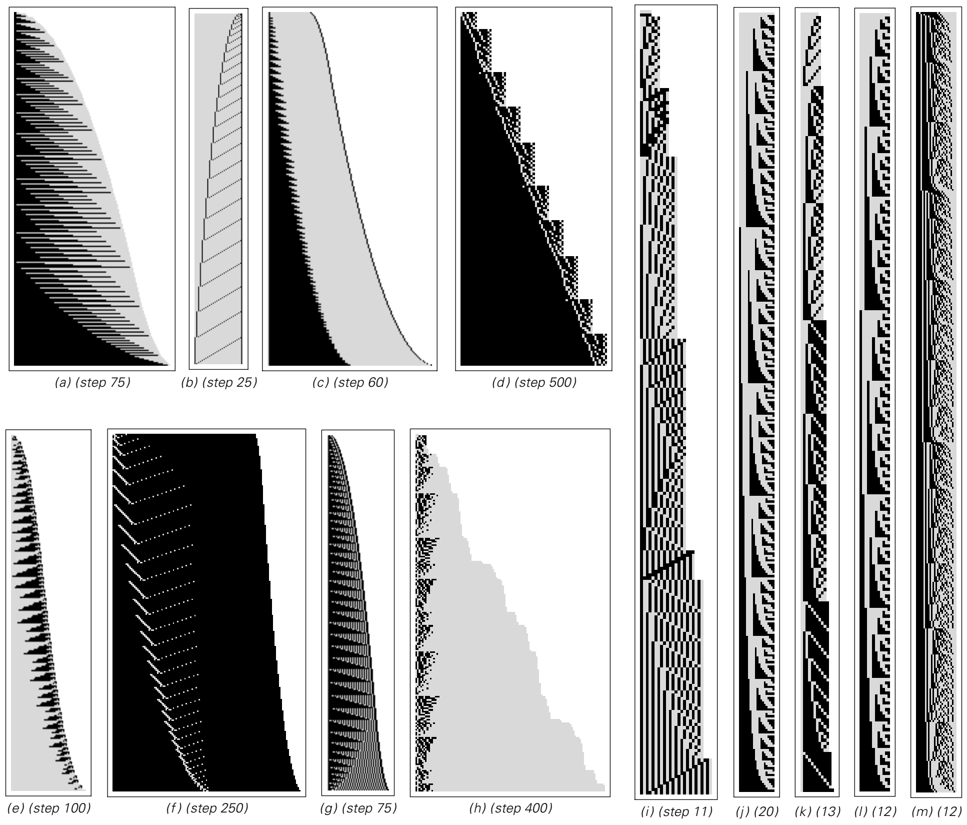
If one looks at many multiway systems, most either grow exponentially quickly, or not at all; slow growth of the kind seen on the facing page is rather rare. And indeed even when such growth leads to a certain amount of apparent randomness it typically in the end seems to exhibit some form of repetition. If one allows more rapid growth, however, then there presumably start to be all sorts of multiway systems that never show any such regularity. But in practice it tends to be rather difficult to study these kinds of multiway systems—since the number of states they generate quickly becomes too large to handle.

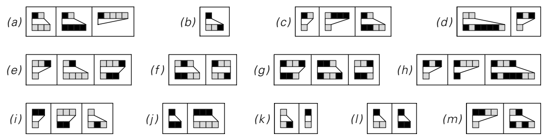
One can get some idea about how such systems behave, however, just by looking at the states that occur at early steps. The picture below shows an example—with ultimately fairly simple nested behavior.




The collections of states generated on successive steps by a simple multiway system with rapid growth shown on page 205. The particular rule used here eventually generates all states beginning with a white cell. At step t there are *Fibonacci[t + 1]* states; a given state with m white cells and n black cells appears at step $2m + n - 1$.

The pictures on the next page show some more examples. Sometimes the set of states that get generated at a particular step show essential repetition—though often with a long period. Sometimes this set in effect includes a large fraction of the possible digit sequences of a given length—and so essentially shows nesting. But in other cases there is at least a hint of considerably more complexity—even though the total number of states may still end up growing quite smoothly.

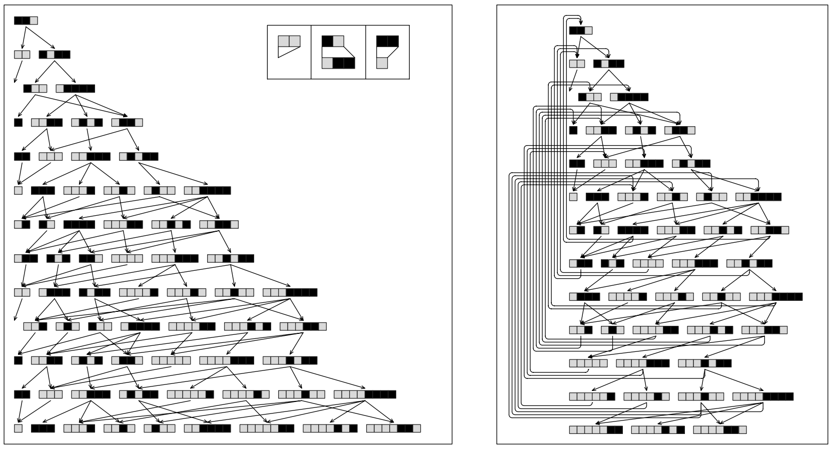




Collections of states generated at particular steps in the evolution of various multiway systems. Rule (k) was shown on the previous page; rules (d) and (f) on page 205.

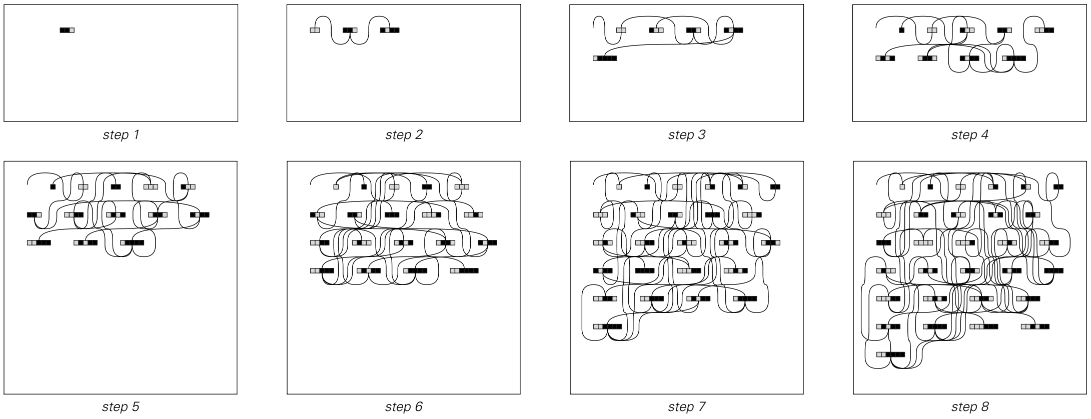
Looking carefully at the pictures of multiway system evolution on previous pages, a feature one notices is that the same sequences often occur on several different steps. Yet it is a consequence of the basic setup for multiway systems that whenever any particular sequence occurs, it must always lead to exactly the same behavior.

So this means that the complete evolution can be represented as in the picture at the top of the facing page, with each sequence shown explicitly only once, and any sequence generated more than once indicated just by an arrow going back to its first occurrence.



The evolution of a multiway system, first with every sequence explicitly shown at each step, and then with every sequence only ever shown once.

But there is no need to arrange the picture like this: for the whole behavior of the multiway system can in a sense be captured just by giving the network of what sequence leads to what other. The picture below shows stages in building up such a network. And what we see is that just as the network systems that we discussed in the previous section can build up their own pattern of connections in space, so also multiway systems can in effect build up their own pattern of connections in time—and this pattern can often be quite complicated.



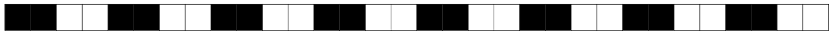
The network built up by the evolution of the multiway system from the top of the page. This network in effect represents a network of connections in time between states of the multiway system.

Systems Based on Constraints

In the course of this book we have looked at many different kinds of systems. But in one respect all these systems have ultimately been set up in the same basic way: they are all based on explicit rules that specify how the system evolves from step to step.

In traditional science, however, it is common to consider systems that are set up in a rather different way: instead of having explicit rules for evolution, the systems are just given constraints to satisfy.

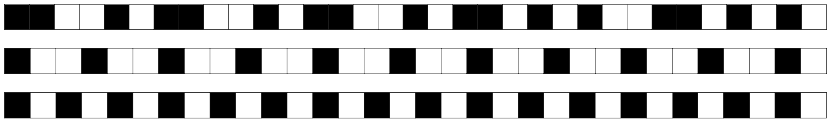
As a simple example, consider a line of cells in which each cell is colored black or white, and in which the arrangement of colors is subject to the constraint that every cell should have exactly one black and one white neighbor. Knowing only this constraint gives no explicit procedure for working out the color of each cell. And in fact it may at first not be clear that there will be any arrangement of colors that can satisfy the constraint. But it turns out that there is—as shown below.



A system consisting of a line of black and white cells whose form is defined by the constraint that every cell should have exactly one black and one white neighbor. The pattern shown is the only possible one that satisfies this constraint. The idea of implicitly determining the behavior of a system by giving constraints that it must satisfy is common in traditional science and mathematics.

And having seen this picture, one might then imagine that there must be many other patterns that would also satisfy the constraint. After all, the constraint is local to neighboring cells, so one might suppose that parts of the pattern sufficiently far apart should always be independent. But in fact this is not true, and instead the system works a bit like a puzzle in which there is only one way to fit in each piece. And in the end it is only the perfectly repetitive pattern shown above that can satisfy the required constraint at every cell.

Other constraints, however, can allow more freedom. Thus, for example, with the constraint that every cell must have at least one neighbor whose color is different from its own, any of the patterns in the picture at the top of the facing page are allowed, as indeed is any pattern that involves no more than two successive cells of the same color.



A system consisting of a line of black and white cells whose form is defined by the constraint that every cell should have at least one neighbor whose color is different from its own. There are many possible arrangements of colors that satisfy this constraint. Some, like the first arrangement above, look quite random. But others, like the second two arrangements above, are simple and repetitive. It turns out that in a one-dimensional system no set of local constraints can force arrangements of more complicated types.

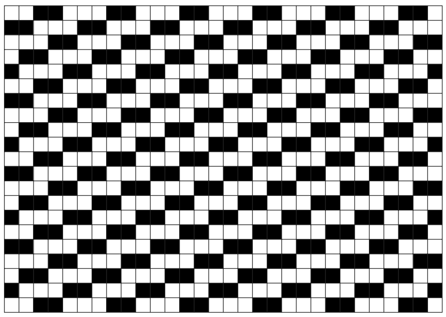
But while the first arrangement of colors shown above looks somewhat random, the last two are simple and purely repetitive.

So what about other choices of constraints? We have seen in this book many examples of systems where simple sets of rules give rise to highly complex behavior. But what about systems based on constraints? Are there simple sets of constraints that can force complex patterns?

It turns out that in one-dimensional systems there are not. For in one dimension it is possible to prove that any local set of constraints that can be satisfied at all can always be satisfied by some simple and purely repetitive arrangement of colors.

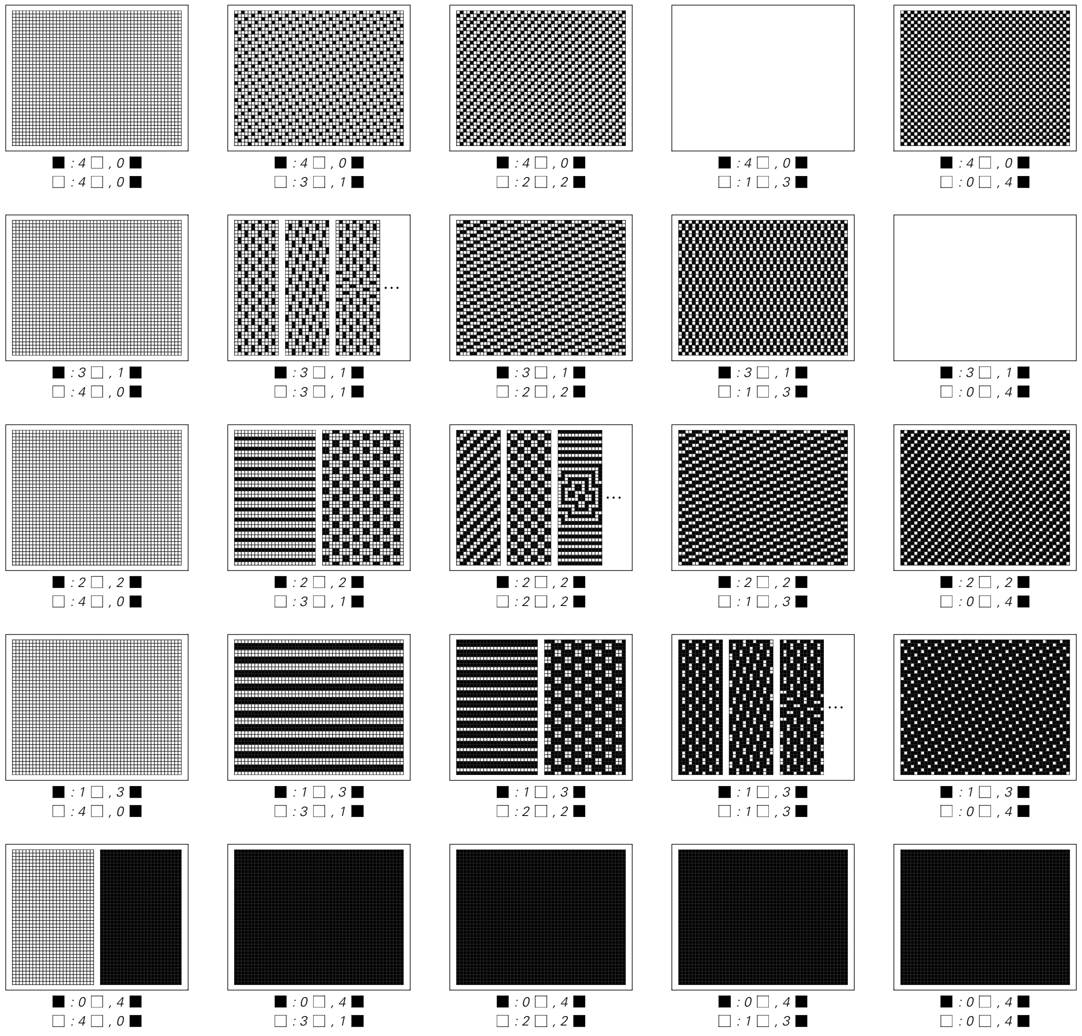
But what about two dimensions? The proof for one dimension breaks down in two dimensions, and so it becomes at least conceivable that a simple set of constraints could force a complex pattern to occur.

As a first example of a two-dimensional system, consider an array of black and white cells in which the constraint is imposed that every black cell should have exactly one black neighbor, and every white cell should have exactly two white neighbors.



A system consisting of a grid of black and white cells defined by the constraint that every black cell should have exactly one black neighbor among its four neighbors, and every white cell should have exactly two white neighbors. The infinite repetitive pattern shown here, together with its rotations and reflections, is the only one that satisfies this constraint. (The picture is assumed to wrap around at each edge.) The pattern can be viewed as a tessellation of 5 × 5 blocks of cells.

As in one dimension, knowing the constraint does not immediately provide a procedure for finding a pattern which satisfies it. But a little experimentation reveals that the simple repetitive pattern above satisfies the constraint, and in fact it is the only pattern to do so.

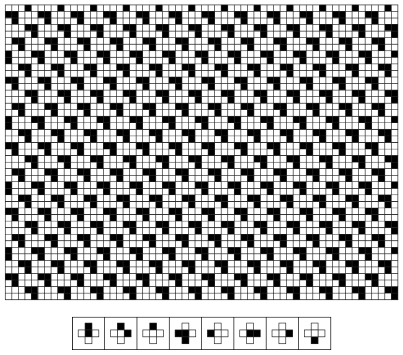


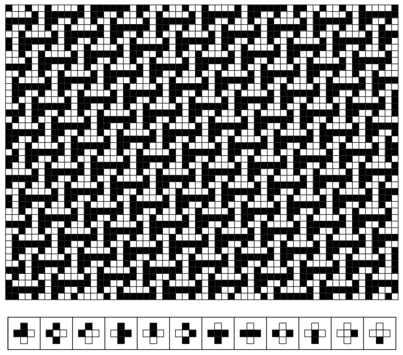
Patterns satisfying constraints which specify that every black cell and every white cell must have a certain fixed number of black and white neighbors. The blank rectangles in the upper right indicate constraints that cannot be satisfied by any pattern whatsoever. Most of the constraints are satisfied by a single pattern, together with its rotations and reflections. In some cases, two distinct patterns are possible, and in a few cases, an infinite set of patterns are possible. In all cases where the constraints can be satisfied at all, a simple repetitive pattern nevertheless suffices.

What about other constraints? The pictures on the facing page show schematically what happens with constraints that require each cell to have various numbers of black and white neighbors.

Several kinds of results are seen. In the two cases shown as blank rectangles on the upper right, there are no patterns at all that satisfy the constraints. But in every other case the constraints can be satisfied, though typically by just one or sometimes two simple infinite repetitive patterns. In the three cases shown in the center a whole range of mixtures of different repetitive patterns are possible. But ultimately, in every case where some pattern can work, a simple repetitive pattern is all that is needed.

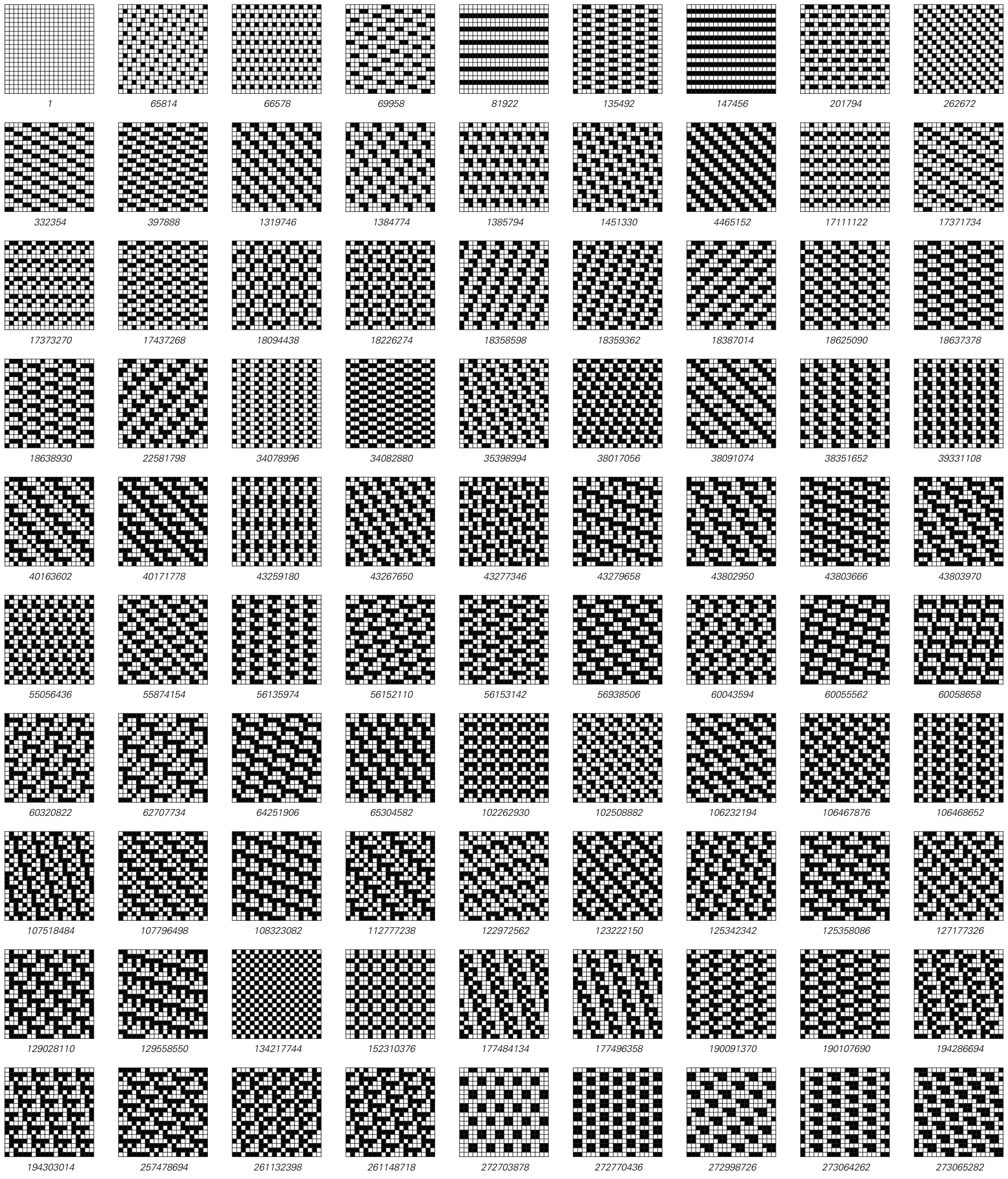
So what about more complicated constraints? The pictures below show examples based on constraints that require the local arrangement of colors around every cell to match a fixed set of possible templates.





Systems specified by the constraint that the local arrangement of colors around every cell must match the fixed set of possible templates shown. Note that these templates apply to every cell, with templates of neighboring cells overlapping. Pattern (a) can be viewed as formed from a tessellation of 5 × 10 blocks of cells; pattern (b) from a tessellation of 24 × 24 blocks. With the numbering scheme for constraints used on the next two pages the cases shown here correspond to 1384774 and 328778790.

There are a total of 4,294,967,296 possible sets of such templates. And of these, 766,979,044 lead to constraints that cannot be satisfied by any pattern. But among the 3,527,988,252 that remain, it turns out that every single one can be satisfied by a simple repetitive pattern. In fact the number of different repetitive patterns that are ever needed is quite small: if a particular constraint can be satisfied by any pattern, then one of the set of 171 repetitive patterns on the next two pages is always sufficient.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17111122
17371734
17373270
17437268
18094438
18226274
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18359362
18387014
18625090
18637378
18638930
22581798
34078996
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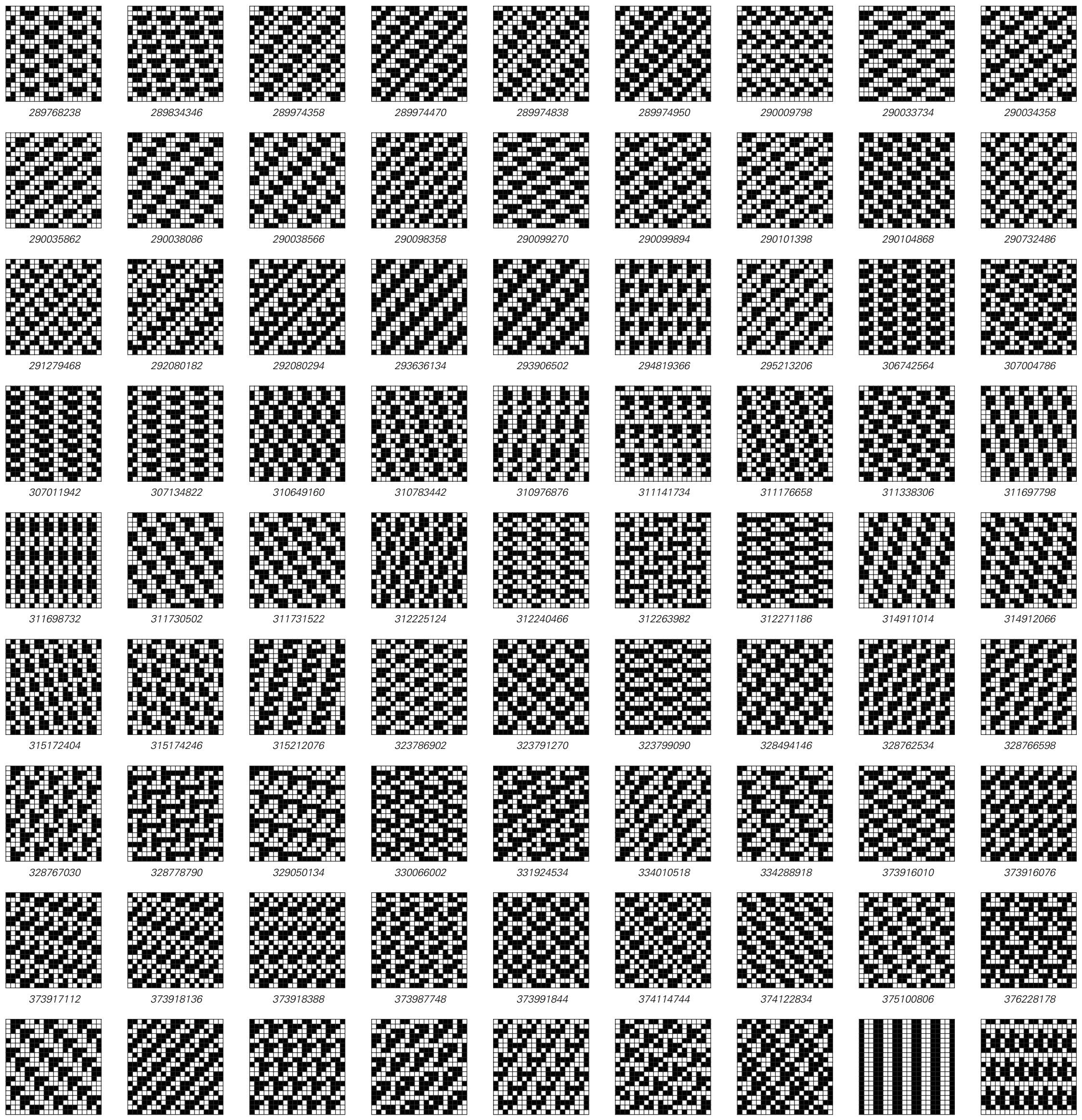
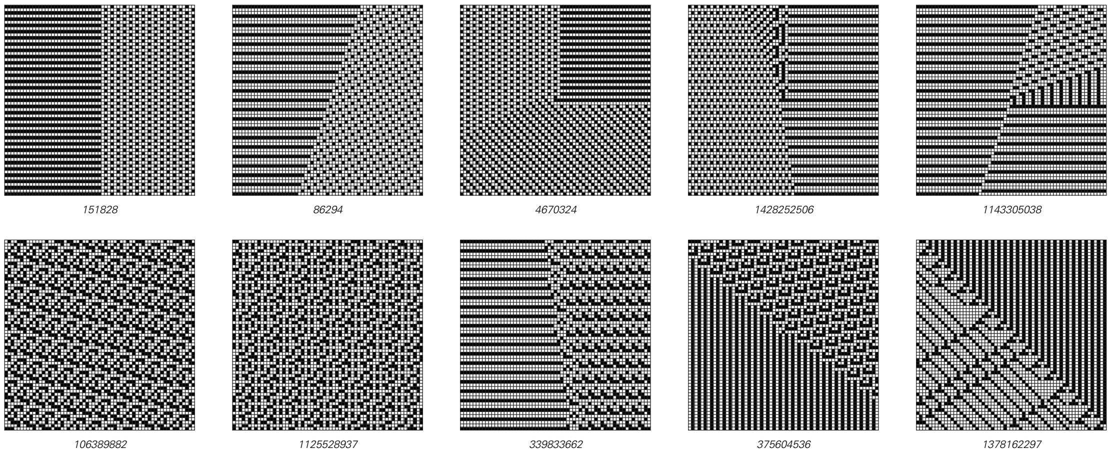


The complete collection of all 171 patterns needed to satisfy constraints of the type shown on the previous page. If none of these 171 patterns satisfy a particular constraint, then it follows that no pattern at all will satisfy the constraint. The patterns are labelled by numbers which specify the minimal constraint which requires the given pattern. Patterns differing by overall reflection, rotation or interchange of black and white are not shown.

So how can one force more complex patterns to occur?

The basic answer is that one must extend at least slightly the kinds of constraints that one considers. And one way to do this is to require not only that the colors around each cell match a set of templates, but also that a particular template from this set must appear at least somewhere in the array of cells.

The pictures below show a few examples of patterns determined by constraints of this kind. A typical feature is that the patterns are divided into several separate regions, often emanating from some kind of center. But at least in all the examples below, the patterns that occur in each individual region are still simple and repetitive.



Examples of patterns produced by systems in which not only must the arrangement of colors in each neighborhood match one of a fixed set of templates, but also a certain template from this set must occur at least once in the pattern. The constraints are numbered as before, and in each picture the template that must occur is shown at the center. Constraint 1125528937 leads to a pattern that repeats in 98 × 98 blocks. The last pattern shown is also repetitive, repeating every 56 cells on the diagonal.

So how can one find constraints that force more complex patterns? To do so has been fairly difficult, and in fact has taken almost as much computational effort as any other single result in this book.

The basic problem is that given a constraint it can be extremely difficult to find out what pattern—if any—will satisfy the constraint.

In a system like a cellular automaton that is based on explicit rules, it is always straightforward to take the rule and apply it to see what pattern is produced. But in a system that is based on constraints, there is no such direct procedure, and instead one must in effect always go outside of the system to work out what patterns can occur.

The most straightforward approach might just be to enumerate every single possible pattern and then see which, if any, of them satisfy a particular constraint. But in systems containing more than just a few cells, the total number of possible patterns is absolutely astronomical, and so enumerating them becomes completely impractical.

A more practical alternative is to build up patterns iteratively, starting with a small region, and then adding new cells in essentially all possible ways, at each stage backtracking if the constraint for the system does not end up being satisfied.

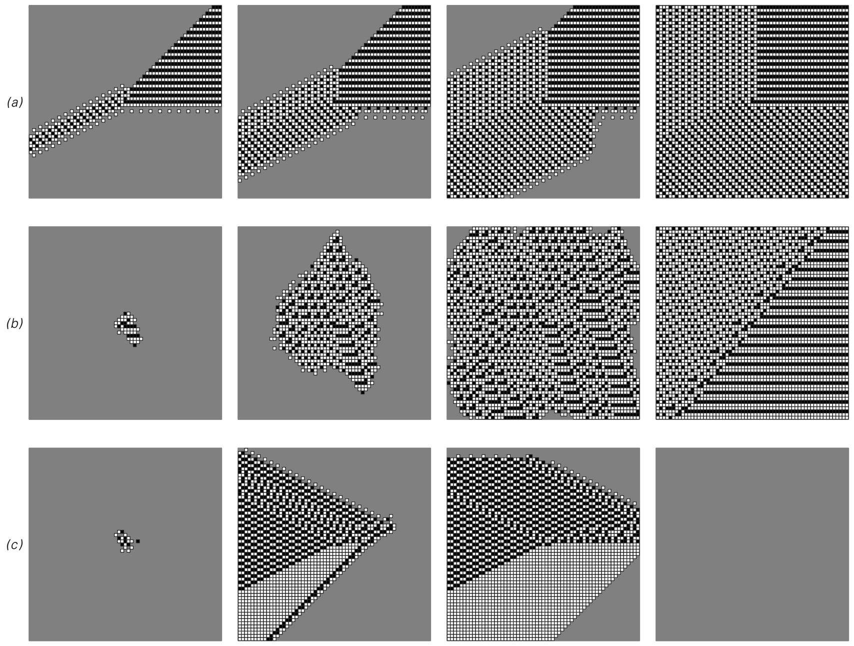
The pictures on the next page show a few sequences of patterns produced by this method. In some cases, there emerge quite quickly simple repetitive patterns that satisfy the constraint. But in other cases, a huge number of possibilities have to be examined in order to find any suitable pattern.

And what if there is no pattern at all that can satisfy a particular constraint? One might think that to demonstrate this would effectively require examining every conceivable pattern on the infinite grid of cells. But in fact, if one can show that there is no pattern that satisfies the constraint in a limited region, then this proves that no pattern can satisfy the constraint on the whole grid. And indeed for many constraints, there are already quite small regions for which it is possible to establish that no pattern can be found.

But occasionally, as in the third picture on the next page, one runs into constraints that can be satisfied for regions containing thousands of cells, but not for the whole grid. And to analyze such cases inevitably requires examining huge numbers of possible patterns.

But with an appropriate collection of tricks, it is in the end feasible to take almost any system of the type discussed here, and determine what pattern, if any, satisfies its constraint.

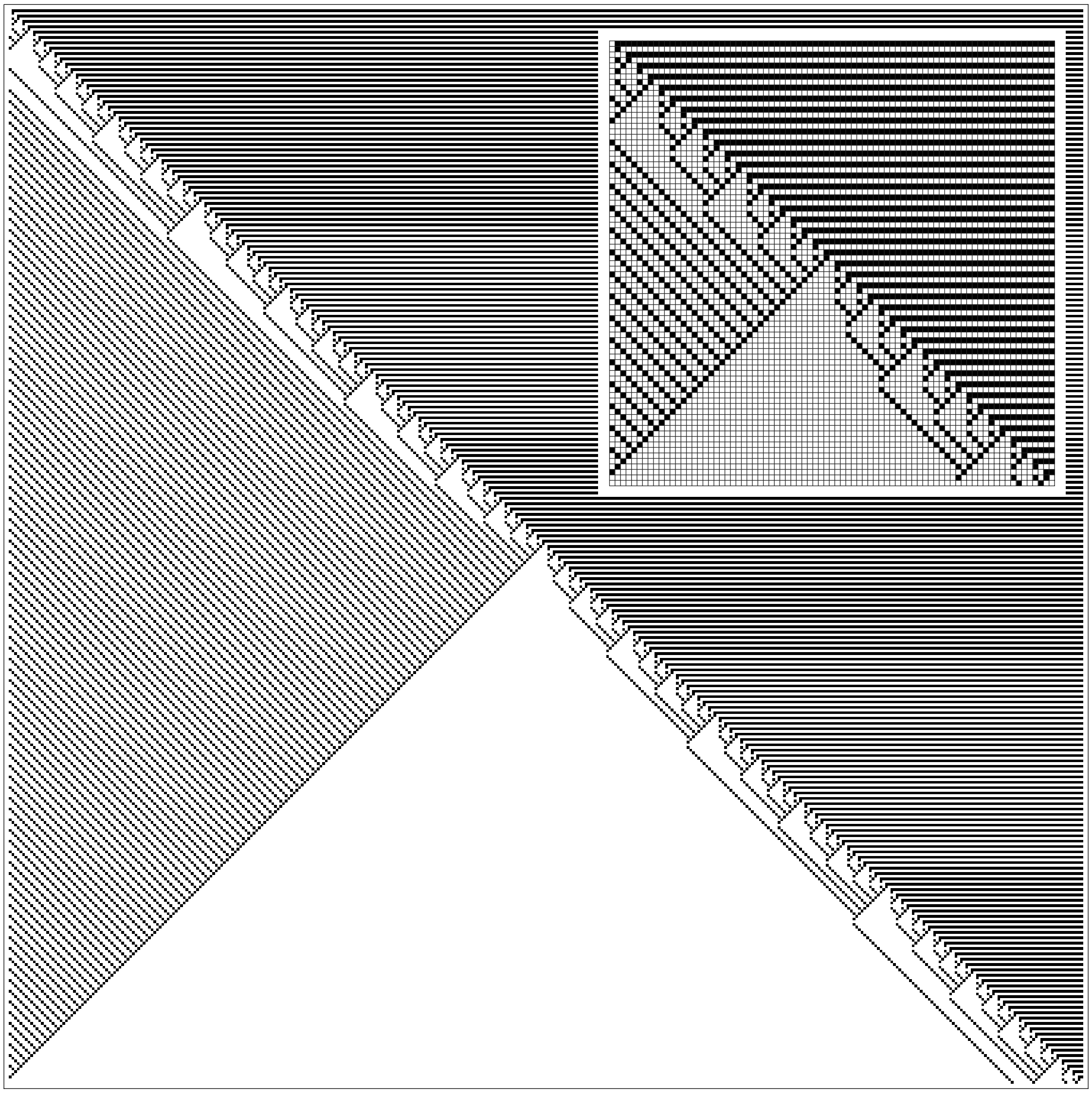
So what kinds of patterns can be needed? In the vast majority of cases, simple repetitive patterns, or mixtures of such patterns, are the only ones that are needed.

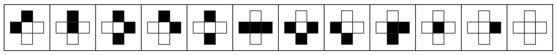


Stages in finding patterns that satisfy constraints (a) 4670324, (b) 373384574, and (c) 387520105. Gray is used to indicate cells whose colors have not yet been determined. The first stage shown in each case corresponds to cells whose colors can be deduced immediately from the presence of a particular template at the center. In case (a) choices for additional cells can be made straightforwardly, and an infinite regular pattern can be built up without any backtracking. In case (b), many choices for additional cells have to be tried, with much backtracking, and in the end the automatic procedure fails to find a repetitive pattern. Nevertheless, as the last stage demonstrates, a repetitive pattern does in fact exist. In case (c), the automatic procedure finds a fairly large and almost regular pattern that satisfies the constraints, but in this case it turns out that no infinite pattern exists.

But if one systematically examines possible constraints in the order shown on pages 214 and 215, then it turns out that after examining more than 18 million of them, one finally discovers the system shown on the facing page. And in this system, unlike all others before it, no repetitive pattern is possible; the only pattern that satisfies the constraint is the non-repetitive nested pattern shown in the picture.

After testing millions of constraints, and tens of billions of candidate patterns, therefore, it is finally possible to establish that a system based on simple constraints of the type discussed here can be forced to exhibit behavior more complex than pure repetition.



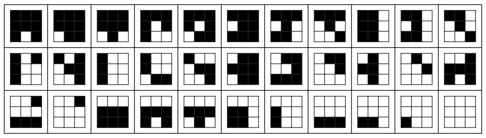
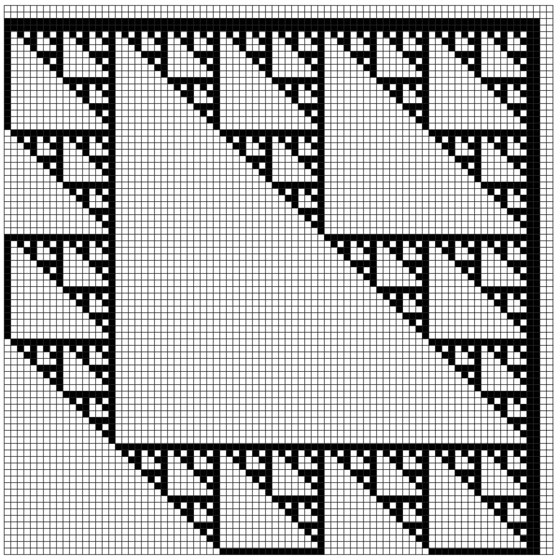


The simplest system based on constraints that is forced to exhibit a non-repetitive pattern. The constraint requires that the arrangement of colors around each cell must match one of the 12 templates shown, and that at least somewhere in the pattern a template containing a pair of stacked black cells must occur. In the numbering scheme used on preceding pages, the constraint is number 18762389. The pattern shown is unique, in that no variations of it, except for trivial translations, will satisfy the constraints. The nested structure on the diagonal essentially corresponds to a progression of base 2 digit sequences for positive and negative numbers.

What about still more complex behavior?

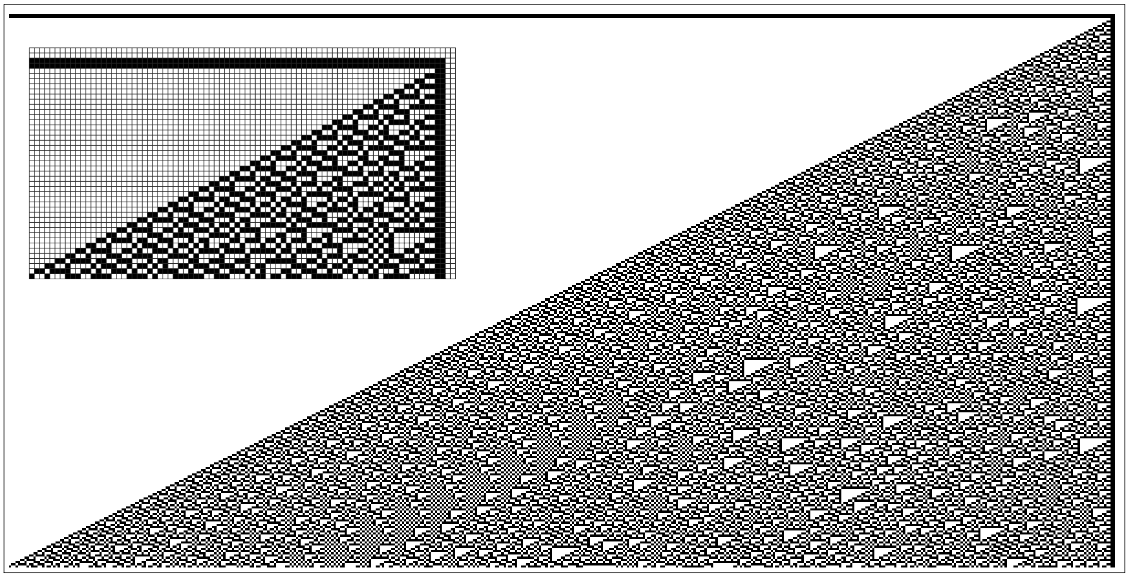
There are altogether 137,438,953,472 constraints of the type shown on page 216. And of the millions of these that I have tested, none have forced anything more complicated than the kind of nested behavior seen on the previous page. But if one extends again the type of constraints one considers, it turns out to become possible to construct examples that force more complex behavior.

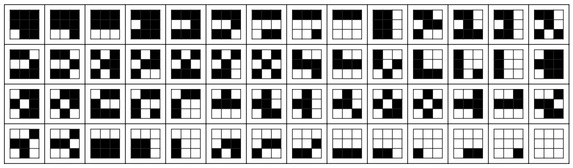
The idea is to set up templates that involve complete 3 × 3 blocks of cells, including diagonal neighbors. The picture below then shows an example of such a system, in which by allowing only a specific set of 33 templates, a nested pattern is forced to occur.

An example of a system based on a constraint involving 3 × 3 templates of cells. In this particular system, only the 33 templates shown above (out of the 512 possible ones) are allowed to occur. This constraint, together with the requirement that the first template must appear at least somewhere, then turns out to force a nested pattern to occur. The system shown was specifically constructed in correspondence with the rule 60 elementary one-dimensional cellular automaton.

What about more complex patterns? Searches have not succeeded in finding anything. But explicit construction, based on correspondence with one-dimensional cellular automata, leads to the example shown at the top of the facing page: a system with 56 allowed templates in which the only pattern satisfying the constraint is a complex and largely random one, derived from the rule 30 cellular automaton.





A system based on a constraint, in which a complex and largely random pattern is forced to occur. The constraint specifies that only the 56 3 × 3 templates shown at left can occur anywhere in the pattern, with the first template appearing at least once. The pattern required to satisfy this constraint corresponds to a shifted version of the one generated by the evolution of the rule 30 elementary one-dimensional cellular automaton.

So finally this shows that it is indeed possible to force complex behavior to occur in systems based on constraints. But from what we have seen in this section such behavior appears to be quite rare: unlike many of the simple rules that we have discussed in this book, it seems that almost all simple constraints lead only to fairly simple patterns.

Any phenomenon based on rules can always ultimately also be described in terms of constraints. But the results of this section indicate that these descriptions can have to be fairly complicated for complex behavior to occur. So the fact that traditional science and mathematics tends to concentrate on equations that operate like constraints provides yet another reason for their failure to identify the fundamental phenomenon of complexity that I discuss in this book.

Two Dimensions and Beyond

Introduction

▪ **Other lattices.** See page 929.

▪ **Page 170 · 1D phenomena.** Among the phenomena that cannot occur in one dimension are those associated with shape, winding and knotting, as well as traditional phase transitions with reversible evolution rules (see page 981).

Cellular Automata

▪ **Implementation.** An *n×n* array of white squares with a single black square in the middle can be generated by

```
PadLeft[{{1}}, {n, n}, 0, Floor[{n, n}/2]]
```

For the 5-neighbor rules introduced on page 170 each step can be implemented by

```
CAStep[rule_, a_] := Map[rule〚10 - #〛 &,
    ListConvolve[{{0, 2, 0}, {2, 1, 2}, {0, 2, 0}}, a, 2], {2}]
```

where *rule* is obtained from the *code* number by *IntegerDigits[code, 2, 10]*.

For the 9-neighbor rules introduced on page 177

```
CAStep[rule_, a_] := Map[rule〚18 - #〛 &,
    ListConvolve[{{2, 2, 2}, {2, 1, 2}, {2, 2, 2}}, a, 2], {2}]
```

where *rule* is given by *IntegerDigits[code, 2, 18]*.

In *d* dimensions with *k* colors, 5-neighbor rules generalize to *(2d+1)*-neighbor rules, with

```
CAStep[{rule_, d_}, a_] :=
  Map[rule〚-1 - #〛 &, a + k AxesTotal[a, d], {d}]
AxesTotal[a_, d_] := Apply[Plus, Map[RotateLeft[a, #] +
      RotateRight[a, #] &, IdentityMatrix[d]]]
```

with *rule* given by *IntegerDigits[code, k, k (2 d (k - 1) + 1)]*.

9-neighbor rules generalize to 3^d-neighbor rules, with

```
CAStep[{rule_, d_}, a_] :=
  Map[rule〚-1 - #〛 &, a + k FullTotal[a, d], {d}]
FullTotal[a_, d_] :=
  Array[RotateLeft[a, {##}] &, Table[3, {d}], -1, Plus] - a
```

with *rule* given by *IntegerDigits[code, k, k ((3^d - 1) (k - 1) + 1)]*.

In 3 dimensions, the positions of black cells can conveniently be displayed using

```
Graphics3D[Map[Cuboid[-Reverse[#]] &, Position[a, 1]]]
```

▪ **General rules.** One can specify the neighborhood for any rule in any dimension by giving a list of the offsets for the cells used to update a given cell. For 1D elementary rules the list is *{{-1}, {0}, {1}}*, while for 2D 5-neighbor rules it is *{{-1, 0}, {0, -1}, {0, 0}, {0, 1}, {1, 0}}*. In this book such offset lists are always taken to be in the order given by *Sort*, so that for range *r* rules in *d* dimensions the order is the same as *Flatten[Array[List, Table[2 r + 1, {d}], -r], d - 1]*. One can specify a neighborhood configuration by giving in the same order as the offset list the color of each cell in the neighborhood. With offset list *os* and *k* colors the possible neighborhood configurations are

```
Reverse[Table[IntegerDigits[i - 1,
      k, Length[os]], {i, k ^ Length[os]}]]
```

(These are shown on page 53 for elementary rules and page 941 for 5-neighbor rules.) If a cellular automaton rule takes the new color of a cell with neighborhood configuration *IntegerDigits[i, k, Length[os]]* to be *u〚i + 1〛*, then one can define its rule number to be *FromDigits[Reverse[u], k]*. A single step in evolution of a general cellular automaton with state *a* and rule number *num* is then given by

```
Map[IntegerDigits[num, k, k ^ Length[os]]〚-1 - #〛 &,
  Apply[Plus, MapIndexed[k ^ (Length[os] - First[#2])
      RotateLeft[a, #1] &, os]], {-1}]
```

or equivalently by

```
Map[IntegerDigits[num, k, k ^ Length[os]]〚-# - 1〛 &,
  ListCorrelate[Fold[ReplacePart[k #1, 1, #2 + r + 1] &,
     Array[0 &, Table[2 r + 1, {d}]], os], a, r + 1], {d}]
```

▪ **Numbers of possible rules.** The table below gives the total number of 2D rules of various types with two possible colors for each cell. Given an initial pattern with a certain symmetry, a rule will maintain that symmetry if the rule is such that every neighborhood equivalent under the symmetry yields the same color of cell. Rules are considered rotationally symmetric in the table below if they preserve any possible rotational symmetry consistent with the underlying arrangement of cells. Totalistic rules depend only on the total number of black cells in a neighborhood; outer totalistic rules (as in the previous note) also depend on the color of the center cell. Growth totalistic rules make any cell that becomes black remain black forever.

In such a rule, given a list of how many neighbors around a given cell (out of *s* possible) make the cell turn black the outer totalistic code for the rule can be obtained from

```
Apply[Plus, 2^Join[2 list, 2 Range[s + 1] - 1]]
```

	5 - neighbor square	9 - neighbor square	hexagonal
general	$2^{32} \simeq 4 \times 10^9$	$2^{512} \simeq 10^{154}$	$2^{128} \simeq 3 \times 10^{38}$
rotationally symmetric	$2^{12} = 4096$	$2^{140} \simeq 10^{42}$	$2^{28} \simeq 3 \times 10^8$
completely symmetric	$2^{12} = 4096$	$2^{102} \simeq 5 \times 10^{30}$	$2^{26} \simeq 7 \times 10^7$
outer totalistic	$2^{10} = 1024$	$2^{18} \simeq 3 \times 10^5$	$2^{14} = 16384$
totalistic	$2^6 = 64$	$2^{10} = 1024$	$2^8 = 256$
growth totalistic	$2^5 = 32$	$2^9 = 512$	$2^7 = 128$

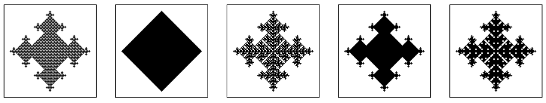
▪ **Symmetric 5-neighbor rules.** Among the 32 possible 5-cell neighborhoods shown for example on page 941 there are 12 classes related by symmetries, given by

```
s = {{1}, {2, 3, 9, 17}, {4, 10, 19, 25},
     {5}, {6, 7, 13, 21}, {8, 14, 23, 29}, {11, 18},
     {12, 20, 26, 27}, {15, 22}, {16, 24, 30, 31}, {28}, {32}}
```

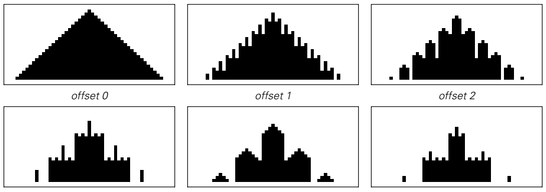
Completely symmetric 5-neighbor rules can be numbered from 0 to 4095, with each digit specifying the new color of the cell for each of these symmetry classes of neighborhoods. Such rule numbers can be converted to general form using

```
FromDigits[Map[Last, Sort[Flatten[Map[Thread,
        Thread[{s, IntegerDigits[n, 2, 12]}]], 1]]], 2]
```

▪ **Growth rules.** The pictures below show examples of rules in which a cell becomes black if it has exactly the specified numbers of black neighbors (the initial conditions used have the minimal number of black cells for growth). The code numbers in these cases are given by $2/3 (4^n - 1) + \text{Apply}[\text{Plus}, 4^{\text{list}}]$ where *n* is the number of neighbors, here 5. (See also the 9-neighbor examples on page 373.)



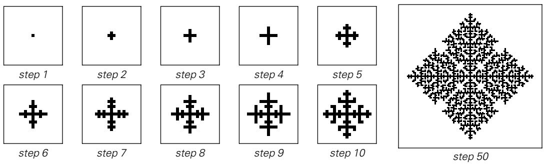
▪ **Page 171 · Code 942 slices.** The following is the result of taking vertical slices through the pattern with a sequence of offsets from the center:



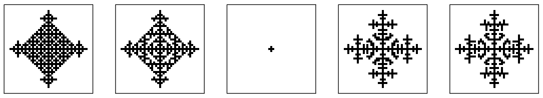
▪ **History.** As indicated on pages 876–878, 2D cellular automata were historically studied more extensively than 1D ones—though rarely with simple initial conditions. The 5-cell neighborhood on page 170 was considered by John von Neumann in 1952; the 9-cell one on page 177 by Edward Moore in 1962. (Both are also common in finite difference approximations in numerical analysis.) (The 7-cell hexagonal neighborhood of page 369 was considered for image processing purposes by Marcel Golay in 1959.) Ever since the invention of the Game of Life around 1970 a remarkable number of hardware and software simulators have been built to watch its evolution. But until after my work in the 1980s simulators for more general 2D cellular automata were rare. A sequence of hardware simulators were nevertheless built starting in the mid-1970s by Tommaso Toffoli and later Norman Margolus. And as mentioned on page 1077, going back to the 1950s some image processing systems have been based on particular families of 2D cellular automaton rules.

▪ **Ulam systems.** Having formulated the system around 1960, Stanislaw Ulam and collaborators (see page 877) in 1967 simulated 120 steps of the process shown below, with black cells after *t* steps occurring at positions

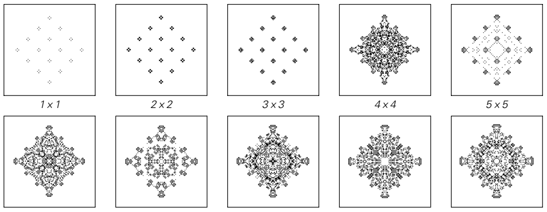
```
Map[First,
  First[Nest[UStep[p[q[r[#1], #2]] &, {{1, 0}, {0, 1}, {-1, 0},
          {0, -1}}, #] &, ({#, #} &)[{{{0, 0}, {0, 0}}}], t]]]
UStep[f_, os_, {a_, b_}] := ({Join[a, #], #} &)[f[Flatten[
      Outer[{#1 + #2, #1} &, Map[First, b], os, 1], 1], a]]
r[c_] := Map[First, Select[Split[Sort[c],
      First[#1] == First[#2] &], Length[#] == 1 &]]
q[c_, a_] := Select[c,
    Apply[And, Map[Function[u, qq[#1, u, a]], a]] &]
p[c_] := Select[c,
    Apply[And, Map[Function[u, pp[#1, u]], c]] &]
pp[{x_, u_}, {y_, v_}] := Max[Abs[x - y]] > 1 || u == v
qq[{x_, u_}, {y_, v_}, a_] := x == y || Max[Abs[x - y]] > 1 ||
    u == y || First[Cases[a, {u, z_} → z]] == y
```



These rules are fairly complicated, and involve more history than ordinary cellular automata. But from the discoveries in this book we now know that much simpler rules can also yield very complicated behavior. And as the pictures below show, this is true even just for parts of the rules above (*s* alone yields outer totalistic code 686 in 2D, and rule 90 in 1D).



Ulam also in 1967 considered the pure 2D cellular automaton with outer totalistic code 12 (though he stated its rule in a complicated way). As shown in the pictures below, when started from blocks of certain sizes this rule yields complex patterns—although nothing like this was noted in 1967.



■ **Limiting shapes.** When growth occurs at the maximum rate the outer boundaries of a cellular automaton pattern reflect the neighborhood involved in its underlying rule (in rough analogy to the Wulff construction for shapes of crystals). When growth occurs at a slower rate, a wide range of polygonal and other shapes can be obtained, as illustrated in the main text.

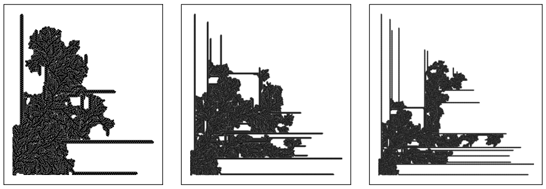
■ **Additive rules.** See page 1092.

■ **Page 174 · Cellular automaton art.** 2D cellular automata can be used to make a wide range of designs for rugs, wallpaper, and similar objects. Repeating squares of pattern can be produced by using periodic boundary conditions. Rules with more than two colors will sometimes be appropriate. For rugs, it is typically desirable to have each cell correspond to more than one tuft, since otherwise with most rules the rug looks too busy. (Compare page 872.)

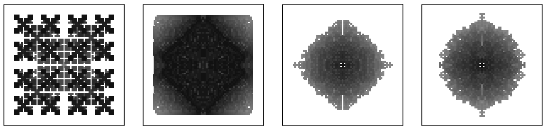
■ **Page 177 · Code 175850.** See also page 980.

■ **Page 178 · Code 746.** The pattern generated is not perfectly circular, as discussed on page 979. Its interior is mostly fixed, but there are scattered small regions that cycle with a variety of periods.

■ **Page 181 · Code 174826.** The pictures below show the upper-right quadrant for more steps. Most of the lines visible are 8 cells across, and grow by 4 cells every 12 steps. They typically survive being hit by more complicated growth from the side. But occasionally runners 3 cells wide will start on the side of a line. And since these go 2 cells every 3 steps they always catch up with lines, producing complicated growth, often terminating the lines.



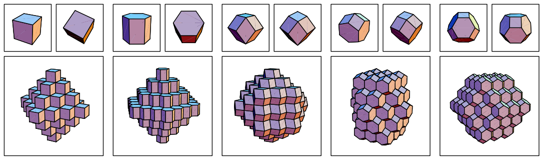
■ **Page 183 · Projections from 3D.** Looking from above, with closer cells shown darker, the following show patterns generated after 30 steps, by (a) the rule at the top of page 183, (b) the rule at the bottom of page 183, (c) the rule where a cell becomes black if exactly 3 out of 26 neighbors were black and (d) the same as (c), but with a 3×3×1 rather than a 3×1×1 initial block of black cells:



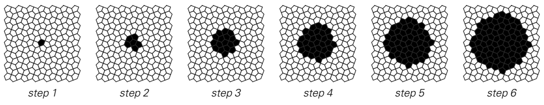
■ **Other geometries.** Systems like cellular automata can readily be set up on any geometrical structure in which a limited number of types of cells can be identified, with every cell of a given type having a similar neighborhood.

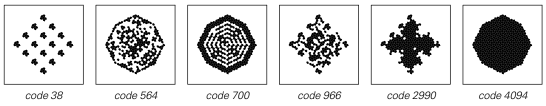
In the simplest case, the cells are all identical, and are laid out in the same orientation in a repetitive array. The centers of the cells form a lattice, with coordinates that are integer multiples of some set of basis vectors. The possible complete symmetries of such lattices are much studied in crystallography. But for the purpose of nearest-neighbor cellular automaton rules, what matters is not detailed geometry, but merely what cells are adjacent to a given cell. This can be determined by looking at the Voronoi region (see page 987) for each point in the lattice. In any given dimension, this region (variously known as a Dirichlet domain or Wigner-Seitz cell, and dual to the primitive cell, first Brillouin zone or Wulff shape) has a limited number of possible overall shapes. The most symmetrical versions of these shapes in 2D are the square (4 neighbors) and hexagon (6) and in 3D (as found by Evgraf Fedorov in 1885) the cube (6), hexagonal prism (8), rhombic dodecahedron (12) (e.g.

face-centered cubic crystals), rhombo-hexagonal or elongated dodecahedron (12) and truncated octahedron or tetradecahedron (14) (e.g. body-centered cubic crystals), as shown below. (In 4D, 8, 16 and 24 nearest neighbors are possible; in higher dimensions possibilities have been investigated in connection with sphere packing.) (Compare pages 1029 and 986.)

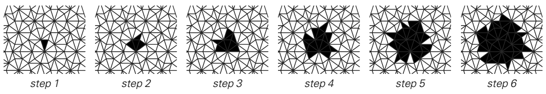


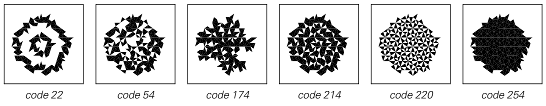
In general, there is no need for individual cells in a cellular automaton to have the same orientation. A triangular lattice is one example where they do not. And indeed, any tiling of congruent figures can readily be used to make a cellular automaton, as illustrated by the pentagonal example below. (Outer totalistic codes specify rules; the first rule makes a particular cell black when any of its five neighbors are black and has code 4094. Note that even though individual cells are pentagonal, large-scale cellular automaton patterns usually have 2-, 4- or 8-fold symmetry.)





There is even no need for the tiling to be repetitive; the picture below shows a cellular automaton on a nested Penrose tiling (see page 932). This tiling has two different shapes of tile, but here both are treated the same by the cellular automaton rule, which is given by an outer totalistic code number. The first example is code 254, which makes a particular cell become black when any of its three neighbors are black. (Large-scale cellular automaton patterns here can have 5-fold symmetry.) (See also page 1027.)





▪ **Networks.** Cellular automata can be set up so that each cell corresponds to a node in a network. (See page 936.) The only requirement is that around each node the network must have the same structure (or at least a limited number of possible structures). For nearest-neighbor rules, it suffices that each node has the same number of connections. For longer-range rules, the network must satisfy constraints of the kind discussed on page 483. (Cayley graphs of groups always have the necessary homogeneity.) If the connections at each node are not labelled, then only totalistic cellular automaton rules can be implemented. Many topological and geometrical properties of the underlying network can affect the overall behavior of a cellular automaton on it.

Turing Machines

▪ **Implementation.** With rules represented as a list of elements of the form *{s, a} → {sp, ap, {dx, dy}}* (*s* is the state of the head and *a* the color of the cell under the head) each step in the evolution of a 2D Turing machine is given by

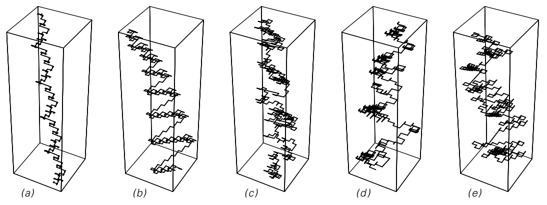
```
TM2DStep[rule_, {s_, tape_, r : {x_, y_}}] :=
  Apply[{#1, ReplacePart[tape, #2, {r}], r + #3} &,
    {s, tape[[x, y]]} /. rule]
```

▪ **History.** At a formal level 2D Turing machines have been studied since at least the 1950s. And on several occasions systems equivalent to specific simple 2D Turing machines have also been constructed. In fact, much as for cellular automata, more explicit experiments have been done on 2D Turing machines than 1D ones. A tradition of early robotics going back to the 1940s—and leading for example to the Logo computer language—involved studying idealizations of mobile turtles. And in 1971 Michael Paterson and John Conway constructed what they described as an idealization of a prehistoric worm, which was essentially a 2D Turing machine in which the state of the head records the direction of the motion taken at each step. Michael Beeler in 1973 used a computer at MIT to investigate all 1296 possible worms with rules of the simplest type on a hexagonal grid, and he found several with fairly complex behavior. But this discovery does not appear to have been followed up, and systems equivalent to simple 2D Turing machines were reinvented again, largely independently, several times in the mid-1980s: by Christopher Langton in 1985 under the name "vants"; by Rudy Rucker in 1987 under the name "turmites"; and by Allen Brady in 1987 under the name "turning machines". The specific 4-state rule

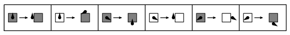
```
{s_, c_} :→ With[{sp = s (2 c - 1) ⅈ},
    {sp, 1 - c, {Re[sp], Im[sp]}}]
```

has been called Langton's ant, and various studies of it were done in the 1990s.

▪ **Visualization.** The pictures below show the 2D position of the head at 500 successive steps for the rules on page 185.



Some 2D Turing machines exhibit elements of randomness at some steps, but then fill in every so often to form simple repetitive patterns. An example is the 3-state rule



▪ **Rules based on turning.** The rules used in the main text specify the displacement of the head at each step in terms of fixed directions in the underlying grid. An alternative is to specify the turns to make at each step in the motion of the head. This is how turtles in the Logo computer language are set up. (Compare the discussion of paths in substitution systems on page 892.)

▪ **2D mobile automata.** Mobile automata can be generalized just like Turing machines. Even in the simplest case, however, with only four neighbors involved there are already $(4k)^{k^5}$ possible rules, or nearly 10^{29} even for $k = 2$.

Substitution Systems and Fractals

▪ **Implementation.** With the rule on page 187 given for example by *{1 → {{1, 0}, {1, 1}}, 0 → {{0, 0}, {0, 0}}}* the result of *t* steps in the evolution of a 2D substitution system from a initial condition such as *{{1}}* is given by

```
SS2DEvolve[rule_, init_, t_] :=
  Nest[Flatten2D[# /. rule] &, init, t]
Flatten2D[list_] :=
  Apply[Join, Map[MapThread[Join, #] &, list]]
```

▪ **Connection with digit sequences.** Just as in the 1D case discussed on page 891, the color of a cell at position *{i, j}* in a 2D substitution system can be determined using a finite automaton from the digit sequences of the numbers *i* and *j*. At step *n*, the complete array of cells is

```
Table[If[FreeQ[Transpose[IntegerDigits[{i, j}, k, n]], form],
    1, 0], {i, 0, k^n - 1}, {j, 0, k^n - 1}]
```

where for the pattern on page 187, $k = 2$ and *form* = *{0, 1}*. For patterns (a) through (f) on page 188, $k = 3$ and *form* is given respectively by (a) *{1, 1}*, (b) *{0 | 2, 0 | 2}*, (c) *{0 | 2, 0 | 2} | {1, 1}*, (d) *{i_, j_} /; j > i*, (e) *{0, 2} | {1, 1} | {2, 0}*, (f) *{0, 2} | {1, 1}*. Note that the excluded pairs of digits are in exact correspondence with the positions of which squares are *0* in the underlying rules for the substitution systems. (See pages 608 and 1091.)

▪ **Page 187 · Sierpiński pattern.** Other ways to generate step *n* of the pattern shown here in various orientations include:

- *Mod[Array[Binomial, {2, 2}^n, 0], 2]* (see pages 611 and 870)
- *1 - Sign[Array[BitAnd, {2, 2}^n, 0]]* (see pages 608 and 871)

-
  ```
  NestList[Mod[RotateLeft[#] + #, 2] &,
    PadLeft[{1}, 2^n], 2^n - 1]
  ```
  (see page 870)

-
  ```
  NestList[Mod[ListConvolve[{1, 1}, #, -1], 2] &,
    PadLeft[{1}, 2^n], 2^n - 1]
  ```
  (see page 870)
- *IntegerDigits[NestList[BitXor[2 #, #] &, 1, 2^n - 1], 2, 2^n]* (see page 906)

-
  ```
  NestList[Mod[Rest[FoldList[Plus, 0, #]], 2] &,
    Table[1, {2^n}], 2^n - 1]
  ```
  (see page 1034)

-
  ```
  Table[PadRight[
      Mod[CoefficientList[(1 + x)^(t-1), x], 2], 2^n - 1], {t, 2^n}]
  ```
  (see pages 870 and 951)

-
  ```
  Reverse[Mod[CoefficientList[Series[1/(1 - (1 + x) y),
        {x, 0, 2^n - 1}, {y, 0, 2^n - 1}], {x, y}], 2]]
  ```
  (see page 1091)

-
  ```
  Nest[Apply[Join, MapThread[
        Join, {{#, #}, {0 #, #}}, 2]] &, {{1}}, n]
  ```
  (compare page 1073)

The positions of black squares can be found from:

-
  ```
  Nest[Flatten[2 # /. {x_, y_} → {{x, y}, {x + 1, y}, {x, y + 1}},
      1] &, {{0, 0}}, n]
  ```

-
  ```
  (Transpose[{Re[#], Im[#]}] &)[
    Flatten[Nest[{2 #, 2 # + 1, 2 # + i} &, {0}, n]]]
  ```
  (compare page 1005)

-
  ```
  Position[Map[Split, NestList[Sort[Flatten[{#, # + 1}]] &,
      {0}, 2^n - 1]], _?(OddQ[Length[#]] &), {2}]
  ```
  (see page 358)

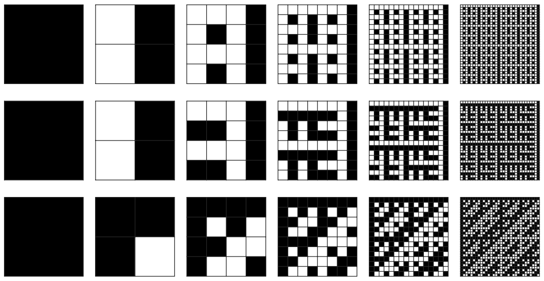
-
  ```
  Flatten[Table[Map[{t, #} &,
    Fold[Flatten[{#1, #1 + #2}] &, 0, Flatten[2^(Position[
      Reverse[IntegerDigits[t, 2]], 1] - 1)]]], {t, 2^n - 1}], 1]
  ```
  (see page 870)

-
  ```
  Map[Map[FromDigits[#, 2] &, Transpose[Partition[#, 2]]] &,
    Position[Nest[{{#, #}, {#}} &, 1, n], 1] - 1]
  ```
  (see page 509)

A formatting hack giving the same visual pattern is

```
DisplayForm[Nest[SubsuperscriptBox[#, #, #] &, "1", n]]
```

▪ **Non-white backgrounds.** The pictures below show substitution systems in which white squares are replaced by blocks which contain black squares. There is still a nested structure but it is usually not visually as obvious as before. (See page 583.)



▪ **Higher-dimensional generalizations.** The state of a *d*-dimensional substitution system can be represented by a nested list of depth *d*. The evolution of the system for *t* steps can be obtained from

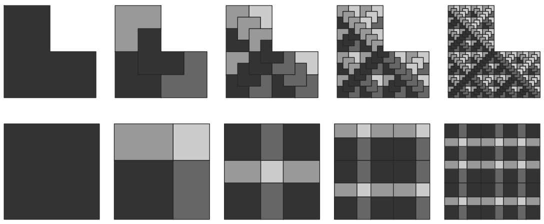
```
SSEvolve[rule_, init_, t_, d_Integer] :=
  Nest[FlattenArray[# /. rule, d] &, init, t]
FlattenArray[list_, d_] :=
  Fold[Function[{a, n}, Map[MapThread[Join, #, n] &,
       a, -{d + 2}]], list, Reverse[Range[d] - 1]]
```

The analog in 3D of the 2D rule on page 187 is

```
{1 → Array[If[LessEqual[##], 0, 1] &, {2, 2, 2}],
  0 → Array[0 &, {2, 2, 2}]}
```

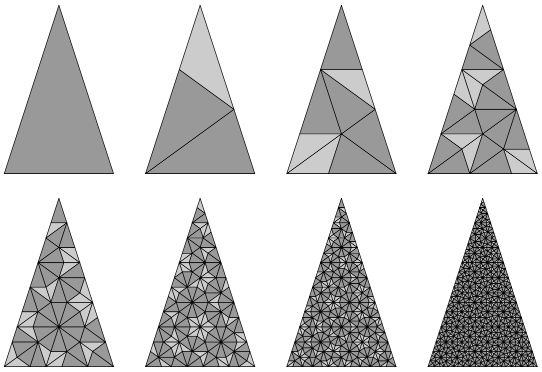
Note that in *d* dimensions, each black cell must be replaced by at least $d + 1$ black cells at each step in order to obtain an object that is not restricted to a dimension $d - 1$ hyperplane.

▪ **Other shapes.** The systems on pages 187 and 188 are based on subdividing squares into smaller squares. But one can also set up substitution systems that are based on subdividing other geometrical figures, as shown below.



The second example involves two distinct shapes: a square and a *GoldenRatio* aspect ratio rectangle. Labelling each shape and orientation with a different color, the behavior of this system can be reproduced with equal-sized squares using the rule *{3 → {{1, 0}, {3, 2}}, 2 → {{1}, {3}}, 1 → {{3, 2}}, 0 → {{3}}}* starting from initial condition *{{3}}*.

▪ **Penrose tilings.** The nested pattern shown below was studied by Roger Penrose in 1974 (see page 943).



The arrangement of triangles at step *t* can be obtained from a substitution system according to

```
With[{ϕ = GoldenRatio}, Nest[# /. a[p_, q_, r_] :→
     With[{s = (p + ϕ q) (2 - ϕ)}, {a[r, s, q], b[r, s, p]}] /.
   b[p_, q_, r_] :→ With[{s = (p + ϕ r) (2 - ϕ)}, {a[p, q, s], b[
       r, s, q]}] &, a[{1/2, Sin[2 π/5] ϕ}, {1, 0}, {0, 0}], t]]
```

This pattern can be viewed as generalizations of the pattern generated by the 1D Fibonacci substitution system (c) on page 83. As discussed on page 903, this 1D sequence can be obtained by looking at how a line with *GoldenRatio* slope cuts through a 2D lattice of squares. Penrose tilings can be obtained by looking at how a 2D plane with slopes based on *GoldenRatio* cuts through a lattice of hypercubes in 5D. The tilings turn out to have approximate 5-fold symmetry. (See also page 943.)

In general, projections onto any regular lattice in any number of dimensions from hyperplanes with any quadratic irrational slopes will yield nested patterns that can be generated by subdividing some shape or another according to a substitution system. Despite some confusion in the literature, however, this procedure can reproduce only a tiny fraction of all possible nested patterns.

▪ **Page 189 · Dragon curve.** The pattern shown here can be obtained in several related ways, including from numbers in base $i - 1$ (see below) and from a doubled version of the paths generated by 1D paperfolding substitution systems (see page 892). Its boundary has fractal dimension $2\,\mathrm{Log}[2, \mathrm{Root}[2 + \#1^2 - \#1^3, 1]] \simeq 1.52$.

▪ **Implementation.** The most convenient approach is to represent each pattern by a list of complex numbers, with the center of each square being given in terms of each complex number *z* by *{Re[z], Im[z]}*. The pattern after *n* steps is then given by *Nest[Flatten[f[#]] &, {0}, n]*, where for the rule on page 189 *f[z_] = 1/2 (1 - ⅈ) {z + 1/2, z - 1/2}* (*f[z_] = (1 - ⅈ) {z + 1, z}* gives a transformed version). For the rule on page 190, *f[z_] = 1/2 (1 - ⅈ) {ⅈ z + 1/2, z - 1/2}*. For rules (a), (b) and (c) (Koch curve) on page 191 the forms of *f[z_]* are respectively:

(0.296 - 0.57 ⅈ) z - 0.067 ⅈ - {1.04, 0.237}

N[1/40 {17 (√3 - ⅈ) z, -24 + 14 z}]

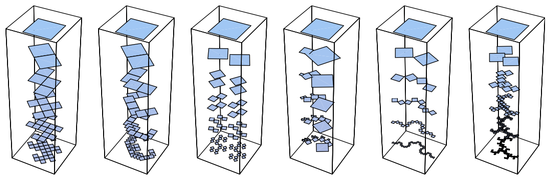
N[(1/2 (1/√3 - 1) (ⅈ + {1, -1}) - ⅈ - (1 + {ⅈ, -ⅈ}/√3) z)/2]

▪ **Connection with digit sequences.** Patterns after *t* steps can be viewed as containing all *t*-digit integers in an appropriate complex base. Thus the patterns on page 189 can be formed from *t*-digit integers in base *ⅈ - 1* containing only digits 0 and 1, as given by

Table[FromDigits[IntegerDigits[s, 2, t], ⅈ - 1], {s, 0, 2^t *- 1}]*

In the particular case of base *ⅈ - q* with digits 0 through q^2, it turns out that for sufficiently large *t* any complex integer can be represented, and will therefore be part of the pattern. (Compare page 1094.)

▪ **Visualization.** The 3D pictures below show successive steps in the evolution of each of the geometric substitution systems from the main text.

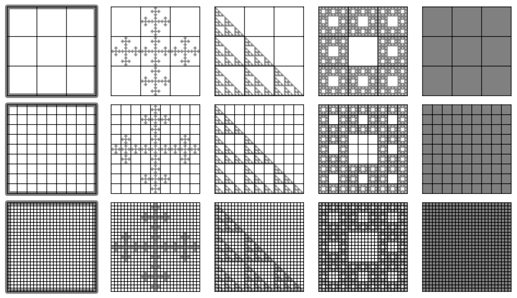


▪ **Parameter space sets.** See pages 407 and 1006 for a discussion of varying parameters in geometrical substitution systems.

▪ **Affine transformations.** Any set of so-called affine transformations that take the vector for each point, multiply it by a fixed matrix and then add a fixed vector, will yield nested patterns similar to those shown in the main text. Linear operations on complex numbers of the kind discussed above correspond geometrically to rotations, translations and rescalings. General affine transformations also allow reflection and skewing. In addition, affine transformations can readily be generalized to any number of dimensions, while complex numbers represent only two dimensions.

▪ **Complex maps.** Many kinds of nonlinear transformations on complex numbers yield nested patterns. Sets of so-called Möbius transformations of the form *z → (a z + b)/(c z + d)* always yield such patterns (and correspond to so-called modular groups when *a d - b c == 1*). Transformations of the form *z → {Sqrt[z - c], -Sqrt[z - c]}* yield so-called Julia sets which form nested patterns for many values of *c* (see note below). In fact, a fair fraction of all possible transformations based on algebraic functions will yield nested patterns. For typically the continuity of such functions implies that only a limited number of shapes not related by limited variations in local magnification can occur at any scale.

▪ **Fractal dimensions.** Certain features of nested patterns can be characterized by so-called fractal dimensions. The pictures below show five patterns with three successively finer grids superimposed. The dimension of a pattern can be computed by looking at how the number of grid squares that have any gray in them varies with the length *a* of the edge of each grid square. In the first case shown, this number varies like $(1/a)^1$ for small *a*, while in the last case, it varies like $(1/a)^2$. In general, if the number varies like $(1/a)^d$, one can take *d* to be the dimension of the pattern. And in the intermediate cases shown, it turns out that *d* has non-integer values.

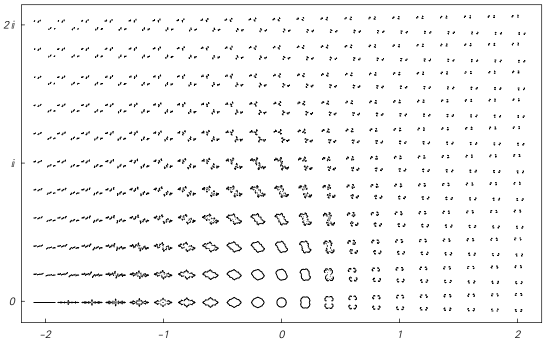


The grid in the pictures above fits over the pattern in a very regular way. But even when this does not happen, the limiting behavior for small *a* is still $(1/a)^d$ for any nested pattern. This form is inevitable if the underlying pattern effectively has the same structure on all scales. For some of the more complex patterns encountered in this book, however, there continues to be different structure on different scales, so that the effective value of *d* fluctuates as the scale changes, and may not converge to any definite value. (Precise definitions of dimension based for example on the maximum ever achieved by *d* will often in general imply formally non-computable values, as in the discussion of page 1138.)

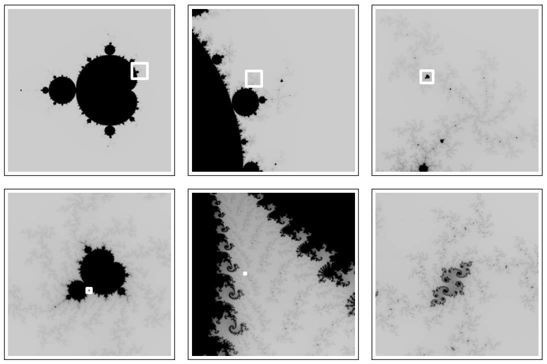
Fractal dimensions characterize some aspects of nested patterns, but patterns with the same dimension can often look very different. One approach to getting better characterizations is to look at each grid square, and to ask not just whether there is any gray in it, but how much. Quantities derived from the mean, variance and other moments of the probability distribution can serve as generalizations of fractal dimension. (Compare page 959.)

▪ **History of fractals.** The idea of using nested 2D shapes in art probably goes back to antiquity; some examples were shown on page 43. In mathematics, nested shapes began to be used at the end of the 1800s, mainly as counterexamples to ideas about continuity that had grown out of work on calculus. The first examples were graphs of functions: the curve on page 918 was discussed by Bernhard Riemann in 1861 and by Karl Weierstrass in 1872. Later came geometrical figures: example (c) on page 191 was introduced by Helge von Koch in 1906, the example on page 187 by Wacław Sierpiński in 1916, examples (a) and (c) on page 188 by Karl Menger in 1926 and the example on page 190 by Paul Lévy in 1937. Similar figures were also produced independently in the 1960s in the course of early experiments with computer graphics, primarily at MIT. From the point of view of mathematics, however, nested shapes tended to be viewed as rare and pathological examples, of no general significance. But the crucial idea that was developed by Benoit Mandelbrot in the late 1960s and early 1970s was that in fact nested shapes can be identified in a great many natural systems and in several branches of mathematics. Using early raster-based computer display technology, Mandelbrot was able to produce striking pictures of what he called fractals. And following the publication of Mandelbrot's 1975 book, interest in fractals increased rapidly. Quantitative comparisons of pure power laws implied by the simplest fractals with observations of natural systems have had somewhat mixed success, leading to the introduction of multifractals with more parameters, but Mandelbrot's general idea of the importance of fractals is now well established in both science and mathematics.

▪ **The Mandelbrot set.** The pictures below show Julia sets produced by the procedure of taking the transformation $z \to \{Sqrt[z-c], -Sqrt[z-c]\}$ discussed above and iterating it starting at $z = 0$ for an array of values of c in the complex plane.

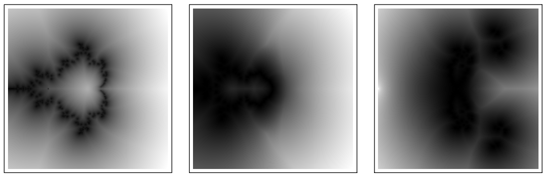


The Mandelbrot set introduced by Benoit Mandelbrot in 1979 is defined as the set of values of c for which such Julia sets are connected. This turns out to be equivalent to the set of values of c for which starting at $z = 0$ the inverse mapping $z \to z^2 + c$ leads only to bounded values of z. The Mandelbrot set turns out to have many intricate features which have been widely reproduced for their aesthetic value, as well as studied by mathematicians. The first picture below shows the overall form of the set; subsequent pictures show successive magnifications of the regions indicated. All parts of the Mandelbrot set are known to be connected. The whole set is not self-similar. However, as seen in the third and fourth pictures, within the set are isolated small copies of the whole set. In addition, as seen in the last picture, near most values of c the boundary of the Mandelbrot set looks very much like the Julia set for that value of c.



On pages 407 and 1006 I discuss parameter space sets that are somewhat analogous to the Mandelbrot set, but whose properties are in many respects much clearer. And from this discussion there emerges the following interpretation of the Mandelbrot set that appears not to be well known but which I find most illuminating. Look at the array of Julia sets and ask for each c whether the Julia set includes the point $z = 0$.

The set of values of c for which it does corresponds exactly to the boundary of the Mandelbrot set. The pictures below show a generalization of this idea, in which gray level indicates the minimum distance *Abs[z - z0]* of any point z in the Julia set from a fixed point z_0. The first picture shows the case $z_0 = 0$, corresponding to the usual Mandelbrot set.



■ **Page 192 · Neighbor-dependent substitution systems.** Given a list of individual replacement rules such as *{{_, 1}, {0, 1}} → {{1, 0}, {1, 1}}*, each step in the evolution shown corresponds to

```
Flatten2D[Partition[list, {2, 2}, 1, -1] /. rule]
```

One can consider rules in which some replacements lead to subdivision of elements but others do not. However, unlike for the 1D case, there will in general in 2D be an arbitrarily large set of different possible neighborhood configurations around any given cell.

■ **Page 192 · Space-filling curves.** One can conveniently scan a finite 2D grid just by going along each successive row in turn. One can scan a quadrant of an infinite grid using the σ function on page 1127, or one can scan a whole grid by for example going in a square spiral that at step t reaches position

```
(1/2 (-1)^# ({1, -1}(Abs[#^2 - t] - #) + #^2 - t - Mod[#, 2]) &)[
  Round[Sqrt[t]]]
```

Network Systems

■ **Implementation.** The nodes in a network system can conveniently be labelled by numbers *1, 2, … n*, and the network obtained at a particular step can be represented by a list of pairs, where the pair at position i gives the numbers corresponding to the nodes reached by following the above and below connections from node i. With this setup, a network consisting of just one node is *{{1, 1}}* and a 1D array of n nodes can be obtained with

```
CyclicNet[n_] := RotateRight[
    Table[Mod[{i - 1, i + 1}, n] + 1, {i, n}]]
```

With above connections represented as *1* and the below connections as *2*, the node reached by following a succession s of connections from node i is given by

```
Follow[list_, i_, s_List] := Fold[list[[#1]][[#2]] &, i, s]
```

The total number of distinct nodes reached by following all possible succession of connections up to length d is given by

```
NeighborNumbers[list_, i_Integer, d_Integer] :=
    Map[Length, NestList[Union[Flatten[list[[#]]]] &,
            Union[list[[i]]], d - 1]]
```

For each such list the rules for the network system then specify how the connections from node i should be rerouted. The rule *{2, 3} → {{2, 1}, {1}}* specifies that when *NeighborNumbers* gives *{2, 3}* for a node i, the connections from that node should become *{Follow[list, i, {2, 1}], Follow[list, i, {1}]}*. The rule *{2, 3} → {{{2, 1}, {1, 1}}, {1}}* specifies that a new node should be inserted in the above connection, and this new node should have connections *{Follow[list, i, {2, 1}], Follow[list, i, {1, 1}]}*. With rules set up in this way, each step in the evolution of a network system is given by

```
NetEvolveStep[{depth_Integer, rule_List}, list_List] := Block[
   {new = {}}, Join[Table[Map[NetEvolveStep1[#, list, i] &,
       Replace[NeighborNumbers[list, i, depth],
        rule]], {i, Length[list]}], new]]

NetEvolveStep1[s : {___Integer}, list_, i_] := Follow[list, i, s]

NetEvolveStep1[{s1 : {___Integer}, s2 : {___Integer}},
    list_, i_] := Length[list] + Length[
       AppendTo[new, {Follow[list, i, s1], Follow[list, i, s2]}]]
```

The set of nodes that can be reached from node i is given by

```
ConnectedNodes[list_, i_] :=
   FixedPoint[Union[Flatten[{#, list[[#]]}]] &, {i}]
```

and disconnected nodes can be removed using

```
RenumberNodes[list_, seq_] :=
   Map[Position[seq, #][[1, 1]] &, list[[seq]], {2}]
```

The sequence of networks obtained on successive steps by applying the rules and then removing all nodes not connected to node number *1* is given by

```
NetEvolveList[rule_, init_, t_Integer] :=
   NestList[(RenumberNodes[#, ConnectedNodes[#, 1]] &)[
         NetEvolveStep[rule, #]] &, init, t]
```

Note that the nodes in each network are not necessarily numbered in the order that they appear on successive lines in the pictures in the main text. Additional information on the origin of each new node must be maintained if this order is to be found.

■ **Rule structure.** For depth 1, the possible results from *NeighborNumbers* are *{1}* and *{2}*. For depth 2, they are *{1, 1}*, *{1, 2}*, *{2, 1}*, *{2, 2}*, *{2, 3}* and *{2, 4}*. In general, each successive element in a list from *NeighborNumbers* cannot be more than twice the previous element.

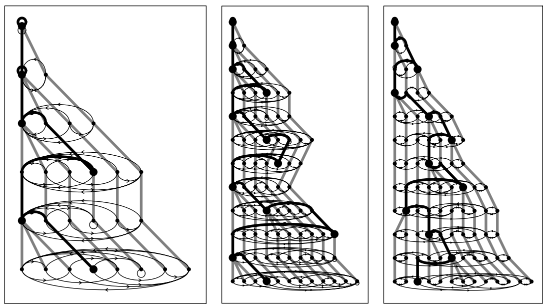
■ **Undirected networks.** Networks with connections that do not have definite directions are discussed at length in Chapter 9, mainly as potential models for space in the universe. The rules for updating such networks turn out to be somewhat more difficult to apply than those for the network systems discussed here.

■ **Page 199 · Computer science.** The networks discussed here can be thought of as very simple analogs of data structures in practical computer programs. The connections correspond to pointers between elements of these data structures. The fact that there are two connections coming from each node is reminiscent of the LISP language, but in the networks considered here there are no leaves corresponding to atoms in LISP. Note that the process of dropping nodes that become disconnected is analogous to so-called "garbage collection" for data structures. The networks considered here are also related to the combinator systems discussed on page 1121.

■ **Page 202 · Properties.** Random behavior seems to occur in a few out of every thousand randomly selected rules of the kind shown here. In case (c), the following gives a list of the numbers of nodes generated up to step *t*:

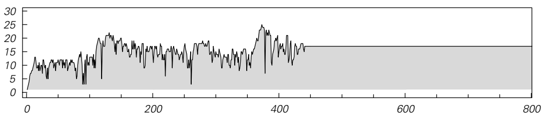
```
FoldList[Plus, 1, Join[{1, 4, 12, 10, -20, 6, 4},
     Map[d, IntegerDigits[Range[4, t - 5], 2]]]]
d[{___, 1}] = 1
d[{1, p : ((0) ..), 0}] :=
  -Apply[Plus, 4 Range[Length[{p}]] - 1] + 6
d[{__, 1, p : ((0) ..), 0}] := d[{1, p, 0}] - 7
d[{___, p : ((1) ..), q : ((0) ...), 1, 0}] :=
  4 Length[{p}] + 3 Length[{q}] + 2
d[{___, p : ((1) ..), 1, 0}] := 4 Length[{p}] + 2
```

■ **Sequential network systems.** In the network systems discussed in the main text, every node is updated in parallel at each step. It is however also possible to consider systems in which there is only a single active node, and operations are performed only on that node at any particular step. The active node can move by following its above or below connections, in a way that is determined by a rule which depends on the local structure of the network. The pictures below show examples of sequential network systems; the path of the active node is indicated by a thick black line.

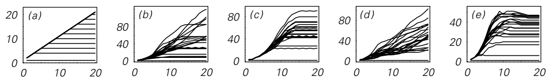


It is rather common for the active node eventually to get stuck at a particular position in the network; the picture below shows the effect of this on the total number of nodes in the last case illustrated above. The rule for this system is

```
{{1, 1} → {{{{}, {1, 1}}, {2}}, 2}, {1, 2} → {{{2, 2}, {{}, {2, 2}}}, 2},
  {2, 1} → {{{}, {2, 2}}, 2}, {2, 2} → {{{1, 2}, {{1}, {2}}}, 1},
  {2, 3} → {{{{1, 2}, {1}}, {{2}, {2, 1}}}, 2},
  {2, 4} → {{{2, 2}, {{2, 1}, {}}}, 1}}
```



■ **Dimensionality of networks.** As discussed on page 479, if a sufficiently large network has a d-dimensional form, then by following r connections in succession from a given node, one should reach about r^d distinct nodes. The plots below show the actual numbers of nodes reached as a function of r for the systems on pages 202 and 203 at steps 1, 10, 20, ..., 200.



■ **Cellular automata on networks.** The cellular automata that we have considered so far all have cells arranged in regular arrays. But one can also set up generalizations in which the cells correspond to nodes in arbitrary networks. Given a network of the kind discussed in the main text of this section, one can assign a color to each node, and then update this color at each step according to a rule that depends on the colors of the nodes to which the connections from that node go. The behavior obtained depends greatly on the form of the network, but with networks of finite size the results are typically like those obtained for other finite size cellular automata of the kind discussed on page 259.

■ **Implementation.** Given a network represented as a list in which element i is $\{a, i, b\}$, where a is the node reached by the above connection from node i, and b is the node reached by the below connection, each step corresponds to

```
NetCAStep[{rule_, net_}, list_] :=
  Map[Replace[#, rule] &, list[[net]]]
```

■ **Boolean networks.** Several lines of development from the cybernetics movement (notably in immunology, genetics and management science) led in the 1960s to a study of random Boolean networks—notably by Stuart Kauffman and Crayton Walker. Such systems are like cellular automata on networks, except for the fact that when they are set up each node has a rule that is randomly chosen from all 2^{2^s} possible ones with s inputs. With $s = 2$ class 2 behavior (see Chapter 6) tends to dominate. But for $s > 2$, the behavior one sees quickly approaches what is typical for a random mapping in which the network representing the evolution of the 2^m states of the m underlying nodes is itself connected essentially randomly (see page 963). (Attempts were made in the 1980s to study phase transitions as a function of s in analogy to ones in percolation and spin glasses.) Note that in almost all work on random Boolean networks averages are in effect taken over possible configurations, making it impossible to see anything like the kind of complex behavior that I discuss in cellular automata and many other systems in this book.

Multiway Systems

▪ **Implementation.** It is convenient to represent the state of a multiway system at each step by a list of strings, where an individual string is for example *"ABBAAB"*. The rules for the multiway system can then be given for example as

```
{"AAB" → " BB", " BA" → " ABB"}
```

The evolution of the system is given by the functions

```
MWStep[rule_List, slist_List] := Union[Flatten[
        Map[Function[s, Map[MWStep1[#, s] &, rule]], slist]]]
MWStep1[p_String → q_String, s_String] :=
   Map[StringReplacePart[s, q, #] &, StringPosition[s, p]]
MWEvolveList[rule_, init_List, t_Integer] :=
   NestList[MWStep[rule, #] &, init, t]
```

An alternative approach uses lists instead of strings, and in effect works by tracing the internal steps that *Mathematica* goes through in trying out possible matchings. With the rule from above written as

```
{{x___, 0, 0, 1, y___} → {x, 1, 1, y},
   {x___, 1, 0, y___} → {x, 0, 1, 1, y}}
```

MWStep can be rewritten as

```
MWStep[rule_List, slist_List] :=
   Union[Flatten[Map[ReplaceList[#, rule] &, slist], 1]]
```

The case shown on page 206 is

```
{"AB" → "", "ABA" → "ABBAB", "ABABBB" → "AAAAABA"}
```

starting with *{"ABABAB"}*. Note that the rules are set up so that a string for which there are no applicable replacements at a given step is simply dropped.

▪ **General properties.** The merging of states (as done above by *Union*) is crucial to the behavior seen. Note that the pictures shown indicate only which states yield which states—not for example in how many ways the rules can be applied to a given state to yield a given new state.

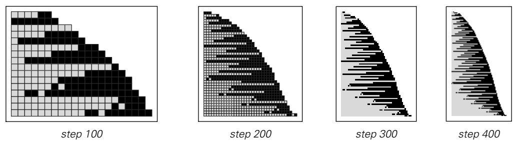
If there was no merging, then if a typical state yielded more than one new state, then inevitably the total number of states would increase exponentially. But when there is merging, this need not occur—making it difficult to give probabilistic estimates of growth rates. Note that a given rule can yield very different growth rates with different initial conditions. Thus, for example, the growth rate for *{"A" → "AA", "AB" → "BA", "BA" → "AB"}* is t^{n+1}, where n is the number of initial *B*'s. With most rules, states that appear at one step can disappear at later steps. But if *"A" → "A"* and its analogs are part of the rule, then every state will always be kept, almost inevitably leading to overall nesting in pictures like those on page 208.

In cases where all strings that appear both in rules and initial conditions are sorted—so that for example *A*'s appear before *B*'s—any string generated will also be sorted, so it can be specified just by giving a list of how many *A*'s and how many *B*'s appear in it. The rule for the system can then be stated in terms of a difference vector—which for *{"BA" → "AAA", "BAA" → "BBBA"}* is *{{2, -1}, {-1, 2}}*. Given a list of string specifications, a step in the evolution of the multiway system corresponds to

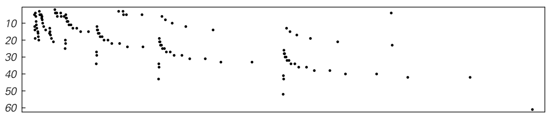
```
Select[Union[Flatten[Outer[Plus, diff, list, 1], 1]],
   Abs[#] == # &]
```

▪ **Page 206 · Properties.** The total number of strings grows approximately quadratically; its differences repeat (offset by 1) with period 1071. The number of new strings generated at successive steps grows approximately linearly; its differences repeat with period 21. The third element of the rule is at first used only on some steps—but after step 50 it appears to be used somewhere in every step.

The pictures below show in stacked form (as on page 208) all sequences generated at various steps of evolution. Note that after just a few steps, the sequences produced always seem to consist of white elements followed by black, with possibly one block of black in the white region. Without this additional block of black, only the first case in the rule can ever apply.



In analogy with page 796 the picture below shows when different strings with lengths up to 10 are reached in the evolution of the system.



Different initial conditions for this multiway system lead to behavior that either dies out (as for *"ABA"*), or grows exponentially forever (as for *"ABAABABA"*).

■ **Frequency of behavior.** Among multiway systems with randomly chosen rules, one finds about equal numbers that grow rapidly and die out completely. A few percent exhibit repetitive behavior, while only one in several million exhibit more complex behavior. One common form of more complex behavior is quadratic growth, with essentially periodic fluctuations superimposed—as on page 206.

■ **History.** Versions of multiway systems have been invented many times in a variety of contexts. In mathematics specific examples of them arose in formal group theory (see below) around the end of the 1800s. Axel Thue considered versions with two-way rules (analogous to semigroups, as discussed below) in 1912, leading to the name semi-Thue systems sometimes being used for general multiway systems. Other names for multiway systems have included string and term rewrite systems, production systems and associative calculi. From the early 1900s various generalizations of multiway systems were used as idealizations of mathematical proofs (see page 1150); multiway systems with explicit pattern variables (such as *s_*) were studied under the name canonical systems by Emil Post starting in the 1920s. Since the 1950s, multiway systems have been widely used as generators of formal languages (see below). Simple analogs of multiway systems have also been used in genetic analysis in biology and in models for particle showers and other branching processes in physics and elsewhere.

■ **Semigroups and groups.** The multiway systems that I discuss can be viewed as representations for generalized versions of familiar mathematical structures. Semigroups are obtained by requiring that rules come in pairs: with each rule such as *"ABB"* → *"BA"* there must also be the reversed rule *"BA"* → *"ABB"*. Such pairs of rules correspond to relations in the semigroup, specifying for example that *"ABB"* is equivalent to *"BA"*. (The operation in the semigroup is concatenation of strings; *""* acts as an identity element, so in fact a monoid is always obtained.) Groups require that not only rules but also symbols come in pairs. Thus, for example, in addition to a symbol *A*, there must be an inverse symbol *a*, with the rules *"Aa"* → *""*, *"aA"* → *""* and their reversals.

In the usual mathematical approach, the objects of greatest interest for many purposes are those collections of sequences that cannot be transformed into each other by any of the rules given. Such collections correspond to distinct elements of the group or semigroup, and in general many different choices of underlying rules may yield the same elements with the same properties. In terms of multiway systems, each of the elements corresponds to a disconnected part of the network formed from all possible sequences.

Given a particular representation of a group or semigroup in terms of rules for a multiway system, an object that is often useful is the so-called Cayley graph—a network where each node is an element of the group, and the connections show what elements are reached by appending each possible symbol to the sequences that represent a given element. The so-called free semigroup has no relations and thus no rules, so that all strings of generators correspond to distinct elements, and the Cayley graph is a tree like the ones shown on page 196. The simplest non-trivial commutative semigroup has rules *"AB"* → *"BA"* and *"BA"* → *"AB"*, so that strings of generators with *A*'s and *B*'s in different orders are equivalent and the Cayley graph is a 2D grid.

For some sets of underlying rules, the total number of distinct elements in a group or semigroup is finite. (Compare page 945.) A major mathematical achievement in the 1980s was the complete classification of all possible so-called simple finite groups that in effect have no factors. (For semigroups no such classification has yet been made.) In each case, there are many different choices of rules that yield the same group (and similar Cayley graphs). And it is known that even fairly simple sets of rules can yield large and complicated groups. The icosahedral group A_5 defined by the rules $x^2 == y^3 == (x\,y)^5 == 1$ has 60 elements. But in the most complicated case a dozen rules yield the Monster Group, where the number of elements is

808017424794512875886459904961710757005754368000000000

(See also pages 945 and 1032.)

Following work in the 1980s and 1990s by Mikhael Gromov and others, it is also known that for groups with randomly chosen underlying rules, the Cayley graph is usually either finite, or has a rapidly branching tree-like structure. But there are presumably also marginal cases that exhibit complex behavior analogous to what we saw in the main text. And indeed for example, despite conjectures to the contrary, it was found in the 1980s by Rostislav Grigorchuk that complicated groups could be constructed in which growth intermediate between polynomial and exponential can occur. (Note that different choices of generators can yield Cayley graphs with different local subgraphs; but the overall structure of a sufficiently large graph for a particular group is always the same.)

■ **Formal languages.** The multiway systems that I discuss are similar to so-called generative grammars in the theory of formal languages. The idea of a generative grammar is that all possible expressions in a particular formal language can be produced by applying in all possible ways the set of replacement rules given by the grammar. Thus, for example, the rules {"x" → "xx", "x" → "(x)", "x" → "()"} starting with "x" will generate all expressions that consist of balanced sequences of parentheses. (Final expressions correspond to those without the "non-terminal" symbol *x*.) The hierarchy described by Noam Chomsky in 1956 distinguishes four kinds of generative grammars (see page 1104):

Regular grammars. The left-hand side of each rule must consist of one non-terminal symbol, and the right-hand side can contain only one non-terminal symbol. An example is {"x" → "xA", "x" → "yB", "y" → "xA"} starting with "x" which generates sequences in which no pair of *B*'s ever appear together. Expressions in regular languages can be recognized by finite automata of the kind discussed on page 957.

Context-free grammars. The left-hand side of each rule must consist of one non-terminal symbol, but the right-hand side can contain several non-terminal symbols. Examples include the parenthesis language mentioned above, {"x" → "AxA", "x" → "B"} starting with "x", and the syntactic definitions of *Mathematica* and most other modern computer languages. Context-free languages can be recognized by a computer using only memory on a single last-in first-out stack. (See pages 1091 and 1103.)

Context-sensitive grammars. The left-hand side of each rule is no longer than the right, but is otherwise unrestricted. An example is {"Ax" → "AAxx", "xA" → "BAA", "xB" → "Bx"} starting with "AAxBA", which generates expressions of the form Table["A", {n}] <> Table["B", {n}] <> Table["A", {n}].

Unrestricted grammars. Any rules are allowed.

(See also page 944.)

■ **Multidimensional multiway systems.** As a generalization of multiway systems based on 1D strings one can consider systems in which rules operate on arbitrary blocks of elements in an array in any number of dimensions. Still more general network substitution systems are discussed on page 508.

■ **Limited size versions.** One can set up multiway systems of limited size by applying transformations cyclically to strings.

■ **Multiway tag systems.** See page 1141.

■ **Multiway systems based on numbers.** One can consider for example the rule $n \to \{n+1, 2n\}$ implemented by

```
NestList[Union[Flatten[{# + 1, 2 #}]] &, {0}, t]
```

In this case there are Fibonacci[t + 2] distinct numbers obtained at step *t*. In general, rules based on simple arithmetic operations yield only simple nested structures. If the numbers *n* are allowed to have both real and imaginary parts then results analogous to those discussed for substitution systems on page 933 are obtained. (Somewhat related systems based on recursive sequences are discussed on page 907. Compare also sorted multiway systems on page 937.)

■ **Non-deterministic systems.** Multiway systems are examples of what are often in computer science called non-deterministic systems. The general idea of a non-deterministic system is to have rules with several possible outcomes, and then to allow each of these outcomes to be followed. Non-deterministic Turing machines are a common example. For most types of systems (such as Turing machines) such non-deterministic versions do not ultimately allow any greater range of computations to be performed than deterministic ones. (But see page 766.)

■ **Fundamental physics.** See page 504.

■ **Game systems.** One can think of positions or configurations in a game as corresponding to nodes in a large network, and the possible moves in the game as corresponding to connections between nodes. Most games have rules which imply that if certain states are reached one player can be forced in the end to lose, regardless of what specific moves they make. And even though the underlying rules in the game may be simple, the pattern of such winning positions is often quite complex. Most games have huge networks whose structure is difficult to visualize (even the network for tic-tac-toe, for example, has 5478 nodes). One example that allows easy visualization is a simplification of several common games known as nim. This has *k* piles of objects, and on alternate steps each of two players takes as many objects as they want from any one of the piles. The winner is the player who manages to take the very last object. With just two piles one player can force the other to lose by arranging that after each of their moves the two piles have equal heights. With more than two piles it was discovered in 1901 that one player can in general force the other to lose by arranging that after each of their moves Apply[BitXor, h] == 0, where *h* is the list of heights. For $k > 1$ this yields a nested pattern, analogous to those shown on page 871. If one allows only specific numbers of objects to be taken at each step a nested pattern is again obtained. With more general rules it seems almost inevitable that much more complicated patterns will occur.

Systems Based on Constraints

▪ **The notion of equations.** In the mathematical framework traditionally used in the exact sciences, laws of nature are usually represented not by explicit rules for evolution, but rather by abstract equations. And in general what such equations do is to specify constraints that systems must satisfy. Sometimes these constraints just relate the state of a system at one time to its state at a previous time. And in such cases, the constraints can usually be converted into explicit evolution rules. But if the constraints relate different features of a system at one particular time, then they cannot be converted into evolution rules. In computer programs and other kinds of discrete systems, explicit evolution rules and implicit constraints usually work very differently. But in traditional continuous mathematics, it turns out that these differences are somewhat obscured. First of all, at a formal level, equations corresponding to these two cases can look very similar. And secondly, the equations are almost always so difficult to deal with at all that distinctions between the two cases are not readily noticed.

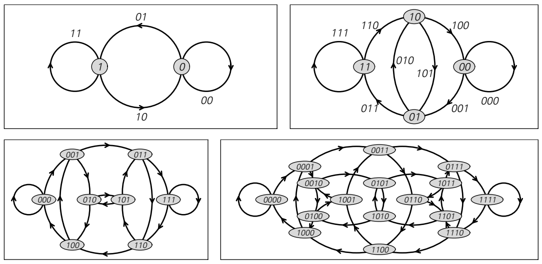
In the language of differential equations—the most widely used models in traditional science—the two cases we are discussing are essentially so-called initial value and boundary value problems, discussed on page 923. And at a formal level, the two cases are so similar that in studying partial differential equations one often starts with an equation, and only later tries to work out whether initial or boundary values are needed in order to get either any solution or a unique solution. For the specific case of second-order equations, it is known in general what is needed. Elliptic equations such as the Laplace equation need boundary values, while hyperbolic and parabolic equations such as the wave equation and diffusion equation need initial values. But for higher-order equations it can be extremely difficult to work out what initial or boundary values are needed, and indeed this has been the subject of much research for many decades.

Given a partial differential equation with initial or boundary values, there is then the question of solving it. To do this on a computer requires constructing a discrete approximation. But it turns out that the standard methods used (such as finite difference and finite element) involve extremely similar computations for initial and for boundary value problems, leaving no trace of the significant differences between these cases that are so obvious in the discrete systems that we discuss in most of this book.

▪ **Linear and nonlinear systems.** A vast number of different applications of traditional mathematics are ultimately based on linear equations of the form $u == m . v$ where u and v are vectors (lists) and m is a matrix (list of lists), all containing ordinary continuous numbers. If v is known then such equations in essence provide explicit rules for computing u. But if only u is known, then the equations can instead be thought of as providing implicit constraints for v. However, it so happens that even in this case v can still be found fairly straightforwardly using *LinearSolve[m, u]*. With vectors of length n it generically takes about n^2 steps to compute u given v, and a little less than n^3 steps to compute v given u (the best known algorithms—which are based on matrix multiplication—currently involve about $n^{2.4}$ steps). But as soon as the original equation is nonlinear, say $u == m_1 . v + m_2 . v^2$, the situation changes dramatically. It still takes only about n^2 steps to compute u given v, but it becomes vastly more difficult to compute v given u, taking perhaps 2^{2^n} steps. (Generically there are 2^n solutions for v, and even for integer coefficients in the range $-r$ to $+r$ already in 95% of cases there are 4 solutions with $n = 2$ as soon as $r \geq 6$.)

▪ **Explanations based on constraints.** In some areas of science it is common to give explanations in terms of constraints rather than mechanisms. Thus, for example, in physics there are so-called variational principles which state that physical systems will behave in ways that minimize or maximize certain quantities. One such principle implies that atoms in molecules will tend to arrange themselves so as to minimize their energy. For simple molecules, this is a useful principle. But for complicated molecules of the kind that are common in living systems, this principle becomes much less useful. In fact, in finding out what configuration such molecules actually adopt, it is usually much more relevant to know how the molecule evolves in time as it is created than which of its configurations formally has minimum energy. (See pages 342 and 1185.)

▪ **Page 211 · 1D constraints.** The constraints in the main text can be thought of as specifying that only some of the k^n possible blocks of cells of length n (with k possible colors for each cell) are allowed. To see the consequences of such constraints consider breaking a sequence of colors into blocks of length n, with each block overlapping by $n - 1$ cells with its predecessor, as in *Partition[list, n, 1]*. If all possible sequences of colors were allowed, then there would be k possibilities for what block could follow a given block, given by *Map[Rest, Table[Append[list, i], {i, 0, k - 1}]]*. The possible sequences of length n blocks that can occur are conveniently represented by possible paths by so-called de Bruijn networks, of the kind shown for $k = 2$ and $n = 2$ through 5 below.



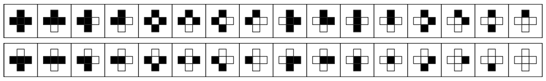
Given the network for a particular n, it is straightforward to see what happens when only certain length n blocks are allowed: one just keeps the arcs in the network that correspond to allowed blocks, and drops all other ones. Then if one can still form an infinite path by going along the arcs that remain, this path will correspond to a pattern that satisfies the constraints. Sometimes there will be a unique such path; in other cases there will be choices that can be made along the path. But the crucial point is that since there are only k^{n-1} nodes in the network, then if any infinite path is possible, there must be such a path that visits the same node and thus repeats itself after at most k^{n-1} cells. The constraint on page 210 has $k = 2$ and $n = 3$; the pattern that satisfies it repeats with period 4, thus saturating the bound. (See also page 266.)

■ **1D cellular automata.** In a cellular automaton with k colors and r neighbors, configurations that are left invariant after t steps of evolution according to the cellular automaton rule are exactly the ones which contain only those length $2r + 1$ blocks in which the center cell is the same before and after the evolution. Such configurations therefore obey constraints of the kind discussed in the main text. As we will see on page 225 some cellular automata evolve to invariant configurations from any initial conditions, but most do not. (See page 954.)

■ **Dynamical systems theory.** Sets of sequences in which a finite collection of blocks are excluded are sometimes known as finite complement languages, or subshifts of finite type. (See page 958.)

■ **Page 215 · 2D constraints.** The constraints shown here are minimal, in the sense that in each case removing any of the allowed templates prevents the constraint from ever being satisfied. Note that constraints which differ only by overall rotation, reflection or interchange of black and white are not explicitly shown. The number of allowed templates out of the total of 32 possible varies from 1 to 15 for the constraints shown, with 12 being the most common. Smaller sets of allowed templates typically seem to lead to constraints that can be satisfied by visually simpler patterns.

■ **Numbering scheme.** The constraint numbered n allows the templates at *Position[IntegerDigits[n, 2, 32], 1]* in the list below. (See also page 927.)



■ **Identifying the 171 patterns.** The number of constraints to consider can be reduced by symmetries, by discarding sets of templates that are supersets of ones already known to be satisfiable, and by requiring that each template in the set be compatible with itself or with at least one other in each of the eight immediately adjacent positions. The remaining constraints can then be analyzed by attempting to build up explicit patterns that satisfy them, as discussed below.

■ **Checking constraints.** A set of allowed templates can be specified by a *Mathematica* pattern of the form $t_1 \mid t_2 \mid t_3$ etc. where the t_i are for example *{{_, 1, _}, {0, 0, 1}, {_, 0, _}}*. To check whether an array *list* contains only arrangements of colors corresponding to allowed templates one can then use

```
SatisfiedQ[list_, allowed_] :=
  Apply[And, Map[MatchQ[#, allowed] &,
     Partition[list, {3, 3}, {1, 1}], {2}], {0, 1}]
```

■ **Representing repetitive patterns.** Repetitive patterns are often most conveniently represented as tessellations of rectangles whose corners overlap. Pattern (a) on page 213 can be specified as

```
{{2, -1, 2, 3}, {{0, 0, 0, 0}, {1, 1, 0, 0}, {1, 0, 0, 0}}}
```

Given this, a complete *nx* by *ny* array filled with this pattern can be constructed from

```
c[{d1_, d2_, d3_, d4_}, {x_, y_}] :=
  With[{d = d1 d2 + d1 d4 + d3 d4},
    Mod[{{d2 x + d4 x + d3 y, d4 x - d1 y}}/d, 1]]
Fill[{dlist_, data_}, {nx_, ny_}] :=
 Array[c[dlist, {##}] &, {nx, ny}] /. Flatten[MapIndexed[
    c[dlist, Reverse[#2]] → #1 &, Reverse[data], {2}], 1]
```

■ **Searching for patterns.** The basic approach to finding a pattern which satisfies a particular constraint on an infinite array of cells is to start with a pattern which satisfies the constraint in a small region, and then to try to extend the pattern. Often the constraint will immediately force a unique extension of the pattern, at least for some distance. But eventually there will normally be places where the pattern is not yet uniquely determined, and so a series of choices have to be made. The procedure used to find the results in this book attempts to extend patterns along a square spiral, making whatever choices are needed, and backtracking if these turn out to be inconsistent with the constraint. At every step in the procedure, regularities are tested for that would imply the possibility of an infinite repetitive pattern. In addition, whenever there is a choice, the first cases to be tried are set up to be ones that tend to extend whatever regularity has developed so far. And when backtracking is needed, the procedure always goes back to the most recent choice that actually affected whatever inconsistency was discovered. And in addition it remembers what has already been worked out, so as to avoid, for example, unnecessarily working out the pattern on the opposite side of the spiral again.

▪ **Undecidability.** The general problem of whether an infinite pattern exists that satisfies a particular constraint is formally undecidable (see page 1139). This means that in general there can be no upper bound on the size of region for which the constraints can be satisfied, even if they are not satisfiable for the complete infinite grid.

▪ **NP completeness.** The problem of whether a pattern can be found that satisfies a constraint even in a finite region is NP-complete. (See page 1145.) This suggests that to determine whether a repetitive pattern with repeating blocks of size *n* exists may in general take a number of steps which grows more rapidly than any polynomial in *n*.

▪ **Enumerating patterns.** Compare page 959.

▪ **Page 219 · Non-periodic pattern.** The color at position *x*,*y* in the pattern is given by

```
a[x_, y_] := Mod[y + 1, 2] /; x + y > 0
a[x_, y_] := 0 /; Mod[x + y, 2] == 1
a[x_, y_] :=
   Mod[Floor[(x - y) 2^((x+y-6)/4)], 2] /; Mod[x + y, 4] == 2
a[x_, y_] := 1 - Sign[Mod[x - y + 2, 2^((-x-y+8)/4)]]
```

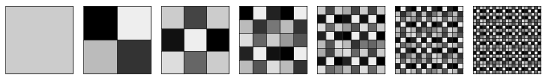
The origin of the *x*,*y* coordinates is the only freedom in this pattern. The nested structure is like the progression of base 2 digit sequences shown on page 117. Negative numbers are effectively represented by complements of digit sequences, much as in typical practical computers. With the procedure described above for finding patterns that satisfy a constraint, generating the pattern shown here is straightforward once the appropriate constraint is identified.

▪ **Other types of constraints.** Constraints based on smaller templates simply require smaller numbers of repetitive patterns: ▦:4; ▦:7; ▦:17; ▦:11; ▦:12. To extend the class of systems considered in the main text, one can increase the size of the templates, or increase the number of possible colors for each cell. For 3×3 templates with two colors extensive randomized searches have failed to discover examples where non-repetitive patterns are forced to occur. Another extension of the constraints in the main text is to require that not just a single template, but every template in the set, must occur somewhere in the pattern. Searches of such systems have also failed to discover examples of forced non-repetitive patterns beyond the one shown in the text.

▪ **Forcing nested patterns.** It is straightforward to find constraints that allow nested patterns; the challenge is to find ones that force such patterns to occur. Many nested patterns (such as the one made by rule 90, for example) contain large areas of uniform white, and it is typically difficult to prevent pure repetition of that area. One approach to finding constraints that can be satisfied only by nested patterns is nevertheless to start from specific nested patterns, look at what templates occur, and then see whether these templates are such that they do not allow any purely repetitive patterns. A convenient way to generate a large class of nested patterns is to use 2D substitution systems of the kind discussed on page 188. But searching all 4 billion or so possible such systems with 2×2 blocks and up to four colors one finds not a single case in which a nested pattern is forced to occur. It can nevertheless be shown that with a sufficiently large number of extra colors any nested pattern can be forced to occur. And it turns out that a result from the mid-1970s by Robert Ammann for a related problem of tiling (see below) allows one to construct a specific system with 16 colors in which constraints of the kind discussed here force a nested pattern to occur. One starts from the substitution system with rules

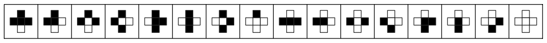
```
{1 → {{3}}, 2 → {{13, 1}, {4, 10}}, 3 → {{15, 1}, {4, 12}},
  4 → {{14, 1}, {2, 9}}, 5 → {{13, 1}, {4, 12}}, 6 → {{13, 1}, {8, 9}},
  7 → {{15, 1}, {4, 10}}, 8 → {{14, 1}, {6, 10}}, 9 → {{14}, {2}},
  10 → {{16}, {7}}, 11 → {{13}, {8}}, 12 → {{16}, {3}},
  13 → {{5, 11}}, 14 → {{2, 9}}, 15 → {{3, 11}}, 16 → {{6, 10}}}
```

This yields the nested pattern below which contains only 51 of the 65,536 possible 2×2 blocks of cells with 16 colors. It then turns out that with the constraint that the only 2×2 arrangements of colors that can occur are ones that match these 51 blocks, one is forced to get the nested pattern below.



▪ **Relation to 2D cellular automata.** The kind of constraints discussed are exactly those that must be satisfied by configurations that remain unchanged in the evolution of a 2D cellular automaton. The argument for this is similar to the one on pages 941 and 954 for 1D cellular automata. The point is that of the 32 5-cell neighborhoods involved in the 2D cellular automaton rule, only some subset will have the property that the center cell remains unchanged after applying the rule. And any configuration which does not change must involve only these subsets. Using the results of this section it then follows that in the evolution of all 2D cellular automata of the type discussed on page 170 there exist purely repetitive configurations that remain unchanged.

■ **Relation to 1D cellular automata.** A picture that shows the evolution of a 1D cellular automaton can be thought of as a 2D array of cells in which the color of each cell satisfies a constraint that relates it to the cells above according to the cellular automaton rule. This constraint can then be represented in terms of a set of allowed templates; the set for rule 30 is as follows:



To reproduce an ordinary picture of cellular automaton evolution, one would have to specify in advance a whole line of black and white cells. Below this line there would then be a unique pattern corresponding to the application of the cellular automaton rule. But above the line, except for reversible rules, there is no guarantee that any pattern satisfying the constraints can exist.

If one specifies no cells in advance, or at most a few cells, as in the systems discussed in the main text, then the issue is different, however. And now it is always possible to construct a repetitive pattern which satisfies the constraints simply by finding repetitive behavior in the evolution of the cellular automaton from a spatially repetitive initial condition.

■ **Non-computable patterns.** It is known to be possible to set up constraints that will force patterns in which finding the color of a particular cell can require doing something like solving a halting problem—which cannot in general be done by any finite computation. (See also page 1139.)

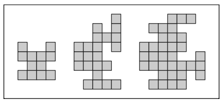
■ **Tiling.** The constraints discussed here are similar to those encountered in covering the plane with tiles of various shapes. Of regular polygons, only squares, triangles and hexagons can be used to do this, and in these cases the tilings are always repetitive. For some time it was believed that any set of tiles that could cover the plane could be arranged to do so repetitively. But in 1964 Robert Berger demonstrated that this was not the case, and constructed a set of about 20,000 tiles that could cover the plane only in a nested fashion. Later Berger reduced the number of tiles needed to 104. Then Raphael Robinson in 1971 reduced the number tiles to six, and in 1974 Roger Penrose showed that just two tiles were necessary. Penrose's tiles can cover the plane only in a nested pattern that can be constructed from a substitution system that successively subdivides each tile, as shown on page 932. (Note that various dissections of these tiles can also be used. The edges of the particular shapes shown should strictly be distinguished in order to prevent trivial periodic arrangements.) The triangles in the construction have angles which are multiples of $\pi/5$, so that the whole tiling has an approximate 5-fold symmetry (see page 994). Repetitive tilings of the plane can only have 3-, 4- or 6-fold symmetry.

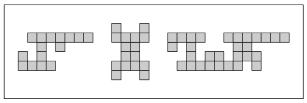
No single shape is known which has the property that it can tile the plane only non-repetitively, although one strongly suspects that one must exist. In 3D, John Conway has found a single biprism that can fill space only in a sequence of layers with an irrational rotation angle between each layer.

In addition, in no case has a simple set of tiles been found which force a pattern more complicated than a nested one. The results on page 221 in this book can be used to constructed a complicated set of tiles with this property, but I suspect that a much simpler set could be found.

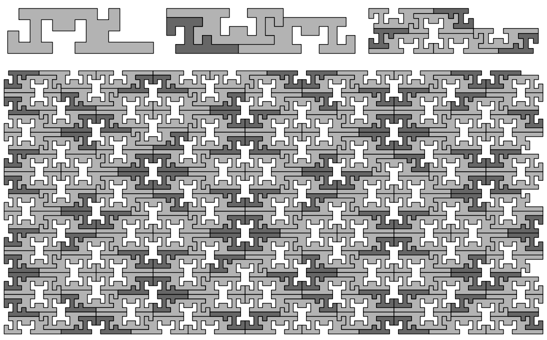
(See also page 1139.)

■ **Polyominoes.** An example of a tiling problem that is in some respects particularly close to the grid-based constraint systems discussed in the main text concerns covering the plane with polyominoes that are formed by gluing collections of squares together. Tiling by polyominoes has been investigated since at least the late 1950s, particularly by Solomon Golomb, but it is only very recently that sets of polyominoes which force non-periodic patterns have been found. The set (a) below was announced by Roger Penrose in 1994; the slightly smaller set (b) was found by Matthew Cook as part of the development of this book.


(a)


(b)

Both of these sets yield nested patterns. Steps in the construction of the pattern for set (b) are shown below. At stage n the number of polyominoes of each type is *Fibonacci[2 n - {2, 0, 1}]/{1, 2, 1}*. Set (a) works in a roughly similar way, but with a considerably more complicated recursion.



■ **Ground states of spin systems.** The constraints discussed in the main text are similar to those that arise in the physics of 2D spin systems. An example of such a system is the so-called Ising model discussed on page 981. The idea in all such systems is to have an array of spins, each of which can be either up or down. The energy associated with each spin is then given by some function which depends on the configuration of neighboring spins. The ground state of the system corresponds to an arrangement of spins with the smallest total energy. In the ordinary Ising model, this ground state is simply all spins up or all spins down. But in generalizations of the Ising model with more complicated energy functions, the conditions to get a state of the lowest possible energy can correspond exactly to the constraints discussed in the main text. And from the results shown one sees that in some cases random-looking ground states should occur. Note that a rather different way to get a somewhat similar ground state is to consider a spin glass, in which the standard Ising model energy function is used, but multiplied by -1 or +1 at random for each spin.

■ **Correspondence systems.** For a discussion of a class of 1D systems based on constraints see page 757.

■ **Sequence equations.** Another way to set up 1D systems based on constraints is by having equations like *Flatten[{x, 1, x, 0, y}] === Flatten[{0, y, 0, y, x}]*, where each variable stands for a list. Fairly simple such equations can force fairly complicated results, although as discussed on page 1141 there are known to be limits to this complexity.

■ **Pattern-avoiding sequences.** As another form of constraint one can require, say, that no pair of identical blocks ever appear together in a sequence, so that the sequence does not match *{___, x__, x__, ___}*. With just two possible elements, no sequence above length 3 can satisfy this constraint. But with $k = 3$ possible elements, there are infinite nested sequences that can, such as the one produced by the substitution system $\{0 \to \{0, 1, 2\}, 1 \to \{0, 2\}, 2 \to \{1\}\}$, starting with $\{0\}$. One can find the sequences of length n that work by using

```
Nest[DeleteCases[Flatten[Map[Table[Append[#, i - 1],
          {i, k}] &, #], 1], {___, x__, x__, ___}] &, {{}}, n]
```

and the number of these grows roughly like $3^{n/4}$.

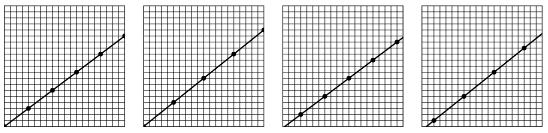
The constraint that no triple of identical blocks appear together turns out to be satisfied by the Thue-Morse nested sequence from page 83—as already noted by Axel Thue in 1906. (The number of sequences that work seems to grow roughly like $2^{n/2}$.)

For any given k, many combinations of blocks will inevitably occur in sufficiently long sequences (compare page 1068). (For example, with $k = 2$, *{___, x__, y__, x__, y__, ___}* always matches any sequence with length more than 18.) But some patterns of blocks can be avoided. And for example it is known that for $k \geq 2$ any pattern with length 6 or more (excluding the ___'s) and only two different variables (say *x__* and *y__*) can always be avoided. But it also known that among the infinite sequences which do this, there are always nested ones (sometimes one has to iterate one substitution rule, then at the end apply once a different substitution rule). With more variables, however, it seems possible that there will be patterns that can be avoided only by sequences with a more complicated structure. And a potential sign of this would be patterns for which the number of sequences that avoid them varies in a complicated way with length.

■ **Formal languages.** Formal languages of the kind discussed on page 938 can be used to define constraints on 1D sequences. The constraints shown on page 210 correspond to special cases of regular languages (see page 940). For both regular and context-free languages the so-called pumping lemmas imply that if any finite sequences satisfy the constraints, then so must an essentially repetitive infinite sequence.

■ **Diophantine equations.** Any algebraic equation—such as $x^3 + x + 1 == 0$—can readily be solved if one allows the variables to have any numerical value. But if one insists that the variables are whole numbers, then the problem is more analogous to the discrete constraints in the main text, and becomes much more difficult. And in fact, even though such so-called Diophantine equations have been studied since well before the time of Diophantus around perhaps 250 AD, only limited results about them are known.

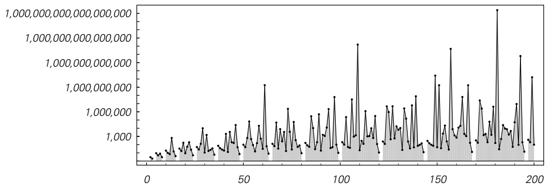
Linear Diophantine equations such as $a x == b y + c$ yield simple repetitive results, as in the pictures below, and can be handled essentially just by knowing *ExtendedGCD[a, b]*.


$3x = 4y$ $4x = 5y$ $3x = 4y + 1$ $4x = 5y + 3$

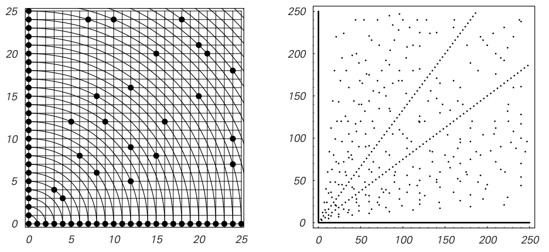
Even the simplest quadratic Diophantine equations can already show much more complex behavior. The equation $x^2 == a y^2$ has no solution except when a is a perfect square. But the Pell equation $x^2 == a y^2 + 1$ (already studied in antiquity) has infinitely many solutions whenever a is positive and not a perfect square. The smallest solution for x is given by

```
Numerator[FromContinuedFraction[
     ContinuedFraction[√a, (If[EvenQ[#], #, 2#] &)[
          Length[Last[ContinuedFraction[√a]]]]]]]
```

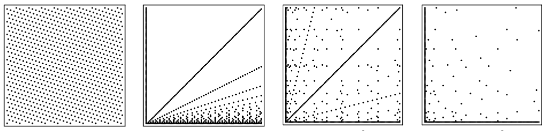
This is plotted below; complicated variation and some very large values are seen (with $a = 61$ for example $x == 1766319049$).



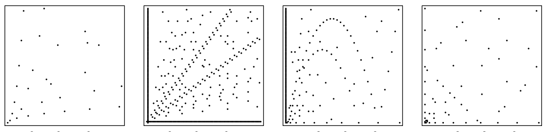
In three variables, the equation $x^2 + y^2 == z^2$ yields so-called Pythagorean triples $\{3, 4, 5\}$, $\{5, 12, 13\}$, etc. And even in this case the set of possible solutions for x and y in the pictures below looks fairly complicated—though after removing common factors, they are in fact just given by $\{x == r^2 - s^2, y == 2 r s, z == r^2 + s^2\}$. (See page 1078.)



The pictures below show the possible solutions for x and y in various Diophantine equations. As in other systems based on numbers, nested patterns are not common—though page 1160 shows how they can in principle be achieved with an equation whose solutions satisfy $Mod[Binomial[x, y], 2] == 1$. (The equation $(2 x + 1) y == z$ also for example has solutions only when z is not of the form 2^j.)



$x + 3 y = 11 z$ $x = y z$ $x y = z^2$ $x y = z^3$



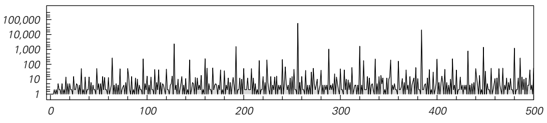
$x^2 + y^2 = z^2 - 1$ $x^2 + y^2 = z^2 + 1$ $x^3 + y^2 = z^2$ $x^3 + y^3 = z^2$

Many Diophantine equations have at most very sparse solutions. And indeed for example Fermat's Last Theorem states that $x^n + y^n == z^n$ can never be satisfied for $n > 2$. With four variables one has for example $3^3 + 4^3 + 5^3 == 6^3$, $1^3 + 6^3 + 8^3 == 9^3$—but with fourth powers the smallest result is $95800^4 + 217519^4 + 414560^4 == 422481^4$.

(See pages 791 and 1164.)

■ **Matrices satisfying constraints.** One can consider for example magic squares, Latin squares (quasigroup multiplication tables), and matrices having the Hadamard property discussed on page 1073. One can also consider matrices whose powers contain certain patterns. (See also page 805.)

■ **Finite groups and semigroups.** Any finite group or semigroup can be thought of as defined by having a multiplication table which satisfies the constraints given on page 887. The total number of semigroups increases faster than exponentially with size in a seemingly quite uniform way. But the number of groups varies in a complicated way with size, as in the picture below. (The peaks are known to grow roughly like $n\wedge(2/27\, Log[2, n]^2)$—intermediate between polynomial and exponential.) As mentioned on page 938, through major mathematical effort, a complete classification of all finite so-called simple groups that in effect have no factors is known. Most such groups come in families that are easy to characterize; a handful of so-called sporadic ones are much more difficult to find. But this classification does not immediately provide a practical way to enumerate all possible groups. (See also pages 938 and 1032.)



■ **Constraints on formulas.** Many standard problems of algebraic computation can be viewed as consisting in finding formulas that satisfy certain constraints. An example is exact solution of algebraic equations. For quadratic equations the standard formula gives solutions for arbitrary coefficients in terms of square roots. Similar formulas in terms of n^{th} roots have been known since the 1500s for equations with degrees n up to 4, although their *LeafCount* starting at $n = 1$ increases like 6, 25, 183, 718. For higher degrees it is known that such general formulas must involve other functions. For degrees 5 and 6 it was shown in the late 1800s that *EllipticTheta* or *Hypergeometric2F1* are sufficient, although for degrees 5 and 6 respectively the necessary formulas have a *LeafCount* in the billions. (Sharing common subexpressions yields a *LeafCount* in the thousands.) (See also page 1129.)