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STEPHEN WOLFRAM
A NEW KIND OF SCIENCE

CHAPTER 12

The Principle of Computational Equivalence

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The Principle of Computational Equivalence

Basic Framework

Following the discussion of the notion of computation in the previous chapter, I am now ready in this chapter to describe a bold hypothesis that I have developed on the basis of the discoveries in this book, and that I call the Principle of Computational Equivalence.

Among principles in science the Principle of Computational Equivalence is almost unprecedentedly broad—for it applies to essentially any process of any kind, either natural or artificial. And its implications are both broad and deep, addressing a host of longstanding issues not only in science, but also in mathematics, philosophy and elsewhere.

The key unifying idea that has allowed me to formulate the Principle of Computational Equivalence is a simple but immensely powerful one: that all processes, whether they are produced by human effort or occur spontaneously in nature, can be viewed as computations.

In our practical experience with computers, we are mostly concerned with computations that have been set up specifically to perform particular tasks. But as I discussed at the beginning of this book there is nothing fundamental that requires a computation to have any such definite purpose. And as I discussed in the previous chapter the process of evolution of a system like a cellular automaton can for example perfectly well be viewed as a computation, even though in a sense all the computation does is generate the behavior of the system.

But what about processes in nature? Can these also be viewed as computations? Or does the notion of computation somehow apply only to systems with abstract elements like, say, the black and white cells in a cellular automaton?

Before the advent of modern computer applications one might have assumed that it did. But now every day we see computations being done with a vast range of different kinds of data—from numbers to text to images to almost anything else. And what this suggests is that it is possible to think of any process that follows definite rules as being a computation—regardless of the kinds of elements it involves.

So in particular this implies that it should be possible to think of processes in nature as computations. And indeed in the end the only unfamiliar aspect of this is that the rules such processes follow are defined not by some computer program that we as humans construct but rather by the basic laws of nature.

But whatever the details of the rules involved the crucial point is that it is possible to view every process that occurs in nature or elsewhere as a computation. And it is this remarkable uniformity that makes it possible to formulate a principle as broad and powerful as the Principle of Computational Equivalence.

Outline of the Principle

Across all the vastly different processes that we see in nature and in systems that we construct one might at first think that there could be very little in common. But the idea that any process whatsoever can be viewed as a computation immediately provides at least a uniform framework in which to discuss different processes.

And it is by using this framework that the Principle of Computational Equivalence is formulated. For what the principle does is to assert that when viewed in computational terms there is a fundamental equivalence between many different kinds of processes.

There are various ways to state the Principle of Computational Equivalence, but probably the most general is just to say that almost all processes that are not obviously simple can be viewed as computations of equivalent sophistication.

And although at first this statement might seem vague and perhaps almost inconsequential, we will see in the course of this chapter that in fact it has many very specific and dramatic implications.

One might have assumed that among different processes there would be a vast range of different levels of computational sophistication. But the remarkable assertion that the Principle of Computational Equivalence makes is that in practice this is not the case, and that instead there is essentially just one highest level of computational sophistication, and this is achieved by almost all processes that do not seem obviously simple.

So what might lead one to this rather surprising idea? An important clue comes from the phenomenon of universality that I discussed in the previous chapter and that has been responsible for much of the success of modern computer technology. For the essence of this phenomenon is that it is possible to construct universal systems that can perform essentially any computation—and which must therefore all in a sense be capable of exhibiting the highest level of computational sophistication.

The most familiar examples of universal systems today are practical computers and general-purpose computer languages. But in the fifty or so years since the phenomenon of universality was first identified, all sorts of types of systems have been found to be able to exhibit universality. Indeed, as I showed in the previous chapter, it is possible for example to get universality in cellular automata, Turing machines, register machines—or in fact in practically every kind of system that I have considered in this book.

So this implies that from a computational point of view even systems with quite different underlying structures will still usually show a certain kind of equivalence, in that rules can be found for them that achieve universality—and that therefore can always exhibit the same level of computational sophistication.

But while this is already a remarkable result, it represents only a first step in the direction of the Principle of Computational

Equivalence. For what the result implies is that in many kinds of systems particular rules can be found that achieve universality and thus show the same level of computational sophistication. But the result says nothing about whether such rules are somehow typical, or are instead very rare and special.

And in practice, almost without exception, the actual rules that have been established to be universal have tended to be quite complex. Indeed, most often they have in effect been engineered out of all sorts of components that are direct idealizations of various elaborate structures that exist in practical digital electronic computers.

And on the basis of traditional intuition it has almost always been assumed that this is somehow inevitable, and that in order to get something as sophisticated as universality there must be no choice but to set up rules that are themselves special and sophisticated.

One of the dramatic discoveries of this book, however, is that this is not the case, and that in fact even extremely simple rules can be universal. Indeed, from our discussion in the previous chapter, we already know that among the 256 very simplest possible cellular automaton rules at least rule 110 and three others like it are universal.

And my strong suspicion is that this is just the beginning, and that in time a fair fraction of other simple rules will also be shown to be universal. For one of the implications of the Principle of Computational Equivalence is that almost any rule whose behavior is not obviously simple should ultimately be capable of achieving the same level of computational sophistication and should thus in effect be universal.

So far from universality being some rare and special property that exists only in systems that have carefully been built to exhibit it, the Principle of Computational Equivalence implies that instead this property should be extremely common. And among other things this means that universality can be expected to occur not only in many kinds of abstract systems but also in all sorts of systems in nature.

And as we shall see in this chapter, this idea already has many important and surprising consequences. But still it is far short of what the full Principle of Computational Equivalence has to say.

For knowing that a particular rule is universal just tells one that it is possible to set up initial conditions that will cause a sophisticated computation to occur. But it does not tell one what will happen if, for example, one starts from typical simple initial conditions.

Yet the Principle of Computational Equivalence asserts that even in such a case, whenever the behavior one sees is not obviously simple, it will almost always correspond to a computation of equivalent sophistication.

So what this means is that even, say, in cellular automata that start from very simple initial conditions, one can expect that those aspects of their behavior that do not look obviously simple will usually correspond to computations of equivalent sophistication.

According to the Principle of Computational Equivalence therefore it does not matter how simple or complicated either the rules or the initial conditions for a process are: so long as the process itself does not look obviously simple, then it will almost always correspond to a computation of equivalent sophistication.

And what this suggests is that a fundamental unity exists across a vast range of processes in nature and elsewhere: despite all their detailed differences every process can be viewed as corresponding to a computation that is ultimately equivalent in its sophistication.

The Content of the Principle

Like many other fundamental principles in science, the Principle of Computational Equivalence can be viewed in part as a new law of nature, in part as an abstract fact and in part as a definition. For in one sense it tells us what kinds of computations can and cannot happen in our universe, yet it also summarizes purely abstract deductions about possible computations, and provides foundations for more general definitions of the very concept of computation.

Without the Principle of Computational Equivalence one might assume that different systems would always be able to perform completely different computations, and that in particular there would be no upper limit on the sophistication of computations that systems with sufficiently complicated structures would be able to perform.

But the discussion of universality in the previous chapter already suggests that this is not the case. For it implies that at least across the kinds of systems that we considered in that chapter there is in fact an upper limit on the sophistication of computations that can be done.

For as we discussed, once one has a universal system such a system can emulate any of the kinds of systems that we considered—even ones whose construction is more complicated than its own. So this means that whatever kinds of computations can be done by the universal system, none of the other systems will ever be able to do computations that have any higher level of sophistication.

And as a result it has often seemed reasonable to define what one means by a computation as being precisely something that can be done by a universal system of the kind we discussed in the previous chapter.

But despite this, at an abstract level one can always imagine having systems that do computations beyond what any of the cellular automata, Turing machines or other types of systems in the previous chapter can do. For as soon as one identifies any such class of computations, one can imagine setting up a system which includes an infinite table of their results.

But even though one can perfectly well imagine such a system, the Principle of Computational Equivalence makes the assertion that no such system could ever in fact be constructed in our actual universe.

In essence, therefore, the Principle of Computational Equivalence introduces a new law of nature to the effect that no system can ever carry out explicit computations that are more sophisticated than those carried out by systems like cellular automata and Turing machines.

So what might make one think that this is true? One important piece of evidence is the success of the various models of natural systems that I have discussed in this book based on systems like cellular automata. But despite these successes, one might still imagine that other systems could exist in nature that are based, say, on continuous mathematics, and which would allow computations more sophisticated than those in systems like cellular automata to be done.

Needless to say, I do not believe that this is the case, and in fact if one could find a truly fundamental theory of physics along the lines I discussed in Chapter 9 it would actually be possible to establish this with complete certainty. For such a theory would have the feature that it could be emulated by a universal system of the type I discussed in the previous chapter—with the result that nowhere in our universe could computations ever occur that are more sophisticated than those carried out by the universal systems we have discussed.

So what about computations that we perform abstractly with computers or in our brains? Can these perhaps be more sophisticated? Presumably they cannot, at least if we want actual results, and not just generalities. For if a computation is to be carried out explicitly, then it must ultimately be implemented as a physical process, and must therefore be subject to the same limitations as any such process.

But as I discussed in the previous section, beyond asserting that there is an upper limit to computational sophistication, the Principle of Computational Equivalence also makes the much stronger statement that almost all processes except those that are obviously simple actually achieve this limit.

And this is related to what I believe is a very fundamental abstract fact: that among all possible systems with behavior that is not obviously simple an overwhelming fraction are universal.

So what would be involved in establishing this fact?

One could imagine doing much as I did early in this book and successively looking at every possible rule for some type of system like a cellular automaton. And if one did this what one would find is that many of the rules exhibit obviously simple repetitive or nested behavior. But as I discovered early in this book, many also do not, and instead exhibit behavior that is often vastly more complex.

And what the Principle of Computational Equivalence then asserts is that the vast majority of such rules will be universal.

If one starts from scratch then it is not particularly difficult to construct rules—though usually fairly complicated ones—that one knows are universal. And from the result in the previous chapter that rule 110 is universal it follows for example that any rule containing this one must also be universal. But if one is just given an arbitrary rule—

and especially a simple one—then it can be extremely difficult to determine whether or not the rule is universal.

As we discussed in the previous chapter, the usual way to demonstrate that a rule is universal is to find a scheme for setting up initial conditions and for decoding output that makes the rule emulate some other rule that is already known to be universal.

But the problem is that in any particular case there is almost no limit on how complicated such a scheme might need to be. In fact, about the only restriction is that the scheme itself should not exhibit universality just in setting up initial conditions and decoding output.

And indeed it is almost inevitable that the scheme will have to be at least somewhat complicated: for if a system is to be universal then it must be able to emulate any of the huge range of other systems that are universal—with the result that specifying which particular such system it is going to emulate for the purposes of a proof will typically require giving a fair amount of information, all of which must somehow be part of the encoding scheme.

It is often even more difficult to prove that a system is not universal than to prove that it is. For what one needs to show is that no possible scheme can be devised that will allow the system to emulate any other universal system. And usually the only way to be sure of this is to have a more or less complete analysis of all possible behavior that the system can exhibit.

If this behavior always has an obvious repetitive or nested form then it will often be quite straightforward to analyze. But as we saw in Chapter 10, in almost no other case do standard methods of perception and analysis allow one to make much progress at all.

As mentioned in Chapter 10, however, I do know of a few systems based on numbers for which a fairly complete analysis can be given even though the overall behavior is not repetitive or nested or otherwise obviously simple. And no doubt some other examples like this do exist. But it is my strong belief—as embodied in the Principle of Computational Equivalence—that in the end the vast majority of systems whose behavior is not obviously simple will turn out to be universal.

If one tries to use some kind of systematic procedure to test whether systems are universal then inevitably there will be three types of outcomes. Sometimes the procedure will successfully prove that a system is universal, and sometimes it will prove that it is not. But very often the procedure will simply come to no definite conclusion, even after spending a large amount of effort.

Yet in almost all such cases the Principle of Computational Equivalence asserts that the systems are in fact universal. And although almost inevitably it will never be easy to prove this in any great generality, my guess is that, as the decades go by, more and more specific rules will end up being proved to exhibit universality.

But even if one becomes convinced of the abstract fact that out of all possible rules that do not yield obviously simple behavior the vast majority are universal, this still does not quite establish the assertion made by the Principle of Computational Equivalence that rules of this kind that appear in nature and elsewhere are almost always universal.

For it could still be that the particular rules that appear are somehow specially selected to be ones that are not universal. And certainly there are all sorts of situations in which rules are constrained to have behavior that is too simple to support universality. Thus, for example, in most kinds of engineering one tends to pick rules whose behavior is simple enough that one can readily predict it. And as I discussed in Chapter 8, something similar seems to happen with rules in biology that are determined by natural selection.

But when there are no constraints that force simple overall behavior, my guess is that most rules that appear in nature can be viewed as being selected in no special way—save perhaps for the fact that the structure of the rules themselves tends to be fairly simple.

And what this means is that such rules will typically show the same features as rules chosen at random from all possibilities—with the result that presumably they do in the end exhibit universality in almost all cases where their overall behavior is not obviously simple.

But even if a wide range of systems can indeed be shown to be universal this is still not enough to establish the full Principle of Computational Equivalence. For the Principle of Computational

Equivalence is concerned not only with the computational sophistication of complete systems but also with the computational sophistication of specific processes that occur within systems.

And when one says that a particular system is universal what one means is that it is possible by choosing appropriate initial conditions to make the system perform computations of essentially any sophistication. But from this there is no guarantee that the vast majority of initial conditions—including perhaps all those that could readily arise in nature—will not just yield behavior that corresponds only to very simple computations.

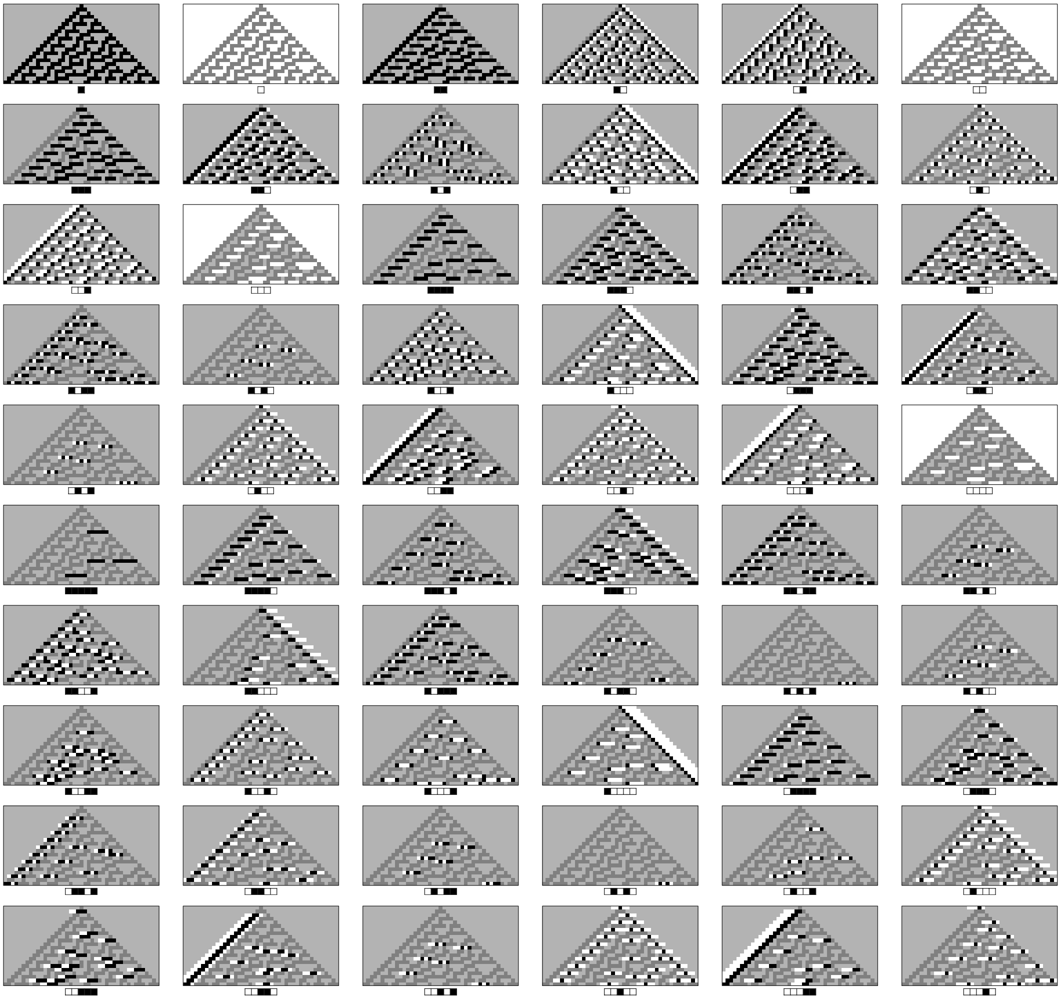
And indeed in the proof of the universality of rule 110 in the previous chapter extremely complicated initial conditions were used to perform even rather simple computations.

But the Principle of Computational Equivalence asserts that in fact even if it comes from simple initial conditions almost all behavior that is not obviously simple will in the end correspond to computations of equivalent sophistication.

And certainly there are all sorts of pictures in this book that lend support to this idea. For over and over again we have seen that simple initial conditions are quite sufficient to produce behavior of immense complexity, and that making the initial conditions more complicated typically does not lead to behavior that looks any different.

Quite often part of the reason for this, as illustrated in the pictures on the facing page, is that even with a single very simple initial condition the actual evolution of a system will generate blocks that correspond to essentially all possible initial conditions. And this means that whatever behavior would be seen with a given overall initial condition, that same behavior will also be seen at appropriate places in the single pattern generated from a specific initial condition.

So this suggests a way of having something analogous to universality in a single pattern instead of in a complete system. The idea would be that a pattern that is universal could serve as a kind of directory of possible computations—with different regions in the pattern giving results for all possible different initial conditions.



Occurrences of progressively longer blocks in the pattern generated by rule 30 starting from a single black cell. So far as I can tell, all possible blocks eventually appear, potentially letting the pattern serve as a kind of directory of all possible computations.

So as a simple example one could imagine having a pattern laid out on a three-dimensional array with each successive vertical plane giving the evolution of some one-dimensional universal system from each of its successive possible initial conditions. And with this setup any computation, regardless of its sophistication, must appear somewhere in the pattern.

In a pattern like the one obtained from rule 30 above different computations are presumably not arranged in any such straightforward way. But I strongly suspect that even though it may be quite impractical to find particular computations that one wants, it is still the case that essentially any possible computation exists somewhere in the pattern.

Much as in the case of universality for complete systems, however, the Principle of Computational Equivalence does not just say that a sophisticated computation will be found somewhere in a pattern produced by a system like rule 30. Rather, it asserts that unless it is obviously simple essentially any behavior that one sees should correspond to a computation of equivalent sophistication.

And in a sense this can be viewed as providing a new way to define the very notion of computation. For it implies that essentially any piece of complex behavior that we see corresponds to a kind of lump of computation that is at some level equivalent.

It is a little like what happens in thermodynamics, where all sorts of complicated microscopic motions are identified as corresponding in some uniform way to a notion of heat.

But computation is both a much more general and much more powerful notion than heat. And as a result, the Principle of Computational Equivalence has vastly richer implications than the laws of thermodynamics—or for that matter, than essentially any single collection of laws in science.

The Validity of the Principle

With the intuition of traditional science the Principle of Computational Equivalence—and particularly many of its implications—might seem almost absurd. But as I have developed more and more new intuition from the discoveries in this book so I have become more and more certain that the Principle of Computational Equivalence must be valid.

But like any principle in science with real content it could in the future always be found that at least some aspect of the Principle of Computational Equivalence is not valid. For as a law of nature the principle could turn out to disagree with what is observed in our universe, while as an abstract fact it could simply represent an incorrect deduction, and even as a definition it could prove not useful or relevant.

But as more and more evidence is accumulated for phenomena that would follow from the principle, so it becomes more and more reasonable to expect that at least in some formulation or another the principle itself must be valid.

As with many fundamental principles the most general statement of the Principle of Computational Equivalence may at first seem quite vague. But almost any specific application of the principle will tend to suggest more specific and precise statements.

Needless to say, it will always be possible to come up with statements that might seem related to the Principle of Computational Equivalence but are not in fact the same. And indeed I suspect this will happen many times over the years to come. For if one tries to use methods from traditional science and mathematics it is almost inevitable that one will be led to statements that are rather different from the actual Principle of Computational Equivalence.

Indeed, my guess is that there is basically no way to formulate an accurate statement of the principle except by using methods from the kind of science introduced in this book. And what this means is that almost any statement that can, for example, readily be investigated by the traditional methods of mathematical proof will tend to be largely irrelevant to the true Principle of Computational Equivalence.

In the course of this book I have made a variety of discoveries that can be interpreted as limited versions of the Principle of Computational Equivalence. And as the years and decades go by, it is my expectation that many more such discoveries will be made. And as these discoveries are absorbed, I suspect that general intuition in science will gradually shift, until in the end the Principle of Computational Equivalence will come to seem almost obvious.

But as of now the principle is far from obvious to most of those whose intuition is derived from traditional science. And as a result all sorts of objections to the principle will no doubt be raised. Some of them will presumably be based on believing that actual systems have less computational sophistication than is implied by the principle, while others will be based on believing that they have more.

But at an underlying level I suspect that the single most common cause of objections will be confusion about various idealizations that are made in traditional models for systems. For even though a system itself may follow the Principle of Computational Equivalence, there is no guarantee that this will also be true of idealizations of the system.

As I discussed at the beginning of Chapter 8, finding a good model for a system is mostly about finding idealizations that are as simple as possible, but that nevertheless still capture the important features of the system. And the point is that in the past there was never a clear idea that computational capabilities of systems might be important, so these were usually not captured correctly when models were made.

Yet one of the characteristics of the kinds of models based on simple programs that I have developed in this book is that they do appear successfully to capture the computational capabilities of a wide range of systems in nature and elsewhere. And in the context of such models what I have discovered is that there is indeed all sorts of evidence for the Principle of Computational Equivalence.

But if one uses the kinds of traditional mathematical models that have in the past been common, things can seem rather different.

For example, many such models idealize systems to the point where their complete behavior can be described just by some simple mathematical formula that relates a few overall numerical quantities. And if one thinks only about this idealization one almost inevitably concludes that the system has very little computational sophistication.

It is also common for traditional mathematical models to suggest too much computational sophistication. For example, as I discussed at the end of Chapter 7, models based on traditional mathematical equations often give constraints on behavior rather than explicit rules for generating behavior.

And if one assumes that actual systems somehow always manage to find ways to satisfy such constraints, one will be led to conclude that these systems must be computationally more sophisticated than any of the universal systems I have discussed—and must thus violate the Principle of Computational Equivalence.

For as I will describe in more detail later in this chapter, an ordinary universal system cannot in any finite number of steps guarantee to be able to tell whether, say, there is any pattern of black and white squares that satisfies some constraint of the type I discussed at the end of Chapter 5. Yet traditional mathematical models often in effect imply that systems in nature can do things like this.

But I explained at the end of Chapter 7 this is presumably just an idealization. For while in simple cases complicated molecules may for example arrange themselves in configurations that minimize energy, the evidence is that in more complicated cases they typically do not. And in fact, what they actually seem to do is instead to explore different configurations by an explicit process of evolution that is quite consistent with the Principle of Computational Equivalence.

One of the features of cellular automata and most of the other computational systems that I have discussed in this book is that they are in some fundamental sense discrete. Yet traditional mathematical models almost always involve continuous quantities. And this has in the past often been taken to imply that systems in nature are able to do computations that are somehow fundamentally more sophisticated than standard computational systems.

But for several reasons I do not believe this conclusion.

For a start, the experience has been that if one actually tries to build analog computers that make use of continuous physical processes they usually end up being less powerful than ordinary digital computers, rather than more so.

And indeed, as I have discussed several times in this book, it is in many cases clear that the whole notion of continuity is just an idealization—although one that happens to be almost required if one wants to make use of traditional mathematical methods.

Fluids provide one obvious example. For usually they are thought of as being described by continuous mathematical equations. But at an underlying level real fluids consist of discrete particles. And this means that whatever the mathematical equations may suggest, the actual ultimate computational capabilities of fluids must be those of a system of discrete particles.

But while it is known that many systems in nature are made up of discrete elements, it is still almost universally believed that there are some things that are fundamentally continuous—notably positions in space and values of quantum mechanical probability amplitudes.

Yet as I discussed in Chapter 9 my strong suspicion is that at a fundamental level absolutely every aspect of our universe will in the end turn out to be discrete. And if this is so, then it immediately implies that there cannot ever ultimately be any form of continuity in our universe that violates the Principle of Computational Equivalence.

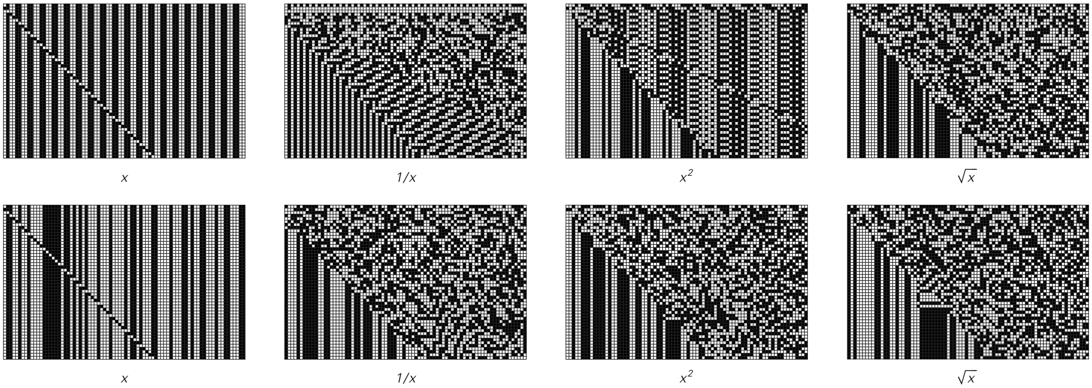
But what if one somehow restricts oneself to a domain where some particular system seems continuous? Can one even at this level perform more sophisticated computations than in a discrete system?

My guess is that for all practical purposes one cannot. Indeed, it is my suspicion that with almost any reasonable set of assumptions even idealized perfectly continuous systems will never in fact be able to perform fundamentally more sophisticated computations.

In a sense the most basic defining characteristic of continuous systems is that they operate on arbitrary continuous numbers. But just to represent every such number in general requires something like an infinite sequence of digits. And so this implies that continuous systems must always in effect be able to operate on infinite sequences.

But in itself this is not particularly remarkable. For even a one-dimensional cellular automaton can be viewed as updating an infinite sequence of cells at every step in its evolution. But one feature of this process is that it is fundamentally local: each cell behaves in a way that is determined purely by cells in a local neighborhood around it.

Yet even the most basic arithmetic operations on continuous numbers typically involve significant non-locality. Thus, for example, when one adds two numbers together there can be carries in the digit sequence that propagate arbitrarily far. And if one computes even a function like $1/x$ almost any digit in x will typically have an effect on almost any digit in the result, as the pictures on the facing page indicate.



Results from mathematical operations on numbers with similar digit sequences. Each successive line in each picture gives the digit sequence obtained by using a value of x in which one successive digit has been reversed. The top row of pictures start from the repetitive base 2 digit sequence of $x = 3/5$; the bottom row of pictures from $x = \pi/4$. The lack of coherence between successive digit sequences in each picture reflects the non-locality of mathematical operations when applied to digit sequences.

But can this detailed kind of phenomenon really be used as the basis for doing fundamentally more sophisticated computations? To compare the general computational capabilities of continuous and discrete systems one needs to find some basic scheme for constructing inputs and decoding outputs that one can use in both types of systems. And the most obvious and practical approach is to require that this always be done by finite discrete processes.

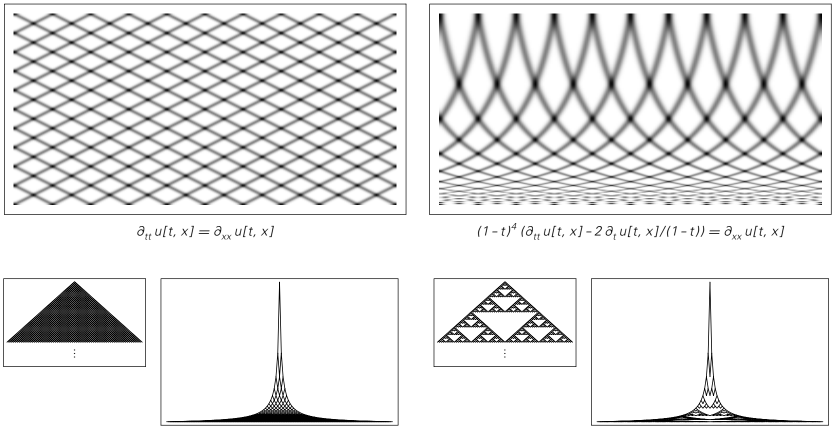
But at least in this case it seems fairly clear that none of the simple functions shown above can for example ever lead to results that go beyond ones that could readily be generated by the evolution of ordinary discrete systems. And the same is presumably true if one works with essentially any of what are normally considered standard mathematical functions. But what happens if one assumes that one can set up a system that not only finds values of such functions but also finds solutions to arbitrary equations involving them?

With pure polynomial equations one can deduce from results in algebra that no fundamentally more sophisticated computations become possible. But as soon as one even allows trigonometric functions, for example, it turns out that it becomes possible to construct equations for which finding a solution is equivalent to finding the outcome of an infinite number of steps in the evolution of a system like a cellular automaton.

And while these particular types of equations have never seriously been proposed as idealizations of actual processes in nature or elsewhere, it turns out that a related phenomenon can presumably occur in differential equations—which represent the most common basis for mathematical models in most areas of traditional science.

Differential equations of the kind we discussed at the end of Chapter 4 work at some level a little like cellular automata. For given the state of a system, they provide rules for determining its state at subsequent times. But whereas cellular automata always evolve only in discrete steps, differential equations instead go through a continuous process of evolution in which time appears just as a parameter.

And by making simple algebraic changes to the way that time enters a differential equation one can often arrange, as in the pictures below, that processes that would normally take an infinite time will actually always occur over only a finite time.



$\partial_{tt} u[t, x] = \partial_{xx} u[t, x]$

$(1 - t)^4 (\partial_{tt} u[t, x] - 2\, \partial_t u[t, x]/(1 - t)) = \partial_{xx} u[t, x]$

Indications of how an infinite amount of computational work can in principle be performed in a finite time in continuous systems like partial differential equations. The top left picture shows a solution to the wave equation. The top right picture shows a solution to an equation obtained from the wave equation by transforming the time variable according to $t \to 1 - 1/t$. The bottom row shows what the same transformation does to patterns of the kind that are generated by simple cellular automata. It is presumably possible to construct partial differential equations that give both the original and transformed versions of these patterns.

So if such processes can correspond to the evolution of systems like cellular automata, then it follows at least formally that differential equations should be able to do in finite time computations that would take a discrete system like a cellular automaton an infinite time to do.

But just as it is difficult to make an analog computer faithfully reproduce many steps in a discrete computation, so also it seems likely that it will be difficult to set up differential equations that for arbitrarily long times successfully manage to emulate the precise behavior of systems like cellular automata. And in fact my suspicion is that to make this work will require taking limits that are rather similar to following the evolution of the differential equations for an infinite time.

So my guess is that even within the formalism of traditional continuous mathematics realistic idealizations of actual processes will never ultimately be able to perform computations that are more sophisticated than the Principle of Computational Equivalence implies.

But what about the process of human thinking? Does it also follow the Principle of Computational Equivalence? Or does it somehow manage to do computations that are more sophisticated than the Principle of Computational Equivalence implies?

There is a great tendency for us to assume that there must be something extremely sophisticated about human thinking. And certainly the fact that present-day computer systems do not emulate even some of its most obvious features might seem to support this view. But as I discussed in Chapter 10, particularly following the discoveries in this book, it is my strong belief that the basic mechanisms of human thinking will in the end turn out to correspond to rather simple computational processes.

So what all of this suggests is that systems in nature do not perform computations that are more sophisticated than the Principle of Computational Equivalence allows. But on its own this is not enough to establish the complete Principle of Computational Equivalence. For the principle also implies a lower limit on computational sophistication—making the assertion that almost any process that is not obviously simple will tend to be equivalent in its computational sophistication.

And one of the consequences of this is that it implies that most systems whose behavior seems complex should be universal. Yet as of now we only know for certain about fairly few systems that are universal, albeit including ones like rule 110 that have remarkably simple rules. And no doubt the objection will be raised that other systems whose behavior seems complex may not in fact be universal.

In particular, it might be thought that the behavior of systems like rule 30—while obviously at least somewhat computationally sophisticated—might somehow be too random to be harnessed to allow complete universality. And although in Chapter 11 I did give a few pieces of evidence that point towards rule 30 being universal, there can still be doubts until this has been proved for certain.

And in fact there is a particularly abstruse result in mathematical logic that might be thought to show that systems can exist that exhibit some features of arbitrarily sophisticated computation, but which are nevertheless not universal. For in the late 1950s a whole hierarchy of systems with so-called intermediate degrees were constructed with the property that questions about the ultimate output from their evolution could not in general be answered by finite computation, but for which the actual form of this output was not flexible enough to be able to emulate a full range of other systems, and thus support universality.

But when one examines the known examples of such systems—all of which have very intricate underlying rules—one finds that even though the particular part of their behavior that is identified as output is sufficiently restricted to avoid universality, almost every other part of their behavior nevertheless does exhibit universality—just as one would expect from the Principle of Computational Equivalence.

So why else might systems like rule 30 fail to be universal? We know from Chapter 11 that systems whose behavior is purely repetitive or purely nested cannot be universal. And so we might wonder whether perhaps some other form of regularity could be present that would prevent systems like rule 30 from being universal.

When we look at the patterns produced by such systems they certainly do not seem to have any great regularity; indeed in most respects they seem far more random than patterns produced by systems like rule 110 that we already know are universal.

But how can we be sure that we are not being misled by limitations in our powers of perception and analysis—and that an extraterrestrial intelligence, for example, might not immediately recognize regularity that would show that universality is impossible?

For as we saw in Chapter 10 the methods of perception and analysis that we normally use cannot detect any form of regularity much beyond repetition or at most nesting. So this means that even if some higher form of regularity is in fact present, we as humans might never be able to tell.

In the history of science and mathematics both repetition and nesting feature prominently. And if there was some common higher form of regularity its discovery would no doubt lead to all sorts of important new advances in science and mathematics.

And when I first started looking at systems like cellular automata I in effect implicitly assumed that some such form of regularity must exist. For I was quite certain that even though I saw behavior that seemed to me complex the simplicity of the underlying rules must somehow ultimately lead to great regularity in it.

But as the years have gone by—and as I have investigated more and more systems and tried more and more methods of analysis—I have gradually come to the conclusion that there is no hidden regularity in any large class of systems, and that instead what the Principle of Computational Equivalence suggests is correct: that beyond systems with obvious regularities like repetition and nesting most systems are universal, and are equivalent in their computational sophistication.

Explaining the Phenomenon of Complexity

Early in this book I described the remarkable discovery that even systems with extremely simple underlying rules can produce behavior that seems to us immensely complex. And in the course of this book, I have shown a great many examples of this phenomenon, and have argued that it is responsible for much of the complexity we see in nature and elsewhere.

Yet so far I have given no fundamental explanation for the phenomenon. But now, by making use of the Principle of Computational Equivalence, I am finally able to do this.

And the crucial point is to think of comparing the computational sophistication of systems that we study with the computational sophistication of the systems that we use to study them.

At first we might assume that our brains and mathematical methods would always be capable of vastly greater computational sophistication than systems based on simple rules—and that as a result the behavior of such systems would inevitably seem to us fairly simple.

But the Principle of Computational Equivalence implies that this is not the case. For it asserts that essentially any processes that are not obviously simple are equivalent in their computational sophistication. So this means that even though a system may have simple underlying rules its process of evolution can still computationally be just as sophisticated as any of the processes we use for perception and analysis.

And this is the fundamental reason that systems with simple rules are able to show behavior that seems to us complex.

At first, one might think that this explanation would depend on the particular methods of perception and analysis that we as humans happen to use. But one of the consequences of the Principle of Computational Equivalence is that it does not. For the principle asserts that the same computational equivalence exists for absolutely any method of perception and analysis that can actually be used.

In traditional science the idealization is usually made that perception and analysis are in a sense infinitely powerful, so that they need not be taken into account when one draws conclusions about a system. But as soon as one tries to deal with systems whose behavior is anything but fairly simple one finds that this idealization breaks down, and it becomes necessary to consider perception and analysis as explicit processes in their own right.

If one studies systems in nature it is inevitable that both the evolution of the systems themselves and the methods of perception and analysis used to study them must be processes based on natural laws. But at least in the recent history of science it has normally been assumed that the evolution of typical systems in nature is somehow much less sophisticated a process than perception and analysis.

Yet what the Principle of Computational Equivalence now asserts is that this is not the case, and that once a rather low threshold has been reached, any real system must exhibit essentially the same level of computational sophistication. So this means that observers will tend to be computationally equivalent to the systems they observe—with the inevitable consequence that they will consider the behavior of such systems complex.

So in the end the fact that we see so much complexity can be attributed quite directly to the Principle of Computational Equivalence, and to the fact that so many of the systems we encounter in practice turn out to be computationally equivalent.

Computational Irreducibility

When viewed in computational terms most of the great historical triumphs of theoretical science turn out to be remarkably similar in their basic character. For at some level almost all of them are based on finding ways to reduce the amount of computational work that has to be done in order to predict how some particular system will behave.

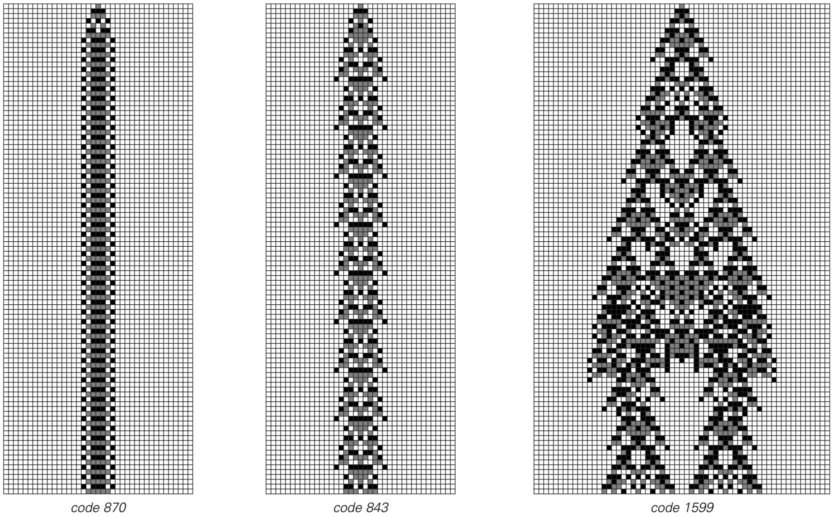
Most of the time the idea is to derive a mathematical formula that allows one to determine what the outcome of the evolution of the system will be without explicitly having to trace its steps.

And thus, for example, an early triumph of theoretical science was the derivation of a formula for the position of a single idealized planet orbiting a star. For given this formula one can just plug in numbers to work out where the planet will be at any point in the future, without ever explicitly having to trace the steps in its motion.

But part of what started my whole effort to develop the new kind of science in this book was the realization that there are many common systems for which no traditional mathematical formulas have ever been found that readily describe their overall behavior.

At first one might have thought this must be some kind of temporary issue, that could be overcome with sufficient cleverness. But from the discoveries in this book I have come to the conclusion that in fact it is not, and that instead it is one of the consequences of a very fundamental phenomenon that follows from the Principle of Computational Equivalence and that I call computational irreducibility.

If one views the evolution of a system as a computation, then each step in this evolution can be thought of as taking a certain amount of computational effort on the part of the system. But what traditional theoretical science in a sense implicitly relies on is that much of this effort is somehow unnecessary—and that in fact it should be possible to find the outcome of the evolution with much less effort.



Examples of computational reducibility and irreducibility in the evolution of cellular automata. The first two rules yield simple repetitive computationally reducible behavior in which the outcome after many steps can readily be deduced without tracing each step. The third rule yields behavior that appears to be computationally irreducible, so that its outcome can effectively be found only by explicitly tracing each step. The cellular automata shown here all have 3-color totalistic rules.

And certainly in the first two examples above this is the case. For just as with the orbit of an idealized planet there is in effect a straightforward formula that gives the state of each system after any number of steps. So even though the systems themselves generate their behavior by going through a whole sequence of steps, we can readily shortcut this process and find the outcome with much less effort.

But what about the third example on the facing page? What does it take to find the outcome in this case? It is always possible to do an experiment and explicitly run the system for a certain number of steps and see how it behaves. But to have any kind of traditional theory one must find a shortcut that involves much less computation.

Yet from the picture on the facing page it is certainly not obvious how one might do this. And looking at the pictures on the next page it begins to seem quite implausible that there could ever in fact be any way to find a significant shortcut in the evolution of this system.

So while the behavior of the first two systems on the facing page is readily seen to be computationally reducible, the behavior of the third system appears instead to be computationally irreducible.

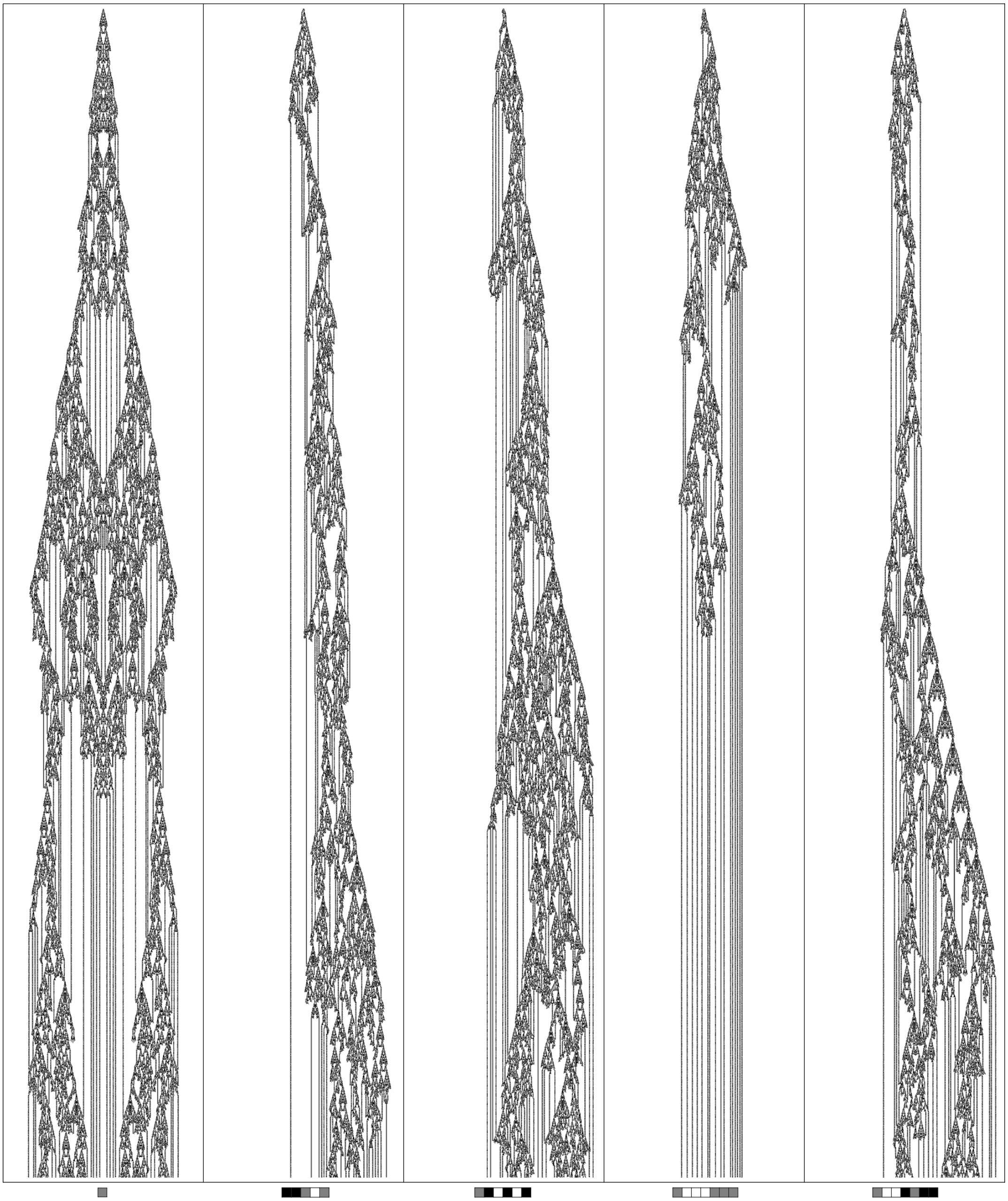
In traditional science it has usually been assumed that if one can succeed in finding definite underlying rules for a system then this means that ultimately there will always be a fairly easy way to predict how the system will behave.

Several decades ago chaos theory pointed out that to have enough information to make complete predictions one must in general know not only the rules for a system but also its complete initial conditions.

But now computational irreducibility leads to a much more fundamental problem with prediction. For it implies that even if in principle one has all the information one needs to work out how some particular system will behave, it can still take an irreducible amount of computational work actually to do this.

Indeed, whenever computational irreducibility exists in a system it means that in effect there can be no way to predict how the system will behave except by going through almost as many steps of computation as the evolution of the system itself.

In traditional science it has rarely even been recognized that there is a need to consider how systems that are used to make predictions actually operate. But what leads to the phenomenon of computational irreducibility is that there is in fact always a fundamental competition



5000 steps in the evolution of the third system from the previous page, starting from several initial conditions. The complexity of the behavior makes it seem inconceivable that there could ever be a procedure that would always immediately find its outcome.

between systems used to make predictions and systems whose behavior one tries to predict.

For if meaningful general predictions are to be possible, it must at some level be the case that the system making the predictions be able to outrun the system it is trying to predict. But for this to happen the system making the predictions must be able to perform more sophisticated computations than the system it is trying to predict.

In traditional science there has never seemed to be much problem with this. For it has normally been implicitly assumed that with our powers of mathematics and general thinking the computations we use to make predictions must be almost infinitely more sophisticated than those that occur in most systems in nature and elsewhere whose behavior we try to predict.

But the remarkable assertion that the Principle of Computational Equivalence makes is that this assumption is not correct, and that in fact almost any system whose behavior is not obviously simple performs computations that are in the end exactly equivalent in their sophistication.

So what this means is that systems one uses to make predictions cannot be expected to do computations that are any more sophisticated than the computations that occur in all sorts of systems whose behavior we might try to predict. And from this it follows that for many systems no systematic prediction can be done, so that there is no general way to shortcut their process of evolution, and as a result their behavior must be considered computationally irreducible.

If the behavior of a system is obviously simple—and is say either repetitive or nested—then it will always be computationally reducible. But it follows from the Principle of Computational Equivalence that in practically all other cases it will be computationally irreducible.

And this, I believe, is the fundamental reason that traditional theoretical science has never managed to get far in studying most types of systems whose behavior is not ultimately quite simple.

For the point is that at an underlying level this kind of science has always tried to rely on computational reducibility. And for example its whole idea of using mathematical formulas to describe behavior makes sense only when the behavior is computationally reducible.

So when computational irreducibility is present it is inevitable that the usual methods of traditional theoretical science will not work. And indeed I suspect the only reason that their failure has not been more obvious in the past is that theoretical science has typically tended to define its domain specifically in order to avoid phenomena that do not happen to be simple enough to be computationally reducible.

But one of the major features of the new kind of science that I have developed is that it does not have to make any such restriction. And indeed many of the systems that I study in this book are no doubt computationally irreducible. And that is why—unlike most traditional works of theoretical science—this book has very few mathematical formulas but a great many explicit pictures of the evolution of systems.

It has in the past couple of decades become increasingly common in practice to study systems by doing explicit computer simulations of their behavior. But normally it has been assumed that such simulations are ultimately just a convenient way to do what could otherwise be done with mathematical formulas.

But what my discoveries about computational irreducibility now imply is that this is not in fact the case, and that instead there are many common systems whose behavior cannot in the end be determined at all except by something like an explicit simulation.

Knowing that universal systems exist already tells one that this must be true at least in some situations. For consider trying to outrun the evolution of a universal system. Since such a system can emulate any system, it can in particular emulate any system that is trying to outrun it. And from this it follows that nothing can systematically outrun the universal system. For any system that could would in effect also have to be able to outrun itself.

But before the discoveries in this book one might have thought that this could be of little practical relevance. For it was believed that except among specially constructed systems universality was rare. And it was also assumed that even when universality was present, very special initial conditions would be needed if one was ever going to perform computations at anything like the level of sophistication involved in most methods of prediction.

But the Principle of Computational Equivalence asserts that this is not the case, and that in fact almost any system whose behavior is not obviously simple will exhibit universality and will perform sophisticated computations even with typical simple initial conditions.

So the result is that computational irreducibility can in the end be expected to be common, so that it should indeed be effectively impossible to outrun the evolution of all sorts of systems.

One slightly subtle issue in thinking about computational irreducibility is that given absolutely any system one can always at least nominally imagine speeding up its evolution by setting up a rule that for example just executes several steps of evolution at once.

But insofar as such a rule is itself more complicated it may in the end achieve no real reduction in computational effort. And what is more important, it turns out that when there is true computational reducibility its effect is usually much more dramatic.

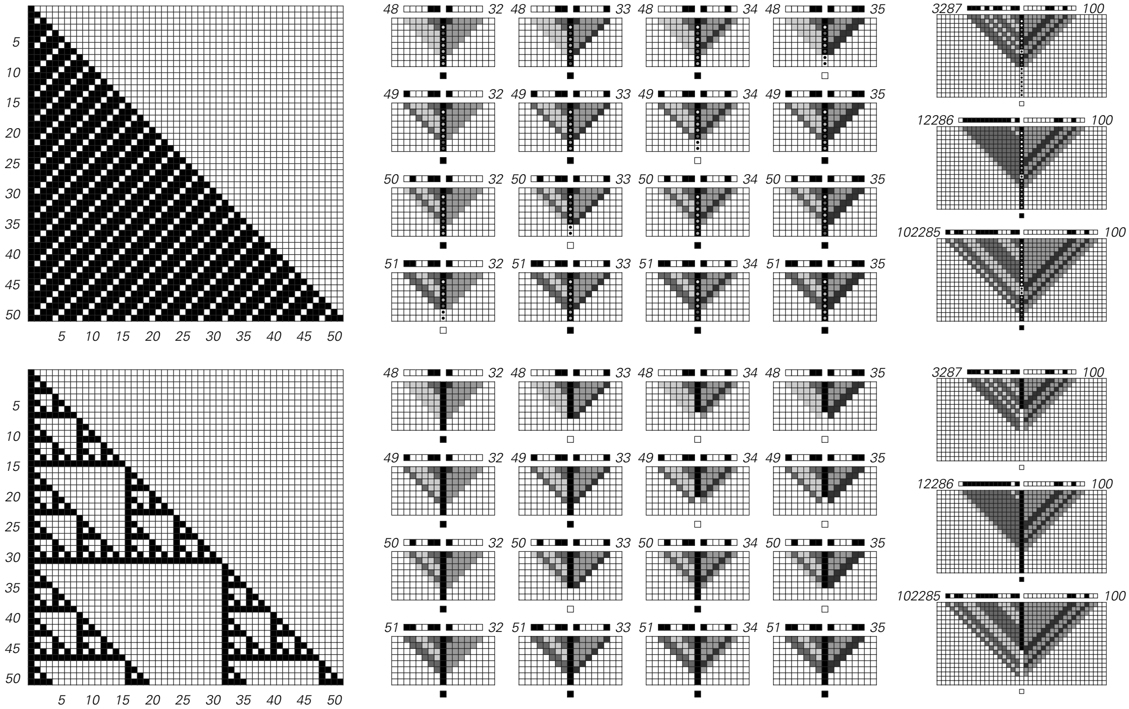
The pictures on the next page show typical examples based on cellular automata that exhibit repetitive and nested behavior. In the patterns on the left the color of each cell at any given step is in effect found by tracing the explicit evolution of the cellular automaton up to that step. But in the pictures on the right the results for particular cells are instead found by procedures that take much less computational effort.

These procedures are again based on cellular automata. But now what the cellular automata do is to take specifications of positions of cells, and then in effect compute directly from these the colors of cells.

The way things are set up the initial conditions for these cellular automata consist of digit sequences of numbers that give positions. The color of a particular cell is then found by evolving for a number of steps equal to the length of these input digit sequences.

And this means for example that the outcome of a million steps of evolution for either of the cellular automata on the left is now determined by just 20 steps of evolution, where 20 is the length of the base 2 digit sequence of the number 1,000,000.

And this turns out to be quite similar to what happens with typical mathematical formulas in traditional theoretical science. For the point of such formulas is usually to allow one to give a number as



Examples of computational reducibility in action. The pictures on the left show patterns produced by the ordinary evolution of cellular automata with elementary rules 188 and 60. The pictures on the right show how colors of particular cells in these patterns can be found with much less computational effort. In each case the position of a cell is specified by a pair of numbers given as base 2 digit sequences in the initial conditions for a cellular automaton. The evolution of the cellular automaton then quickly determines what the color of the cell at that position in the pattern on the left will be. For rule 188 the cellular automaton that does this involves 12 colors; for rule 60 it involves 6. In general, to find the color of a cell after *t* steps of rule 188 or rule 60 evolution takes about *Log[2, t]* steps. Compare page 608.

input, and then to compute directly something that corresponds, say, to the outcome of that number of steps in the evolution of a system.

In traditional mathematics it is normally assumed that once one has an explicit formula involving standard mathematical functions then one can in effect always evaluate this formula immediately.

But evaluating a formula—like anything else—is a computational process. And unless some digits effectively never matter, this process cannot normally take less steps than there are digits in its input.

Indeed, it could in principle be that the process could take a number of steps proportional to the numerical value of its input. But if this were so, then it would mean that evaluating the formula would require as much effort as just tracing each step in the original process whose outcome the formula was supposed to give.

And the crucial point that turns out to be the basis for much of the success of traditional theoretical science is that in fact most standard mathematical functions can be evaluated in a number of steps that is far smaller than the numerical value of their input, and that instead normally grows only slowly with the length of the digit sequence of their input.

So the result of this is that if there is a traditional mathematical formula for the outcome of a process then almost always this means that the process must show great computational reducibility.

In practice, however, the vast majority of cases for which traditional mathematical formulas are known involve behavior that is ultimately either uniform or repetitive. And indeed, as we saw in Chapter 10, if one uses just standard mathematical functions then it is rather difficult even to reproduce many simple examples of nesting.

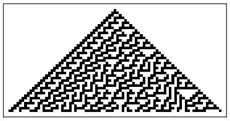
But as the pictures on the facing page and in Chapter 10 illustrate, if one allows more general kinds of underlying rules then it becomes quite straightforward to set up procedures that with very little computational effort can find the color of any element in any nested pattern.

So what about more complex patterns, like the rule 30 cellular automaton pattern at the bottom of the page?

When I first generated such patterns I spent a huge amount of time trying to analyze them and trying to find a procedure that would allow me to compute directly the color of each cell. And indeed it was the fact that I was never able to make much progress in doing this that first led me to consider the possibility that there could be a phenomenon like computational irreducibility.

And now, what the Principle of Computational Equivalence implies is that in fact almost any system whose behavior is not obviously simple will tend to exhibit computational irreducibility.

But particularly when the underlying rules are simple there is often still some superficial computational reducibility. And so, for example, in the rule 30 pattern on the right one can tell whether a cell at a given position has any chance of not being white just by doing a



An example of a pattern where it is difficult to compute directly the color of a particular cell.

very short computation that tests whether that position lies outside the center triangular region of the pattern. And in a class 4 cellular automaton such as rule 110 one can readily shortcut the process of evolution for at least a limited number of steps in places where there happen to be only a few well-separated localized structures present.

And indeed in general almost any regularities that we manage to recognize in the behavior of a system will tend to reflect some kind of computational reducibility in this behavior.

If one views the pattern of behavior as a piece of data, then as we discussed in Chapter 10 regularities in it allow a compressed description to be found. But the existence of a compressed description does not on its own imply computational reducibility. For any system that has simple rules and simple initial conditions—including for example rule 30—will always have such a description.

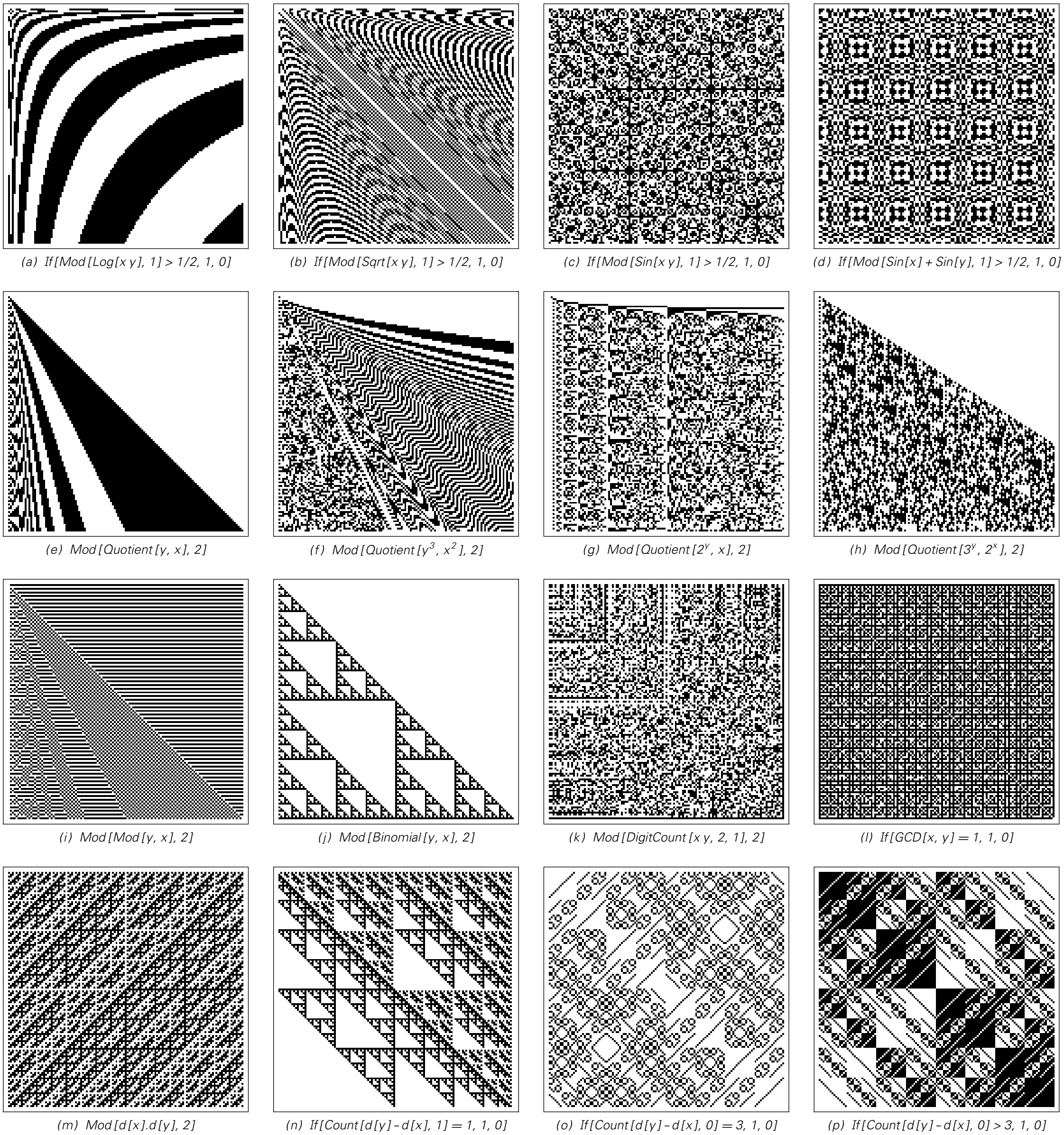
But what makes there be computational reducibility is when only a short computation is needed to find from the compressed description any feature of the actual behavior.

And it turns out that the kinds of compressed descriptions that can be obtained by the methods of perception and analysis that we use in practice and that we discussed in Chapter 10 all essentially have this property. So this is why regularities that we recognize by these methods do indeed reflect the presence of computational reducibility.

But as we saw in Chapter 10, in almost any case where there is not just repetitive or nested behavior, our normal powers of perception and analysis recognize very few regularities—even though at some level the behavior we see may still be generated by extremely simple rules.

And this supports the assertion that beyond perhaps some small superficial amount of computational reducibility a great many systems are in the end computationally irreducible. And indeed this assertion explains, at least in part, why our methods of perception and analysis cannot be expected to go further in recognizing regularities.

But if behavior that we see looks complex to us, does this necessarily mean that it can exhibit no computational reducibility? One way to try to get an idea about this is just to construct patterns



(a) If[Mod[Log[x y], 1] > 1/2, 1, 0]

(b) If[Mod[Sqrt[x y], 1] > 1/2, 1, 0]

(c) If[Mod[Sin[x y], 1] > 1/2, 1, 0]

(d) If[Mod[Sin[x] + Sin[y], 1] > 1/2, 1, 0]

(e) Mod[Quotient[y, x], 2]

(f) Mod[Quotient[y^3, x^2], 2]

(g) Mod[Quotient[2^y, x], 2]

(h) Mod[Quotient[3^y, 2^x], 2]

(i) Mod[Mod[y, x], 2]

(j) Mod[Binomial[y, x], 2]

(k) Mod[DigitCount[x y, 2, 1], 2]

(l) If[GCD[x, y] = 1, 1, 0]

(m) Mod[d[x].d[y], 2]

(n) If[Count[d[y] - d[x], 1] = 1, 1, 0]

(o) If[Count[d[y] - d[x], 0] = 3, 1, 0]

(p) If[Count[d[y] - d[x], 0] > 3, 1, 0]

Examples of patterns set up so that a short computation can be used to determine the color of each cell from the numbers representing its position. Most such patterns look to us quite simple, but the examples shown here were specifically chosen to be ones that look more complicated. In most of them fairly standard mathematical functions are used, but in unusual combinations. In every picture both *x* and *y* run from 1 to 127. *d[n]* stands for *IntegerDigits[n, 2, 7]*. (h) is equivalent to digit sequences of powers of 3 in base 2 (see page 120). (j) is essentially Pascal's triangle (see page 611). (l) was discussed on page 613. (m) is a nested pattern seen on page 583. The only pattern that is known to be obtainable by evolving down the page according to a simple local rule is (j), which corresponds to the rule 60 elementary cellular automaton.

where we explicitly set up the color of each cell to be determined by some short computation from the numbers that represent its position.

When we look at such patterns most of them appear to us quite simple. But as the pictures on the previous page demonstrate, it turns out to be possible to find examples where this is not so, and where instead the patterns appear to us at least somewhat complex.

But for such patterns to yield meaningful examples of computational reducibility it must also be possible to produce them by some process of evolution—say by repeated application of a cellular automaton rule. Yet for the majority of cases shown here there is at least no obvious way to do this.

I have however found one class of systems—already mentioned in Chapter 10—whose behavior does not appear simple, but nevertheless turns out to be computationally reducible, as in the pictures on the facing page. However, I strongly suspect that systems like this are very rare, and that in the vast majority of cases where the behavior that we see in nature and elsewhere appears to us complex it is in the end indeed associated with computational irreducibility.

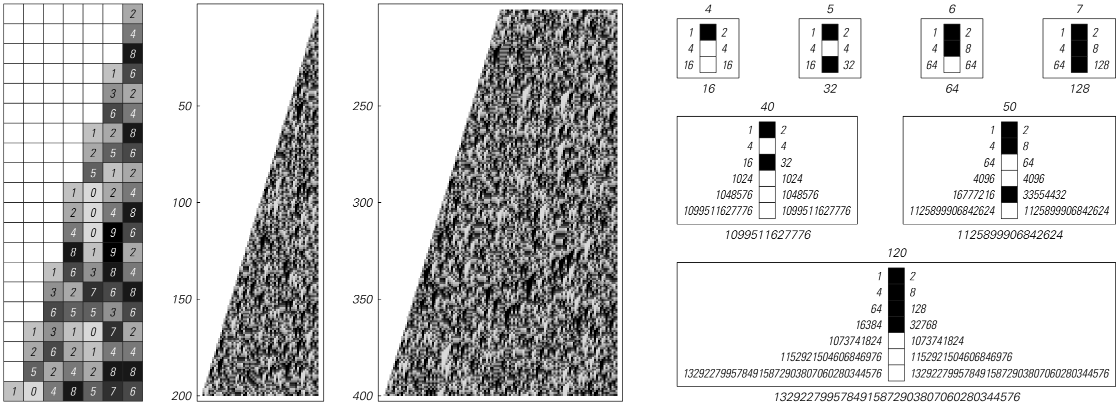
So what does this mean for science?

In the past it has normally been assumed that there is no ultimate limit on what science can be expected to do. And certainly the progress of science in recent centuries has been so impressive that it has become common to think that eventually it should yield an easy theory—perhaps a mathematical formula—for almost anything.

But the discovery of computational irreducibility now implies that this can fundamentally never happen, and that in fact there can be no easy theory for almost any behavior that seems to us complex.

It is not that one cannot find underlying rules for such behavior. Indeed, as I have argued in this book, particularly when they are formulated in terms of programs I suspect that such rules are often extremely simple. But the point is that to deduce the consequences of these rules can require irreducible amounts of computational effort.

One can always in effect do an experiment, and just watch the actual behavior of whatever system one wants to study. But what one



A system whose behavior looks complex but still turns out to be computationally reducible. The system is a cellular automaton with 10 possible colors for each cell. But it can also be viewed as a system based on numbers, in which successive rows are the base 10 digit sequences of successive powers of 2. And it turns out that there is a fast way to compute row *n* just from the base 2 digit sequence of *n*, as the pictures on the right illustrate. This procedure is based on the standard repeated squaring method of finding 2^n by starting from 2, and then successively squaring the numbers one gets, multiplying by 2 if the corresponding base 2 digit in *n* is 1. Using this procedure one can certainly compute the color of any cell on row *n* by doing about *n Log[n]*^{3} operations—instead of the n^2 needed if one carried out the cellular automaton evolution explicitly.

cannot in general do is to find an easy theory that will tell one without much effort what every aspect of this behavior will be.

So given this, can theoretical science still be useful at all?

The answer is definitely yes. For even in its most traditional form it can often deal quite well with those aspects of behavior that happen to be simple enough to be computationally reducible. And since one can never know in advance how far computational reducibility will go in a particular system it is always worthwhile at least to try applying the traditional methods of theoretical science.

But ultimately if computational irreducibility is present then these methods will fail. Yet there are still often many reasons to want to use abstract theoretical models rather than just doing experiments on actual systems in nature and elsewhere. And as the results in this book suggest, by using the right kinds of models much can be achieved.

Any accurate model for a system that exhibits computational irreducibility must at some level inevitably involve computations that are as sophisticated as those in the system itself. But as I have shown in this book even systems with very simple underlying rules can still perform computations that are as sophisticated as in any system.

And what this means is that to capture the essential features even of systems with very complex behavior it can be sufficient to use models that have an extremely simple basic structure. Given these models the only way to find out what they do will usually be just to run them. But the point is that if the structure of the models is simple enough, and fits in well enough with what can be implemented efficiently on a practical computer, then it will often still be perfectly possible to find out many consequences of the model.

And that, in a sense, is what much of this book has been about.

The Phenomenon of Free Will

Ever since antiquity it has been a great mystery how the universe can follow definite laws while we as humans still often manage to make decisions about how to act in ways that seem quite free of obvious laws.

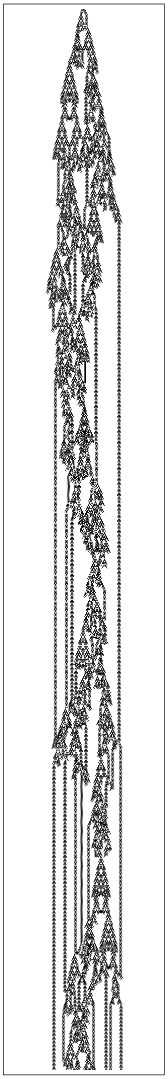
But from the discoveries in this book it finally now seems possible to give an explanation for this. And the key, I believe, is the phenomenon of computational irreducibility.

For what this phenomenon implies is that even though a system may follow definite underlying laws its overall behavior can still have aspects that fundamentally cannot be described by reasonable laws.

For if the evolution of a system corresponds to an irreducible computation then this means that the only way to work out how the system will behave is essentially to perform this computation—with the result that there can fundamentally be no laws that allow one to work out the behavior more directly.

And it is this, I believe, that is the ultimate origin of the apparent freedom of human will. For even though all the components of our brains presumably follow definite laws, I strongly suspect that their overall behavior corresponds to an irreducible computation whose outcome can never in effect be found by reasonable laws.

And indeed one can already see very much the same kind of thing going on in a simple system like the cellular automaton on the left. For



A cellular automaton whose behavior seems to show an analog of free will. Even though its underlying laws are definite—and simple—the behavior is complicated enough that many aspects of it seem to follow no definite laws. (The rule used is the same as on page 740.)

even though the underlying laws for this system are perfectly definite, its overall behavior ends up being sufficiently complicated that many aspects of it seem to follow no obvious laws at all.

And indeed if one were to talk about how the cellular automaton seems to behave one might well say that it just decides to do this or that—thereby effectively attributing to it some sort of free will.

But can this possibly be reasonable? For if one looks at the individual cells in the cellular automaton one can plainly see that they just follow definite rules, with absolutely no freedom at all.

But at some level the same is probably true of the individual nerve cells in our brains. Yet somehow as a whole our brains still manage to behave with a certain apparent freedom.

Traditional science has made it very difficult to understand how this can possibly happen. For normally it has assumed that if one can only find the underlying rules for the components of a system then in a sense these tell one everything important about the system.

But what we have seen over and over again in this book is that this is not even close to correct, and that in fact there can be vastly more to the behavior of a system than one could ever foresee just by looking at its underlying rules. And fundamentally this is a consequence of the phenomenon of computational irreducibility.

For if a system is computationally irreducible this means that there is in effect a tangible separation between the underlying rules for the system and its overall behavior associated with the irreducible amount of computational work needed to go from one to the other.

And it is in this separation, I believe, that the basic origin of the apparent freedom we see in all sorts of systems lies—whether those systems are abstract cellular automata or actual living brains.

But so in the end what makes us think that there is freedom in what a system does? In practice the main criterion seems to be that we cannot readily make predictions about the behavior of the system.

For certainly if we could, then this would show us that the behavior must be determined in a definite way, and so cannot be free. But at least with our normal methods of perception and analysis one typically needs rather simple behavior for us actually to be able to identify overall rules that let us make reasonable predictions about it.

Yet in fact even in living organisms such behavior is quite common. And for example particularly in lower animals there are all sorts of cases where very simple and predictable responses to stimuli are seen. But the point is that these are normally just considered to be unavoidable reflexes that leave no room for decisions or freedom.

Yet as soon as the behavior we see becomes more complex we quickly tend to imagine that it must be associated with some kind of underlying freedom. For at least with traditional intuition it has always seemed quite implausible that any real unpredictability could arise in a system that just follows definite underlying rules.

And so to explain the behavior that we as humans exhibit it has often been assumed that there must be something fundamentally more going on—and perhaps something unique to humans.

In the past the most common belief has been that there must be some form of external influence from fate—associated perhaps with the intervention of a supernatural being or perhaps with configurations of celestial bodies. And in more recent times sensitivity to initial conditions and quantum randomness have been proposed as more appropriate scientific explanations.

But much as in our discussion of randomness in Chapter 6 nothing like this is actually needed. For as we have seen many times in this book even systems with quite simple and definite underlying rules can produce behavior so complex that it seems free of obvious rules.

And the crucial point is that this happens just through the intrinsic evolution of the system—without the need for any additional input from outside or from any sort of explicit source of randomness.

And I believe that it is this kind of intrinsic process—that we now know occurs in a vast range of systems—that is primarily responsible for the apparent freedom in the operation of our brains.

But this is not to say that everything that goes on in our brains has an intrinsic origin. Indeed, as a practical matter what usually seems to happen is that we receive external input that leads to some train of thought which continues for a while, but then dies out until we get more input. And often the actual form of this train of thought is influenced by memory we have developed from inputs in the past—making it not necessarily repeatable even with exactly the same input.

But it seems likely that the individual steps in each train of thought follow quite definite underlying rules. And the crucial point is then that I suspect that the computation performed by applying these rules is often sophisticated enough to be computationally irreducible—with the result that it must intrinsically produce behavior that seems to us free of obvious laws.

Undecidability and Intractability

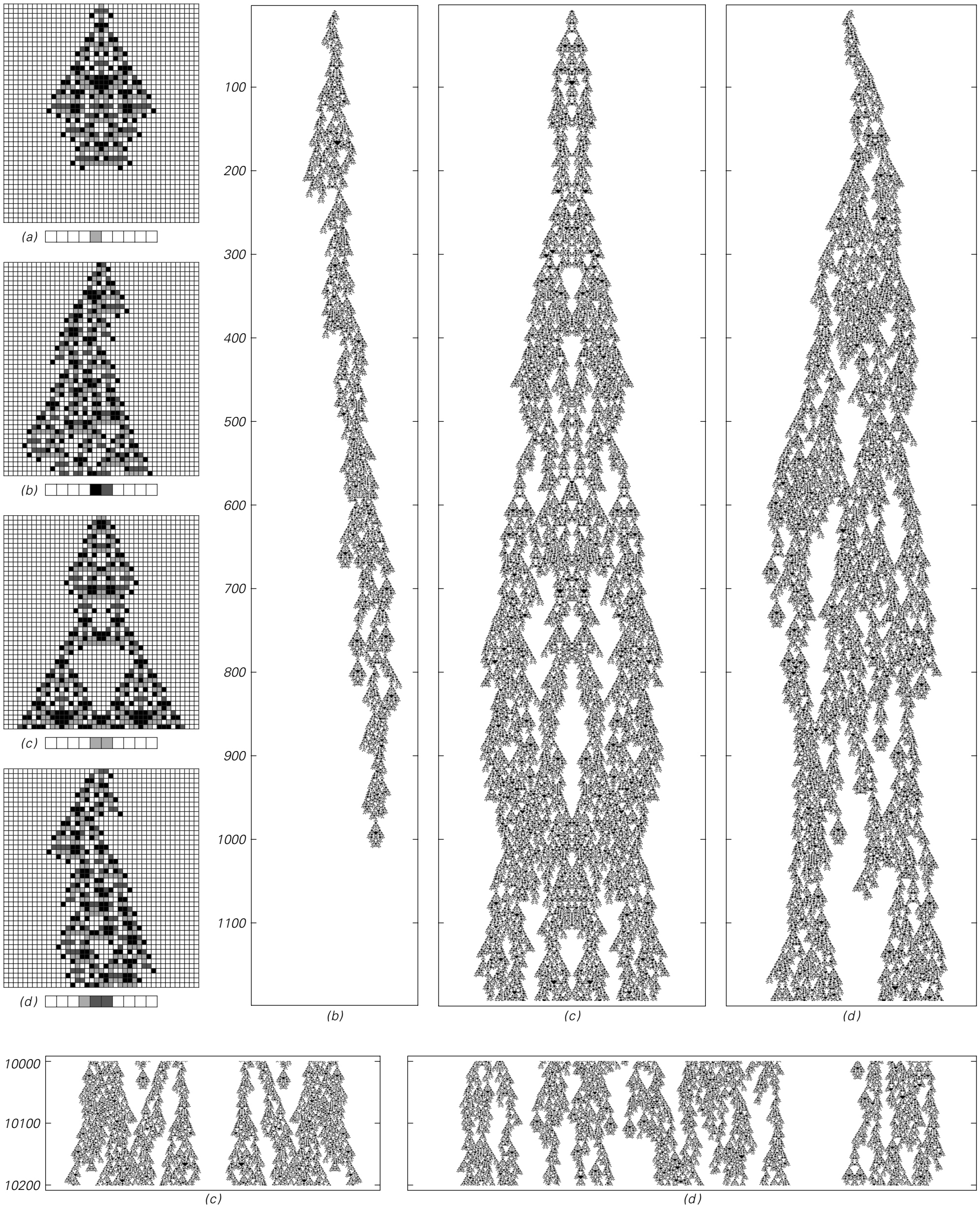
Computational irreducibility is a very general phenomenon with many consequences. And among these consequences are various phenomena that have been widely studied in the abstract theory of computation.

In the past it has normally been assumed that these phenomena occur only in quite special systems, and not, for example, in typical systems with simple rules or of the kind that might be seen in nature. But what my discoveries about computational irreducibility now suggest is that such phenomena should in fact be very widespread, and should for example occur in many systems in nature and elsewhere.

In this chapter so far I have mostly been concerned with ongoing processes of computation, analogous to ongoing behavior of systems in nature and elsewhere. But as a theoretical matter one can ask what the final outcome of a computation will be, after perhaps an infinite number of steps. And if one does this then one encounters the phenomenon of undecidability that was identified in the 1930s.

The pictures on the next page show an example. In each case knowing the final outcome is equivalent to deciding what will eventually happen to the pattern generated by the cellular automaton evolution. Will it die out? Will it stabilize and become repetitive? Or will it somehow continue to grow forever?

One can try to find out by running the system for a certain number of steps and seeing what happens. And indeed in example (a) this approach works well: in only 36 steps one finds that the pattern



Cellular automaton evolution illustrating the phenomenon of undecidability. Pattern (a) dies out after 36 steps; pattern (b) takes 1017 steps. But what the final outcome in cases (c) and (d) will be is not clear after even a million steps. And in general there appears to be no finite computation that can guarantee to determine the final outcome of the evolution after an infinite number of steps. The cellular automaton rule used is a 4-color totalistic one with code 1004600. Whether a pattern in a cellular automaton ever dies out can be viewed as analogous to a version of the halting problem for Turing machines.

dies out. But already in example (b) it is not so easy. One can go for 1000 steps and still not know what is going to happen. And only after 1017 steps does it finally become clear that the pattern in fact dies out.

So what about examples (c) and (d)? What happens to these? After a million steps neither has died out; in fact they are respectively 31,000 and 39,718 cells wide. And after 10 million steps both are still going, now 339,028 and 390,023 cells wide. But even having traced the evolution this far, one still has no idea what its final outcome will be.

And in any system the only way to be able to guarantee to know this in general is to have some way to shortcut the evolution of the system, and to be able to reduce to a finite computation what takes the system an infinite number of steps to do.

But if the behavior of the system is computationally irreducible—as I suspect is so for the cellular automaton on the facing page and for many other systems with simple underlying rules—then the point is that ultimately no such shortcut is possible. And this means that the general question of what the system will ultimately do can be considered formally undecidable, in the sense there can be no finite computation that will guarantee to decide it.

For any particular initial condition it may be that if one just runs the system for a certain number of steps then one will be able to tell what it will do. But the crucial point is that there is no guarantee that this will work: indeed there is no finite amount of computation that one can always be certain will be enough to answer the question of what the system does after an infinite number of steps.

That this is the case has been known since the 1930s. But it has normally been assumed that the effects of such undecidability will rarely be seen except in special and complicated circumstances. Yet what the picture on the facing page illustrates is that in fact undecidability can have quite obvious effects even with a very simple underlying rule and very simple initial conditions.

And what I suspect is that for almost any system whose behavior seems to us complex almost any non-trivial question about what the system does after an infinite number of steps will be undecidable. So, for example, it will typically be undecidable whether the evolution of the system from some particular initial condition will ever generate a specific arrangement of cell colors—or whether it will yield a pattern that is, say, ultimately repetitive or ultimately nested.

And if one asks whether any initial conditions exist that lead, for example, to a pattern that does not die out, then this too will in general be undecidable—though in a sense this is just an immediate consequence of the fact that given a particular initial condition one cannot tell whether or not the pattern it produces will ever die out.

But what if one just looks at possible sequences—as might be used for initial conditions—and asks whether any of them satisfy some constraint? Even if the constraint is easy to test it turns out that there can again be undecidability. For there may be no limit on how far one has to go to be sure that out of the infinite number of possible sequences there are really none that satisfy the constraint.

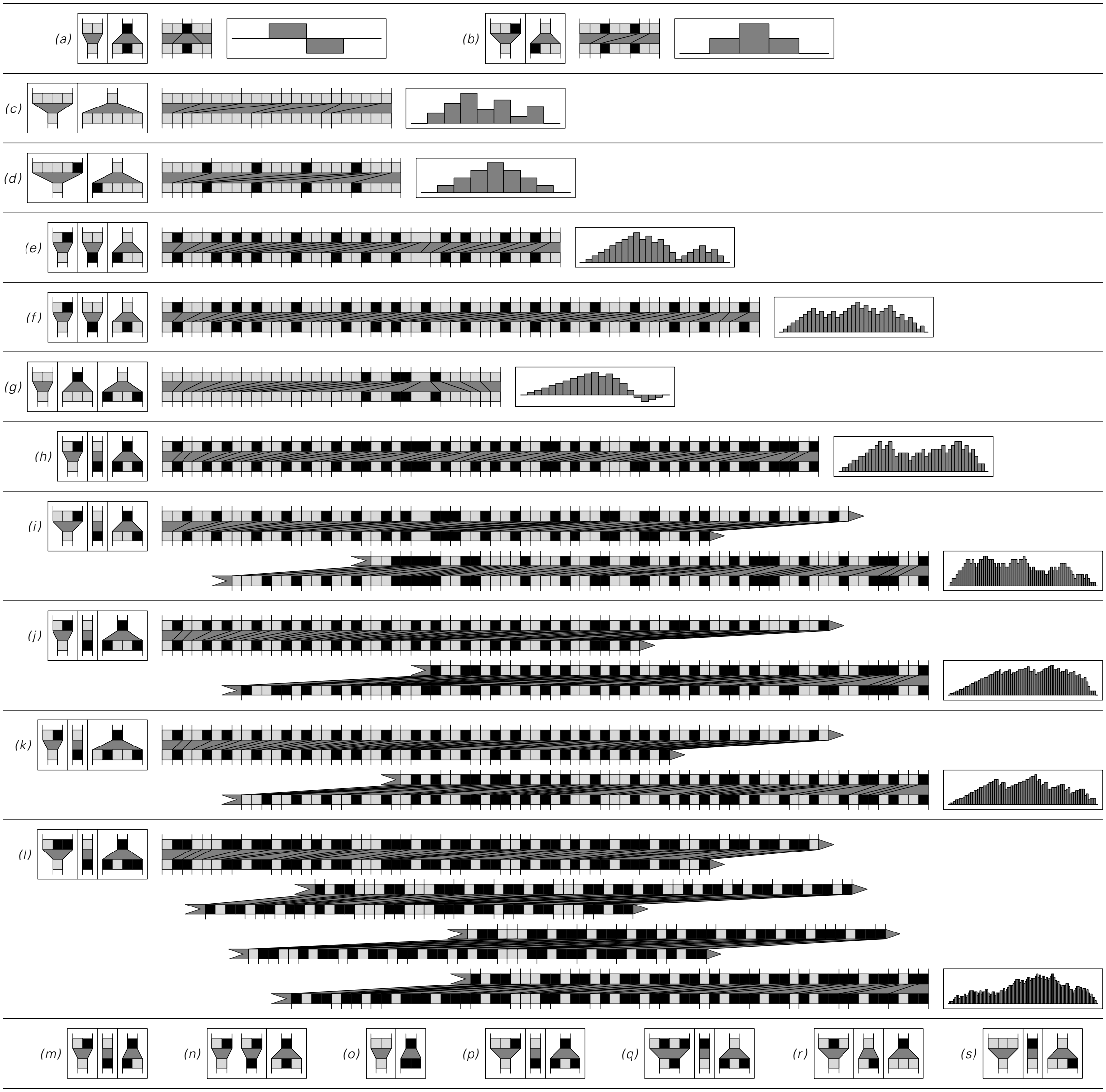
The pictures on the facing page show a simple example of this. The idea is to pick a set of pairs of upper and lower blocks, and then to ask whether there is any sequence of such pairs that satisfies the constraint that the upper and lower strings formed end up being in exact correspondence.

When there are just two kinds of pairs it turns out to be quite straightforward to answer this question. For if any sequence is going to satisfy the constraint one can show that there must already be a sequence of limited length that does so—and if necessary one can find this sequence by explicitly looking at all possibilities.

But as soon as there are more than two pairs things become much more complicated, and as the pictures on the facing page demonstrate, even with very short blocks remarkably long and seemingly quite random sequences can be required in order to satisfy the constraints.

And in fact I strongly suspect that even with just three pairs there is already computational irreducibility, so that in effect the only way to answer the question of whether the constraints can be satisfied is explicitly to trace through some fraction of all arbitrarily long sequences—making this question in general undecidable.

And indeed whenever the question one has can somehow involve looking at an infinite number of steps, or elements, or other things, it



Examples of a class of one-dimensional constraints where it is in general undecidable whether they can be satisfied. The constraints require that concatenating in some order the blocks shown should yield identical upper and lower strings. In cases (a)–(l) the constraints can be satisfied, and the minimal strings which do so are shown. The plots to the right give the successive differences in length between upper and lower strings when each new block is added; that this difference reaches zero reflects the fact that the constraint is satisfied. Cases (m)–(s) show constraints that cannot be satisfied by strings of any finite length. When the constraints involve more than two blocks there seems in general to be no upper limit on how long a string one may need to consider to tell whether the constraints can be satisfied. Pictures (a), (b), (h) and (j) show the longest minimal strings needed for any of the 4096, 16384, 65536 and 262144 constraints involving blocks with totals of 7, 8, 9 and 10 elements. The general problem of satisfying constraints of the kind shown here is known as the Post Correspondence Problem. Finding the systems on this page required constructing—by computer and otherwise—an immense number of proofs of the impossibility of satisfying particular constraints.

turns out that such a question is almost inevitably undecidable if it is asked about a system that exhibits computational irreducibility.

So what about finite questions?

Such questions can ultimately always be answered by finite computations. But when computational irreducibility is present such computations can be forced to have a certain level of difficulty which sometimes makes them quite intractable.

When one does practical computing one tends to assess the difficulty of a computation by seeing how much time it takes and perhaps how big a program it involves and how much memory it needs.

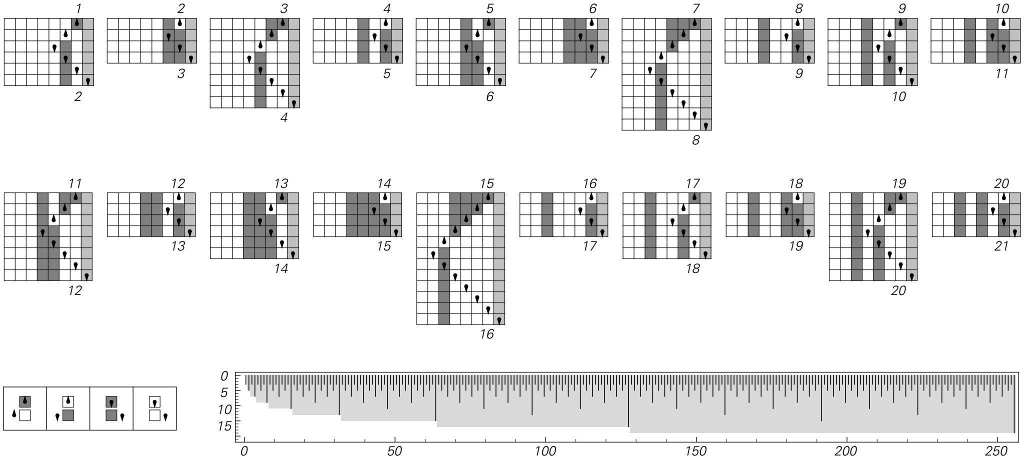
But normally one has no way to tell whether the scheme one has for doing a particular computation is the most efficient possible. And in the past there have certainly been several instances when new algorithms have suddenly allowed all sorts of computations to be done much more efficiently than had ever been thought possible before.

Indeed, despite great efforts in the field of computational complexity theory over the course of several decades almost no firm lower bounds on the difficulty of computations have ever been established. But using the methods of this book it turns out to be possible to begin to get at least a few results.

The key is to consider very small programs. For with such programs it becomes realistic to enumerate every single one of a particular kind, and then just to see explicitly which is the most efficient at performing some specific computation.

In the past such an approach would not have seemed sensible, for it was normally assumed that programs small enough to make it work would only ever be able to do rather trivial computations. But what my discoveries have shown is that in fact even very small programs can be quite capable of doing all sorts of sophisticated computations.

As a first example—based on a rather simple computation—the picture at the top of the facing page shows a Turing machine set up to add 1 to any number. The input to the Turing machine is the base 2 digit sequence for the number. The head of the machine starts at the right-hand end of this sequence, and the machine runs until its head first goes further to the right—at which point the machine stops, with



Examples of the behavior of a simple Turing machine that does the computation of adding 1 to a number. The number is given as a base 2 digit sequence; the Turing machine runs until its head hits the gray stripe on the right. The plot shows the number of steps that this takes as a function of the input number *x*. The result turns out to be given by *2 IntegerExponent[x + 1, 2] + 3*, which has a maximum of *2 n + 3*, where *n* is the length of the digit sequence of *x*, or *Floor[Log[2, x]]*. The average for a given length of input does not increase with *n*, and is always precisely 5.

whatever sequence of digits are left behind being taken to be the output of the computation.

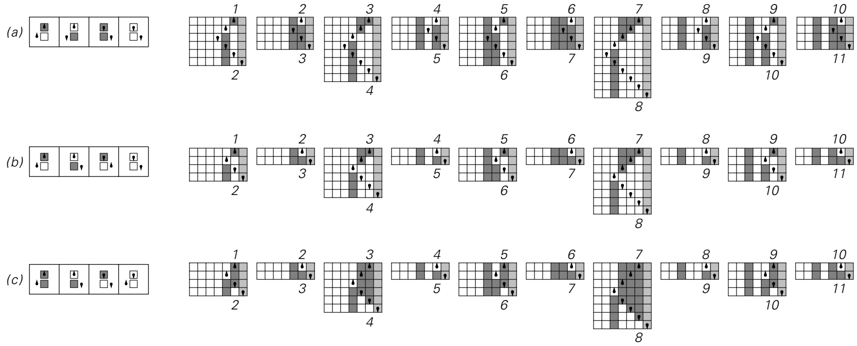
And what the pictures above show is that with this particular machine the number of steps needed to finish the computation varies greatly between different inputs. But if one looks just at the absolute maximum number of steps for any given length of input one finds an exactly linear increase with this length.

So are there other ways to do the same computation in a different number of steps? One can readily enumerate all 4096 possible Turing machines with 2 states and 2 colors. And it turns out that of these exactly 17 perform the computation of adding 1 to a number.

Each of them works in a slightly different way, but all of them follow one of the three schemes shown at the top of the next page—and all of them end up exhibiting the same overall linear increase in number of steps with length of input.

So what about other computations?

It turns out that there are 351 different functions that can be computed by one or more of the 4096 Turing machines with 2 states



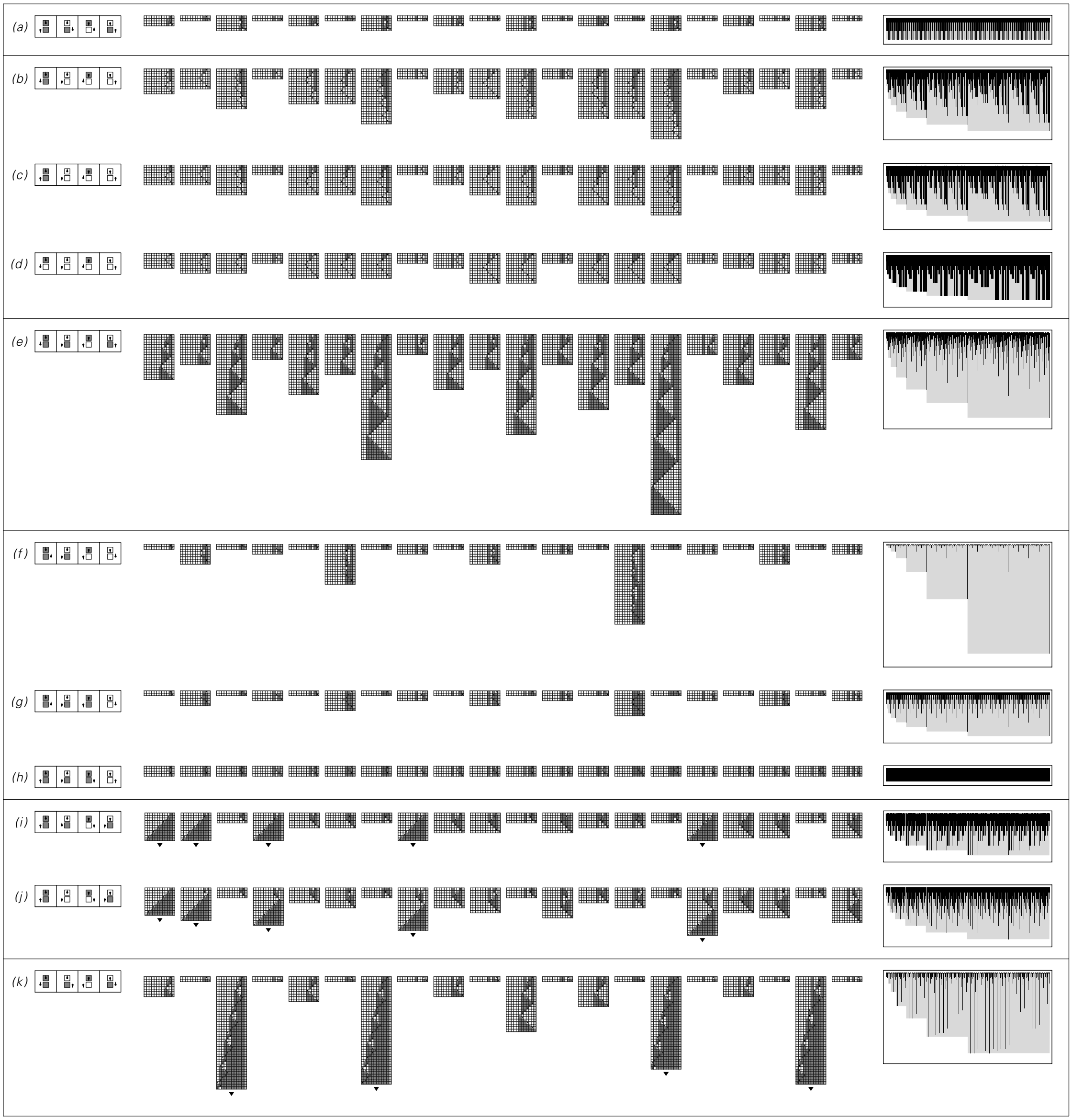
The three schemes for adding 1 to a number that are used by Turing machines with 2 states and 2 colors. All show the same linear growth in maximum number of steps as their size of input increases. This growth can be viewed as a consequence of potentially having to propagate carry digits from one end of the input number to the other. The machines shown are numbered 445, 461 and 1512.

and 2 colors. And as the pictures on the facing page show, different Turing machines can take very different numbers of steps to do the computations they do.

Turing machine (a), for example, always finishes its computation after at most 5 steps, independent of the length of its input. But in most of the other Turing machines shown, the maximum number of steps needed generally increases with the length of the input.

Turing machines (b), (c) and (d) are ones that always compute the same function. But while this means that for a given input each of them yields the same output, the pictures demonstrate that they usually take a different number of steps to do so. Nevertheless, if one looks at the maximum number of steps needed for any given length of input one finds that this still always increases exactly linearly—just as for the Turing machines that add 1 shown at the top of this page.

So are there cases in which there is more rapid growth? Turing machine (e) shows an example in which the maximum number of steps grows like the square of the length of the input. And it turns out that at least among 2-state 2-color Turing machines this is the only one that computes the function it computes—so that at least if one wants to use a program this simple there is no faster way to do the computation.



Examples of computations being done by Turing machines with two states and two colors. Evolution from a succession of initial conditions is shown corresponding to inputs of numbers from 1 to 20. Each block of Turing machines yields the same output for a given input. A computation is taken to be complete when the head of the Turing machine goes further to the right than it was at the beginning. The plots show how many steps this takes for successive inputs with lengths up to 9. The maximum for input of length *n* is (a) *5*, (b) $6n+3$, (c) $4n+3$, (d) $2n+3$, (e) $2n^2+8n+7$, (f) $2^{n+1}-1$ (though the average is $n+2$), (g) $2n+1$, (h) *3*, (i) $2n+1$, (j) $4n-1$, (k) roughly $2.5n^2$. In cases (i), (j) and (k) there are some inputs for which the head goes further and further to the left, and the Turing machine never halts. The machines shown are numbered 3279, 1285, 3333, 261, 1447, 1953, 1969, 3517, 3246, 3374, 1507.

So are there computations that take still longer to do? In Turing machine (f) the maximum number of steps increases exponentially with the length of the input. But unlike in example (e), this Turing machine is not the only one that computes the function it computes. And in fact both (g) and (h) compute the same function—but in a linearly increasing and constant number of steps respectively.

So what about other Turing machines? In general there is no guarantee that a particular Turing machine will ever even complete a computation in a finite number of steps. For as happens with several inputs in examples (i) and (j) the head may end up simply going further and further to the left—and never get to the point on the right that is needed for the computation to be considered complete.

But if one ignores inputs where this happens, then at least in examples (i) and (j) the maximum number of steps still grows in a very systematic linear way with the length of the input.

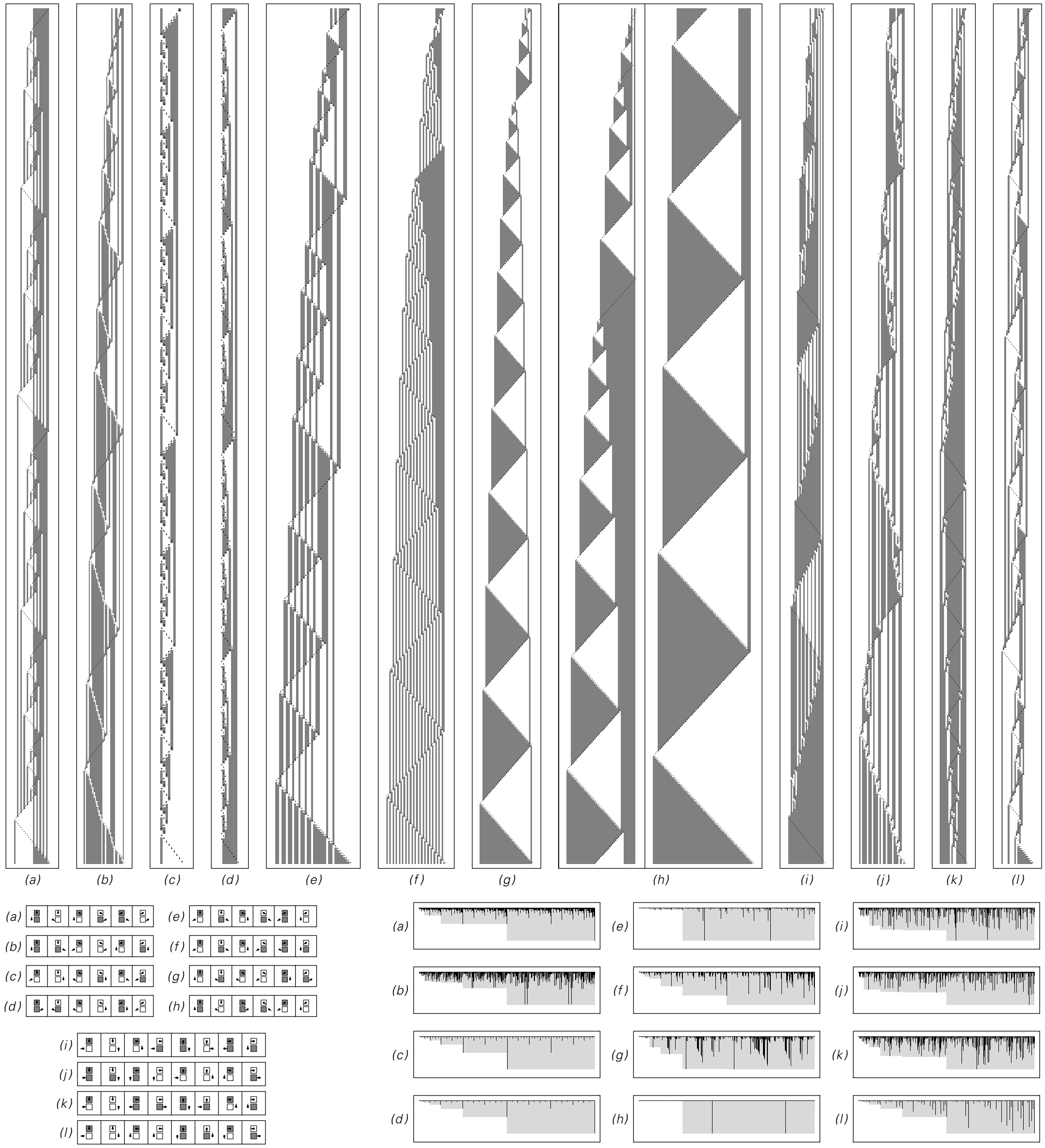
In example (k), however, there is more irregular growth. But once again the maximum number of steps in the end just increases like the square of the length of the input. And indeed if one looks at all 4096 Turing machines with 2 states and 2 colors it turns out that the only rates of growth that one ever sees are linear, square and exponential.

And of the six examples where exponential growth occurs, all of them are like example (f) above—so that there is another 2-state 2-color Turing machine that computes the same function, but without the maximum number of steps increasing at all with input length.

So what happens if one considers more complicated Turing machines? With 3 states and 2 colors there are a total of 2,985,984 possible machines. And it turns out that there are about 33,000 distinct functions that one or more of these machines computes.

Most of the time the fastest machine at computing a given function again exhibits linear or at most quadratic growth. But the facing page shows some cases where instead it exhibits exponential growth.

And indeed in a few cases the growth seems to be even faster. Example (h) is the most extreme among 3-state 2-color Turing machines: with the size 7 input 106 it already takes 1,978,213,883 steps



Examples of Turing machines with 3 and 4 states in which the maximum number of steps before a computation is finished grows at least exponentially with the length of the input. In all cases no Turing machines with the same number of states compute the same functions in fewer steps. In case (h) the number of steps grows so rapidly that only two peaks are seen in the plot. The top row of pictures are all scaled to be exactly the same height, even though the initial conditions cannot be chosen to make the number of steps in each case anything more than roughly the same. The machines have numbers: 582285, 657939, 2018806, 2868668, 2138664, 2139050, 132527, 600720, 3374234978, 1806221583, 1232059922, 3238044559. Cases like (c) and (d) show nested behavior reminiscent of a counter which generates digit sequences of successive integers.

to generate its output, and in general with size n input it may be able to take more than 2^{2^n} steps.

But what if one allows Turing machines with more complicated rules? With 4-state 2-color rules it turns out to be possible to generate the same output as examples (c) and (d) in just a fixed number of steps. But for none of the other 3-state 2-color Turing machines shown do 4-state rules offer any speedup.

Nevertheless, if one looks carefully at examples (a) through (h) each of them shows large regions of either repetitive or nested behavior. And it seems likely that this reflects computational reducibility that should make it possible for sufficiently complicated programs to generate the same output in fewer than exponentially many steps.

But looking at 4-state 2-color Turing machines examples (i) through (l) again appear to exhibit roughly exponential growth. Yet now—much as for the 4-state Turing machines in Chapter 3—the actual behavior seen does not show any obvious computational reducibility.

So this suggests that even though they may be specified by very simple rules there are indeed Turing machine computations that cannot actually be carried out except by spending an amount of computational effort that can increase exponentially with the length of input.

And certainly if one allows no more than 4-state 2-color Turing machines I have been able to establish by explicitly searching all 4 billion or so possible rules that there is absolutely no way to speed up the computations in pictures (i) through (l).

But what about with other kinds of systems?

Once one has a system that is universal it can in principle be made to do any computation. But the question is at what rate. And without special optimization a universal Turing machine will for example typically just operate at some fixed fraction of the speed of any specific Turing machine that it is set up to emulate.

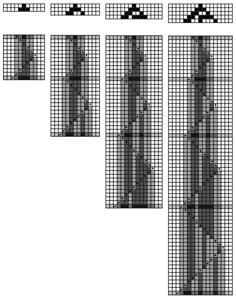
And if one looks at different computers and computer languages practical experience tends to suggest that at least at the level of issues like exponential growth the rate at which a given computation can be done is ultimately rather similar in almost every such system.

But one might imagine that across the much broader range of computational systems that I have considered in this book—and that presumably occur in nature—there could nevertheless still be great differences in the rates at which given computations can be done.

Yet from what we saw in Chapter 11 one suspects that in fact there are not. For in the course of that chapter it became clear that almost all the very varied systems in this book can actually be made to emulate each other in a quite comparable number of steps.

Indeed often we found that it was possible to emulate every step in a particular system by just a fixed sequence of steps in another system. But if the number of elements that can be updated in one step is sufficiently different this tends to become impossible.

And thus for example the picture on the right shows that it can take t^2 steps for a Turing machine that updates just one cell at each step to build up the same pattern as a one-dimensional cellular automaton builds up in t steps by updating every cell in parallel.

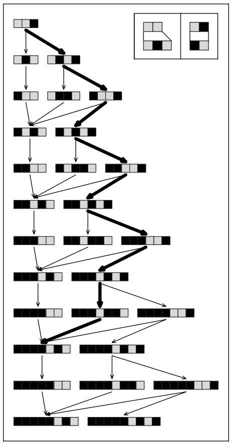


To emulate t steps in the evolution of the cellular automaton takes the Turing machine $2t^2+5t-6$ steps.

And in d dimensions it is common for it to take, say, t^{d+1} steps for one system to emulate t steps of evolution of another.

But can it take an exponential number of steps? Certainly if one has a substitution system that yields exponentially many elements then to reproduce all these elements with an ordinary Turing machine will take exponentially many steps. And similarly if one has a multiway system that yields exponentially many strings then to reproduce all these will again take exponentially many steps.

But what if one asks only about some limited feature of the output—say whether some particular string appears after t steps of evolution of the multiway system? Given a specific path like the one in the picture on the right it takes an ordinary Turing machine not much more than t steps to test whether the path yields the desired string.



A Turing machine can quickly test the highlighted path but could take exponentially long to test all paths.

But how long can it take for a Turing machine to find out whether any path in the multiway system manages to produce the string? If the Turing machine in effect had to examine each of the perhaps exponentially many paths in turn then this could take exponentially many steps. But the celebrated P=NP question in computational complexity theory asks whether in general there is some way to get such an answer in a number of steps that increases not exponentially but only like a power.

And although it has never been established for certain it seems by now likely that in most meaningful senses there is not. So what this implies is that to answer questions about the t-step behavior of a multiway system can take any ordinary Turing machine a number of steps that increases faster than any power of t.

So how common is this kind of phenomenon? One can view asking about possible outcomes in a multiway system as like asking about possible ways to satisfy a constraint. And certainly a great many practical problems can be formulated in terms of constraints.

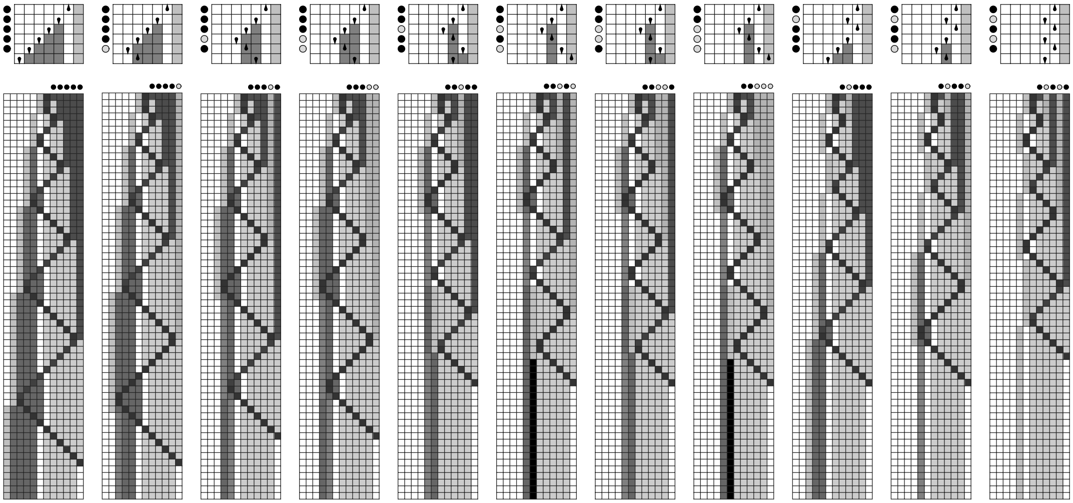
But how do such problems compare to each other? The Principle of Computational Equivalence suggests that those that seem difficult should somehow tend to be equivalent. And indeed it turns out that over the course of the past few decades a rather large number of such problems have in fact all been found to be so-called NP-complete.

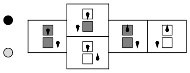
What this means is that these problems exhibit a kind of analog of universality which makes it possible with less than exponential effort to translate any instance of any one of them into an instance of any other. So as an example the picture on the facing page shows how one type of problem about a so-called non-deterministic Turing machine can be translated to a different type of problem about a cellular automaton.

Much like a multiway system, a non-deterministic Turing machine has rules that allow multiple choices to be made at each step, leading to multiple possible paths of evolution. And an example of an NP-complete problem is then whether any of these paths satisfy the constraint that, say, after a particular number of steps, the head of the Turing machine has ever gone further to the right than it starts.

The top row in the picture on the facing page shows the first few of the exponentially many possible paths obtained by making successive sequences of choices in a particular non-deterministic Turing machine. And in the example shown, one sees that for two of these paths the head goes to the right, so that the overall constraint is satisfied.

So what about the cellular automaton below in the picture? Given a particular initial condition its evolution is completely





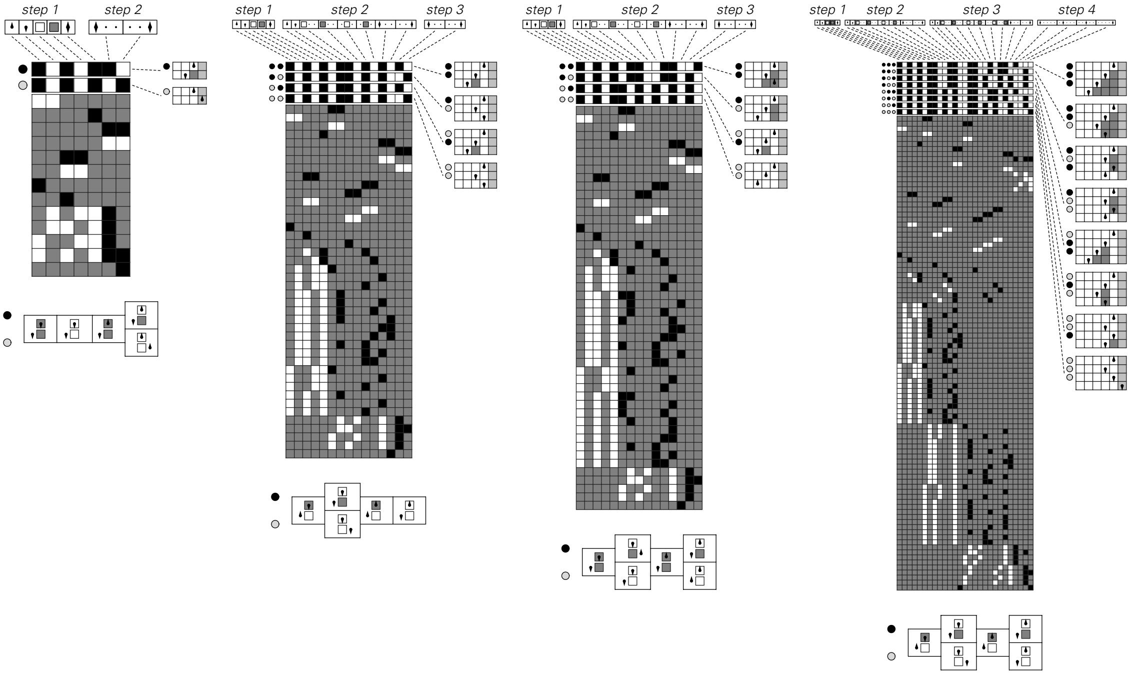
Translation between an NP-complete problem about non-deterministic Turing machines and about cellular automata. The top row shows how a particular non-deterministic Turing machine behaves with successive sequences of choices for rules to apply. The bottom row shows how a cellular automaton can be made to emulate this behavior when given a succession of different initial conditions. The cellular automaton is set up to produce a vertical black stripe if the head of the Turing machine ever goes further to the right than it starts—as it does in cases 6 and 8. The left part of each cellular automaton configuration emulates the actual evolution of the Turing machine; a specification of which rules should be applied at each step is progressively fetched from the right and delivered to the position of the head. Given particular initial conditions for the Turing machine the problem of whether the head ever goes further to the right than it starts is thus equivalent to the problem of whether the cellular automaton ever produces a vertical black stripe given particular initial conditions on its left. The cellular automaton takes $2t^2 + t$ steps to emulate t steps of evolution in the Turing machine. It involves a total of 19 colors.

deterministic. But what the picture shows is that with successive initial conditions it emulates each possible path in the non-deterministic Turing machine.

And so what this means is that the problem of finding whether initial conditions exist that make the cellular automaton produce a certain outcome is equivalent to the non-deterministic Turing machine problem above—and is therefore in general NP-complete.

So what about other kinds of problems?

The picture on the next page shows the equivalence between the classic problem of satisfiability and the non-deterministic Turing machine problem at the top of this page. In satisfiability what one does is to start with a collection of rows of black, white and gray squares. And then what one asks is whether any sequence of just black and



Translation between the NP-complete problem of halting in a non-deterministic Turing machine and the classic NP-complete problem of satisfiability. In satisfiability one sets up a collection of rows of black, white and gray squares, then asks whether there exists any sequence of black and white squares that satisfies the constraint that on every row the color of at least one square agrees with the color of the corresponding square in the sequence. Each row can be viewed as a term in a conjunctive normal form Boolean expression, with each column corresponding to a different variable. When a given square on a particular row is black or white it indicates that a variable or its negation appear in that term. The translation from the Turing machine problem is achieved by representing the behavior of the Turing machine by saying which of a sequence of elementary statements are true about it at each step: whether the head is in one state or another, whether the cell under the head is black or white, and whether the head is at each of the possible positions it can be in. The Boolean expression then gives constraints on which of these statements can simultaneously be true. In the first two pictures, for example, the first row corresponds to the constraint that on the first step of Turing machine evolution, the head cannot simultaneously be in an up and a down state. About the first half of the terms in each Boolean expression correspond to similar general constraints about the operation of Turing machines. There are then a few terms that specify the particular initial conditions used here, followed by terms that give the rule for the Turing machine that is used. The very last term makes the statement that the Turing machine halts. As the pictures indicate, each possible path of evolution for the Turing machine then corresponds to a possible assignment of truth values to the variables associated with each elementary statement. And if there is any path that leads the Turing machine to halt the Boolean expression will be satisfiable. This is the case in the first and fourth examples shown, but not in the other two. In general, it is possible to represent t steps in the evolution of a non-deterministic Turing machine by a Boolean expression with at most t^3 terms in t^2 variables. A version of the translation shown here was what launched the study of NP completeness in the early 1970s.

white squares exists that satisfies the constraint that on every row there is at least one square whose color agrees with the color of the corresponding square in the sequence.

To see the equivalence to questions about Turing machines one imagines breaking the description of the behavior of a Turing machine into a sequence of elementary statements: whether the head is in a particular state on a particular step, whether a certain cell has a particular color, and so on. The underlying rules for the Turing machine then define constraints on which sequences of such statements can be true. And in the picture on the facing page almost every row of black, white and gray squares corresponds to one such constraint.

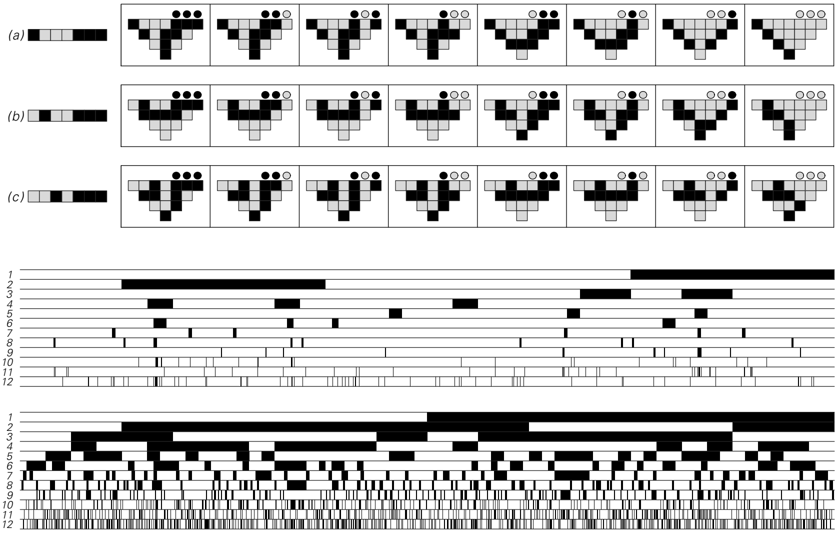
The last row, however, represents the further constraint that the head of the Turing machine must at some point go further to the right than it starts. And this means that to ask whether there is any sequence in the satisfiability problem that obeys all the constraints is equivalent to finding the answer to the Turing machine problem described above.

Starting from satisfiability it is possible to show that all sorts of well-known computational problems in discrete mathematics are NP-complete. And in addition almost any undecidable problem that involves simple constraints—such as the correspondence problem on page 757—turns out to be NP-complete if restricted to finite cases.

In studying the phenomenon of NP completeness what has mostly been done in the past is to try to construct particular instances of rather general problems that exhibit equivalence to other problems. But almost always what is actually constructed is quite complicated—and certainly not something one would expect to occur at all often.

Yet on the basis of intuition from the Principle of Computational Equivalence I strongly suspect that in most cases there are already quite simple instances of general NP-complete problems that are just as difficult as any NP-complete problem. And so, for example, I suspect that it does not take a cellular automaton nearly as complicated as the one on page 767 for it to be an NP-complete problem to determine whether initial conditions exist that lead to particular behavior.

Indeed, my expectation is that asking about possible outcomes of *t* steps of evolution will already be NP-complete even for the rule 30 cellular automaton, as illustrated below.



Example of a simple problem that I suspect is NP-complete. The problem is to determine whether right-hand cells in the initial conditions for rule 30 can be filled in so as to produce a vertical black stripe of a certain height at the bottom of the center column formed after *t* steps of evolution. The pictures at the top show that in case (a) stripes up to height 3 can be produced, in case (b) up to height 2, and in case (c) only up to height 1. The pictures at the bottom indicate in black for which of the 2^{t+1} successive left-hand sequences of $t + 1$ cells it is impossible to get stripes of respectively heights 1 and 2. The apparent randomness of these patterns reflects the likely difficulty of the problem. The problem is related to issues of rule 30 cryptanalysis discussed on page 603.

Just as with the Turing machines of pages 761 and 763 there will be a certain density of cases where the problem is fairly easy to solve. But it seems likely that as one increases *t*, no ordinary Turing machine or cellular automaton will ever be able to guarantee to solve the problem in a number of steps that grows only like some power of *t*.

Yet even so, there could still in principle exist in nature some other kind of system that would be able to do this. And for example one might imagine that this would be possible if one were able to use exponentially small components. But almost all the evidence we have suggests that in our actual universe there are limits on the sizes and densities of components that we can ever expect to manipulate.

In present-day physics the standard mathematical formalism of quantum mechanics is often interpreted as suggesting that quantum systems work like multiway systems, potentially following many paths in parallel. And indeed within the usual formalism one can construct quantum computers that may be able to solve at least a few specific problems exponentially faster than ordinary Turing machines.

But particularly after my discoveries in Chapter 9, I strongly suspect that even if this is formally the case, it will still not turn out to be a true representation of ultimate physical reality, but will instead just be found to reflect various idealizations made in the models used so far.

And so in the end it seems likely that there really can in some fundamental sense be an almost exponential difference in the amount of computational effort needed to find the behavior of a system with given particular initial conditions, and to solve the inverse problem of determining which if any initial conditions yield particular behavior.

In fact, my suspicion is that such a difference will exist in almost any system whose behavior seems to us complex. And among other things this then implies many fundamental limits on the processes of perception and analysis that we discussed in Chapter 10.

Such limits can ultimately be viewed as being consequences of the phenomenon of computational irreducibility. But a much more direct consequence is one that we have discussed before: that even given a particular initial condition it can require an irreducible amount of computational work to find the outcome after a given number of steps of evolution.

One can specify the number of steps t that one wants by giving the sequence of digits in t. And for systems with sufficiently simple behavior—say repetitive or nested—the pictures on page 744 indicate that one can typically determine the outcome with an amount of effort that is essentially proportional to the length of this digit sequence.

But the point is that when computational irreducibility is present, one may in effect explicitly have to follow each of the t steps of evolution—again requiring exponentially more computational work.

Implications for Mathematics and Its Foundations

Much of what I have done in this book has been motivated by trying to understand phenomena in nature. But the ideas that I have developed are general enough that they do not apply just to nature. And indeed in this section what I will do is to show that they can also be used to provide important new insights on fundamental issues in mathematics.

At some rather abstract level one can immediately recognize a basic similarity between nature and mathematics: for in nature one knows that fairly simple underlying laws somehow lead to the rich and complex behavior we see, while in mathematics the whole field is in a sense based on the notion that fairly simple axioms like those on the facing page can lead to all sorts of rich and complex results.

So where does this similarity come from? At first one might think that it must be a consequence of nature somehow intrinsically following mathematics. For certainly early in its history mathematics was specifically set up to capture certain simple aspects of nature.

But one of the starting points for the science in this book is that when it comes to more complex behavior mathematics has never in fact done well at explaining most of what we see every day in nature.

Yet at some level there is still all sorts of complexity in mathematics. And indeed if one looks at a presentation of almost any piece of modern mathematics it will tend to seem quite complex. But the point is that this complexity typically has no obvious relationship to anything we see in nature. And in fact over the past century what has been done in mathematics has mostly taken increasing pains to distance itself from any particular correspondence with nature.

So this suggests that the overall similarity between mathematics and nature must have a deeper origin. And what I believe is that in the end it is just another consequence of the very general Principle of Computational Equivalence that I discuss in this chapter.

For both mathematics and nature involve processes that can be thought of as computations. And then the point is that all these computations follow the Principle of Computational Equivalence, so

$$\begin{gathered}
a \wedge b = b \wedge a \\
a \vee b = b \vee a \\
a \wedge (b \vee \neg b) = a \\
a \vee (b \wedge \neg b) = a \\
a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\
a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)
\end{gathered}$$

basic logic (standard axioms)

$$\begin{gathered}
a \vee b = b \vee a \\
a \vee (b \vee c) = (a \vee b) \vee c \\
\neg(\neg a \vee b) \vee \neg(\neg a \vee \neg b) = a
\end{gathered}$$

basic logic (Huntington axioms)

$$\begin{gathered}
a \vee b = b \vee a \\
a \vee (b \vee c) = (a \vee b) \vee c \\
\neg(\neg(a \vee b) \vee \neg(a \vee \neg b)) = a
\end{gathered}$$

basic logic (Robbins axioms)

$$\begin{gathered}
(a \barwedge a) \barwedge (a \barwedge a) = a \\
a \barwedge (b \barwedge (b \barwedge b)) = a \barwedge a \\
(a \barwedge (b \barwedge c)) \barwedge (a \barwedge (b \barwedge c)) = \\
((b \barwedge b) \barwedge a) \barwedge ((c \barwedge c) \barwedge a)
\end{gathered}$$

basic logic (Sheffer axioms)

$$\begin{gathered}
(a \barwedge a) \barwedge (a \barwedge b) = a \\
a \barwedge (a \barwedge b) = a \barwedge (b \barwedge b) \\
a \barwedge (a \barwedge (b \barwedge c)) = b \barwedge (b \barwedge (a \barwedge c))
\end{gathered}$$

basic logic (shorter axioms)

$$((a \barwedge b) \barwedge c) \barwedge (a \barwedge ((a \barwedge c) \barwedge a)) = c$$

basic logic (shortest axioms)

basic logic, $x\_ \wedge y\_ \longrightarrow x\_$, $x\_ \longrightarrow \forall_{y\_}\, x\_$, $x\_ \longrightarrow x\_ \wedge \#\,\&$, and ...

$$\begin{gathered}
\forall_{a\_}\,(b\_ \Rightarrow c\_) \Rightarrow (\forall_{a\_}\, b\_ \Rightarrow \forall_{a\_}\, c\_) \\
a\_ \Rightarrow \forall_{b\_}\, a\_ \text{ /; FreeQ[a, b]} \\
\exists_{a\_}\, a\_ == b\_ \text{ /; FreeQ[b, a]} \\
a\_ == b\_ \Rightarrow (c\_ \Rightarrow d\_) \text{ /; FreeQ[c, } \forall\_\,\_\text{] \&\&} \\
\text{MatchQ[d, c /. } a \longrightarrow a \mid b\text{]}
\end{gathered}$$

predicate logic

predicate logic and ...

$$\begin{gathered}
0 \neq \Delta a \\
\Delta a == \Delta b \Rightarrow a == b \\
a + 0 == a \\
a + \Delta b == \Delta(a + b) \\
a \times 0 == 0 \\
a \times \Delta b == (a \times b) + a \\
a \neq 0 \Rightarrow \exists_b\, a == \Delta b
\end{gathered}$$

reduced arithmetic (Robinson axioms)

$$\begin{gathered}
0 \neq \Delta a \\
\Delta a == \Delta b \Rightarrow a == b \\
a + 0 == a \\
a + \Delta b == \Delta(a + b) \\
a \times 0 == 0 \\
a \times \Delta b == (a \times b) + a \\
(\alpha\_ |_{a \to 0} \wedge \forall_b\,(\alpha\_ |_{a \to b} \Rightarrow \alpha\_ |_{a \to \Delta b})) \Rightarrow \\
\forall_b\, \alpha\_ |_{a \to b} \text{ /; FreeQ[}\alpha\text{, b]}
\end{gathered}$$

arithmetic (Peano axioms)

$$a \cdot (b \cdot c) == (a \cdot b) \cdot c$$

semigroup theory

$$\begin{gathered}
a \cdot (b \cdot c) == (a \cdot b) \cdot c \\
a \cdot 1 == a \\
1 \cdot a == a
\end{gathered}$$

monoid theory

$$\begin{gathered}
a \cdot (b \cdot c) == (a \cdot b) \cdot c \\
a \cdot 1 == a \\
a \cdot \bar{a} == 1
\end{gathered}$$

group theory (standard axioms)

$$a \mathbin{\bar{\circ}} ((((a \mathbin{\bar{\circ}} a) \mathbin{\bar{\circ}} b) \mathbin{\bar{\circ}} c) \mathbin{\bar{\circ}} (((a \mathbin{\bar{\circ}} a) \mathbin{\bar{\circ}} a) \mathbin{\bar{\circ}} c)) == b$$

group theory (shorter axioms)

$$a \cdot \overline{b \cdot (((c \cdot \bar{c}) \cdot \overline{d \cdot b}) \cdot a)} == d$$

group theory (shorter axioms)

$$\begin{gathered}
a \cdot (b \cdot c) == (a \cdot b) \cdot c \\
a \cdot 1 == a \\
a \cdot \bar{a} == 1 \\
a \cdot b == b \cdot a
\end{gathered}$$

commutative group theory (standard axioms)

$$a \mathbin{\bar{\circ}} (b \mathbin{\bar{\circ}} (c \mathbin{\bar{\circ}} (a \mathbin{\bar{\circ}} b))) == c$$

commutative group theory (shorter axioms)

$$((a \cdot b) \cdot c) \cdot \overline{a \cdot c} == b$$

commutative group theory (shorter axioms)

$$\begin{gathered}
a \oplus (b \oplus c) == (a \oplus b) \oplus c \\
a \oplus 0 == a \\
a \oplus \bar{a} == 0 \\
a \oplus b == b \oplus a \\
a \otimes (b \otimes c) == (a \otimes b) \otimes c \\
a \otimes (b \oplus c) == (a \otimes b) \oplus (a \otimes c) \\
a \otimes b == b \otimes a
\end{gathered}$$

ring theory

$$\begin{gathered}
a \oplus (b \oplus c) == (a \oplus b) \oplus c \\
a \oplus 0 == a \\
a \oplus \bar{a} == 0 \\
a \oplus b == b \oplus a \\
a \otimes (b \otimes c) == (a \otimes b) \otimes c \\
a \otimes (b \oplus c) == (a \otimes b) \oplus (a \otimes c) \\
a \otimes b == b \otimes a \\
a \otimes 1 == a \\
a \neq 0 \Rightarrow a \otimes a^{-1} == 1 \\
0 \neq 1
\end{gathered}$$

field theory

$a + (b + c) == (a + b) + c$	$a \neq 0 \Rightarrow a \times a^{-1} == 1$
$a + 0 == a$	$(a > b \wedge b > c) \Rightarrow a > c$
$a + (-a) == 0$	$a > b \Rightarrow a \neq b$
$a + b == b + a$	$a > b \vee a == b \vee b > a$
$a \times (b \times c) == (a \times b) \times c$	$a > b \Rightarrow a + c > b + c$
$a \times (b + c) == (a \times b) + (a \times c)$	$(a > b \wedge c > 0) \Rightarrow a \times c > b \times c$
$a \times b == b \times a$	
$a \times 1 == a$	$1 > 0$
$(\exists_a\, \alpha\_ \wedge \exists_b\, \forall_a\,(\alpha\_ \Rightarrow a > b)) \Rightarrow \exists_b\, \forall_c\,(c > b \Leftrightarrow \exists_a\,(\alpha\_ \wedge c > a))$ /; FreeQ[α, c | b]	

real algebra (Tarski axioms)

Axiom systems for traditional mathematics. It is from the axiom systems on this page and the next that most of the millions of theorems in the literature of mathematics have ultimately been derived. Note that in several cases axiom systems are given here in much shorter forms than in standard mathematics textbooks. (See also the definitions on the next page.)

predicate logic and ...

$(a \wr b \wr a) \Rightarrow a == b$
$((a \wr b \wr c) \wedge (b \wr d \wr c)) \Rightarrow (a \wr b \wr d)$
$((a \wr b \wr c) \wedge (a \wr b \wr d) \wedge a \neq b) \Rightarrow ((a \wr c \wr d) \vee (a \wr d \wr c))$
$\overset{\frown}{ab} \equiv \overset{\frown}{ba}$
$\overset{\frown}{ab} \equiv \overset{\frown}{cc} \Rightarrow a == b$
$(\overset{\frown}{ab} \equiv \overset{\frown}{cd} \wedge \overset{\frown}{ab} \equiv \overset{\frown}{ef}) \Rightarrow \overset{\frown}{cd} \equiv \overset{\frown}{ef}$
$\exists_a\, (((b \wr c \wr d) \wedge (e \wr d \wr f)) \Rightarrow ((b \wr a \wr e) \wedge (f \wr c \wr a)))$
$\exists_a\, \exists_b\, (((c \wr d \wr e) \wedge (f \wr d \wr g) \wedge c \neq d) \Rightarrow ((c \wr g \wr a) \wedge (c \wr f \wr b) \wedge (a \wr e \wr b)))$
$(\overset{\frown}{ab} \equiv \overset{\frown}{cd} \wedge \overset{\frown}{be} \equiv \overset{\frown}{df} \wedge \overset{\frown}{ag} \equiv \overset{\frown}{ch} \wedge \overset{\frown}{bg} \equiv \overset{\frown}{dh} \wedge (a \wr b \wr e) \wedge (c \wr d \wr f) \wedge a \neq b) \Rightarrow$ $\overset{\frown}{eg} \equiv \overset{\frown}{fh}$
$\exists_a\, ((b \wr c \wr a) \wedge \overset{\frown}{ca} \equiv \overset{\frown}{de})$
$\exists_a\, \exists_b\, \exists_c\, (\neg (a \wr b \wr c) \wedge \neg (b \wr c \wr a) \wedge \neg (c \wr a \wr b))$
$(\overset{\frown}{ab} \equiv \overset{\frown}{ac} \wedge \overset{\frown}{db} \equiv \overset{\frown}{dc} \wedge \overset{\frown}{eb} \equiv \overset{\frown}{ec} \wedge b \neq c) \Rightarrow ((a \wr d \wr e) \vee (d \wr e \wr a) \vee (e \wr a \wr d))$
$\exists_a\, \forall_b\, \forall_c\, ((\alpha\_ \wedge \beta\_) \Rightarrow (a \wr b \wr c)) \Rightarrow$ $\exists_d\, \forall_b\, \forall_c\, ((\alpha\_ \wedge \beta\_) \Rightarrow (b \wr d \wr c))\ /;\ \mathit{FreeQ}[\alpha,\ a \vert c \vert d]\ \&\&\ \mathit{FreeQ}[\beta,\ a \vert b \vert d]$

Euclidean plane geometry

$a \cdot (b \cdot c) == (a \cdot b) \cdot c$
$a \cdot \triangleright a == a$
$\triangleleft a \cdot a == a$
$a \cdot \Box == \Box$
$\Box \cdot a == \Box$
$\triangleright \Box == \Box$
$\triangleleft \Box == \Box$
$(a \neq \Box \wedge b \neq \Box) \Rightarrow (a \cdot b \neq \Box \Leftrightarrow \triangleright a == \triangleleft b)$

elementary category theory

$\forall_a\, (a \in b \Leftrightarrow a \in c) \Rightarrow b == c$	*(extensionality)*
$\neg\, a \in \emptyset$	*(empty set)*
$a \in \{b, c\} \Leftrightarrow (a == b \vee a == c)$	*(pairing)*
$a \in \bigcup b \Leftrightarrow \exists_c\, (c \in b \wedge a \in c)$	*(union)*
$\exists_a\, \forall_b\, (b \in a \Leftrightarrow \forall_c\, (c \in b \Rightarrow c \in d))$	*(power set)*
$\exists_a\, \forall_b\, (b \in a \Leftrightarrow (b \in c \wedge \alpha\_))\ /;\ \mathit{FreeQ}[\alpha,\ a]$	*(subset)*
$\exists_a\, (\emptyset \in a \wedge \forall_b\, (b \in a \Rightarrow \bigcup \{b, \{b\}\} \in a))$	*(infinity)*
$\forall_a\, (a \in b \Rightarrow \forall_c\, \forall_d\, ((\alpha\_ \vert_{e \to c} \wedge \alpha\_ \vert_{e \to d}) \Rightarrow c == d)) \Rightarrow$ $\exists_f\, \forall_g\, (g \in f \Leftrightarrow \exists_a\, (a \in b \wedge \alpha\_ \vert_{e \to g}))\ /;\ \mathit{FreeQ}[\alpha,\ c \vert d \vert f \vert g]$	*(replacement)*

$(\neg\, \emptyset \in a \wedge \forall_b\, \forall_c\, ((b \in a \wedge c \in a \wedge b \neq c) \Rightarrow b \cap c == \emptyset)) \Rightarrow$ $\exists_d\, \forall_b\, (b \in a \Rightarrow \exists_e\, d \cap b == \{e\})$	*(choice)*

$a \neq \emptyset \Rightarrow \exists_b\, (b \in a \wedge a \cap b == \emptyset)$	*(regularity)*

set theory

$\exists_{a\_}\, b\_ \longrightarrow \neg\, \forall_a\, \neg\, b$
$a\_ \Rightarrow b\_ \longrightarrow \neg\, a \vee b$
$a\_ \Leftrightarrow b\_ \longrightarrow (a \Rightarrow b) \wedge (b \Rightarrow a)$
$a\_ \cap b\_ == c\_ \longrightarrow$ $\forall_n\, (n \in c \Leftrightarrow (n \in a \wedge n \in b))$
$\{a\_\} \longrightarrow \{a, a\}$
$a\_ \vert_{b\_ \to c\_} \longrightarrow \forall_b\, (b == c \Rightarrow a)$
$a\_ \subseteq b\_ \longrightarrow \forall_n\, (n \in a \Rightarrow n \in b)$
$a\_ \neq b\_ \longrightarrow \neg\, (a == b)$

definitions

$\wedge$	*and*
$\vee$	*or*
$\neg$	*not*
$\bar{\wedge}$	*nand*
$\forall$	*for all*
$\exists$	*there exists*
Δ	*next integer*
$\cdot$	*composition*
$\bar{\circ}$	*inverse composition*
$\overline{\square}$	*inverse*
$\oplus$	*generalized addition*
$\otimes$	*generalized multiplication*
$\square^{-1}$	*reciprocal*
$\triangleleft$	*left identity morphism*
$\triangleright$	*right identity morphism*
$\Box$	*morphism mismatch*
$(\square \wr \square \wr \square)$	*is between*
$\overset{\frown}{\square\square} \equiv \overset{\frown}{\square\square}$	*segments are congruent*
$\in$	*element of*
$\emptyset$	*empty set*
$\{\square, \square\}$	*pair*
$\bigcup$	*union*
$\mathbb{X}$	*set of all points*
$\mathbb{O}$	*set of all open sets*
$\mathbb{R}$	*set of all real numbers*

typical interpretations

set theory and ...

$a \in \mathbb{O} \Rightarrow a \subseteq \mathbb{X}$
$\emptyset \in \mathbb{O} \wedge \mathbb{X} \in \mathbb{O}$
$(a \in \mathbb{O} \wedge b \in \mathbb{O} \wedge a \cap b == c) \Rightarrow c \in \mathbb{O}$
$a \subseteq \mathbb{O} \Rightarrow \bigcup a \in \mathbb{O}$

general topology

real algebra with all objects restricted to $\mathbb{R}$
$(a \subseteq \mathbb{R} \wedge a \neq \emptyset \wedge \exists_b\, (b \in \mathbb{R} \wedge \forall_c\, (c \in a \Rightarrow c > b))) \Rightarrow$ $\exists_b\, (b \in \mathbb{R} \wedge \forall_d\, (d \in \mathbb{R} \Rightarrow (d > b \Leftrightarrow \exists_c\, (c \in a \wedge d > c))))$

real analysis

Further axiom systems for traditional mathematics. The typical interpretations are relevant for applications, though not for formal derivation of theorems. The last two axioms listed for set theory are usually considered optional.

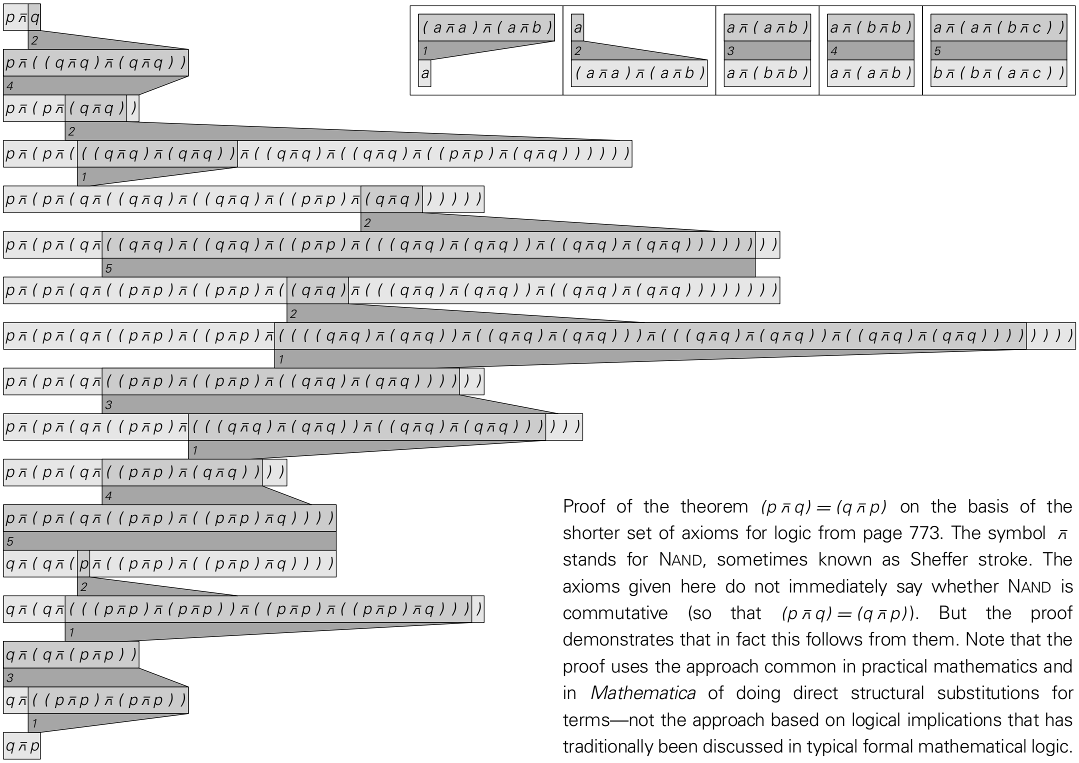
that they ultimately tend to be equivalent in their computational sophistication—and thus show all sorts of similar phenomena.

And what we will see in this section is while some of these phenomena correspond to known features of mathematics—such as Gödel's Theorem—many have never successfully been recognized.

But just what basic processes are involved in mathematics?

Ever since antiquity mathematics has almost defined itself as being concerned with finding theorems and giving their proofs. And in any particular branch of mathematics a proof consists of a sequence of steps ultimately based on axioms like those of the previous two pages.

The picture below gives a simple example of how this works in basic logic. At the top right are axioms specifying certain fundamental equivalences between logic expressions. A proof of the equivalence $p \barwedge q = q \barwedge p$ between logic expressions is then formed by applying these axioms in the particular sequence shown.



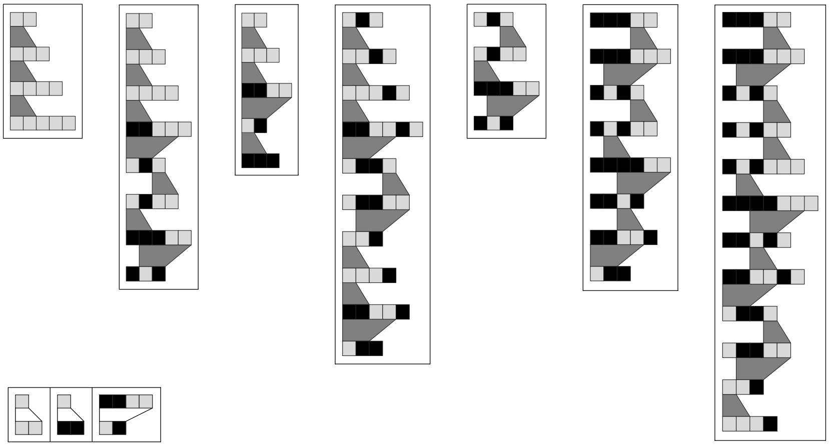
(a⊼a)⊼(a⊼b)
1
a
a
2
(a⊼a)⊼(a⊼b)
a⊼(a⊼b)
3
a⊼(b⊼b)
a⊼(b⊼b)
4
a⊼(a⊼b)
a⊼(a⊼(b⊼c))
5
b⊼(b⊼(a⊼c))
p⊼q
2
p⊼((q⊼q)⊼(q⊼q))
4
p⊼(p⊼(q⊼q))
2
p⊼(p⊼(((q⊼q)⊼(q⊼q))⊼((q⊼q)⊼((q⊼q)⊼((p⊼p)⊼(q⊼q))))))
1
p⊼(p⊼(q⊼((q⊼q)⊼((q⊼q)⊼((p⊼p)⊼(q⊼q))))))
2
p⊼(p⊼(q⊼((q⊼q)⊼((q⊼q)⊼((p⊼p)⊼(((q⊼q)⊼(q⊼q))⊼((q⊼q)⊼(q⊼q))))))))
5
p⊼(p⊼(q⊼((p⊼p)⊼((p⊼p)⊼((q⊼q)⊼(((q⊼q)⊼(q⊼q))⊼((q⊼q)⊼(q⊼q))))))))
2
p⊼(p⊼(q⊼((p⊼p)⊼((((q⊼q)⊼(q⊼q))⊼((q⊼q)⊼(q⊼q)))⊼(((q⊼q)⊼(q⊼q))⊼((q⊼q)⊼(q⊼q)))))))
1
p⊼(p⊼(q⊼((p⊼p)⊼((p⊼p)⊼((q⊼q)⊼(q⊼q))))))
3
p⊼(p⊼(q⊼((p⊼p)⊼(((q⊼q)⊼(q⊼q))⊼((q⊼q)⊼(q⊼q))))))
1
p⊼(p⊼(q⊼((p⊼p)⊼(q⊼q))))
4
p⊼(p⊼(q⊼((p⊼p)⊼((p⊼p)⊼q))))
5
q⊼(q⊼(p⊼((p⊼p)⊼((p⊼p)⊼q))))
2
q⊼(q⊼(((p⊼p)⊼(p⊼p))⊼((p⊼p)⊼((p⊼p)⊼q))))
1
q⊼(q⊼(p⊼p))
3
q⊼((p⊼p)⊼(p⊼p))
1
q⊼p

Proof of the theorem $(p \barwedge q) = (q \barwedge p)$ on the basis of the shorter set of axioms for logic from page 773. The symbol $\barwedge$ stands for Nand, sometimes known as Sheffer stroke. The axioms given here do not immediately say whether Nand is commutative (so that $(p \barwedge q) = (q \barwedge p)$). But the proof demonstrates that in fact this follows from them. Note that the proof uses the approach common in practical mathematics and in *Mathematica* of doing direct structural substitutions for terms—not the approach based on logical implications that has traditionally been discussed in typical formal mathematical logic.

In most kinds of mathematics there are all sorts of additional details, particularly about how to determine which parts of one or more previous expressions actually get used at each step in a proof. But much as in our study of systems in nature, one can try to capture the essential features of what can happen by using a simple idealized model.

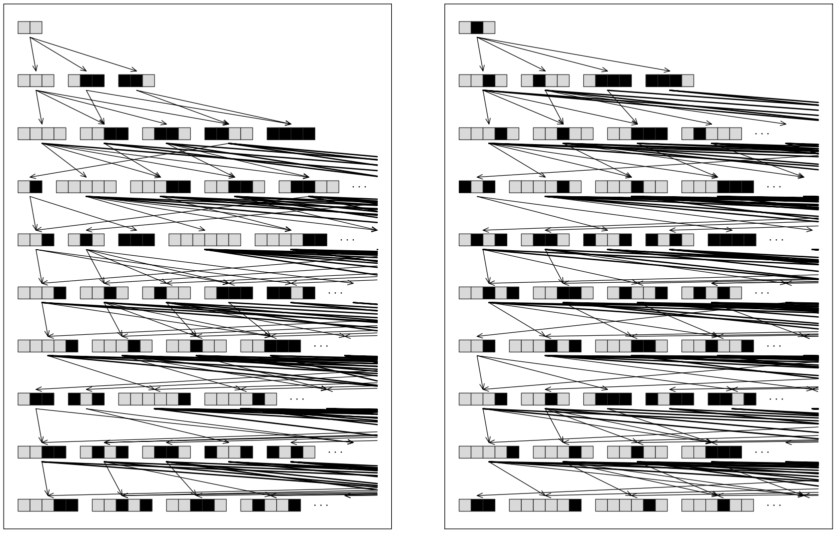
And so for example one can imagine representing a step in a proof just by a string of simple elements such as black and white squares. And one can then consider the axioms of a system as defining possible transformations from one sequence of these elements to another—just like the rules in the multiway systems we discussed in Chapter 5.

The pictures below show how proofs of theorems work with this setup. Each theorem defines a connection between strings, and proving the theorem consists in finding a series of transformations—each associated with an axiom—that lead from one string to another.



Simple idealizations of proofs in mathematics. The rules on the left in effect correspond to axioms that specify valid transformations between strings of black and white elements. The proofs above then show how one string—say □□—can be transformed into another—say □□□□□—by using the axioms. Typically there are many different proofs that can be given of a particular theorem; here in each case the ones shown are examples of the shortest possible proofs. The system shown is an example of a general substitution system of the kind discussed on page 497. Note that the fifth theorem □■□ → ■□■ occurs in effect as a lemma in the second theorem □□ → ■□■.

But just as in the multiway systems in Chapter 5 one can also consider an explicit process of evolution, in which one starts from a particular string, then at each successive step one applies all possible transformations, so that in the end one builds up a whole network of connections between strings, as in the pictures below.



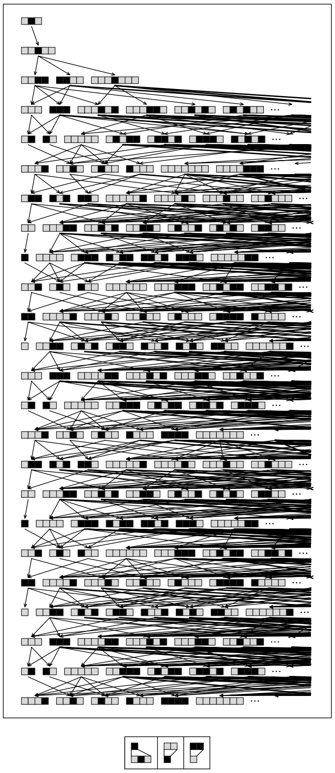
The result of applying the same transformations as on the facing page—but in all possible ways, corresponding to the evolution of a multiway system that represents all possible theorems that can be derived from the axioms. With the axioms used here, the total number of strings grows by a factor of roughly 1.7 at each step; on the last steps shown there are altogether 237 and 973 strings respectively.

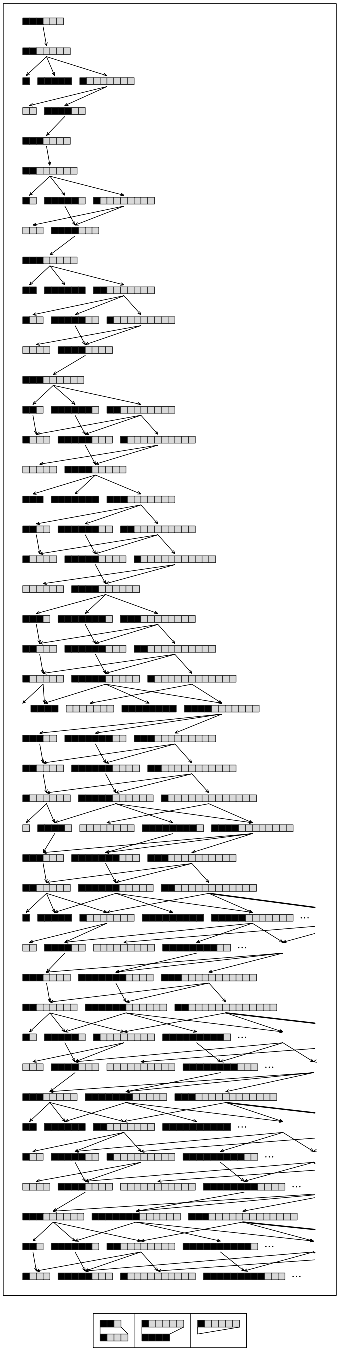
In a sense such a network can then be thought of as representing the whole field of mathematics that can be derived from whatever set of axioms one is using—with every connection between strings corresponding to a theorem, and every possible path to a proof.

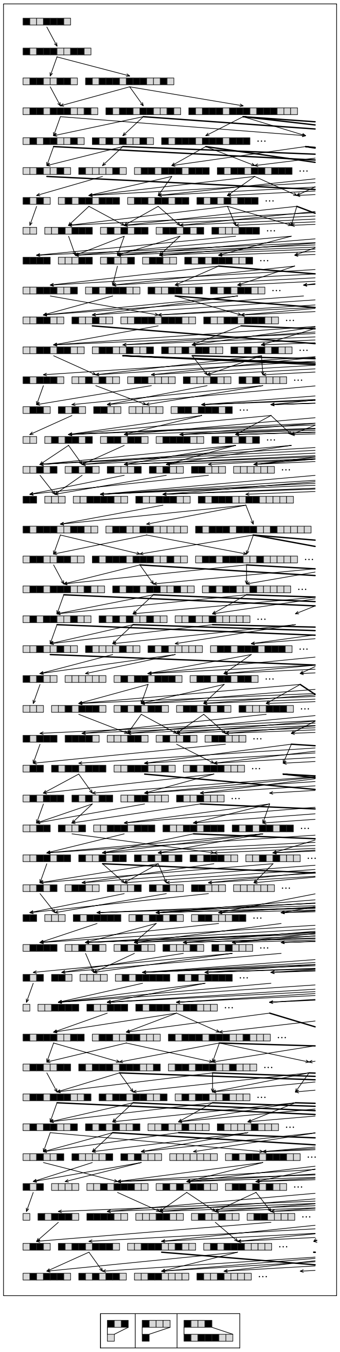
But can networks like the ones above really reflect mathematics as it is actually practiced? For certainly the usual axioms in every traditional area of mathematics are significantly more complicated than any of the multiway system rules used above.

But just like in so many other cases in this book, it seems that even systems whose underlying rules are remarkably simple are already able to capture many of the essential features of mathematics.

An obvious observation in mathematics is that proofs can be difficult to do. One might at first assume that any theorem that is easy







Three examples of multiway systems that show the analog of long proofs. In each case a string consisting of a single white element is eventually generated—but this takes respectively 12, 28 and 34 steps to happen. The first multiway system actually generates all strings in the end (not least since it yields the lemmas ■ → ■■ and ■ → □)—and in fact strings of length $n > 2$ appear after at most $2n + 7$ steps. The second multiway system generates only the $n + 1$ strings where black comes before white—and all of these strings appear after at most $7n$ steps. The third multiway system generates a complicated collection of strings; the numbers of lengths up to 8 are 1, 2, 4, 8, 14, 22, 34, 45. All the strings generated have an even number of black elements.

to state will also be easy to prove. But experience suggests that this is far from correct. And indeed there are all sorts of well-known examples—such as Fermat's Last Theorem and the Four-Color Theorem—in which a theorem that is easy to state seems to require a proof that is immensely long.

So is there an analog of this in multiway systems? It turns out that often there is, and it is that even though a string may be short it may nevertheless take a great many steps to reach.

If the rules for a multiway system always increase string length then it is inevitable that any given string that is ever going to be generated must appear after only a limited number of steps. But if the rules can both increase and decrease string length the story is quite different, as the picture on the facing page illustrates. And often one finds that even a short string can take a rather large number of steps to produce.

But are all these steps really necessary? Or is it just that the rule one has used is somehow inefficient, and there are other rules that generate the short strings much more quickly?

Certainly one can take the rules for any multiway system and add transformations that immediately generate particular short strings. But the crucial point is that like so many other systems I have discussed in this book there are many multiway systems that I suspect are computationally irreducible—so that there is no way to shortcut their evolution, and no general way to generate their short strings quickly.

And what I believe is that essentially the same phenomenon operates in almost every area of mathematics. Just like in multiway systems, one can always add axioms to make it easier to prove particular theorems. But I suspect that ultimately there is almost always computational irreducibility, and this makes it essentially inevitable that there will be short theorems that only allow long proofs.

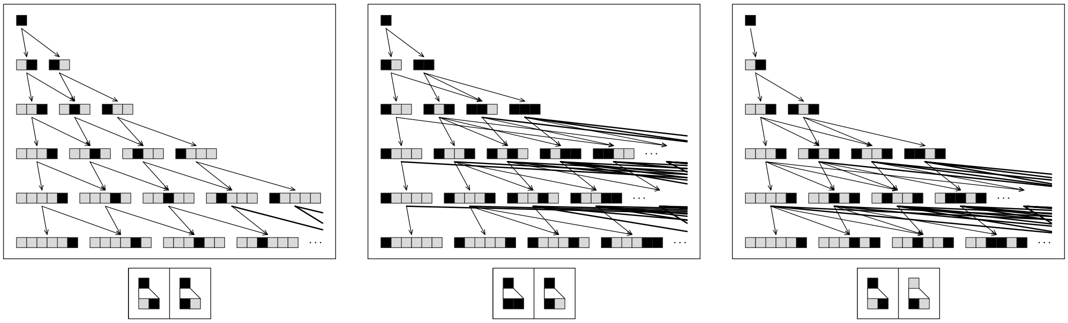
In the previous section we saw that computational irreducibility tends to make infinite questions undecidable. So for example the question of whether a particular string will ever be generated in the evolution of a multiway system—regardless of how long one waits—is in general undecidable. And similarly it can be undecidable whether any proof—regardless of length—exists for a specific result in a mathematical system with particular axioms.

So what are the implications of this?

Probably the most striking arise when one tries to apply traditional ideas of logic—and particularly notions of true and false.

The way I have set things up, one can find all the statements that can be proved true in a particular axiom system just by starting with an expression that represents "true" and then using the rules of the axiom system, as in the picture on the facing page.

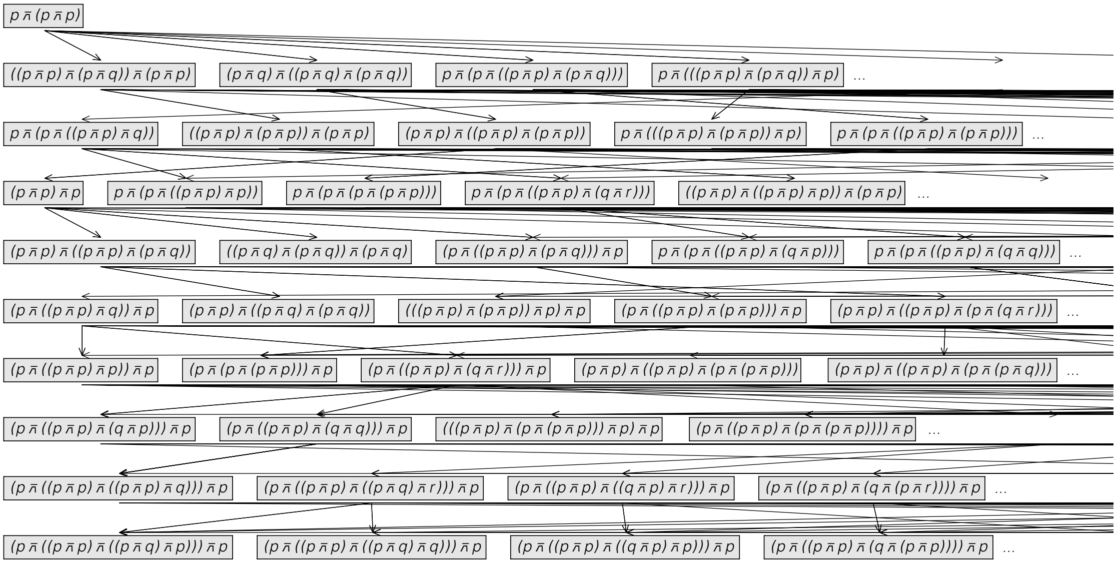
In a multiway system, one can imagine identifying "true" with a string consisting of a single black element. And this would mean that every string in networks like the ones below should correspond to a statement that can be proved true in the axiom system used.



Multiway systems starting from a single black element that represents TRUE. All strings that appear can be thought of as statements that are true according to the axioms represented by the multiway system rules. One can take negation to be the operation that interchanges black and white. This then means that the first multiway system represents an inconsistent axiom system, since on step 2, both □■ and its negation ■□ appear. The other two multiway systems are consistent, so that they never generate both a string and its negation. The third one, however, is incomplete, since for example it never generates either □□ or its negation ■■. The second one, however, is both complete and consistent: it generates all strings that begin with ■, but none that begin with □.

But is this really reasonable? In traditional logic there is always an operation of negation which takes any true statement, and makes it into a false one, and vice versa. And in a multiway system, one possible way negation might work is just to reverse the colors of the elements in a string. But this then leads to a problem in the first picture above.

For the picture implies that both □■ and its negation ■□ can be proved to be true statements. But this cannot be correct. And so what


p ⊼ (p ⊼ p)
((p ⊼ p) ⊼ (p ⊼ q)) ⊼ (p ⊼ p)
(p ⊼ q) ⊼ ((p ⊼ q) ⊼ (p ⊼ q))
p ⊼ (p ⊼ ((p ⊼ p) ⊼ (p ⊼ q)))
p ⊼ (((p ⊼ p) ⊼ (p ⊼ q)) ⊼ p)
...
p ⊼ (p ⊼ ((p ⊼ p) ⊼ q))
((p ⊼ p) ⊼ (p ⊼ p)) ⊼ (p ⊼ p)
(p ⊼ p) ⊼ ((p ⊼ p) ⊼ (p ⊼ p))
p ⊼ (((p ⊼ p) ⊼ (p ⊼ p)) ⊼ p)
p ⊼ (p ⊼ ((p ⊼ p) ⊼ (p ⊼ p)))
...
(p ⊼ p) ⊼ p
p ⊼ (p ⊼ ((p ⊼ p) ⊼ p))
p ⊼ (p ⊼ (p ⊼ (p ⊼ p)))
p ⊼ (p ⊼ ((p ⊼ p) ⊼ (q ⊼ r)))
((p ⊼ p) ⊼ ((p ⊼ p) ⊼ p)) ⊼ (p ⊼ p)
...
(p ⊼ p) ⊼ ((p ⊼ p) ⊼ (p ⊼ q))
((p ⊼ q) ⊼ (p ⊼ q)) ⊼ (p ⊼ q)
(p ⊼ ((p ⊼ p) ⊼ (p ⊼ q))) ⊼ p
p ⊼ (p ⊼ ((p ⊼ p) ⊼ (q ⊼ p)))
p ⊼ (p ⊼ ((p ⊼ p) ⊼ (q ⊼ q)))
...
(p ⊼ ((p ⊼ p) ⊼ q)) ⊼ p
(p ⊼ p) ⊼ ((p ⊼ q) ⊼ (p ⊼ q))
(((p ⊼ p) ⊼ (p ⊼ p)) ⊼ p) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ (p ⊼ p))) ⊼ p
(p ⊼ p) ⊼ ((p ⊼ p) ⊼ (p ⊼ (q ⊼ r)))
...
(p ⊼ ((p ⊼ p) ⊼ p)) ⊼ p
(p ⊼ (p ⊼ (p ⊼ p))) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ (q ⊼ r))) ⊼ p
(p ⊼ p) ⊼ ((p ⊼ p) ⊼ (p ⊼ (p ⊼ p)))
(p ⊼ p) ⊼ ((p ⊼ p) ⊼ (p ⊼ (p ⊼ q)))
...
(p ⊼ ((p ⊼ p) ⊼ (q ⊼ p))) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ (q ⊼ q))) ⊼ p
(((p ⊼ p) ⊼ (p ⊼ (p ⊼ p))) ⊼ p) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ (p ⊼ (p ⊼ p)))) ⊼ p
...
(p ⊼ ((p ⊼ p) ⊼ ((p ⊼ p) ⊼ q))) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ ((p ⊼ q) ⊼ r))) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ ((q ⊼ p) ⊼ r))) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ (q ⊼ (p ⊼ r)))) ⊼ p
...
(p ⊼ ((p ⊼ p) ⊼ ((p ⊼ q) ⊼ p))) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ ((p ⊼ q) ⊼ q))) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ ((q ⊼ p) ⊼ p))) ⊼ p
(p ⊼ ((p ⊼ p) ⊼ (q ⊼ (p ⊼ p)))) ⊼ p
...

The network of statements that can be proved true using the axiom system for logic from page 775. $p \barwedge (p \barwedge p)$ is the simplest representation for TRUE when logic is set up using the NAND operator $\barwedge$. Each arrow indicates an equivalence established by applying a single axiom. On each row only statements that have not appeared before are given. The statements are sorted so that the simplest are first. Note that some fairly simple statements do not show up for at least several rows. The total number of statements on successive rows grows faster than exponentially; for the first few it is 1, 6, 91, 2180, 76138. If continued forever the network would eventually include all possible true statements (tautologies) of logic (see also page 818). Other simple axiom systems for logic like those on page 808 yield networks similar to the one shown.

this means is that with the setup used the underlying axiom system is inconsistent. So what about the other multiway systems on the facing page? At least with the strings one can see in the pictures there are no inconsistencies. But what about with longer strings? For the particular rules shown it is fairly easy to demonstrate that there are never inconsistencies. But in general it is not possible to do this, for after some given string has appeared, it can for example be undecidable whether the negation of that particular string ever appears.

So what about the axiom systems normally used in actual mathematics? None of those on pages 773 and 774 appear to be inconsistent. And what this means is that the set of statements that can be proved true will never overlap with the set that can be proved false.

But can every possible statement that one might expect to be true or false actually in the end be proved either true or false?

In the early 1900s it was widely believed that this would effectively be the case in all reasonable mathematical axiom systems. For at the time there seemed to be no limit to the power of mathematics, and no end to the theorems that could be proved.

But this all changed in 1931 when Gödel's Theorem showed that at least in any finitely-specified axiom system containing standard arithmetic there must inevitably be statements that cannot be proved either true or false using the rules of the axiom system.

This was a great shock to existing thinking about the foundations of mathematics. And indeed to this day Gödel's Theorem has continued to be widely regarded as a surprising and rather mysterious result.

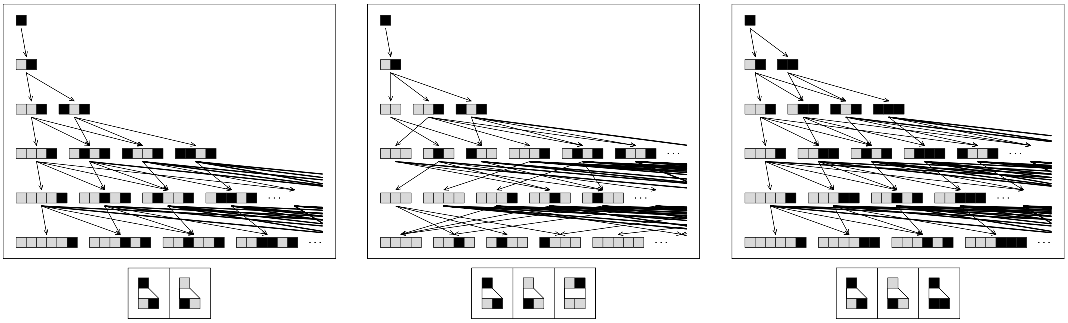
But the discoveries in this book finally begin to make it seem inevitable and actually almost obvious. For it turns out that at some level it can be viewed as just yet another consequence of the very general Principle of Computational Equivalence.

So what is the analog of Gödel's Theorem for multiway systems? Given the setup on page 780 one can ask whether a particular multiway system is complete in the sense that for every possible string the system eventually generates either that string or its negation.

And one can see that in fact the third multiway system is incomplete, since by following its rules one can never for example generate either □□ or its negation ■■. But what if one extends the rules by adding more transformations, corresponding to more axioms? Can one always in the end make the system complete?

If one is not quite careful, one will generate too many strings, and inevitably get inconsistencies where both a string and its negation appear, as in the second picture on the facing page. But at least if one only has to worry about a limited number of steps, it is always possible to set things up so as to get a system that is both complete and consistent, as in the third picture on the facing page.

And in fact in the particular case shown on the facing page it is fairly straightforward to find rules that make the system always complete and consistent. But knowing how to do this requires having behavior that is in a sense simple enough that one can foresee every aspect of it.



The effect of adding transformations to the rules for a multiway system. The first multiway system is incomplete, in the sense that for some strings, it generates neither the string nor its negation. The second multiway system yields more strings—but introduces inconsistency, since it can generate both □■□ and its negation ■□■. The third multiway system is however both complete and consistent: for every string it eventually generates either that string or its negation.

Yet if a system is computationally irreducible this will inevitably not be possible. For at any point the system will always in effect be able to do more things that one did not expect. And this means that in general one will not be able to construct a finite set of axioms that can be guaranteed to lead to ultimate completeness and consistency.

And in fact it turns out that as soon as the question of whether a particular string can ever be reached is undecidable it immediately follows that there must be either incompleteness or inconsistency. For to say that such a question is undecidable is to say that it cannot in general be answered by any procedure that is guaranteed to finish.

But if one had a system that was complete and consistent then it is easy to come up with such a procedure: one just runs the system until either one reaches the string one is looking for or one reaches its negation. For the completeness of the system guarantees that one must always reach one or the other, while its consistency implies that reaching one allows one to conclude that one will never reach the other.

So the result of this is that if the evolution of a multiway system is computationally irreducible—so that questions about its ultimate behavior are undecidable—the system cannot be both complete and consistent. And if one assumes consistency then it follows that there must be strings where neither the string nor its negation can be reached—corresponding to the fact that statements must exist that cannot be proved either true or false from a given set of axioms.

But what does it take to establish that such incompleteness will actually occur in a specific system?

The basic way to do it is to show that the system is universal.

But what exactly does universality mean for something like an axiom system? In effect what it means is that any question about the behavior of any other universal system can be encoded as a statement in the axiom system—and if the answer to the question can be established by watching the evolution of the other universal system for any finite number of steps then it must also be able to be established by giving a proof of finite length in the axiom system.

So what axiom systems in mathematics are then universal?

Basic logic is not, since at least in principle one can always determine the truth of any statement in this system by the finite—if perhaps exponentially long—procedure of trying all possible combinations of truth values for the variables that appear in it.

And essentially the same turns out to be the case for pure predicate logic, in which one just formally adds "for all" and "there exists" constructs. But as soon as one also puts in an abstract function or relation with more than one argument, one gets universality.

And indeed the basis for Gödel's Theorem is the result that the standard axioms for basic integer arithmetic support universality.

Set theory and several other standard axiom systems can readily be made to reproduce arithmetic, and are therefore also universal. And the same is true of group theory and other algebraic systems like ring theory.

If one puts enough constraints on the axioms one uses, one can eventually prevent universality—and in fact this happens for commutative group theory, and for the simplified versions of both real algebra and geometry on pages 773 and 774.

But of the axiom systems actually used in current mathematics research every single one is now known to be universal.

From page 773 we can see that many of these axiom systems can be stated in quite simple ways. And in the past it might have seemed hard to believe that systems this simple could ever be universal, and thus in a sense be able to emulate essentially any system.

But from the discoveries in this book this now seems almost inevitable. And indeed the Principle of Computational Equivalence implies that beyond some low threshold almost any axiom system should be expected to be universal.

So how does universality actually work in the case of arithmetic?

One approach is illustrated in the picture on the next page. The idea is to set up an arithmetic statement that can be proved true if the evolution of a cellular automaton from a given initial condition makes a given cell be a given color at a given step, and can be proved false if it does not.

By changing numbers in this arithmetic statement one can then in effect sample different aspects of the cellular automaton evolution. And with the cellular automaton being a universal one such as rule 110 this implies that the axioms of arithmetic can support universality.

Such universality then implies Gödel's Theorem and shows that there must exist statements about arithmetic that cannot ever be proved true or false from its normal axioms.

So what are some examples of such statements?

The original proof of Gödel's Theorem was based on considering the particular self-referential statement "this statement is unprovable".

At first it does not seem obvious that such a statement could ever be set up as a statement in arithmetic. But if it could then one can see that it would immediately follow that—as the statement says—it cannot be proved, since otherwise there would be an inconsistency.

And in fact the main technical difficulty in the original proof of Gödel's Theorem had to do with showing—by doing what amounted to establishing the universality of arithmetic—that the statement could indeed meaningfully be encoded as a statement purely in arithmetic.

But at least with the original encoding used, the statement would be astronomically long if written out in the notation of page 773. And from this result, one might imagine that unprovability would never be relevant in any practical situation in mathematics.

But does one really need to have such a complicated statement in order for it to be unprovable from the axioms of arithmetic?

$$
\begin{aligned}
&(-3 x_6 + x_7 + x_8)^2 + (2^{1+x_3 (1+x_1+2 x_3)} x_2 - 2 x_4 - x_{10} + x_{11})^2 + (-2 x_8 - x_9 + x_{10} + x_{11})^2 + (1 - 2^{(1+x_3)(x_1+2 x_3)} + x_4 + x_{12})^2 + \\
&\quad (1 - 2^{x_1} + x_2 + x_{13})^2 + (1 - 2^{x_1} + x_5 + x_{14})^2 + (-x_4 + 2^{x_3} x_5 + 2^{x_1+2 x_3} x_6 + 2^{x_1+x_3} x_{15} + x_{16})^2 + (1 - 2^{x_3} + x_{15} + x_{17})^2 + \\
&\quad (1 - 2^{x_3} + x_{16} + x_{18})^2 + (-x_6 - 2 x_7 + x_9 + x_{19})^2 + (-(2 + 2^{x_6})^{x_6} + (1 + 2^{x_6})^{x_7} (1 + 2 x_{20} + (1 + 2^{x_6}) x_{21}) + x_{22})^2 + (1 - (1 + 2^{x_6})^{x_7} + x_{22} + x_{23})^2 + \\
&\quad (1 - 2^{x_6} + 2 x_{20} + x_{24})^2 + (-(2 + 4^{x_6})^{2 x_6} + (1 + 4^{x_6})^{x_7} (1 + 2 x_{25} + (1 + 4^{x_6}) x_{26}) + x_{27})^2 + (1 - (1 + 4^{x_6})^{x_7} + x_{27} + x_{28})^2 + \\
&\quad (1 - 4^{x_6} + 2 x_{25} + x_{29})^2 + (-(2 + 2^{x_8})^{x_8} + (1 + 2^{x_8})^{x_6} (1 + 2 x_{30} + (1 + 2^{x_8}) x_{31}) + x_{32})^2 + (1 - (1 + 2^{x_8})^{x_6} + x_{32} + x_{33})^2 + \\
&\quad (1 - 2^{x_8} + 2 x_{30} + x_{34})^2 + (-(2 + 2^{x_8})^{x_8} + (1 + 2^{x_8})^{2 x_6} (1 + 2 x_{35} + (1 + 2^{x_8}) x_{36}) + x_{37})^2 + (1 - (1 + 2^{x_8})^{2 x_6} + x_{37} + x_{38})^2 + \\
&\quad (1 - 2^{x_8} + 2 x_{35} + x_{39})^2 + (-(2 + 2^{x_6})^{x_6} + (1 + 2^{x_6})^{x_9} (1 + 2 x_{40} + (1 + 2^{x_6}) x_{41}) + x_{42})^2 + (1 - (1 + 2^{x_6})^{x_9} + x_{42} + x_{43})^2 + \\
&\quad (1 - 2^{x_6} + 2 x_{40} + x_{44})^2 + (-(2 + 4^{x_7})^{2 x_7} + (1 + 4^{x_7})^{x_9} (1 + 2 x_{45} + (1 + 4^{x_7}) x_{46}) + x_{47})^2 + (1 - (1 + 4^{x_7})^{x_9} + x_{47} + x_{48})^2 + \\
&\quad (1 - 4^{x_7} + 2 x_{45} + x_{49})^2 + (-(2 + 2^{x_{19}})^{x_{19}} + (1 + 2^{x_{19}})^{x_6} (1 + 2 x_{50} + (1 + 2^{x_{19}}) x_{51}) + x_{52})^2 + (1 - (1 + 2^{x_{19}})^{x_6} + x_{52} + x_{53})^2 + \\
&\quad (1 - 2^{x_{19}} + 2 x_{50} + x_{54})^2 + (-(2 + 2^{x_{19}})^{x_{19}} + (1 + 2^{x_{19}})^{2 x_7} (1 + 2 x_{55} + (1 + 2^{x_{19}}) x_{56}) + x_{57})^2 + (1 - (1 + 2^{x_{19}})^{2 x_7} + x_{57} + x_{58})^2 + \\
&\quad (1 - 2^{x_{19}} + 2 x_{55} + x_{59})^2 + (-(2 + 2^{x_9})^{x_9} + (1 + 2^{x_9})^{x_{10}} (1 + 2 x_{60} + (1 + 2^{x_9}) x_{61}) + x_{62})^2 + (1 - (1 + 2^{x_9})^{x_{10}} + x_{62} + x_{63})^2 + (1 - 2^{x_9} + 2 x_{60} + x_{64})^2 + \\
&\quad (-(2 + 4^{x_8})^{2 x_8} + (1 + 4^{x_8})^{x_{10}} (1 + 2 x_{65} + (1 + 4^{x_8}) x_{66}) + x_{67})^2 + (1 - (1 + 4^{x_8})^{x_{10}} + x_{67} + x_{68})^2 + (1 - 4^{x_8} + 2 x_{65} + x_{69})^2 + \\
&\quad (-(2 + 2^{x_{11}})^{x_{11}} + (1 + 2^{x_{11}})^{x_9} (1 + 2 x_{70} + (1 + 2^{x_{11}}) x_{71}) + x_{72})^2 + (1 - (1 + 2^{x_{11}})^{x_9} + x_{72} + x_{73})^2 + (1 - 2^{x_{11}} + 2 x_{70} + x_{74})^2 + \\
&\quad (-(2 + 2^{x_{11}})^{x_{11}} + (1 + 2^{x_{11}})^{2 x_8} (1 + 2 x_{75} + (1 + 2^{x_{11}}) x_{76}) + x_{77})^2 + (1 - (1 + 2^{x_{11}})^{2 x_8} + x_{77} + x_{78})^2 + (1 - 2^{x_{11}} + 2 x_{75} + x_{79})^2 = 0
\end{aligned}
$$

x_1 (initial width)	1
x_2 (initial state)	1
x_3 (steps)	1
x_4 (evolution)	22
x_5	1
x_6	2
x_7	0
x_8	6
x_9	0
x_{10}	0
x_{11}	12
x_{12}	41
x_{13}	0
x_{14}	0
x_{15}	1
⋮	

x_1 (initial width)	1
x_2 (initial state)	1
x_3 (steps)	2
x_4 (evolution)	4508
x_5	1
x_6	140
x_7	8
x_8	412
x_9	0
x_{10}	0
x_{11}	824
x_{12}	28259
x_{13}	0
x_{14}	0
x_{15}	3
⋮	

x_1 (initial width)	1
x_2 (initial state)	1
x_3 (steps)	3
x_4 (evolution)	17177704
x_5	1
x_6	134200
x_7	2096
x_8	400504
x_9	32
x_{10}	32
x_{11}	801008
x_{12}	251257751
x_{13}	0
x_{14}	0
x_{15}	6
⋮	

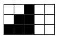
x_1 (initial width)	1
x_2 (initial state)	1
x_3 (steps)	4
x_4 (evolution)	1105983545840
x_5	1
x_6	2160124112
x_7	8437888
x_8	6471934448
x_9	32768
x_{10}	32768
x_{11}	12943868896
x_{12}	34078388542991
x_{13}	0
x_{14}	0
x_{15}	15
⋮	

x_1 (initial width)	3
x_2 (initial state)	5
x_3 (steps)	4
x_4 (evolution)	1409438147512048
x_5	7
x_6	688202220464
x_7	940049184
x_8	2063666612208
x_9	805306880
x_{10}	805306880
x_{11}	4127333224416
x_{12}	34619358871451919
x_{13}	2
x_{14}	0
x_{15}	13
⋮	

$x_4 = 22 =$

```
0 1 0
1 1 0
```

$x_4 = 4508 =$



```
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0 1 1 0 0
1 1 1 0 0
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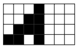

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1 1 0 1 0 0 0
```

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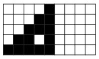

```
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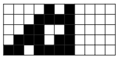
0 0 0 1 1 0 0 0 0
0 0 1 1 1 0 0 0 0
0 1 1 0 1 0 0 0 0
1 1 1 1 1 0 0 0 0
```

$x_4 = 1409438147512048 =$



```
0 0 0 0 1 0 1 0 0 0 0
0 0 0 1 1 1 1 0 0 0 0
0 0 1 1 0 0 1 0 0 0 0
0 1 1 1 0 1 1 0 0 0 0
1 1 0 1 1 1 1 0 0 0 0
```

Universality in arithmetic, illustrated by an integer equation whose solutions in effect emulate the rule 110 universal cellular automaton from Chapter 11. The equation has many solutions, but all of them satisfy the constraint that the variables x_1 through x_4 must encode possible initial conditions and evolution histories for rule 110. If one fills in fixed values for x_1, x_2 and x_3, then only one value for x_4 is ever possible—corresponding to the evolution history of rule 110 for x_3 steps starting from a width x_1 initial condition given by the digit sequence of x_2. In general any statement about the possible behavior of rule 110 can be encoded as a statement in arithmetic about solutions to the equation. So for example if one fills in values for x_1, x_2 and x_4, but not x_3, then the statement that the equation has no solution for any x_3 corresponds to a statement that rule 110 can never exhibit certain behavior, even after any number of steps. But the universality of rule 110 implies that such statements must in general be undecidable. So from this it follows that in at least some instances the axioms of arithmetic can never be used to give a finite proof of whether or not the statement is true. The construction shown here can be viewed as providing a simple proof of Gödel's Theorem on the existence of unprovable statements in arithmetic. Note that the equation shown is a so-called exponential Diophantine one, in which some variables appear in exponents. At the cost of considerably more complication—and using for example 2154 variables—it is possible to avoid this. The equation above can however already be viewed as capturing the essence of what is needed to demonstrate the general unsolvability of Diophantine equations and Hilbert's Tenth Problem.

Over the past seventy years a few simpler examples have been constructed—mostly with no obviously self-referential character.

But usually these examples have involved rather sophisticated and obscure mathematical constructs—most often functions that are somehow set up to grow extremely rapidly. Yet at least in principle there should be examples that can be constructed based just on statements that no solutions exist to particular integer equations.

If an integer equation such as $x^2 = y^3 + 12$ has a definite solution such as $x = 47$, $y = 13$ in terms of particular finite integers then this fact can certainly be proved using the axioms of arithmetic. For it takes only a finite calculation to check the solution, and this very calculation can always in effect be thought of as a proof.

But what if the equation has no solutions? To test this explicitly one would have to look at an infinite number of possible integers. But the point is that even so, there can still potentially be a finite mathematical proof that none of these integers will work.

And sometimes the proof may be straightforward—say being based on showing that one side of the equation is always odd while the other is always even. In other cases the proof may be more difficult—say being based on establishing some large maximum size for a solution, then checking all integers up to that size.

And the point is that in general there may in fact be absolutely no proof that can be given in terms of the normal axioms of arithmetic.

So how can one see this?

The picture on the facing page shows that one can construct an integer equation whose solutions represent the behavior of a system like a cellular automaton. And the way this works is that for example one variable in the equation gives the number of steps of evolution, while another gives the outcome after that number of steps.

So with this setup, one can specify the number of steps, then solve for the outcome after that number of steps. But what if for example one instead specifies an outcome, then tries to find a solution for the number of steps at which this outcome occurs?

If in general one was able to tell whether such a solution exists then it would mean that one could always answer the question of whether, say, a particular pattern would ever die out in the evolution of a given cellular automaton. But from the discussion of the previous section we know that this in general is undecidable.

So it follows that it must be undecidable whether a given integer equation of some particular general form has a solution. And from the arguments above this in turn implies that there must be specific integer equations that have no solutions but where this fact cannot be proved from the normal axioms of arithmetic.

So how ultimately can this happen?

At some level it is a consequence of the involvement of infinity. For at least in a universal system like arithmetic any question that is entirely finite can in the end always be answered by a finite procedure.

But what about questions that somehow ask, say, about infinite numbers of possible integers? To have a finite way to address questions like these is often in the end the main justification for setting up typical mathematical axiom systems in the first place.

For the point is that instead of handling objects like integers directly, axiom systems can just give abstract rules for manipulating statements about them. And within such statements one can refer, say, to infinite sets of integers just by a symbol like s.

And particularly over the past century there have been many successes in mathematics that can be attributed to this basic kind of approach. But the remarkable fact that follows from Gödel's Theorem is that whatever one does there will always be cases where the approach must ultimately fail. And it turns out that the reason for this is essentially the phenomenon of computational irreducibility.

For while simple infinite quantities like $1/0$ or the total number of integers can readily be summarized in finite ways—often just by using symbols like ∞ and $\aleph_0$—the same is not in general true of all infinite processes. And in particular if an infinite process is computationally irreducible then there cannot in general be any useful finite summary of what it does—since the existence of such a summary would imply computational reducibility.

So among other things this means that there will inevitably be questions that finite proofs based on axioms that operate within ordinary computational systems will never in general be able to answer.

And indeed with integer equations, as soon as one has a general equation that is universal, it typically follows that there will be specific instances in which the absence of solutions—or at least of solutions of some particular kind—can never be proved on the basis of the normal axioms of arithmetic.

For several decades it has been known that universal integer equations exist. But the examples that have actually been constructed are quite complicated—like the one on page 786—with the simplest involving 9 variables and an immense number of terms.

Yet from the discoveries in this book I am quite certain that there are vastly simpler examples that exist—so that in fact there are in the end rather simple integer equations for which the absence of solutions can never be proved from the normal axioms of arithmetic.

If one just starts looking at sequences of integer equations—as on the next page—then in the very simplest cases it is usually fairly easy to tell whether a particular equation will have any solutions. But this rapidly becomes very much more difficult. For there is often no obvious pattern to which equations ultimately have solutions and which do not. And even when equations do have solutions, the integers involved can be quite large. So, for example, the smallest solution to $x^2 = 61\,y^2 + 1$ is $x = 1766319049$, $y = 226153980$, while the smallest solution to $x^3 + y^3 = z^3 + 2$ is $x = 1214928$, $y = 3480205$, $z = 3528875$.

Integer equations such as $a\,x + b\,y + c\,z = d$ that have only linear dependence on any variable were largely understood even in antiquity. Quadratic equations in two variables such as $x^2 = a\,y^2 + b$ were understood by the 1800s. But even equations such as $x^2 = a\,y^3 + b$ were not properly understood until the 1980s. And with equations that have higher powers or more variables questions of whether solutions exist quickly end up being unsolved problems of number theory.

It has certainly been known for centuries that there are questions about integer equations and other aspects of number theory that are easy to state, yet seem very hard to answer. But in practice it has almost

$2x+3y=1$	□	
$2x+3y=2$	□	
$2x+3y=3$	□	
$2x+3y=4$	□	
$2x+3y=5$	$x=1$	$y=1$
$2x+3y=6$	□	
$2x+3y=7$	$x=2$	$y=1$
$2x+3y=8$	$x=1$	$y=2$
$2x+3y=9$	$x=3$	$y=1$
$2x+3y=10$	$x=2$	$y=2$
$2x+3y=11$	$x=1$	$y=3$
$2x+3y=12$	$x=3$	$y=2$
$2x+3y=13$	$x=2$	$y=3$
$2x+3y=14$	$x=1$	$y=4$
$2x+3y=15$	$x=3$	$y=3$

$x^2=y^2+1$	□	
$x^2=y^2+2$	□	
$x^2=y^2+3$	$x=2$	$y=1$
$x^2=y^2+4$	□	
$x^2=y^2+5$	$x=3$	$y=2$
$x^2=y^2+6$	□	
$x^2=y^2+7$	$x=4$	$y=3$
$x^2=y^2+8$	$x=3$	$y=1$
$x^2=y^2+9$	$x=5$	$y=4$
$x^2=y^2+10$	□	
$x^2=y^2+11$	$x=6$	$y=5$
$x^2=y^2+12$	$x=4$	$y=2$
$x^2=y^2+13$	$x=7$	$y=6$
$x^2=y^2+14$	□	
$x^2=y^2+15$	$x=4$	$y=1$
$x^2=y^2+16$	$x=5$	$y=3$

$x^2=y^2+1$	□	
$x^2=2y^2+1$	$x=3$	$y=2$
$x^2=3y^2+1$	$x=2$	$y=1$
$x^2=4y^2+1$	□	
$x^2=5y^2+1$	$x=9$	$y=4$
$x^2=6y^2+1$	$x=5$	$y=2$
$x^2=7y^2+1$	$x=8$	$y=3$
$x^2=8y^2+1$	$x=3$	$y=1$
$x^2=9y^2+1$	□	
$x^2=10y^2+1$	$x=19$	$y=6$
$x^2=11y^2+1$	$x=10$	$y=3$
$x^2=12y^2+1$	$x=7$	$y=2$
$x^2=13y^2+1$	$x=649$	$y=180$
$x^2=14y^2+1$	$x=15$	$y=4$
$x^2=15y^2+1$	$x=4$	$y=1$
$x^2=16y^2+1$	□	
$x^2=17y^2+1$	$x=33$	$y=8$
$x^2=18y^2+1$	$x=17$	$y=4$
$x^2=19y^2+1$	$x=170$	$y=39$
$x^2=20y^2+1$	$x=9$	$y=2$

$x^2=y^3-20$	$x=14$	$y=6$
$x^2=y^3-19$	$x=18$	$y=7$
$x^2=y^3-18$	$x=3$	$y=3$
$x^2=y^3-17$	□	
$x^2=y^3-16$	□	
$x^2=y^3-15$	$x=7$	$y=4$
$x^2=y^3-14$	□	
$x^2=y^3-13$	$x=70$	$y=17$
$x^2=y^3-12$	□	
$x^2=y^3-11$	$x=4$	$y=3$
$x^2=y^3-10$	□	
$x^2=y^3-9$	□	
$x^2=y^3-8$	□	
$x^2=y^3-7$	$x=1$	$y=2$
$x^2=y^3-6$	□	
$x^2=y^3-5$	□	
$x^2=y^3-4$	$x=2$	$y=2$
$x^2=y^3-3$	□	
$x^2=y^3-2$	$x=5$	$y=3$
$x^2=y^3-1$	□	
$x^2=y^3$	$x=1$	$y=1$
$x^2=y^3+1$	$x=3$	$y=2$
$x^2=y^3+2$	□	
$x^2=y^3+3$	$x=2$	$y=1$
$x^2=y^3+4$	□	
$x^2=y^3+5$	□	
$x^2=y^3+6$	□	
$x^2=y^3+7$	□	
$x^2=y^3+8$	$x=3$	$y=1$
$x^2=y^3+9$	$x=6$	$y=3$
$x^2=y^3+10$	□	
$x^2=y^3+11$	□	
$x^2=y^3+12$	$x=47$	$y=13$
$x^2=y^3+13$	□	
$x^2=y^3+14$	□	
$x^2=y^3+15$	$x=4$	$y=1$
$x^2=y^3+16$	□	
$x^2=y^3+17$	$x=5$	$y=2$
$x^2=y^3+18$	$x=19$	$y=7$
$x^2=y^3+19$	$x=12$	$y=5$
$x^2=y^3+20$	□	

$x^2=y^3+1$	$x=3$	$y=2$
$x^2=2y^3+1$	□	
$x^2=3y^3+1$	$x=2$	$y=1$
$x^2=4y^3+1$	□	
$x^2=5y^3+1$	□	
$x^2=6y^3+1$	$x=7$	$y=2$
$x^2=7y^3+1$	□	
$x^2=8y^3+1$	$x=3$	$y=1$
$x^2=9y^3+1$	□	
$x^2=10y^3+1$	$x=9$	$y=2$

$x^3=y^4-20xy-1$	$x=10$	$y=7$
$x^3=y^4-19xy-1$	$x=3$	$y=4$
$x^3=y^4-18xy-1$	$x=75$	$y=26$
$x^3=y^4-17xy-1$		
$x^3=y^4-16xy-1$		
$x^3=y^4-15xy-1$	$x=624$	$y=125$
$x^3=y^4-14xy-1$		
$x^3=y^4-13xy-1$		
$x^3=y^4-12xy-1$	$x=3$	$y=2$
$x^3=y^4-11xy-1$		
$x^3=y^4-10xy-1$		
$x^3=y^4-9xy-1$	$x=80$	$y=27$
$x^3=y^4-8xy-1$	$x=12$	$y=7$
$x^3=y^4-7xy-1$	$x=1$	$y=2$
$x^3=y^4-6xy-1$	$x=15$	$y=8$
$x^3=y^4-5xy-1$		
$x^3=y^4-4xy-1$	$x=30$	$y=13$
$x^3=y^4-3xy-1$		
$x^3=y^4-2xy-1$		
$x^3=y^4-xy-1$		
$x^3=y^4-1$	□	
$x^3=y^4+xy-1$	$x=1$	$y=1$
$x^3=y^4+2xy-1$	$x=3$	$y=2$
$x^3=y^4+3xy-1$	$x=5$	$y=3$
$x^3=y^4+4xy-1$	$x=2$	$y=1$
$x^3=y^4+5xy-1$		
$x^3=y^4+6xy-1$		
$x^3=y^4+7xy-1$		
$x^3=y^4+8xy-1$	$x=20$	$y=9$
$x^3=y^4+9xy-1$	$x=3$	$y=1$
$x^3=y^4+10xy-1$		
$x^3=y^4+11xy-1$	$x=5$	$y=2$
$x^3=y^4+12xy-1$		
$x^3=y^4+13xy-1$		
$x^3=y^4+14xy-1$		
$x^3=y^4+15xy-1$		
$x^3=y^4+16xy-1$	$x=4$	$y=1$
$x^3=y^4+17xy-1$		
$x^3=y^4+18xy-1$	$x=8$	$y=3$
$x^3=y^4+19xy-1$		
$x^3=y^4+20xy-1$		

$x^2=y^5+3$	$x=2$	$y=1$
$x^2=y^5+y+3$	$x=2537$	$y=23$
$x^2=y^5+2y+3$		
$x^2=y^5+3y+3$		
$x^2=y^5+4y+3$		
$x^2=y^5+5y+3$	$x=3$	$y=1$
$x^2=y^5+6y+3$		
$x^2=y^5+7y+3$	$x=7$	$y=2$
$x^2=y^5+8y+3$		
$x^2=y^5+9y+3$		

$x^3+y^3=z^2+1$	$x=1$	$y=1$	$z=1$
$x^3+y^3=z^2+2$	$x=107$	$y=232$	$z=3703$
$x^3+y^3=z^2+3$	$x=1$	$y=3$	$z=5$
$x^3+y^3=z^2+4$	$x=5$	$y=12$	$z=43$
$x^3+y^3=z^2+5$	$x=1$	$y=2$	$z=2$
$x^3+y^3=z^2+6$	$x=7$	$y=24$	$z=119$
$x^3+y^3=z^2+7$	$x=2$	$y=2$	$z=3$
$x^3+y^3=z^2+8$	$x=1$	$y=2$	$z=1$
$x^3+y^3=z^2+9$	$x=3$	$y=7$	$z=19$
$x^3+y^3=z^2+10$	$x=2$	$y=3$	$z=5$

$x^3+y^3=z^3-20$	$x=107$	$y=137$	$z=156$
$x^3+y^3=z^3-19$	$x=14$	$y=16$	$z=19$
$x^3+y^3=z^3-18$	$x=1$	$y=2$	$z=3$
$x^3+y^3=z^3-17$	$x=103$	$y=111$	$z=135$
$x^3+y^3=z^3-16$	$x=10$	$y=12$	$z=14$
$x^3+y^3=z^3-15$	$x=262$	$y=265$	$z=332$
$x^3+y^3=z^3-14$	□		
$x^3+y^3=z^3-13$	□		
$x^3+y^3=z^3-12$	$x=5725013$	$y=9019406$	$z=9730705$
$x^3+y^3=z^3-11$	$x=2$	$y=2$	$z=3$
$x^3+y^3=z^3-10$	$x=3$	$y=3$	$z=4$
$x^3+y^3=z^3-9$	$x=52$	$y=216$	$z=217$
$x^3+y^3=z^3-8$	$x=16$	$y=12$	$z=18$
$x^3+y^3=z^3-7$	$x=605809$	$y=680316$	$z=812918$
$x^3+y^3=z^3-6$	$x=1$	$y=1$	$z=2$
$x^3+y^3=z^3-5$	□		
$x^3+y^3=z^3-4$	□		
$x^3+y^3=z^3-3$			
$x^3+y^3=z^3-2$	$x=5$	$y=6$	$z=7$
$x^3+y^3=z^3-1$	$x=6$	$y=8$	$z=9$
$x^3+y^3=z^3$	□		
$x^3+y^3=z^3+1$	$x=1$	$y=2$	$z=2$
$x^3+y^3=z^3+2$	$x=1214928$	$y=3480205$	$z=3528875$
$x^3+y^3=z^3+3$	$x=4$	$y=4$	$z=5$
$x^3+y^3=z^3+4$	□		
$x^3+y^3=z^3+5$	□		
$x^3+y^3=z^3+6$	$x=10529$	$y=60248$	$z=60355$
$x^3+y^3=z^3+7$	$x=32$	$y=104$	$z=105$
$x^3+y^3=z^3+8$	$x=1$	$y=2$	$z=1$
$x^3+y^3=z^3+9$	$x=2097$	$y=11305$	$z=11329$
$x^3+y^3=z^3+10$	$x=130$	$y=141$	$z=171$
$x^3+y^3=z^3+11$	$x=297$	$y=619$	$z=641$
$x^3+y^3=z^3+12$	$x=7$	$y=10$	$z=11$
$x^3+y^3=z^3+13$	□		
$x^3+y^3=z^3+14$	□		
$x^3+y^3=z^3+15$	$x=2$	$y=2$	$z=1$
$x^3+y^3=z^3+16$	$x=2429856$	$y=6960410$	$z=7057750$
$x^3+y^3=z^3+17$	$x=25$	$y=50$	$z=52$
$x^3+y^3=z^3+18$	$x=94$	$y=101$	$z=123$
$x^3+y^3=z^3+19$	$x=26$	$y=76$	$z=77$
$x^3+y^3=z^3+20$	$x=1$	$y=3$	$z=2$

universally been assumed that with the continued development of mathematics any of these questions could in the end be answered.

However, what Gödel's Theorem shows is that there must always exist some questions that cannot ever be answered using the normal axioms of arithmetic. Yet the fact that the few known explicit examples have been extremely complicated has made this seem somehow fundamentally irrelevant for the actual practice of mathematics.

But from the discoveries in this book it now seems quite certain that vastly simpler examples also exist. And it is my strong suspicion that in fact of all the current unsolved problems seriously studied in number theory a fair fraction will in the end turn out to be questions that cannot ever be answered using the normal axioms of arithmetic.

If one looks at recent work in number theory, most of it tends to be based on rather sophisticated methods that do not obviously depend only on the normal axioms of arithmetic. And for example the elaborate proof of Fermat's Last Theorem that has been developed may make at least some use of axioms that come from fields like set theory and go beyond the normal ones for arithmetic.

But so long as one stays within, say, the standard axiom systems of mathematics on pages 773 and 774, and does not in effect just end up implicitly adding as an axiom whatever result one is trying to prove, my strong suspicion is that one will ultimately never be able to go much further than one can purely with the normal axioms of arithmetic.

And indeed from the Principle of Computational Equivalence I strongly believe that in general undecidability and unprovability will start to occur in practically any area of mathematics almost as soon as one goes beyond the level of questions that are always easy to answer.

But if this is so, why then has mathematics managed to get as far as it has? Certainly there are problems in mathematics that have remained unsolved for long periods of time. And I suspect that many of these will in fact in the end turn out to involve undecidability and

◀ Smallest solutions for various sequences of integer (or so-called Diophantine) equations. □ indicates that it can be proved that no solution exists. A blank indicates that I know only that no solution exists below a billion. Methods for resolving some of the equations in the first column were known in antiquity; all had been resolved by the 1800s. Practical methods for resolving the so-called elliptic curve equations in the second column were developed only in the 1980s. No general methods are yet known for most of the other equations given—and some classes of them may in fact show undecidability.

unprovability. But the issue remains why such phenomena have not been much more obvious in everyday work in mathematics.

At some level I suspect the reason is quite straightforward: it is that like most other fields of human inquiry mathematics has tended to define itself to be concerned with just those questions that its methods can successfully address. And since the main methods traditionally used in mathematics have revolved around doing proofs, questions that involve undecidability and unprovability have inevitably been avoided.

But can this really be right? For at least in the past century mathematics has consistently given the impression that it is concerned with questions that are somehow as arbitrary and general as possible.

But one of the important conclusions from what I have done in this book is that this is far from correct. And indeed for example traditional mathematics has for the most part never even considered most of the kinds of systems that I discuss in this book—even though they are based on some of the very simplest rules possible.

So how has this happened? The main point, I believe, is that in both the systems it studies and the questions it asks mathematics is much more a product of its history than is usually realized.

And in fact particularly compared to what I do in this book the vast majority of mathematics practiced today still seems to follow remarkably closely the traditions of arithmetic and geometry that already existed even in Babylonian times.

It is a fairly recent notion that mathematics should even try to address arbitrary or general systems. For until not much more than a century ago mathematics viewed itself essentially just as providing a precise formulation of certain aspects of everyday experience—mainly those related to number and space.

But in the 1800s, with developments such as non-Euclidean geometry, quaternions, group theory and transfinite numbers it began to be assumed that the discipline of mathematics could successfully be applied to any abstract system, however arbitrary or general.

Yet if one looks at the types of systems that are actually studied in mathematics they continue even to this day to be far from as general as possible. Indeed at some level most of them can be viewed as having been arrived at by the single rather specific approach of starting from some known set of theorems, then trying to find systems that are progressively more general, yet still manage to satisfy these theorems.

And given this approach, it tends to be the case that the questions that are considered interesting are ones that revolve around whatever theorems a system was set up to satisfy—making it rather likely that these questions can themselves be addressed by similar theorems, without any confrontation with undecidability or unprovability.

But what if one looks at other kinds of systems?

One of the main things I have done in this book is in a sense to introduce a new approach to generalization in which one considers systems that have simple but completely arbitrary rules—and that are not set up with any constraint about what theorems they should satisfy.

But if one has such a system, how does one decide what questions are interesting to ask about it? Without the guidance of known theorems, the obvious thing to do is just to look explicitly at how the system behaves—perhaps by making some kind of picture.

And if one does this, then what I have found is that one is usually immediately led to ask questions that run into phenomena like undecidability. Indeed, from my experiments it seems that almost as soon as one leaves behind the constraints of mathematical tradition undecidability and unprovability become rather common.

As the picture on the next page indicates, it is quite straightforward to set up an axiom system that deals with logical statements about a system like a cellular automaton. And within such an axiom system one can ask questions such as whether the cellular automaton will ever behave in a particular way after any number of steps.

But as we saw in the previous section, such questions are in general undecidable. And what this means is that there will inevitably be cases of them for which no proof of a particular answer can ever be given within whatever axiom system one is using.

So from this one might conclude that as soon as one looks at cellular automata or other kinds of systems beyond those normally studied in mathematics it must immediately become effectively impossible to make progress using traditional mathematical methods.

⟨a, b ⋄ c, d⟩ == ⟨a, b, c⟩ ⋄ ⟨b, c, d⟩
⟨a ⋄ b, c, d⟩ == ⟨b, c, d⟩
⟨a, b, c ⋄ d⟩ == ⟨a, b, c⟩
a ⋄ (b ⋄ c) == (a ⋄ b) ⋄ c

basic axioms

⟨■, ■, ■⟩ == □
⟨■, ■, □⟩ == ■
⟨■, □, ■⟩ == ■
⟨■, □, □⟩ == □
⟨□, ■, ■⟩ == ■
⟨□, ■, □⟩ == ■
⟨□, □, ■⟩ == ■
⟨□, □, □⟩ == □

rule 110 axioms

(↓ a)⟶⟨□, (□ ⋄ a) ⋄ □, □⟩

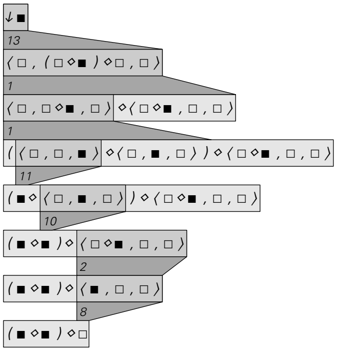
definition

O a ⇔ O ((□ ⋄ a) ⋄ □)
O a ⇒ O ↓ a

advanced axioms

■	*black cell*
□	*white cell*
⋄	*concatenation*
⟨⟩	*update*
↓	*evolution step*
O	*state occurs*

typical interpretations



↓ ■ == (■ ⋄ ■) ⋄ □
↓ ↓ ■ == ↓ ((■ ⋄ ■) ⋄ □)
↓ ↓ ■ == (■ ⋄ ((■ ⋄ ■) ⋄ □)) ⋄ □
O ■ ⇒ O ((■ ⋄ ■) ⋄ □)
O ■ ⇒ O ((■ ⋄ ((■ ⋄ ■) ⋄ □)) ⋄ □)
O ■ ⇒ $\exists_a$ $\exists_b$ O (a ⋄ (((■ ⋄ ((■ ⋄ ■) ⋄ □)) ⋄ □) ⋄ b))

provable statements

O ■ ⇒ $\exists_a$ $\exists_b$ O (a ⋄ (((□ ⋄ ((■ ⋄ □) ⋄ ■)) ⋄ □) ⋄ b))

unprovable statement

An axiom system for statements about the rule 110 cellular automaton. The top statement above makes the assertion that the outcome after one step of evolution from a single black cell has a particular form. A proof of this statement is shown to the left. All the statements in the top block above can be proved true from the axiom system. The statement at the bottom, however, cannot be proved either true or false. The axioms given are set up using predicate logic.

But in fact, in the fifteen years or so since I first emphasized the importance of cellular automata all sorts of traditional mathematical work has actually been done on them. So how has this been possible?

The basic point is that the work has tended to concentrate on particular aspects of cellular automata that are simple enough to avoid undecidability and unprovability. And typically it has achieved this in one of two ways: either by considering only very specific cases that have been observed or constructed to be simple, or by looking at things in so much generality that only rather simple properties ever survive.

So for example when presented with the 256 elementary cellular automaton patterns shown on page 55 mathematicians in my experience have two common responses: either to single out specific patterns that have a simple repetitive or perhaps nested form, or to generalize and look not at individual patterns, but rather at aggregate properties obtained say by evolving from all possible initial conditions.

And about questions that concern, for example, the structure of a pattern that looks to us complex, the almost universal reaction is that such questions can somehow not be of any real mathematical interest.

Needless to say, in the framework of the new kind of science in this book, such questions are now of great interest. And my results suggest that if one is ever going to study many important phenomena that occur in nature one will also inevitably run into them. But to traditional mathematics they seem uninteresting and quite alien.

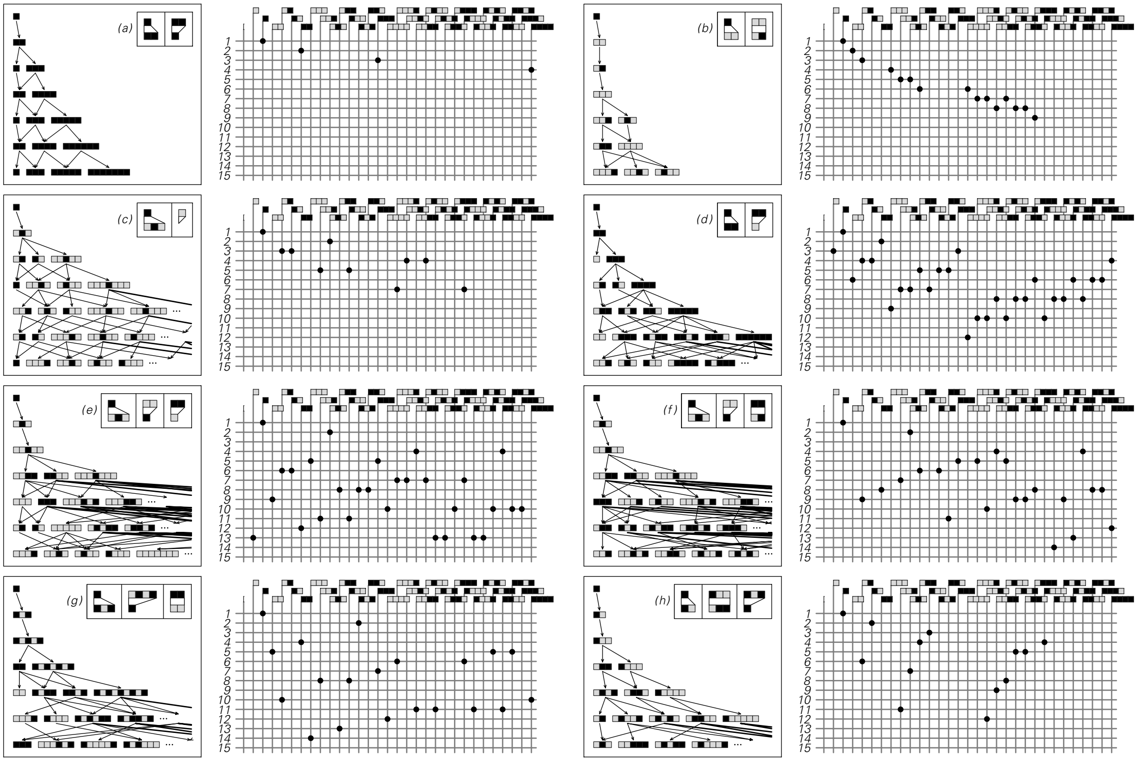
As I said above, it is at some level not surprising that questions will be considered interesting in a particular field only if the methods of that field can say something useful about them. But this I believe is ultimately why there have historically been so few signs of undecidability or unprovability in mathematics. For any kinds of questions in which such phenomena appear are usually not amenable to standard methods of mathematics based on proof, and as a result such questions have inevitably been viewed as being outside what should be considered interesting for mathematics.

So how then can one set up a reasonable idealization for mathematics as it is actually practiced? The first step—much as I discussed earlier in this section—is to think not so much about systems that might be described by mathematics as about the internal processes associated with proof that go on inside mathematics.

A proof must ultimately be based on an axiom system, and one might have imagined that over the course of time mathematics would have sampled a wide range of possible axiom systems. But in fact in its historical development mathematics has normally stuck to only rather few such systems—each one corresponding essentially to some identifiable field of mathematics, and most given on pages 773 and 774.

So what then happens if one looks at all possible simple axiom systems—much as we looked, say, at all possible simple cellular automata earlier in this book? To what extent does what one sees capture the features of mathematics? With axiom systems idealized as multiway systems the pictures on the next page show some results.

In some cases the total number of theorems that can ever be proved is limited. But often the number of theorems increases rapidly with the length of proof—and in most cases an infinite number of theorems can eventually be proved. And given experience with mathematics an obvious question to ask in such cases is to what extent the system is consistent, or complete, or both.



Plots showing which possible strings get generated in the first 15 steps of evolution in various multiway systems. Each string that is generated can be thought of as a theorem derived from the set of axioms represented by the rules of the multiway system. A dot shows at which step a given string first appears—and indicates the shortest proof of the theorem that string represents. In most cases, many strings are never produced—so that there are many possible statements that simply do not follow from the axioms given. Thus for example in first case shown only strings containing nothing but black elements are ever produced.

But to formulate such a question in a meaningful way one needs a notion of negation. In general, negation is just some operation that takes a string and yields another, giving back the original if it is applied a second time. Earlier in this section we discussed cases in which negation simply reverses the color of each element in a string. And as a generalization of this one can consider cases in which negation can be any operation that preserves lengths of strings.

And in this case it turns out that the criterion for whether a system is complete and consistent is simply that exactly half the possible strings of a given length are eventually generated if one starts from the string representing "true".

For if more than half the strings are generated, then somewhere both a string and its negation would have to appear, implying that the system must be inconsistent. And similarly, if less than half the strings are generated, there must be some string for which neither that string nor its negation ever appear, implying that the system is incomplete.

The pictures on the next page show the fractions of strings of given lengths that are generated on successive steps in various multiway systems. In general one might have to wait an arbitrarily large number of steps to find out whether a given string will ever be generated. But in practice after just a few steps one already seems to get a reasonable indication of the overall fraction of strings that will ever be generated.

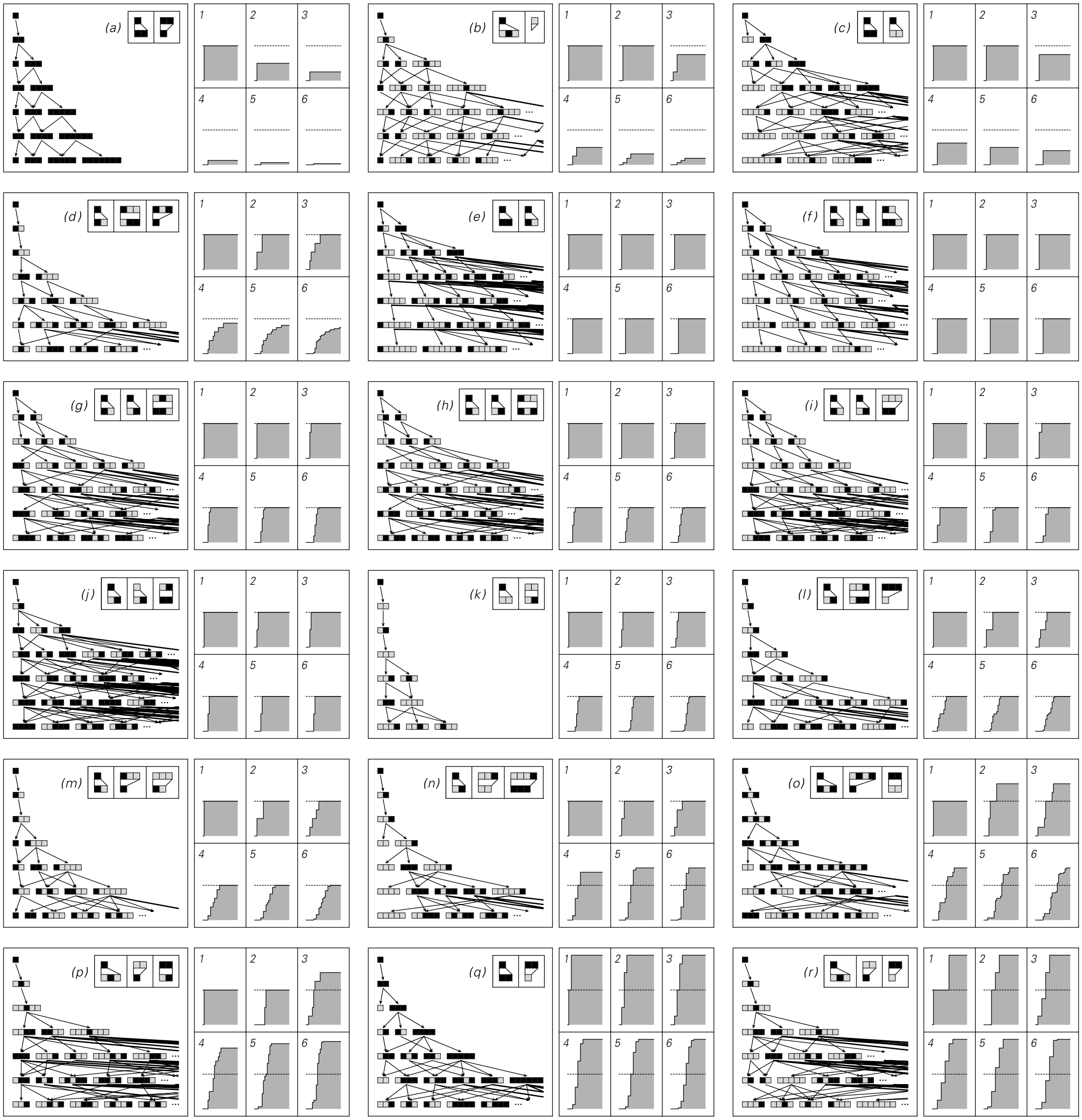
And what one sees is that there is a broad distribution: from cases in which very few strings can be generated—corresponding to a very incomplete axiom system—to cases in which all or almost all strings can be generated—corresponding to a very inconsistent axiom system.

So where in this distribution do the typical axiom systems of ordinary mathematics lie? Presumably none are inconsistent. And a few—like basic logic and real algebra—are both complete and consistent, so that in effect they lie right in the middle of the distribution. But most are known to be incomplete. And as we discussed above, this is inevitable as soon as universality is present.

But just how incomplete are they? The answer, it seems, is typically not very. For if one looks at axiom systems that are widely used in mathematics they almost all tend to be complete enough to prove at least a fair fraction of statements either true or false.

So why should this be? I suspect that it has to do with the fact that in mathematics one usually wants axiom systems that one can think of as somehow describing definite kinds of objects—about which one then expects to be able to establish all sorts of definite statements.

And certainly if one looks at the history of mathematics most basic axiom systems have been arrived at by starting with objects—such as finite integers or finite sets—then trying to find collections of axioms that somehow capture the relevant properties of these objects.



Examples of multiway systems that generate different fractions of possible strings, and in effect range from being highly incomplete to highly inconsistent. The plots show what fraction of strings of a given length have been produced by each of the first 25 steps in the evolution of each multiway system. If less than half the strings of a given length are ever produced, this means that there must be some strings where neither the string nor its negation can be proved, indicating incompleteness. But if more than half the strings are produced, there must be cases where both a string and its negation can be proved, indicating inconsistency. Rules (f) through (i), however, produce exactly half the strings of any given length, and can be considered complete and consistent.

But one feature is that normally the resulting axiom system is in a sense more general than the objects one started from. And this is why for example one can often use the axiom system to extrapolate to infinite situations. But it also means that it is not clear whether the axiom system actually describes only the objects one wants—or whether for example it also describes all sorts of other quite different objects.

One can think of an axiom system—say one of those listed on pages 773 and 774—as giving a set of constraints that any object it describes must satisfy. But as we saw in Chapter 5, it is often possible to satisfy a single set of constraints in several quite different ways.

And when this happens in an axiom system it typically indicates incompleteness. For as soon as there are just two objects that both satisfy the constraints but for which there is some statement that is true about one but false about the other it immediately follows that at least this statement cannot consistently be proved true or false, and that therefore the axiom system must be incomplete.

One might imagine that if one were to add more axioms to an axiom system one could always in the end force there to be only one kind of object that would satisfy the constraints of the system. But as we saw earlier, as soon as there is universality it is normally impossible to avoid incompleteness. And if an axiom system is incomplete there must inevitably be different kinds of objects that satisfy its constraints. For given any statement that cannot be proved from the axioms there must be distinct objects for which it is true, and for which it is false.

If an axiom system is far from complete—so that a large fraction of statements cannot be proved true or false—then there will typically be many different kinds of objects that are easy to specify and all satisfy the constraints of the system but for which there are fairly obvious properties that differ. But if an axiom system is close to complete—so that the vast majority of statements can be proved true or false—then it is almost inevitable that the different kinds of objects that satisfy its constraints must differ only in obscure ways.

And this is presumably the case in the standard axiom system for arithmetic from page 773. Originally this axiom system was intended to describe just ordinary integers. But Gödel's Theorem showed that it is incomplete, so that there must be more than one kind of object that can satisfy its constraints. Yet it is rather close to being complete—since as we saw earlier one has to go through at least millions of statements before finding ones that it cannot prove true or false.

And this means that even though there are objects other than the ordinary integers that satisfy the standard axioms of arithmetic, they are quite obscure—in fact, so much so that none have ever yet actually been constructed with any real degree of explicitness. And this is why it has been reasonable to think of the standard axiom system of arithmetic as being basically just about ordinary integers.

But if instead of this standard axiom system one uses the reduced axiom system from page 773—in which the usual axiom for induction has been weakened—then the story is quite different. There is again incompleteness, but now there is much more of it, for even statements as simple as $x + y = y + x$ and $x + 0 = x$ cannot be proved true or false from the axioms. And while ordinary integers still satisfy all the constraints, the system is sufficiently incomplete that all sorts of other objects with quite different properties also do. So this means that the system is in a sense no longer about any very definite kind of mathematical object—and presumably that is why it is not used in practice in mathematics.

At this juncture it should perhaps be mentioned that in their raw form quite a few well-known axiom systems from mathematics are actually also far from complete. An example of this is the axiom system for group theory given on page 773. But the point is that this axiom system represents in a sense just the beginning of group theory. For it yields only those theorems that hold abstractly for any group.

Yet in doing group theory in practice one normally adds axioms that in effect constrain one to be dealing say with a specific group rather than with all possible groups. And the result of this is that once again one typically has an axiom system that is at least close to complete.

In basic arithmetic and also usually in fields like group theory the underlying objects that one imagines describing can at some level be manipulated—and understood—in fairly concrete ways. But in a field like set theory this is less true. Yet even in this case an attempt has historically been made to get an axiom system that somehow describes definite kinds of objects. But now the main way this has been done is by progressively adding axioms so as to get closer to having a system that is complete—with only a rather vague notion of just what underlying objects one is really expecting to describe.

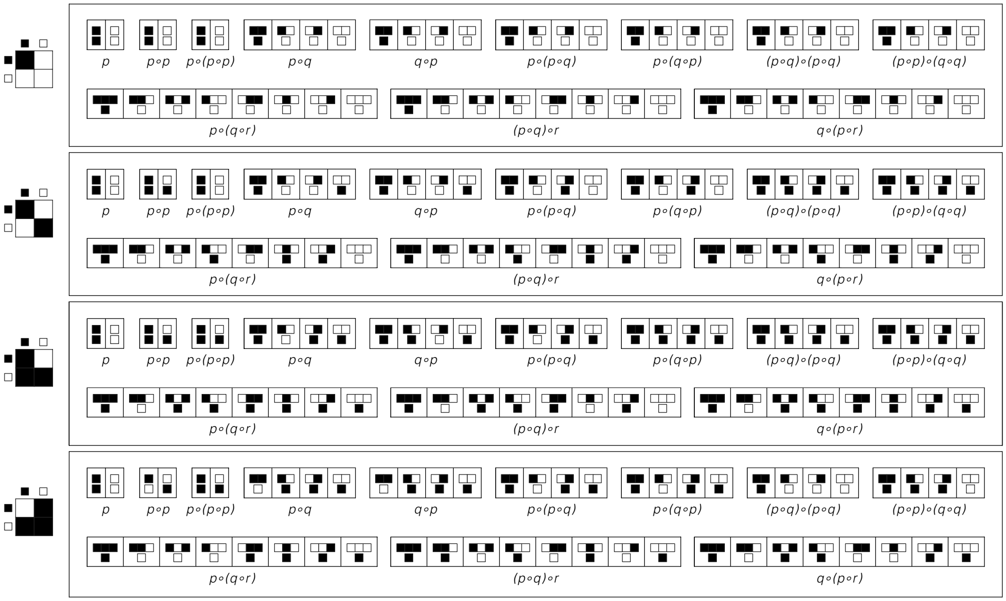
In studying basic processes of proof multiway systems seem to do well as minimal idealizations. But if one wants to study axiom systems that potentially describe definite objects it seems to be somewhat more convenient to use what I call operator systems. And indeed the version of logic used on page 775—as well as many of the axiom systems on pages 773 and 774—are already set up essentially as operator systems.

The basic idea of an operator system is to work with expressions such as $(p \circ q) \circ ((q \circ r) \circ p)$ built up using some operator $\circ$, and then to consider for example what equivalences may exist between such expressions. If one has an operator whose values are given by some finite table then it is always straightforward to determine whether expressions are equivalent. For all one need do, as in the pictures at the top of the next page, is to evaluate the expressions for all possible values of each variable, and then to see whether the patterns of results one gets are the same.

And in this way one can readily tell, for example, that the first operator shown is idempotent, so that $p \circ p = p$, while both the first two operators are associative, so that $(p \circ q) \circ r = p \circ (q \circ r)$, and all but the third operator are commutative, so that $p \circ q = q \circ p$. And in principle one can use this method to establish any equivalence that exists between any expressions with an operator of any specific form.

But the crucial idea that underlies the traditional approach to mathematical proof is that one should also be able to deduce such results just by manipulating expressions in purely symbolic form, using the rules of an axiom system, without ever having to do anything like filling in explicit values of variables.

And one advantage of this approach is that at least in principle it allows one to handle operators—like those found in many areas of mathematics—that are not based on finite tables. But even for operators given by finite tables it is often difficult to find axiom systems that can successfully reproduce all the results for a particular operator.

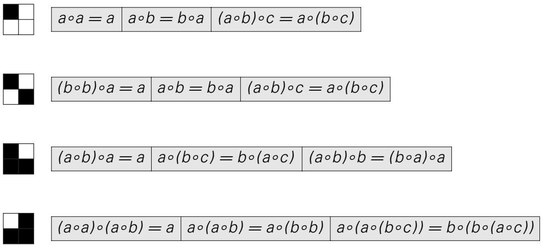


Values of expressions obtained by using operators of various forms. For each expression the sequence of values for every possible combination of values of variables is shown. Two expressions are equivalent when this sequence of values is the same. With black and white interpreted as True and False, the forms of operators shown here correspond respectively to And, Equal, Implies and Nand. (The first argument to each operator is shown on the left; the second on top.) The arrays of values generated can be thought of as being like truth tables.

With the way I have set things up, any axiom system is itself just a collection of equivalence results. So the question is then which equivalence results need to be included in the axiom system in order that all other equivalence results can be deduced just from these.

In general this can be undecidable—for there is no limit on how long even a single proof might need to be. But in some cases it turns out to be possible to establish that a particular set of axioms can successfully generate all equivalence results for a given operator—and indeed the picture at the top of the facing page shows examples of this for each of the four operators in the picture above.

So if two expressions are equivalent then by applying the rules of the appropriate axiom system it must be possible to get from one to the other—and in fact the picture on page 775 shows an example of how



$a\circ a = a$ | $a\circ b = b\circ a$ | $(a\circ b)\circ c = a\circ(b\circ c)$

$(b\circ b)\circ a = a$ | $a\circ b = b\circ a$ | $(a\circ b)\circ c = a\circ(b\circ c)$

$(a\circ b)\circ a = a$ | $a\circ(b\circ c) = b\circ(a\circ c)$ | $(a\circ b)\circ b = (b\circ a)\circ a$

$(a\circ a)\circ(a\circ b) = a$ | $a\circ(a\circ b) = a\circ(b\circ b)$ | $a\circ(a\circ(b\circ c)) = b\circ(b\circ(a\circ c))$

Axiom systems that can be used to derive all the equivalences between expressions that involve operators with the forms shown. Each axiom can be applied in either direction—as in the picture on page 775, with each variable standing for any expression, as in a *Mathematica* pattern. The operators shown are AND, EQUAL, IMPLIES and NAND. They yield respectively junctional, equivalential, implicational and full propositional or sentential calculus (ordinary logic).

this can be done for the fourth axiom system above. But if one removes just a single axiom from any of the axiom systems above then it turns out that they no longer work, and for example they cannot establish the equivalence result stated by whichever axiom one has removed.

In general one can think of axioms for an operator system as giving constraints on the form of the operator. And if one is going to reproduce all the equivalences that hold for a particular form then these constraints must in effect be such as to force that form to occur.

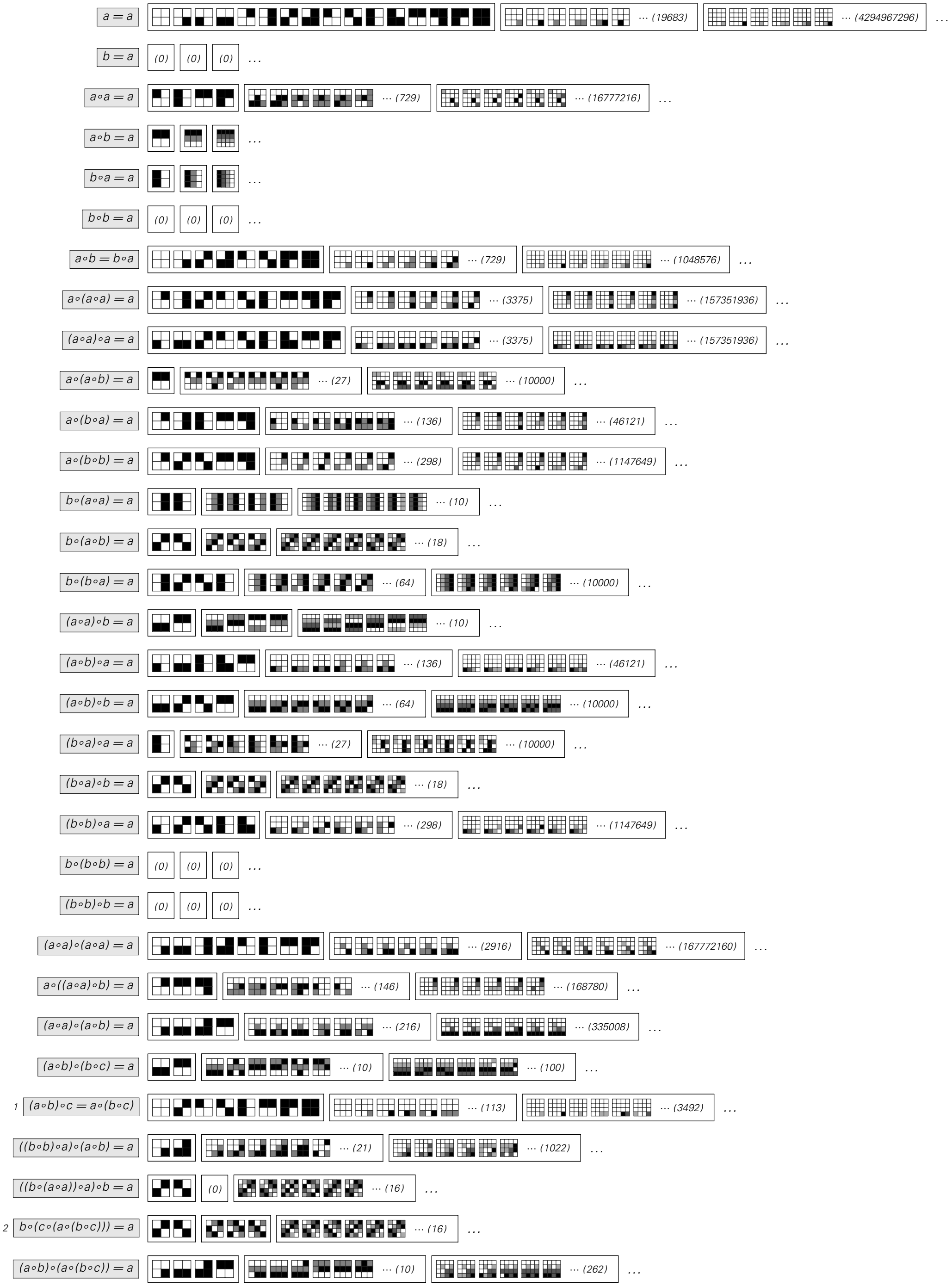
So what happens in general for arbitrary axiom systems? Do they typically force the operator to have a particular form, or not?

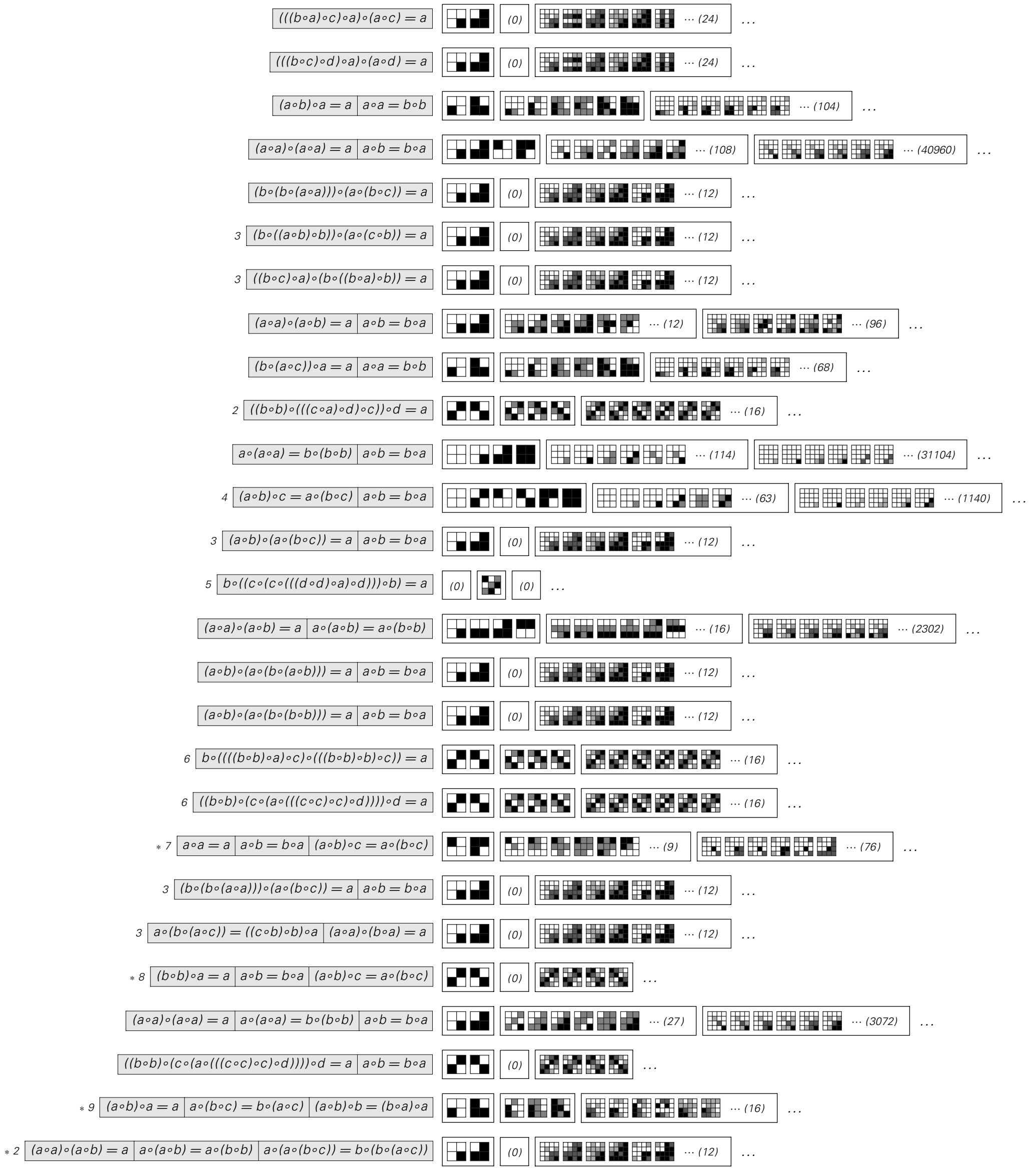
The pictures on the next two pages show which forms of operators are allowed by various different axiom systems. The successive blocks of results in each case give the forms allowed with progressively more possible values for each variable.

Indicated by stars near the bottom of the picture are the four axiom systems from the top of this page. And for each of these only a limited number of forms are allowed—all of which ultimately turn out to be equivalent to just the single forms shown on the facing page.

But what about other axiom systems? Every axiom system must allow an operator of at least some form. But what the pictures on the next two pages show is that the vast majority of axiom systems actually allow operators with all sorts of different forms.

And what this means is that these axiom systems are in a sense not really about operators of any particular form. And so in effect they are also far from complete—for they can prove only equivalence results that hold for every single one of the various operators they allow.


a = a … (19683) … (4294967296) …
b = a (0) (0) (0) …
a∘a = a … (729) … (16777216) …
a∘b = a …
b∘a = a …
b∘b = a (0) (0) (0) …
a∘b = b∘a … (729) … (1048576) …
a∘(a∘a) = a … (3375) … (157351936) …
(a∘a)∘a = a … (3375) … (157351936) …
a∘(a∘b) = a … (27) … (10000) …
a∘(b∘a) = a … (136) … (46121) …
a∘(b∘b) = a … (298) … (1147649) …
b∘(a∘a) = a … (10) …
b∘(a∘b) = a … (18) …
b∘(b∘a) = a … (64) … (10000) …
(a∘a)∘b = a … (10) …
(a∘b)∘a = a … (136) … (46121) …
(a∘b)∘b = a … (64) … (10000) …
(b∘a)∘a = a … (27) … (10000) …
(b∘a)∘b = a … (18) …
(b∘b)∘a = a … (298) … (1147649) …
b∘(b∘b) = a (0) (0) (0) …
(b∘b)∘b = a (0) (0) (0) …
(a∘a)∘(a∘a) = a … (2916) … (167772160) …
a∘((a∘a)∘b) = a … (146) … (168780) …
(a∘a)∘(a∘b) = a … (216) … (335008) …
(a∘b)∘(b∘c) = a … (10) … (100) …
1 (a∘b)∘c = a∘(b∘c) … (113) … (3492) …
((b∘b)∘a)∘(a∘b) = a … (21) … (1022) …
((b∘(a∘a))∘a)∘b = a (0) … (16) …
2 b∘(c∘(a∘(b∘c))) = a … (16) …
(a∘b)∘(a∘(b∘c)) = a … (10) … (262) …


(((b∘a)∘c)∘a)∘(a∘c) = a (0) ··· (24) …
(((b∘c)∘d)∘a)∘(a∘d) = a (0) ··· (24) …
(a∘b)∘a = a | a∘a = b∘b ··· (104) …
(a∘a)∘(a∘a) = a | a∘b = b∘a ··· (108) ··· (40960) …
(b∘(b∘(a∘a)))∘(a∘(b∘c)) = a (0) ··· (12) …
3 (b∘((a∘b)∘b))∘(a∘(c∘b)) = a (0) ··· (12) …
3 ((b∘c)∘a)∘(b∘((b∘a)∘b)) = a (0) ··· (12) …
(a∘a)∘(a∘b) = a | a∘b = b∘a ··· (12) ··· (96) …
(b∘(a∘c))∘a = a | a∘a = b∘b ··· (68) …
2 ((b∘b)∘(((c∘a)∘d)∘c))∘d = a ··· (16) …
a∘(a∘a) = b∘(b∘b) | a∘b = b∘a ··· (114) ··· (31104) …
4 (a∘b)∘c = a∘(b∘c) | a∘b = b∘a ··· (63) ··· (1140) …
3 (a∘b)∘(a∘(b∘c)) = a | a∘b = b∘a (0) ··· (12) …
5 b∘((c∘(c∘(((d∘d)∘a)∘d)))∘b) = a (0) (0) …
(a∘a)∘(a∘b) = a | a∘(a∘b) = a∘(b∘b) ··· (16) ··· (2302) …
(a∘b)∘(a∘(b∘(a∘b))) = a | a∘b = b∘a (0) ··· (12) …
(a∘b)∘(a∘(b∘(b∘b))) = a | a∘b = b∘a (0) ··· (12) …
6 b∘((((b∘b)∘a)∘c)∘(((b∘b)∘b)∘c)) = a ··· (16) …
6 ((b∘b)∘(c∘(a∘(((c∘c)∘c)∘d))))∘d = a ··· (16) …
∗ 7 a∘a = a | a∘b = b∘a | (a∘b)∘c = a∘(b∘c) ··· (9) ··· (76) …
3 (b∘(b∘(a∘a)))∘(a∘(b∘c)) = a | a∘b = b∘a (0) ··· (12) …
3 a∘(b∘(a∘c)) = ((c∘b)∘b)∘a | (a∘a)∘(b∘a) = a (0) ··· (12) …
∗ 8 (b∘b)∘a = a | a∘b = b∘a | (a∘b)∘c = a∘(b∘c) (0) …
(a∘a)∘(a∘a) = a | a∘(a∘a) = b∘(b∘b) | a∘b = b∘a ··· (27) ··· (3072) …
((b∘b)∘(c∘(a∘(((c∘c)∘c)∘d))))∘d = a | a∘b = b∘a (0) …
∗ 9 (a∘b)∘a = a | a∘(b∘c) = b∘(a∘c) | (a∘b)∘b = (b∘a)∘a ··· (16) …
∗ 2 (a∘a)∘(a∘b) = a | a∘(a∘b) = a∘(b∘b) | a∘(a∘(b∘c)) = b∘(b∘(a∘c)) (0) ··· (12) …

Forms of a binary operator satisfying the constraints of a series of different axiom systems. The successive blocks of results in each case show forms of the operator allowed with 2, 3 and 4 possible elements. Note that with 3 and 4 elements, only forms inequivalent under interchange of element labels are shown. Representations of notable systems in mathematics are: (1) semigroup theory, (2) commutative group theory, (3) basic logic, (4) commutative semigroup theory, (5) squag theory, (6) group theory, (7) junctional calculus, (8) equivalential calculus and (9) implicational calculus. In each case the operator forms shown correspond to possible semigroups, commutative groups, systems of logic (Boolean algebras), etc. with 2, 3 and 4 possible elements. The operator forms shown can be thought of as giving multiplication tables. In model theory, these forms are usually called the models of an axiom system.

So if one makes a list of all possible axiom systems—say starting with the simplest—where in such a list should one expect to see axiom systems that correspond to traditional areas of mathematics?

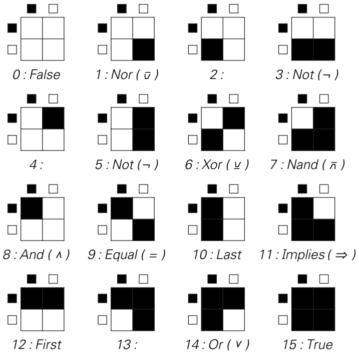
Most axiom systems as they are given in typical textbooks are sufficiently complicated that they will not show up at all early. And in fact the only immediate exception is the axiom system $\{(a \circ b) \circ c = a \circ (b \circ c)\}$ for what are known as semigroups—which ironically are usually viewed as rather advanced mathematical objects.

But just how complicated do the axiom systems for traditional areas of mathematics really need to be? Often it seems that they can be vastly simpler than their textbook forms. And so, for example, as page 773 indicates, interpreting the $\circ$ operator as division, $\{a \circ (b \circ (c \circ (a \circ b))) = c\}$ is known to be an axiom system for commutative group theory, and $\{a \circ ((((a \circ a) \circ b) \circ c) \circ (((a \circ a) \circ a) \circ c)) = b\}$ for general group theory.

So what about basic logic? How complicated an axiom system does one need for this? Textbook discussions of logic mostly use axiom systems at least as complicated as the first one on page 773. And such axiom systems not only involve several axioms—they also normally involve three separate operators: AND (∧), OR (∨) and NOT (¬).

But is this in fact the only way to formulate logic?

As the picture below shows, there are 16 different possible operators that take two arguments and allow two values, say true and false. And of these AND, OR and NOT are certainly the most commonly used in both everyday language and most of mathematics.

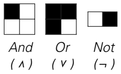


Logical functions of two arguments and their common names. Black stands for TRUE; white for FALSE. AND, OR, NOT, and IMPLIES are widely used in traditional logic. EQUAL (if and only if) is common in more mathematical settings, while XOR is widespread in discrete mathematics. NAND and NOR are mostly used only in circuit design and in a few foundational studies of logic. The first argument for each function appears on the left in the picture; the second argument on top. The functions are numbered like 2-neighbor analogs of the cellular automaton rules of page 53.

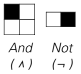
But at least at a formal level, logic can be viewed simply as a theory of functions that take on two possible values given variables with two possible values. And as we discussed on page 616, any such function can be represented as a combination of AND, OR and NOT.

But the table below demonstrates that as soon as one goes beyond the familiar traditions of language and mathematics there are other operators that can also just as well be used as primitives. And indeed it has been known since before 1900 that both NAND and NOR on their own work—a fact I already used on pages 617 and 775.

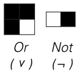
0	$\neg a \land a$	1	$\neg (a \lor b)$	2	$\neg a \land b$	3	$\neg a$
4	$\neg b \land a$	5	$\neg b$	6	$\neg (a \land b) \land (a \lor b)$	7	$\neg (a \land b)$
8	$a \land b$	9	$a \land b \lor \neg (a \lor b)$	10	b	11	$\neg a \lor b$
12	a	13	$\neg b \lor a$	14	$a \lor b$	15	$\neg a \lor a$



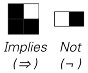
0	$\neg a \land a$	1	$\neg a \land \neg b$	2	$\neg a \land b$	3	$\neg a$
4	$\neg b \land a$	5	$\neg b$	6	$\neg (\neg a \land \neg b) \land \neg (a \land b)$	7	$\neg (a \land b)$
8	$a \land b$	9	$\neg (\neg a \land b) \land \neg (\neg b \land a)$	10	b	11	$\neg (\neg b \land a)$
12	a	13	$\neg (\neg a \land b)$	14	$\neg (\neg a \land \neg b)$	15	$\neg (\neg a \land a)$



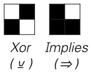
0	$\neg (\neg a \lor a)$	1	$\neg (a \lor b)$	2	$\neg (\neg b \lor a)$	3	$\neg a$
4	$\neg (\neg a \lor b)$	5	$\neg b$	6	$\neg (\neg a \lor b) \lor \neg (\neg b \lor a)$	7	$\neg a \lor \neg b$
8	$\neg (\neg a \lor \neg b)$	9	$\neg (\neg a \lor \neg b) \lor \neg (a \lor b)$	10	b	11	$\neg a \lor b$
12	a	13	$\neg b \lor a$	14	$a \lor b$	15	$\neg a \lor a$



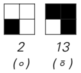
0	$\neg (a \Rightarrow a)$	1	$\neg (\neg a \Rightarrow b)$	2	$\neg (b \Rightarrow a)$	3	$\neg a$
4	$\neg (a \Rightarrow b)$	5	$\neg b$	6	$(a \Rightarrow b) \Rightarrow \neg (b \Rightarrow a)$	7	$a \Rightarrow \neg b$
8	$\neg (a \Rightarrow \neg b)$	9	$\neg ((a \Rightarrow b) \Rightarrow \neg (b \Rightarrow a))$	10	b	11	$a \Rightarrow b$
12	a	13	$b \Rightarrow a$	14	$\neg a \Rightarrow b$	15	$a \Rightarrow a$



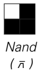
0	$a \veebar a$	1	$(a \Rightarrow b) \veebar b$	2	$((a \Rightarrow b) \Rightarrow b) \veebar a$	3	$(a \Rightarrow a) \veebar a$
4	$((a \Rightarrow b) \Rightarrow b) \veebar b$	5	$(a \Rightarrow a) \veebar b$	6	$a \veebar b$	7	$(a \Rightarrow b) \veebar a$
8	$(((a \Rightarrow b) \Rightarrow b) \veebar a) \veebar b$	9	$((a \Rightarrow a) \veebar a) \veebar b$	10	b	11	$a \Rightarrow b$
12	a	13	$b \Rightarrow a$	14	$(a \Rightarrow b) \Rightarrow b$	15	$a \Rightarrow a$



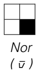
0	$a \circ a$	1	$a \circ (a \bar{\circ} b)$	2	$a \circ b$	3	$a \circ (a \bar{\circ} a)$
4	$b \circ a$	5	$b \circ (a \bar{\circ} a)$	6	$a \circ b \bar{\circ} (b \bar{\circ} a)$	7	$a \circ b \bar{\circ} b$
8	$(a \circ b) \circ b$	9	$(a \circ b) \circ (b \bar{\circ} a)$	10	b	11	$b \bar{\circ} a$
12	a	13	$a \bar{\circ} b$	14	$a \bar{\circ} (a \bar{\circ} b)$	15	$a \bar{\circ} a$



0	$((a \bar{\land} a) \bar{\land} a) \bar{\land} ((a \bar{\land} a) \bar{\land} a)$	1	$((a \bar{\land} a) \bar{\land} (b \bar{\land} b)) \bar{\land} ((a \bar{\land} a) \bar{\land} a)$	2	$((a \bar{\land} a) \bar{\land} a) \bar{\land} ((a \bar{\land} a) \bar{\land} b)$	3	$a \bar{\land} a$
4	$((a \bar{\land} a) \bar{\land} a) \bar{\land} ((a \bar{\land} b) \bar{\land} a)$	5	$b \bar{\land} b$	6	$((a \bar{\land} a) \bar{\land} b) \bar{\land} ((a \bar{\land} b) \bar{\land} a)$	7	$a \bar{\land} b$
8	$(a \bar{\land} b) \bar{\land} (a \bar{\land} b)$	9	$((a \bar{\land} a) \bar{\land} (b \bar{\land} b)) \bar{\land} (a \bar{\land} b)$	10	b	11	$(a \bar{\land} b) \bar{\land} a$
12	a	13	$(a \bar{\land} a) \bar{\land} b$	14	$(a \bar{\land} a) \bar{\land} (b \bar{\land} b)$	15	$(a \bar{\land} a) \bar{\land} a$



0	$(a \bar{\lor} a) \bar{\lor} a$	1	$a \bar{\lor} b$	2	$(a \bar{\lor} b) \bar{\lor} a$	3	$a \bar{\lor} a$
4	$(a \bar{\lor} a) \bar{\lor} b$	5	$b \bar{\lor} b$	6	$((a \bar{\lor} a) \bar{\lor} (b \bar{\lor} b)) \bar{\lor} (a \bar{\lor} b)$	7	$((a \bar{\lor} a) \bar{\lor} (b \bar{\lor} b)) \bar{\lor} ((a \bar{\lor} a) \bar{\lor} a)$
8	$(a \bar{\lor} a) \bar{\lor} (b \bar{\lor} b)$	9	$((a \bar{\lor} a) \bar{\lor} b) \bar{\lor} ((a \bar{\lor} b) \bar{\lor} a)$	10	b	11	$((a \bar{\lor} a) \bar{\lor} a) \bar{\lor} ((a \bar{\lor} a) \bar{\lor} b)$
12	a	13	$((a \bar{\lor} a) \bar{\lor} a) \bar{\lor} ((a \bar{\lor} b) \bar{\lor} a)$	14	$(a \bar{\lor} b) \bar{\lor} (a \bar{\lor} b)$	15	$((a \bar{\lor} a) \bar{\lor} a) \bar{\lor} ((a \bar{\lor} a) \bar{\lor} a)$



Functions that can be used to formulate logic. In each case the minimal combinations of primitive functions necessary to reproduce each of the 16 logical functions of two arguments is given. From these any possible logical function with any number of arguments can be obtained. Most textbook treatments of logic use AND, OR, and NOT as primitive functions. NAND and NOR are the only primitive functions that work on their own.

So this means that logic can be set up using just a single operator. But how complicated an axiom system does it then need? The first box in the picture below shows that the direct translation of the standard textbook AND, OR, NOT axiom system from page 773 is very complicated.

(a) $(a\circ b)\circ(a\circ b) = (b\circ a)\circ(b\circ a)$ $(a\circ a)\circ(b\circ b) = (b\circ b)\circ(a\circ a)$ $(a\circ((b\circ b)\circ((b\circ b)\circ(b\circ b))))\circ(a\circ((b\circ b)\circ((b\circ b)\circ(b\circ b)))) = a$ $(a\circ a)\circ(((b\circ(b\circ b))\circ(b\circ(b\circ b)))\circ((b\circ(b\circ b))\circ(b\circ(b\circ b)))) = a$ $a\circ b = ((a\circ b)\circ(a\circ b))\circ((a\circ b)\circ(a\circ b))$ $(a\circ((b\circ b)\circ(c\circ c)))\circ(a\circ((b\circ b)\circ(c\circ c))) = (((a\circ b)\circ(a\circ b))\circ((a\circ b)\circ(a\circ b)))\circ(((a\circ c)\circ(a\circ c))\circ((a\circ c)\circ(a\circ c)))$ $(a\circ a)\circ(((b\circ c)\circ(b\circ c))\circ((b\circ c)\circ(b\circ c))) = (((a\circ a)\circ(b\circ b))\circ((a\circ a)\circ(c\circ c)))\circ(((a\circ a)\circ(b\circ b))\circ((a\circ a)\circ(c\circ c)))$

(b) $(a\circ a)\circ(a\circ a) = a$ $a\circ b = b\circ a$ $a\circ((b\circ c)\circ(b\circ c)) = b\circ((a\circ c)\circ(a\circ c))$ $(a\circ b)\circ(a\circ(b\circ b)) = a$

(c) $(a\circ a)\circ(a\circ a) = a$ $a\circ(b\circ(b\circ b)) = a\circ a$ $(a\circ(b\circ c))\circ(a\circ(b\circ c)) = ((b\circ b)\circ a)\circ((c\circ c)\circ a)$

(d) $(a\circ a)\circ(a\circ b) = a$ $a\circ(a\circ b) = a\circ(b\circ b)$ $a\circ(a\circ(b\circ c)) = b\circ(b\circ(a\circ c))$

(e) $a\circ(b\circ(a\circ c)) = ((c\circ b)\circ b)\circ a$ $(a\circ a)\circ(b\circ a) = a$

(f) $(a\circ b)\circ(a\circ(b\circ c)) = a$ $a\circ b = b\circ a$

(g) $((b\circ c)\circ a)\circ(b\circ((b\circ a)\circ b)) = a$

(h) $(b\circ((a\circ b)\circ b))\circ(a\circ(c\circ b)) = a$

Axiom systems for basic logic (propositional calculus) formulated in terms of NAND ($\barwedge$). The number of operators that occur in these axiom systems is respectively 94, 17, 17, 13, 9, 6, 6, 6. System (a) is a translation of the standard textbook one given on page 773 in terms of AND, OR and NOT. (b) is based on the Robbins axioms from page 773. (c) is the Sheffer axiom system. (e) is the Meredith axiom system. The other axiom systems were found for this book. (d) was used on page 775. (g) and (h) are as short as is possible. Each axiom system given applies equally well to NOR as well as NAND.

But boxes (b) and (c) show that known alternative axiom systems for logic reduce the size of the axiom system by about a factor of ten. And some further reduction is achieved by manipulating the resulting axioms—leading to the axiom system used above and given in box (d).

But can one go still further? And what happens for example if one just tries to search simple axiom systems for ones that work?

One can potentially test axiom systems by seeing what operators satisfy their constraints, as on page 805. The first non-trivial axiom system that even allows the NAND operator is $\{(a\circ a)\circ(a\circ a) = a\}$. And the first axiom system for which NAND and NOR are the only operators allowed that involve 2 possible values is $\{((b\circ b)\circ a)\circ(a\circ b) = a\}$.

But if one now looks at operators involving 3 possible values then it turns out that this axiom system allows ones not equivalent to NAND

and Nor. And this means that it cannot successfully reproduce all the results of logic. Yet if any axiom system with just a single axiom is going to be able to do this, the axiom must be of the form $\{\ldots = a\}$.

With up to 6 Nands and 2 variables none of the 16,896 possible axiom systems of this kind work even up to 3-value operators. But with 6 Nands and 3 variables, 296 of the 288,684 possible axiom systems work up to 3-value operators, and 100 work up to 4-value operators.

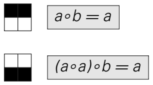
And of the 25 of these that are not trivially equivalent, it then turns out that the two given as (g) and (h) on the facing page can actually be proved as on the next two pages to be axiom systems for logic—thus showing that in the end quite remarkable simplification can be achieved relative to ordinary textbook axiom systems.

If one looks at axiom systems of the form $\{\ldots = a,\ a \circ b = b \circ a\}$ the first one that one finds that allows only Nand and Nor with 2-value operators is $\{(a \circ a) \circ (a \circ a) = a,\ a \circ b = b \circ a\}$. But as soon as one uses a total of just 6 Nands, one suddenly finds that out of the 3402 possibilities with 3 variables 32 axiom systems equivalent to case (f) above all end up working all the way up to at least 4-value operators. And in fact it then turns out that (f) indeed works as an axiom system for logic.

So what this means is that if one were just to go through a list of the simplest few thousand axiom systems one would already be quite likely to find one that represents logic.

In human intellectual history logic has had great significance. But if one looks just at axiom systems is there anything obviously special about the ones for logic? My guess is that unless one asks about very specific details there is really not—and that standard logic is in a sense distinguished in the end only by its historical context.

One feature of logic is that its axioms effectively describe a single specific operator. But it turns out that there are all sorts of other axioms that also do this. I gave three examples on page 803, and in the picture on the right I give two more very simple examples. Indeed, given many forms of operator there are always axiom systems that can be found to describe it.



Axiom systems that reproduce equivalence results for the forms of operators shown.

L1 (a ((a a) a)) (a ((a a) a))

= A (((((a a) a) (a ((a a) a))) (a ((a a) a))) (((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a)))) (a ((a a) a))

= A (((((a a) a) (a ((a a) a))) (a ((a a) a))) (((a a) a) (a ((a a) a)))) (a ((a a) a))

= A (((((a a) a) (a ((a a) a))) (a ((a a) a))) a) (a ((a a) a))

= A ((a (a ((a a) a))) a) (a ((a a) a))

= A a

L2 (a a) ((a ((a a) a)) a)

= A (a a) ((a ((a a) a)) (((a ((a a) a)) a) (a ((a a) a))))

= L1 (a a) ((a ((a a) a)) (((a ((a a) a)) ((a ((a a) a)) (a ((a a) a)))) (a ((a a) a))))

= L1 (((a ((a a) a)) (a ((a a) a))) ((a ((a a) a)) (a ((a a) a)))) ((a ((a a) a)) (((a ((a a) a)) ((a ((a a) a)) (a ((a a) a)))) (a ((a a) a))))

= A (a ((a a) a)) (a ((a a) a))

= L1 a

L3 (a b) ((a a) (((a a) b) (a a)))

= L2 (((a a) ((a ((a a) a)) a)) b) ((a a) (((a a) b) (a a)))

= A b

L4 ((a ((a b) a)) d) (b ((b d) b))

= A ((a ((a b) a)) d) (b ((b d) (((a c) b) (a ((a b) a)))))

= A ((a ((a b) a)) d) (b (((((a c) b) (a ((a b) a))) d) (((a c) b) (a ((a b) a)))))

= A ((a ((a b) a)) d) ((((a c) b) (a ((a b) a))) (((((a c) b) (a ((a b) a))) d) (((a c) b) (a ((a b) a)))))

= L3 (((((a c) b) (a ((a b) a))) ((((a c) b) ((a c) b)) (((((a c) b) ((a c) b)) (a ((a b) a))) (((a c) b) ((a c) b))))) d) ((((a c) b) (a ((a b) a))) (((((a c) b) (a ((a b) a))) d) (((a c) b) (a ((a b) a)))))

= A d

L5 (a ((a a) a)) ((a ((a a) a)) a)

= A (a ((a a) a)) ((a ((a a) a)) (((a ((a a) a)) a) (a ((a a) a))))

= A (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) (((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a)))) ((a ((a a) a)) (((a ((a a) a)) a) (a ((a a) a))))

= A (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) (((a a) a) (a ((a a) a)))) ((a ((a a) a)) (((a ((a a) a)) a) (a ((a a) a))))

= A (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) a) ((a ((a a) a)) (((a ((a a) a)) a) (a ((a a) a))))

= A (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) (((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))))) a) ((a ((a a) a)) (((a ((a a) a)) a) (a ((a a) a))))

= L4 a

L6 (a ((a a) a)) (a (a ((a a) a)))

= L1 (a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a)))

= A ((((a ((a a) a)) ((a ((a a) a)) a)) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) (((a ((a a) a)) ((a ((a a) a)) a)) ((((a ((a a) a)) ((a ((a a) a)) a)) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) ((a ((a a) a)) a))))

= A (a ((a a) a)) (((a ((a a) a)) ((a ((a a) a)) a)) ((((a ((a a) a)) ((a ((a a) a)) a)) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) ((a ((a a) a)) a))))

= L5 (a ((a a) a)) (a ((((a ((a a) a)) ((a ((a a) a)) a)) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) ((a ((a a) a)) a))))

= L5 (a ((a a) a)) (a ((a ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) ((a ((a a) a)) a))))

= L1 (a ((a a) a)) (a ((a ((a ((a a) a)) (a (a ((a a) a))))) ((a ((a a) a)) ((a ((a a) a)) a))))

= L1 (a ((a a) a)) (a ((a ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) ((a ((a a) a)) a))))

= L4 (a ((a a) a)) (a (((((a ((a a) a)) a) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) ((a ((a a) a)) a))))

= A (a ((a a) a)) (a ((a ((a a) a)) ((a ((a a) a)) ((a ((a a) a)) a))))

L7 (a ((a a) a)) (a (a ((a a) a)))

= L6 (a ((a a) a)) (a ((a ((a a) a)) ((a ((a a) a)) ((a ((a a) a)) a))))

= L5 (a ((a a) a)) (a ((a ((a a) a)) a))

= A (a ((a a) a)) (a (((((a a) a) (a ((a a) a))) ((a a) (((a a) (a ((a a) a))) (a a)))) a))

= A (a ((a a) a)) (a ((a ((a a) (((a a) (a ((a a) a))) (a a)))) a))

= L3 ((a (a ((a a) a))) ((a a) (((a a) (a ((a a) a))) (a a)))) (a ((a ((a a) (((a a) (a ((a a) a))) (a a)))) a))

= A (a a) (((a a) (a ((a a) a))) (a a))

L8 ((a a) (((a a) (a ((a a) a))) (a a))) ((a a) (((a a) (a ((a a) a))) (a a)))

= L7 ((a ((a a) a)) (a (a ((a a) a)))) ((a ((a a) a)) (a (a ((a a) a))))

= L1 ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a)))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))

= L1 a ((a a) a)

L9 (a ((a a) a)) (a (a ((a a) a)))

= L1 (a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a)))

= L8 (((a ((a a) a)) (a ((a a) a))) ((((a ((a a) a)) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a))))) (((a ((a a) a)) (a ((a a) a))) ((((a ((a a) a)) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a)))))

= L1 (a ((((a ((a a) a)) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a))))) (a ((((a ((a a) a)) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a)))))

L10 (a a) (((a a) (a ((a a) a))) (a a))

= L7 (a ((a a) a)) (a (a ((a a) a)))

= L9 (a ((((a ((a a) a)) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a))))) (a ((((a ((a a) a)) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a)))))

= L1 (a ((a ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a))))) (a ((a ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a)))))

L11 (a a) (((a a) (a ((a a) a))) (a a))

= L10 (a ((a ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a))))) (a ((a ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a)))))

= L1 (a ((a ((a ((a a) a)) (a (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a))))) (a ((a ((a ((a a) a)) (a (a ((a a) a))))) ((a ((a a) a)) (a ((a a) a)))))

= L7 (a ((a ((a a) (((a a) (a ((a a) a))) (a a)))) ((a ((a a) a)) (a ((a a) a))))) (a ((a ((a a) (((a a) (a ((a a) a))) (a a)))) ((a ((a a) a)) (a ((a a) a)))))

L12 (a a) (((a a) (a ((a a) a))) (a a))

= L11 (a ((a ((a a) (((a a) (a ((a a) a))) (a a)))) ((a ((a a) a)) (a ((a a) a))))) (a ((a ((a a) (((a a) (a ((a a) a))) (a a)))) ((a ((a a) a)) (a ((a a) a)))))

= A (a (((((a a) a) (a ((a a) a))) ((a a) (((a a) (a ((a a) a))) (a a)))) ((a ((a a) a)) (a ((a a) a))))) (a (((((a a) a) (a ((a a) a))) ((a a) (((a a) (a ((a a) a))) (a a)))) ((a ((a a) a)) (a ((a a) a)))))

= A (a ((a ((a a) a)) ((a ((a a) a)) (a ((a a) a))))) (a ((a ((a a) a)) ((a ((a a) a)) (a ((a a) a)))))

= L1 (a ((a ((a a) a)) a)) (a ((a ((a a) a)) a))

L13 (a ((a a) a)) (a (a ((a a) a)))

= L7 (a a) (((a a) (a ((a a) a))) (a a))

= L12 (a ((a ((a a) a)) a)) (a ((a ((a a) a)) a))

L14 (a ((a ((a a) a)) a)) (a ((a ((a a) a)) a))

= L13 (a ((a a) a)) (a (a ((a a) a)))

= L1 (a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a)))

= A ((((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) ((((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a)))))

= A ((((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) (((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))))) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) ((((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a)))))

= L4 (a ((a a) a)) (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) ((((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))) ((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a)))))

= L4 (a ((a a) a)) (((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a))) ((a ((a a) a)) ((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a)))))

= A (a ((a a) a)) (((((a a) a) (a ((a a) a))) (a ((a a) a))) ((a ((a a) a)) ((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a)))))

= A (a ((a a) a)) ((a (a ((a a) a))) ((a ((a a) a)) ((((a a) a) ((((a a) a) (a ((a a) a))) ((a a) a))) (a ((a a) a)))))

= A (a ((a a) a)) ((a (a ((a a) a))) ((a ((a a) a)) ((((a a) a) (a ((a a) a))) (a ((a a) a)))))

L15 (a ((a ((a a) a)) a)) (a ((a ((a a) a)) a))

= L14 (a ((a a) a)) ((a (a ((a a) a))) ((a ((a a) a)) ((((a a) a) (a ((a a) a))) (a ((a a) a)))))

= A (a ((a a) a)) ((a (a ((a a) a))) ((a ((a a) a)) (a (a ((a a) a)))))

= L13 (a ((a a) a)) ((a (a ((a a) a))) ((a ((a ((a a) a)) a)) (a ((a ((a a) a)) a))))

L16 (a ((a ((a a) a)) a)) (a ((a ((a a) a)) a))

= L15 (a ((a a) a)) ((a (a ((a a) a))) ((a ((a ((a a) a)) a)) (a ((a ((a a) a)) a))))

= L12 (a ((a a) a)) ((a (a ((a a) a))) ((a a) (((a a) (a ((a a) a))) (a a))))

= L3 (a ((a a) a)) (a ((a a) a))

L17 (a ((a a) a)) (a (a ((a a) a)))

= L13 (a ((a ((a a) a)) a)) (a ((a ((a a) a)) a))

= L16 (a ((a a) a)) (a ((a a) a))

= L1 a

L18 a ((a a) a)

= L1 ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a)))) ((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a))))

= L16 ((a ((a a) a)) (((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a)))) (a ((a a) a)))) ((a ((a a) a)) (((a ((a a) a)) (((a ((a a) a)) (a ((a a) a))) (a ((a a) a)))) (a ((a a) a))))

= L1 ((a ((a a) a)) (((a ((a a) a)) (a (a ((a a) a)))) (a ((a a) a)))) ((a ((a a) a)) (((a ((a a) a)) (a (a ((a a) a)))) (a ((a a) a))))

= L17 ((a ((a a) a)) (a (a ((a a) a)))) ((a ((a a) a)) (a (a ((a a) a))))

= L17 a a

L19 (a a) (a (a a))

= L18 (a a) (a (a ((a a) a)))

= L18 (a ((a a) a)) (a (a ((a a) a)))

= L17 a

L20 a (a ((a ((a a) a)) a))

= L18 a (a ((a ((a ((a a) a)) a)) a))

= L2 ((a a) ((a ((a a) a)) a)) (a ((a ((a ((a a) a)) a)) a))

= A (a ((a a) a)) a

L21 a (a a)

= L18 a (a ((a a) a))

= L18 a (a ((a ((a a) a)) a))

= L20 (a ((a a) a)) a

L22 (a a) (a a)

= L18 (a ((a a) a)) (a ((a a) a))

= L1 a

T1 (a a) (a a)

= L22 a

L23 (a a) a

= L18 (a ((a a) a)) a

= L21 a (a a)

L24 ((a b) a) (a a)

= L18 ((a b) a) (a ((a a) a))

= A a

L25 a ((a ((a b) (a b))) a)

= A (((a b) ((a b) (a b))) (a ((a ((a b) (a b))) a))) ((a b) (((a b) (a ((a ((a b) (a b))) a))) (a b)))

= A ((a b) (a b)) ((a b) (((a b) (a ((a ((a b) (a b))) a))) (a b)))

= L19 ((a b) (a b)) ((a b) (((((a b) (a b)) ((a b) ((a b) (a b)))) (a ((a ((a b) (a b))) a))) (a b)))

= A ((a b) (a b)) ((a b) (((((a b) (a b)) ((((a b) (a b)) (a ((a (a b)) a))) ((a b) (a b)))) (a ((a ((a b) (a b))) a))) (a b)))

= L24 ((a b) (a b)) ((a b) (((((((a b) (a b)) ((((a b) (a b)) (a ((a (a b)) a))) ((a b) (a b)))) (a ((a (a b)) a))) (((a b) (a b)) ((((a b) (a b)) (a ((a (a b)) a))) ((a b) (a b))))) ((((a b) (a b)) ((((a b) (a b)) (a ((a (a b)) a))) ((a b) (a b)))) (((a b) (a b)) ((((a b) (a b)) (a ((a (a b)) a))) ((a b) (a b)))))) (a ((a ((a b) (a b))) a))) (a b)))

= A ((a b) (a b)) ((a b) ((((a ((a (a b)) a)) ((((a b) (a b)) ((((a b) (a b)) (a ((a (a b)) a))) ((a b) (a b)))) (((a b) (a b)) ((((a b) (a b)) (a ((a (a b)) a))) ((a b) (a b)))))) (a ((a ((a b) (a b))) a))) (a b)))

= A ((a b) (a b)) ((a b) ((((a ((a (a b)) a)) ((((a b) (a b)) ((a b) ((a b) (a b)))) (((a b) (a b)) ((a b) ((a b) (a b)))))) (a ((a ((a b) (a b))) a))) (a b)))

= L19 ((a b) (a b)) ((a b) ((((a ((a (a b)) a)) ((a b) (a b))) (a ((a ((a b) (a b))) a))) (a b)))

= A ((a b) (a b)) ((a b) (((a b) (a b)) (a b)))

L26 a ((a ((a b) (a b))) a)

= L25 ((a b) (a b)) ((a b) (((a b) (a b)) (a b)))

= A ((a b) (a b)) (((((a b) (a b)) (a b)) ((a b) (((a b) (a b)) (a b)))) (((a b) (a b)) ((((a b) (a b)) ((a b) (((a b) (a b)) (a b)))) ((a b) (a b)))))

= A ((a b) (a b)) ((a b) (((a b) (a b)) ((((a b) (a b)) ((a b) (((a b) (a b)) (a b)))) ((a b) (a b)))))

= L12 ((a b) (a b)) ((a b) (((a b) (((a b) (((a b) (a b)) (a b))) (a b))) ((a b) (((a b) (((a b) (a b)) (a b))) (a b)))))

= L16 ((a b) (a b)) ((a b) (((a b) (((a b) (a b)) (a b))) ((a b) (((a b) (a b)) (a b)))))

= L1 ((a b) (a b)) ((a b) (a b))

L27 a ((a ((a b) (a b))) a)

= L26 ((a b) (a b)) ((a b) (a b))

= L22 a b

L28 a

= L24 ((a b) a) (a a)

= L27 ((a b) a) ((((a b) a) ((((a b) a) (a a)) (((a b) a) (a a)))) ((a b) a))

= L24 ((a b) a) ((((a b) a) (a a)) ((a b) a))

L29 ((a b) a) (a ((a b) a))

= L24 ((a b) a) ((((a b) a) (a a)) ((a b) a))

= L28 a

L30 (a b) (a a)

= L27 (a ((a ((a b) (a b))) a)) (a a)

= L29 (a ((a ((a b) (a b))) a)) ((((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))) (((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))))

= L29 (a ((a ((a b) (a b))) a)) ((((a ((a b) (a b))) a) ((((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))) ((a ((a b) (a b))) a))) (((a ((a b) (a b))) a) ((((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))) ((a ((a b) (a b))) a))))

= A (((((a ((a b) (a b))) a) ((((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))) ((a ((a b) (a b))) a))) (a ((a ((a b) (a b))) a))) (((a ((a b) (a b))) a) ((((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))) ((a ((a b) (a b))) a)))) ((((a ((a b) (a b))) a) ((((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))) ((a ((a b) (a b))) a))) (((a ((a b) (a b))) a) ((((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))) ((a ((a b) (a b))) a))))

= L24 ((a ((a b) (a b))) a) ((((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))) ((a ((a b) (a b))) a))

= L29 ((a ((a b) (a b))) a) (a ((a ((a b) (a b))) a))

= L29 a

L31 b ((a b) (a b))

= L3 ((a b) ((a a) (((a a) b) (a a)))) ((a b) (a b))

= L30 a b

L32 a ((a b) (a b))

= L30 ((a b) (a a)) ((a b) (a b))

= L30 a b

L33 a ((a b) a)

= L32 a ((a ((a b) (a b))) a)

= L27 a b

L34 ((a b) c) (b c)

= L33 ((a b) c) (b ((b c) b))

= L33 ((a ((a b) a)) c) (b ((b c) b))

= L4 c

L35 b a

= L31 a ((b a) (b a))

= L33 a ((a ((b a) (b a))) a)

= L31 a ((b a) a)

L36 b (b (a b))

= L33 b (b (a ((a b) a)))

= L33 b (b ((b (a ((a b) a))) b))

= A (((a ((a b) a)) b) (a ((a b) a))) (b ((b (a ((a b) a))) b))

= A (((a ((a b) a)) (((a ((a b) a)) b) (a ((a b) a)))) (a ((a b) a))) (b ((b (a ((a b) a))) b))

= L4 a ((a b) a)

= L33 a b

L37 (a b) a

= L36 a (a ((a b) a))

= L33 a (a b)

L38 (b a) a

= L36 a (a ((b a) a))

= L35 a (b a)

L39 (b b) (b (a b))

= L38 (b b) ((a b) b)

= L33 (b b) ((a b) (((a b) b) (a b)))

= L3 (b b) ((a b) (((a b) ((a b) ((a a) (((a a) b) (a a))))) (a b)))

= L3 (((a b) ((a a) (((a a) b) (a a)))) ((a b) ((a a) (((a a) b) (a a))))) ((a b) (((a b) ((a b) ((a a) (((a a) b) (a a))))) (a b)))

= A (a b) ((a a) (((a a) b) (a a)))

= L3 b

L40 a

= L39 (a a) (a ((b a) a))

= L38 (a a) (a (a (b a)))

= L36 (a a) (b a)

L41 a

= L39 (a a) (a ((a b) a))

= L33 (a a) (a b)

L42 b a

= L41 b ((a a) (a b))

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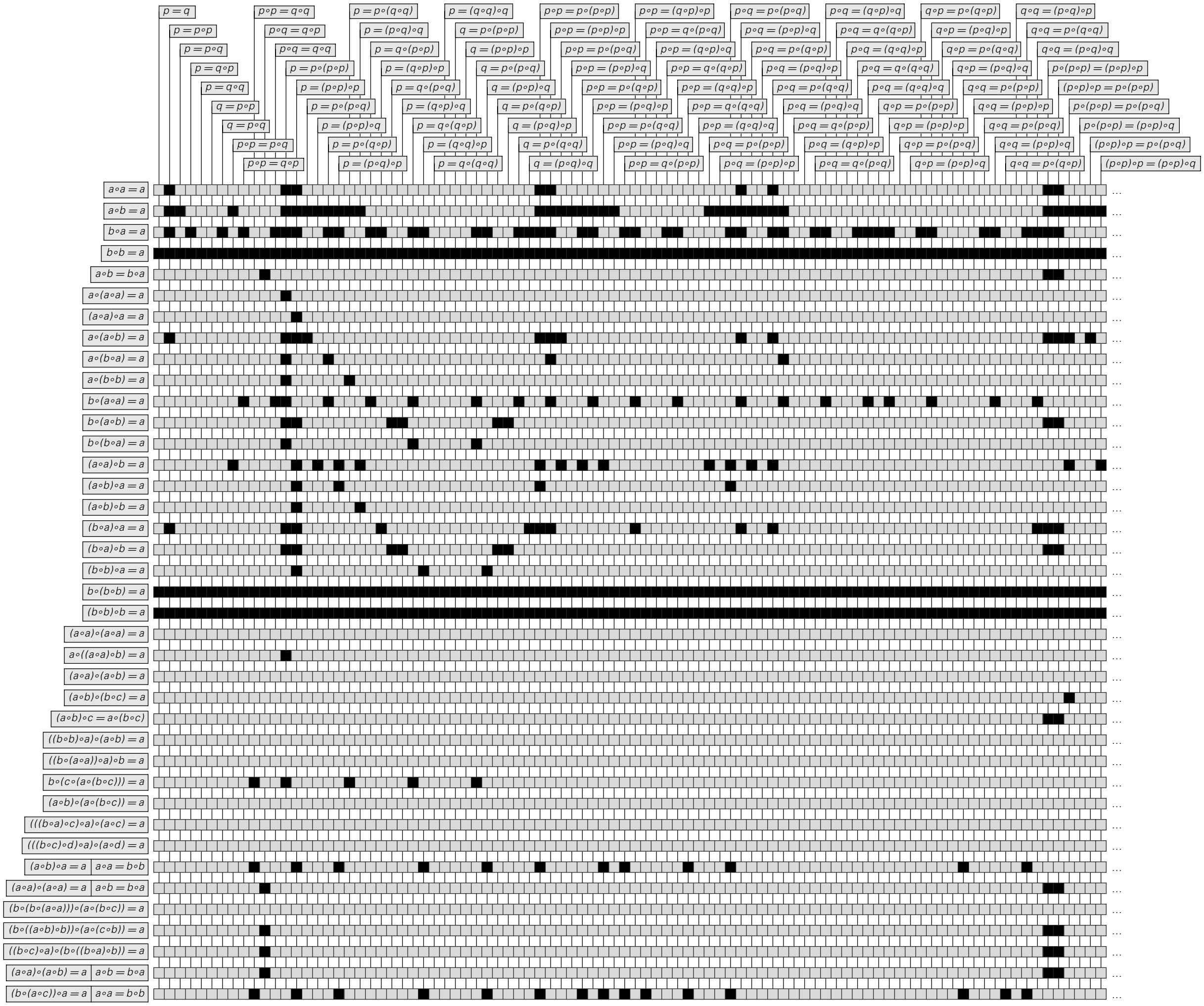
= L33 b((aa)(a((ab)a)))
= L33 b((aa)(((aa)(a((ab)a)))(aa)))
= A (((aa)b)(a((ab)a)))((aa)(((aa)(a((ab)a)))(aa)))
= A a((ab)a)
= L33 ab
L43 (ac)((ab)c)
= L42 ((ab)c)(ac)
= L33 ((ab)c)(a((ac)a))
= A c
L44 (bc)((ab)c)
= L42 ((ab)c)(bc)
= L34 c
L45 (ba)((ac)b)
= L42 (ab)((ac)b)
= L43 b
L46 (ba)(a(bc))
= L42 (ba)((bc)a)
= L43 a
L47 bc
= L46 ((ab)(bc))((bc)((ab)c))
= L44 ((ab)(bc))c
= L42 c((ab)(bc))
L48 ab
= L45 ((ab)(ca))(((ca)b)(ab))
= L42 ((ab)(ca))((ab)((ca)b))
= L44 ((ab)(ca))b
L49 b((ab)(ca))
= L42 ((ab)(ca))b
= L48 ab
L50 (ab)c
= L43 (a((ab)c))((ac)((ab)c))
= L43 (a((ab)c))c
= L42 c(a((ab)c))
L51 a(b(ab))
= L42 a((ab)b)
= L42 ((ab)b)a
= L50 a((ab)(((ab)b)a))
= L42 a((ba)(((ab)b)a))
= L44 aa
L52 (ba)(ab)
= L50 (ab)(b((ba)(ab)))
= L47 (ab)(ab)
L53 (aa)((ba)(ba))
= L44 (aa)((ba)(((aa)(ba))((((aa)(ba))(aa))(ba))))
= L50 (aa)((((aa)(ba))(aa))(ba))
= L42 (aa)((ba)(((aa)(ba))(aa)))
= L42 (aa)((ba)((aa)((aa)(ba))))
= L40 (aa)((ba)((aa)a))
= L42 (aa)((ba)(a(aa)))
= L47 a(aa)
L54 ((ab)(ab))((ab)(ab))
= L52 ((ba)(ab))((ba)(ab))
= L52 ((ab)(ba))((ba)(ab))
= L52 ((ab)(ba))((ab)(ab))
L55 ab
= L22 ((ab)(ab))((ab)(ab))
= L54 ((ab)(ba))((ab)(ab))
= L52 ((ba)(ba))((ab)(ab))
L56 a(b(bb))
= L53 a((bb)((ab)(ab)))
= L42 a(((ab)(ab))(bb))
= L40 a(((ab)(ab))(((bb)(ab))((bb)(ab))))
= L53 a((ab)((ab)(ab)))
= L42 a(((ab)(ab))(ab))
= L32 a(((ab)(ab))(a((ab)(ab))))
= L51 aa
T2 a(b(bb))
= L56 aa
L57 ((aa)(((ab)(ab))c))((aa)(((ab)(ab))c))
= L56 ((aa)(((ab)(d(dd)))c))((aa)(((ab)(d(dd)))c))
= L56 ((a(d(dd)))(((ab)(d(dd)))c))((a(d(dd)))(((ab)(d(dd)))c))
= L56 ((a(d(dd)))(((ab)(d(dd)))c))(d(dd))
= L42 (d(dd))((a(d(dd)))(((ab)(d(dd)))c))
= L42 (d(dd))((((ab)(d(dd)))c)(a(d(dd))))
= L42 ((((ab)(d(dd)))c)(a(d(dd))))(d(dd))
= L46 ((((ab)(d(dd)))c)(a(d(dd))))(((ab)(d(dd)))((d(dd))((ab)(d(dd)))))
= L33 ((((ab)(d(dd)))c)(a((a(d(dd)))a)))(((ab)(d(dd)))((d(dd))((ab)(d(dd)))))
= A ((((ab)(d(dd)))c)(a((a(d(dd)))a)))(((ab)(d(dd)))((((ab)(d(dd)))(a((a(d(dd)))a)))((ab)(d(dd)))))
= A a((a(d(dd)))a)
= L33 a(d(dd))
L58 (bb)(((bc)(bc))d)
= L22 (((bb)(((bc)(bc))d))((bb)(((bc)(bc))d)))(((bb)(((bc)(bc))d))((bb)(((bc)(bc))d)))
= L57 (b(a(aa)))(b(a(aa)))
= L56 (bb)(bb)
L59 (aa)(((ab)(ab))c)
= L58 (aa)(aa)
= L22 a
L60 a
= L59 (aa)(((a((ba)(cb)))(a((ba)(cb))))((d((a((ba)(cb)))(a((ba)(cb)))))(ed)))
= L49 (aa)(d((a((ba)(cb)))(a((ba)(cb)))))
= L49 (aa)(d((ba)(ba)))
L61 c((ac)(((ab)c)((ab)c)))
= L42 c((((ab)c)((ab)c))(ac))
= L42 ((((ab)c)((ab)c))(ac))c
= L46 ((((ab)c)((ab)c))(ac))(((ab)c)(c((ab)c)))
= L33 ((((ab)c)((ab)c))(a((ac)a)))(((ab)c)(c((ab)c)))
= A ((((ab)c)((ab)c))(a((ac)a)))(((ab)c)((((ab)c)(a((ac)a)))((ab)c)))
= A a((ac)a)
= L33 ac
L62 (aa)b
= L61 b(((aa)b)((((aa)(((ac)(ac))d))b)(((aa)(((ac)(ac))d))b)))
= L59 b(((aa)b)((ab)(ab)))
= L50 b(((ab)(ab))((aa)(((aa)b)((ab)(ab)))))
= L45 b(((ab)(ab))((((ab)((bc)a))((ab)((bc)a)))(((aa)b)((ab)(ab)))))
= L59 b(ab)
L63 a(ab)
= L42 a(ba)
= L62 (bb)a
L64 a(bc)
= L45 ((a(bc))c)((ca)(a(bc)))
= L42 ((a(bc))c)((ca)((bc)a))
= L44 ((a(bc))c)a
L65 a(bc)
= L64 ((a(bc))c)a
= L42 a((a(bc))c)
= L42 a(c(a(bc)))
L66 ac
= L59 ((ac)(ac))((((ac)(ca))((ac)(ca)))b)
= L52 ((ac)(ac))((((ca)(ca))((ca)(ca)))b)
= L22 ((ac)(ac))((ca)b)
L67 (ab)(ab)
= L59 (((ab)(ab))((ab)(ab)))(((((ab)(ab))((ba)(ba)))(((ab)(ab))((ba)(ba))))c)
= L55 (((ab)(ab))((ab)(ab)))(((ba)(ba))c)
= L22 (ab)(((ba)(ba))c)
L68 a(((bc)(ba))((bc)(ba)))
= L42 (((bc)(ba))((bc)(ba)))a
= L63 a(a((bc)(ba)))
= L33 a(a((bc)(b((ba)b))))
= L33 a(a((bc)(((bc)(b((ba)b)))(bc))))
= A a((((bc)a)(b((ba)b)))((bc)(((bc)(b((ba)b)))(bc))))
= A a(b((ba)b))
= L33 a(ba)
= L62 (bb)a
L69 (bc)a
= L22 (((bc)(bc))((bc)(bc)))a
= L68 a(((((bc)(bc))((cb)(cb)))(((bc)(bc))a))((((bc)(bc))((cb)(cb)))(((bc)(bc))a)))
= L55 a(((cb)(((bc)(bc))a))((cb)(((bc)(bc))a)))
L70 (bc)a
= L69 a(((cb)(((bc)(bc))a))((cb)(((bc)(bc))a)))
= L67 a(((cb)(cb))((cb)(cb)))
= L22 a(cb)
L71 ((bc)(bc))a
= L68 a((((bc)(cb))((bc)a))(((bc)(cb))((bc)a)))
= L52 a((((cb)(cb))((bc)a))(((cb)(cb))((bc)a)))
= L66 a((cb)(cb))
L72 (ba)(((bc)a)((bc)a))
= L3 (a((ba)(((bc)a)((bc)a))))((aa)(((aa)((ba)(((bc)a)((bc)a))))(aa)))
= L33 (a((ba)(((bc)a)((bc)a))))((aa)((ba)(((bc)a)((bc)a))))
= L61 (ba)((aa)((ba)(((bc)a)((bc)a))))
L73 (ba)(((bc)a)((bc)a))
= L72 (ba)((aa)((ba)(((bc)a)((bc)a))))
= L60 (ba)a
= L70 a(ab)
= L63 (bb)a
L74 (aa)c
= L73 (ac)(((ab)c)((ab)c))
= L50 (((ab)c)((ab)c))(a((ac)(((ab)c)((ab)c))))
= L73 (((ab)c)((ab)c))(a((aa)c))
= L22 (((ab)c)((ab)c))(((aa)(aa))((aa)c))
= L41 (((ab)c)((ab)c))(aa)
L75 (aa)((c(ab))(c(ab)))
= L71 (((ab)c)((ab)c))(aa)
= L74 (aa)c
L76 (b(ac))(aa)
= L22 (((b(ac))(b(ac)))((b(ac))(b(ac))))(aa)
= L63 (aa)((aa)((b(ac))(b(ac))))
= L75 (aa)((aa)b)
L77 ((ab)(ab))(ca)
= L75 ((ab)(ab))(((ca)((ab)c))((ca)((ab)c)))
= L45 ((ab)(ab))(cc)
L78 ((bc)(bc))a
= L45 ((bc)(bc))((ab)((bc)a))
= L70 (((bc)a)(ab))((bc)(bc))
= L22 (((((bc)a)(ab))(((bc)a)(ab)))((((bc)a)(ab))(((bc)a)(ab))))((bc)(bc))
= L63 ((bc)(bc))(((bc)(bc))((((bc)a)(ab))(((bc)a)(ab))))
= L76 (((((bc)a)(ab))(((bc)a)(ab)))((bc)(ab)))((bc)(bc))
= L70 ((bc)(bc))(((bc)(ab))((((bc)a)(ab))(((bc)a)(ab))))
= L73 ((bc)(bc))(((bc)(bc))(ab))
= L63 ((ab)(ab))((bc)(bc))
= L47 ((ab)(ab))((c((ab)(bc)))(c((ab)(bc))))
= L75 ((ab)(ab))c
= L71 c((ba)(ba))
L79 a(c((ab)(ab)))
= L42 a(c((ba)(ba)))
= L78 a(((bc)(bc))a)
= L62 (((bc)(bc))((bc)(bc)))a
= L22 (bc)a
L80 a((ba)c)
= L70 (c(ba))a
= L79 a((ba)((ac)(ac)))
= L42 a(((ac)(ac))(ba))
= L77 a(((ac)(ac))(bb))
= L78 a(c((a(bb))(a(bb))))
= L79 ((bb)c)a
L81 ((ca)(ab))((ca)(ab))
= L40 ((((ca)(ab))((ca)(ab)))(((ca)(ab))((ca)(ab))))((aa)(((ca)(ab))((ca)(ab))))
= L75 ((((ca)(ab))((ca)(ab)))(((ca)(ab))((ca)(ab))))((aa)(ca))
= L22 ((ca)(ab))((aa)(ca))
= L70 ((aa)(ca))((ab)(ca))
= L40 a((ab)(ca))
= L70 ((ca)(ab))a
= L79 a((ab)((a(ca))(a(ca))))
= L70 a(((a(ca))(a(ca)))(ba))
= L77 a(((a(ca))(a(ca)))(bb))
= L78 a((ca)((a(bb))(a(bb))))
= L79 ((bb)(ca))a
T3 ((bb)a)((cc)a)
= L42 ((bb)a)(a(cc))
= L42 (a(cc))((bb)a)
= L22 (((aa)(aa))(cc))((bb)a)
= L80 ((bb)a)(((aa)((bb)a))(cc))
= L70 ((cc)((aa)((bb)a)))((bb)a)
= L81 (((aa)((bb)a))(((bb)a)c))(((aa)((bb)a))(((bb)a)c))
= L40 (a(((bb)a)c))(a(((bb)a)c))
= L80 ((((bb)(bb))c)a)((((bb)(bb))c)a)
= L42 ((c((bb)(bb)))a)((c((bb)(bb)))a)
= L40 ((((c((bb)(bb)))(c((bb)(bb))))((bb)(c((bb)(bb)))))a)((((c((bb)(bb)))(c((bb)(bb))))((bb)(c((bb)(bb)))))a)
= L60 ((((c((bb)(bb)))(c((bb)(bb))))b)a)((((c((bb)(bb)))(c((bb)(bb))))b)a)
= L42 ((b((c((bb)(bb)))(c((bb)(bb)))))a)((b((c((bb)(bb)))(c((bb)(bb)))))a)
= L78 ((((cb)(cb))((bb)(bb)))a)((((cb)(cb))((bb)(bb)))a)
= L40 ((((cb)(cb))((b((bb)(cb)))(b((bb)(cb)))))a)((((cb)(cb))((b((bb)(cb)))(b((bb)(cb)))))a)
= L65 ((((cb)(cb))((b((cb)(b((bb)(cb)))))(b((cb)(b((bb)(cb)))))))a)((((cb)(cb))((b((cb)(b((bb)(cb)))))(b((cb)(b((bb)(cb)))))))a)
= L75 ((((cb)(cb))b)a)((((cb)(cb))b)a)
= L78 ((b((cb)(cb)))a)((b((cb)(cb)))a)
= L31 ((cb)a)((cb)a)
= L70 (a(bc))(a(bc))
```

A proof that the axiom system $\{((b\circ c)\circ a)\circ(b\circ((b\circ a)\circ b)) = a\}$ given as example (g) on page 808 can reproduce the Sheffer axiom system (c), and is thus a complete axiom system for logic. The proof involves taking the original axiom A and using it to establish a sequence of lemmas Ln, from which it is eventually possible to prove the three Sheffer axioms Tn. In each part of the proof each line can be obtained from the previous one just as on page 775 by applying the axiom or lemma indicated. Explicit $\bar{\wedge}$ operators have been omitted to allow expressions to be printed more compactly. The proof shown takes a total of 343 steps, and involves intermediate expressions with as many as 128 NANDs. It is quite possible that the proof could be considerably shortened. Note that any proof can always be recast without lemmas, but will usually then be much longer.

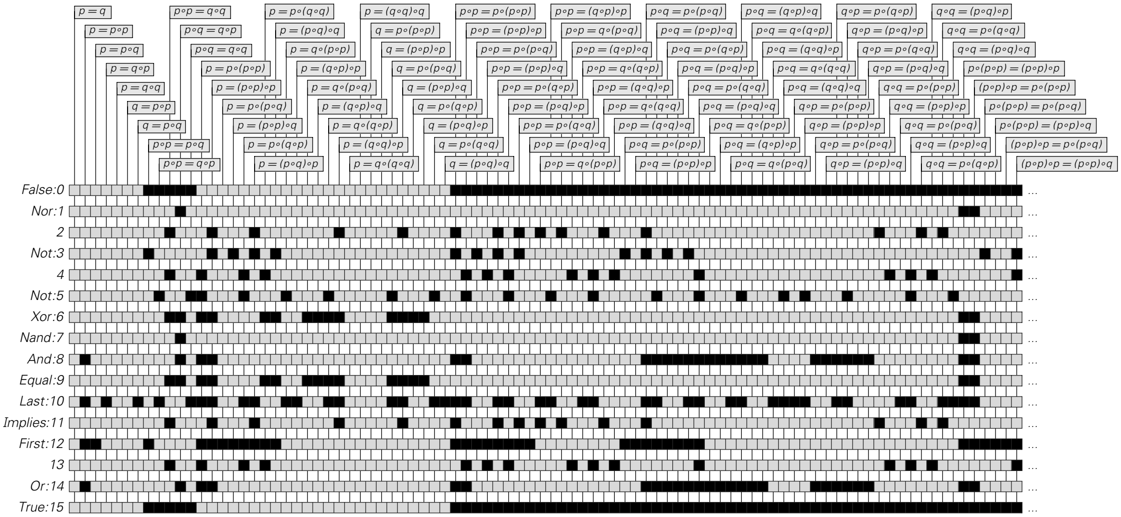
So what about patterns of theorems? Does logic somehow stand out when one looks at these? The picture below shows which possible simple equivalence theorems hold in systems from page 805.

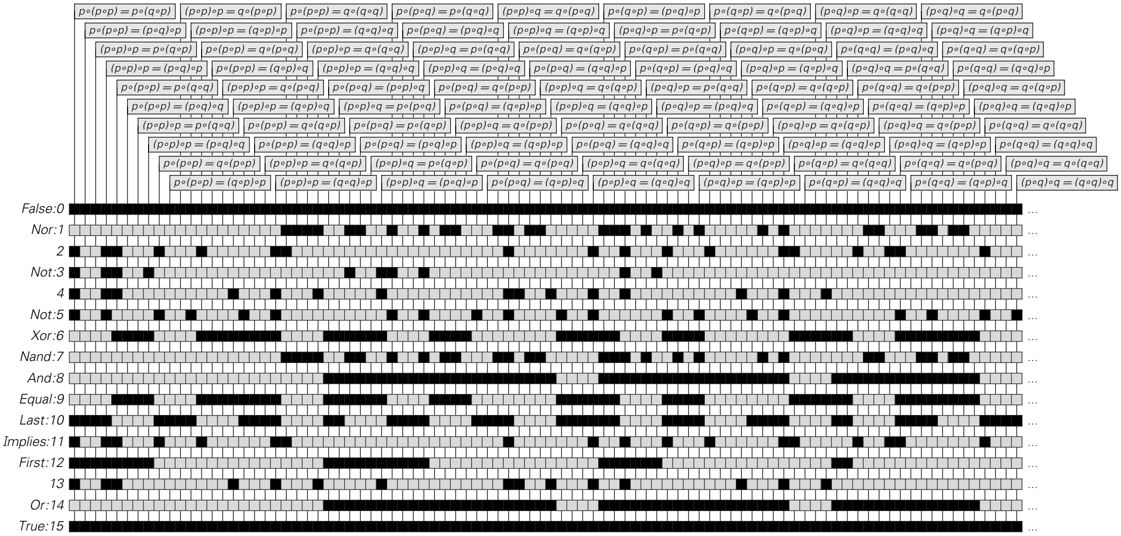
And comparing with page 805 one sees that typically the more forms of operator are allowed by the constraints of an axiom system, the fewer equivalence results hold in that axiom system.



Theorems that can be proved on the basis of simple axiom systems from page 805. A black square indicates that a particular theorem holds in a particular axiom system. In general the question of whether a given theorem holds is undecidable, but the particular theorems given here happen to be simple enough that results for them can with some effort be established with certainty.

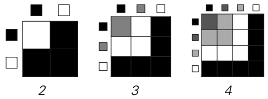
So what happens if essentially just a single form of operator is allowed? The pictures below show results for the 16 forms from page 806, and among these one sees that logic yields the fewest theorems.

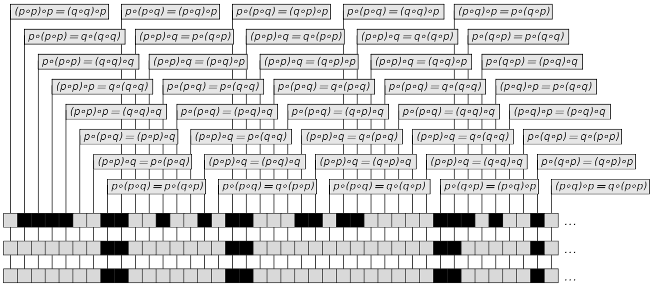




Theorems that hold with operators of each of the forms shown on page 806. NAND and NOR yield the smallest number of theorems.

But if one considers for example analogs of logic for variables with more than two possible values, the picture below shows that one immediately gets systems with still fewer theorems.





Theorems in analogs of logic that allow different numbers of truth values. Statements like $p = \neg\neg p$ do not hold in general with more than 2 truth values.

So what about proofs? Is there something about these that is somehow special in the case of ordinary logic?

In the axiom systems on page 803 the typical lengths of proofs seem to increase from one system to the next, so that they end up being longest for the last axiom system, which corresponds to logic.

But if one picks a different axiom system for logic—say one of the others on page 808—then the length of a particular proof will usually change. But since one can always just start by proving the new axioms, the change can only be by a fixed amount. And as it turns out, even the simplest axiom system (f) given on page 808 seems to allow fairly short proofs of at least most short theorems.

But as one tries to prove progressively longer theorems it appears that whatever axiom system one uses for logic the lengths of proofs can increase as fast as exponentially. A crucial point, however, is that for theorems of a given length there is always a definite upper limit on the length of proof needed. Yet once again this is not something unique to logic. Indeed, it turns out that this must always be the case for any axiom system—like those on page 803—that ends up allowing essentially only operators of a single form.

So what about other axiom systems?

The very simplest ones on pages 805 and 812 seem to yield proofs that are always comparatively short. But when one looks at axiom systems that are even slightly more complicated the proofs of anything but the simplest results can get much longer—making it in practice often difficult to tell whether a given result can actually even be proved at all.

And this is in a sense just another example of the same basic phenomenon that we already saw early in this section in multiway systems, and that often seems to occur in real mathematics: that even if a theorem is short to state, its proof can be arbitrarily long.

And this I believe is ultimately a reflection of the Principle of Computational Equivalence. For the principle suggests that most axiom systems whose consequences are not obviously simple will tend to be universal. And this means that they will exhibit computational irreducibility and undecidability—and will allow no general upper limit to be placed on how long a proof could be needed for any given result.

As I discussed earlier, most of the common axiom systems in traditional mathematics are known to be universal—basic logic being one of the few exceptions. But one might have assumed that to achieve their universality these axiom systems would have to be specially set up with all sorts of specific sophisticated features.

Yet from the results of this book—as embodied in the Principle of Computational Equivalence—we now know that this is not the case, and that in fact universality should already be rather common even among very simple axiom systems, like those on page 805.

And indeed, while operator systems and multiway systems have many superficial differences, I suspect that when it comes to universality they work very much the same. So in either idealization, one should not have to go far to get axiom systems that exhibit universality—just like most of the ones in traditional mathematics.

But once one has reached an axiom system that is universal, why should one in a sense ever have to go further? After all, what it means for an axiom system to be universal is that by setting up a suitable encoding it must in principle be possible to make that axiom system reproduce any other possible axiom system.

But the point is that the kinds of encodings that are normally used in mathematics are in practice rather limited. For while it is common, say, to take a problem in geometry and reformulate it as a problem in algebra, this is almost always done just by setting up a direct translation between the objects one is describing—usually in effect just by renaming the operators used to manipulate them.

Yet to take full advantage of universality one must consider not only translations between objects but also translations between complete proofs. And if one does this it is indeed perfectly possible, say, to program arithmetic to reproduce any proof in set theory. In fact, all one need do is to encode the axioms of set theory in something like the arithmetic equation system of page 786.

But with the notable exception of Gödel's Theorem these kinds of encodings are not normally used in mathematics. So this means that even when universality is present realistic idealizations of mathematics must still distinguish different axiom systems.

So in the end what is it that determines which axiom systems are actually used in mathematics? In the course of this section I have discussed a few criteria. But in the end history seems to be the only real determining factor. For given almost any general property that one can pick out in axiom systems like those on pages 773 and 774 there typically seem to be all sorts of operator and multiway systems—often including some rather simple ones—that share the exact same property.

So this leads to the conclusion that there is in a sense nothing fundamentally special about the particular axiom systems that have traditionally been used in mathematics—and that in fact there are all sorts of other axiom systems that could perfectly well be used as foundations for what are in effect new fields of mathematics—just as rich as the traditional ones, but without the historical connections.

So what about existing fields of mathematics? As I mentioned earlier in this section, I strongly believe that even within these there are fundamental limitations that have implicitly been imposed on what has actually been studied. And most often what has happened is that there are only certain kinds of questions or statements that have been considered of real mathematical interest.

The picture on the facing page shows a rather straightforward version of this. It lists in order a large number of theorems from basic logic, highlighting just those few that are considered interesting enough by typical textbooks of logic to be given explicit names.

1 $a = a \land a$	2 $a = a \lor a$	$a \land a = a \lor a$	3 $a \land b = b \land a$	4 $a \lor b = b \lor a$	5 $a = \lnot \lnot a$
$a \land a = \lnot \lnot a$	$a \lor a = \lnot \lnot a$	$\lnot a = \lnot (a \land a)$	$\lnot a = \lnot (a \lor a)$	$a = (a \land a) \land a$	$a = (a \lor a) \land a$
$a = a \land (a \land a)$	$a = a \land (a \lor a)$	$a = a \land a \lor a$	$a = (a \lor a) \lor a$	$a = a \lor a \land a$	$a = a \lor (a \lor a)$
6 $a = a \land (a \lor b)$	7 $a = a \lor a \land b$	$a = (a \lor b) \land a$	$a = a \land (b \lor a)$	$a = a \land b \lor a$	$a = a \lor b \land a$
$a = (b \lor a) \land a$	$a = b \land a \lor a$	$\lnot a = \lnot a \land \lnot a$	$\lnot a = \lnot a \lor \lnot a$	$\lnot a \land a = a \land \lnot a$	$\lnot a \lor a = a \lor \lnot a$
$\lnot (a \land a) = \lnot (a \lor a)$	$\lnot \lnot a = (a \land a) \land a$	$\lnot \lnot a = (a \lor a) \land a$	$\lnot \lnot a = a \land (a \land a)$	$\lnot \lnot a = a \land (a \lor a)$	$\lnot \lnot a = a \land a \lor a$
$\lnot \lnot a = (a \lor a) \lor a$	$\lnot \lnot a = a \lor a \land a$	$\lnot \lnot a = a \lor (a \lor a)$	$\lnot \lnot a = a \land (a \lor b)$	$\lnot \lnot a = a \lor a \land b$	$\lnot \lnot a = (a \lor b) \land a$
$\lnot \lnot a = a \land (b \lor a)$	$\lnot \lnot a = a \land b \lor a$	$\lnot \lnot a = a \lor b \land a$	8 $\lnot a \land a = \lnot b \land b$	$a \land \lnot a = \lnot b \land b$	$\lnot a \land a = b \land \lnot b$
$a \land \lnot a = b \land \lnot b$	9 $\lnot a \lor a = \lnot b \lor b$	$a \lor \lnot a = \lnot b \lor b$	$\lnot a \lor a = b \lor \lnot b$	$a \lor \lnot a = b \lor \lnot b$	$\lnot \lnot a = (b \lor a) \land a$
$\lnot \lnot a = b \land a \lor a$	$a \land \lnot b = \lnot b \land a$	$\lnot a \land b = b \land \lnot a$	$a \lor \lnot b = \lnot b \lor a$	$\lnot a \lor b = b \lor \lnot a$	$\lnot (a \land b) = \lnot (b \land a)$
$\lnot (a \lor b) = \lnot (b \lor a)$	$a \land a = (a \land a) \land a$	$a \lor a = (a \land a) \land a$	$a \land a = (a \lor a) \land a$	$a \lor a = (a \lor a) \land a$	$a \land a = a \land (a \land a)$
$a \lor a = a \land (a \land a)$	$a \land a = a \land (a \lor a)$	$a \lor a = a \land (a \lor a)$	$a \land a = a \land a \lor a$	$a \lor a = a \land a \lor a$	$a \land a = (a \lor a) \lor a$
$a \lor a = (a \lor a) \lor a$	$a \land a = a \lor a \land a$	$a \lor a = a \lor a \land a$	$a \land a = a \lor (a \lor a)$	$a \lor a = a \lor (a \lor a)$	$a \land a = a \land (a \lor b)$
$a \lor a = a \land (a \lor b)$	$a \land a = a \lor a \land b$	$a \lor a = a \lor a \land b$	$a \land a = (a \lor b) \land a$	$a \lor a = (a \lor b) \land a$	$a \land a = a \land (b \lor a)$
$a \lor a = a \land (b \lor a)$	$a \land a = a \land b \lor a$	$a \lor a = a \land b \lor a$	$a \land a = a \lor b \land a$	$a \lor a = a \lor b \land a$	$a \land a = (b \lor a) \land a$
$a \lor a = (b \lor a) \land a$	$a \land a = b \land a \lor a$	$a \lor a = b \land a \lor a$	$a \land b = (a \land a) \land b$	$a \land b = (a \lor a) \land b$	$a \land b = a \land (a \land b)$
$a \lor b = a \land a \lor b$	$a \lor b = (a \lor a) \lor b$	$a \lor b = a \lor (a \lor b)$	$a \land b = (a \land b) \land a$	$a \land b = a \land (b \land a)$	$a \lor b = (a \lor b) \lor a$
$a \lor b = a \lor (b \lor a)$	$a \land b = (a \land b) \land b$	$a \land b = a \land (b \land b)$	$a \land b = a \land (b \lor b)$	$a \lor b = (a \lor b) \lor b$	$a \lor b = a \lor b \land b$
$a \lor b = a \lor (b \lor b)$	$a \land b = (b \land a) \land a$	$a \land b = b \land (a \land a)$	$a \land b = b \land (a \lor a)$	$a \lor b = (b \lor a) \lor a$	$a \lor b = b \lor a \land a$
$a \lor b = b \lor (a \lor a)$	$a \land b = (b \land a) \land b$	$a \land b = b \land (a \land b)$	$a \lor b = (b \lor a) \lor b$	$a \lor b = b \lor (a \lor b)$	$a \land b = (b \land b) \land a$
$a \land b = (b \lor b) \land a$	$a \land b = b \land (b \land a)$	$a \lor b = b \land b \lor a$	$a \lor b = (b \lor b) \lor a$	$a \lor b = b \lor (b \lor a)$	$\lnot (a \land a) = \lnot a \land \lnot a$
$\lnot (a \lor a) = \lnot a \land \lnot a$	$\lnot (a \land a) = \lnot a \lor \lnot a$	$\lnot (a \lor a) = \lnot a \lor \lnot a$	10 $\lnot (a \lor b) = \lnot a \land \lnot b$	$\lnot (a \land b) = \lnot a \lor \lnot b$	$\lnot (a \lor b) = \lnot b \land \lnot a$
$\lnot (a \land b) = \lnot b \lor \lnot a$	$\lnot a \land \lnot a = \lnot a \lor \lnot a$	$\lnot a \land \lnot b = \lnot b \land \lnot a$	$\lnot a \lor \lnot b = \lnot b \lor \lnot a$	$(a \land a) \land a = (a \lor a) \land a$	$(a \land a) \land a = a \land (a \land a)$
$(a \lor a) \land a = a \land (a \land a)$	$(a \land a) \land a = a \land (a \lor a)$	$(a \lor a) \land a = a \land (a \lor a)$	$a \land (a \land a) = a \land (a \lor a)$	$(a \land a) \land a = a \land a \lor a$	$(a \lor a) \land a = a \land a \lor a$
$a \land (a \land a) = a \land a \lor a$	$a \land (a \lor a) = a \land a \lor a$	$(a \land a) \land a = (a \lor a) \lor a$	$(a \lor a) \land a = (a \lor a) \lor a$	$a \land (a \land a) = (a \lor a) \lor a$	$a \land (a \lor a) = (a \lor a) \lor a$
$a \land a \lor a = (a \lor a) \lor a$	$(a \land a) \land a = a \lor a \land a$	$(a \lor a) \land a = a \lor a \land a$	$a \land (a \land a) = a \lor a \land a$	$a \land (a \lor a) = a \lor a \land a$	$a \land a \lor a = a \lor a \land a$
$(a \lor a) \lor a = a \lor a \land a$	$(a \land a) \land a = a \lor (a \lor a)$	$(a \lor a) \land a = a \lor (a \lor a)$	$a \land (a \land a) = a \lor (a \lor a)$	$a \land (a \lor a) = a \lor (a \lor a)$	$a \land a \lor a = a \lor (a \lor a)$
$(a \lor a) \lor a = a \lor (a \lor a)$	$a \lor a \land a = a \lor (a \lor a)$	$(a \land a) \land a = a \land (a \lor b)$	$(a \lor a) \land a = a \land (a \lor b)$	$a \land (a \land a) = a \land (a \lor b)$	$a \land (a \lor a) = a \land (a \lor b)$
$a \land a \lor a = a \land (a \lor b)$	$(a \lor a) \lor a = a \land (a \lor b)$	$a \lor a \land a = a \land (a \lor b)$	$a \lor (a \lor a) = a \land (a \lor b)$	$(a \land a) \land a = a \lor a \land b$	$(a \lor a) \land a = a \lor a \land b$
$a \land (a \land a) = a \lor a \land b$	$a \land (a \lor a) = a \lor a \land b$	$a \land a \lor a = a \lor a \land b$	$(a \lor a) \lor a = a \lor a \land b$	$a \lor a \land a = a \lor a \land b$	$a \lor (a \lor a) = a \lor a \land b$

⋮ *50 lines*

$a \land b \lor b = b \land (b \lor a)$	$(a \lor b) \lor b = b \land b \lor a$	$a \lor b \land b = b \land b \lor a$	$a \lor (b \lor b) = b \land b \lor a$	$(a \lor b) \lor b = (b \lor b) \lor a$	$a \lor b \land b = (b \lor b) \lor a$
$a \lor (b \lor b) = (b \lor b) \lor a$	$(a \lor b) \land b = b \lor b \land a$	$a \land b \lor b = b \lor b \land a$	$(a \lor b) \lor b = b \lor (b \lor a)$	$a \lor b \land b = b \lor (b \lor a)$	$a \lor (b \lor b) = b \lor (b \lor a)$
$(a \lor b) \land b = (b \land b) \land b$	$a \land b \lor b = (b \land b) \land b$	$(a \lor b) \land b = (b \lor b) \land b$	$a \land b \lor b = (b \lor b) \land b$	$(a \lor b) \land b = b \land (b \land b)$	$a \land b \lor b = b \land (b \land b)$
$(a \lor b) \land b = b \land (b \lor b)$	$a \land b \lor b = b \land (b \lor b)$	$(a \lor b) \land b = b \land b \lor b$	$a \land b \lor b = b \land b \lor b$	$(a \lor b) \land b = (b \lor b) \lor b$	$a \land b \lor b = (b \lor b) \lor b$
$(a \lor b) \land b = b \lor b \land b$	$a \land b \lor b = b \lor b \land b$	$(a \lor b) \land b = b \lor (b \lor b)$	$a \land b \lor b = b \lor (b \lor b)$	$(a \lor b) \land b = b \land (b \lor c)$	$a \land b \lor b = b \land (b \lor c)$
$(a \lor b) \land b = b \lor b \land c$	$a \land b \lor b = b \lor b \land c$	$(a \lor b) \land b = (b \lor c) \land b$	$a \land b \lor b = (b \lor c) \land b$	$(a \lor b) \land b = b \land (c \lor b)$	$a \land b \lor b = b \land (c \lor b)$
$(a \lor b) \land b = b \land c \lor b$	$a \land b \lor b = b \land c \lor b$	$(a \lor b) \land b = b \lor c \land b$	$a \land b \lor b = b \lor c \land b$	$(a \lor b) \land b = (c \lor b) \land b$	$a \land b \lor b = (c \lor b) \land b$
$(a \lor b) \land b = c \land b \lor b$	$a \land b \lor b = c \land b \lor b$	11 $(a \land b) \land c = a \land (b \land c)$	12 $(a \lor b) \lor c = a \lor (b \lor c)$	$(a \land b) \land c = (a \land c) \land b$	$a \land (b \land c) = (a \land c) \land b$
$(a \land b) \land c = a \land (c \land b)$	$a \land (b \land c) = a \land (c \land b)$	$a \land (b \lor c) = a \land (c \lor b)$	$(a \lor b) \lor c = (a \lor c) \lor b$	$a \lor (b \lor c) = (a \lor c) \lor b$	$a \lor b \land c = a \lor c \land b$
$(a \lor b) \lor c = a \lor (c \lor b)$	$a \lor (b \lor c) = a \lor (c \lor b)$	$(a \land b) \land c = (b \land a) \land c$	$a \land (b \land c) = (b \land a) \land c$	$(a \lor b) \land c = (b \lor a) \land c$	$(a \land b) \land c = b \land (a \land c)$

⋮ *392 lines*

$a \land (b \lor c) = (a \land a) \land (c \lor b)$	$a \land (b \lor c) = (a \lor a) \land (c \lor b)$	$a \lor b \land c = a \land a \lor c \land b$	$a \lor b \land c = (a \lor a) \lor c \land b$	$(a \lor b) \lor c = a \land a \lor (c \lor b)$
$(a \lor b) \lor c = (a \lor a) \lor (c \lor b)$	$a \lor (b \lor c) = a \land a \lor (c \lor b)$	$a \lor (b \lor c) = (a \lor a) \lor (c \lor b)$	$(a \land b) \land c = (a \land b) \land (a \land c)$	$a \land (b \land c) = (a \land b) \land (a \land c)$
13 $a \land (b \lor c) = a \land b \lor a \land c$	14 $a \lor b \land c = (a \lor b) \land (a \lor c)$	$(a \lor b) \lor c = (a \lor b) \lor (a \lor c)$	$a \lor (b \lor c) = (a \lor b) \lor (a \lor c)$	$(a \land b) \land c = (a \land b) \land (b \land c)$
$a \land (b \land c) = (a \land b) \land (b \land c)$	$(a \lor b) \lor c = (a \lor b) \lor (b \lor c)$	$a \lor (b \lor c) = (a \lor b) \lor (b \lor c)$	$(a \land b) \land c = (a \land b) \land (c \land a)$	$a \land (b \land c) = (a \land b) \land (c \land a)$

⋮

The theorems of basic logic written out in order of increasing complexity. Those considered interesting enough to name in typical textbooks are highlighted. The theorems are respectively: (1), (2) idempotence (laws of tautology) of And and Or, (3), (4) commutativity of And and Or, (5) law of double negation, (6), (7) absorption (redundancy) laws, (8) law of noncontradiction (definition of False), (9) law of excluded middle (definition of True), (10) de Morgan's law, (11), (12) associativity of And and Or, (13), (14) distributive laws. With the exception of the second distributive law, it turns out that the highlighted theorems are exactly the ones that cannot be derived from preceding theorems in the list. The distributive laws appear at positions 2813 and 2814 in the list; it takes a long proof to obtain the second one from preceding theorems.

But what determines which theorems these will be? One might have thought that it would be purely a matter of history. But actually looking at the list of theorems it always seems that the interesting ones are in a sense those that show the least unnecessary complication.

And indeed if one starts from the beginning of the list one finds that most of the theorems can readily be derived from simpler ones earlier in the list. But there are a few that cannot—and that therefore provide in a sense the simplest statements of genuinely new information. And remarkably enough what I have found is that these theorems are almost exactly the ones highlighted on the previous page that have traditionally been identified as interesting.

So what happens if one applies the same criterion in other settings? The picture below shows as an example theorems from the formulation of logic discussed above based on NAND.

$a \barwedge b = b \barwedge a$	$a = (a \barwedge a) \barwedge (a \barwedge a)$	$a = (a \barwedge a) \barwedge (a \barwedge b)$	$a = (a \barwedge a) \barwedge (b \barwedge a)$
$a = (a \barwedge b) \barwedge (a \barwedge a)$	$a = (b \barwedge a) \barwedge (a \barwedge a)$	$(a \barwedge a) \barwedge a = a \barwedge (a \barwedge a)$	$(a \barwedge a) \barwedge a = (b \barwedge b) \barwedge b$
$a \barwedge (a \barwedge a) = (b \barwedge b) \barwedge b$	$(a \barwedge a) \barwedge a = b \barwedge (b \barwedge b)$	$a \barwedge (a \barwedge a) = b \barwedge (b \barwedge b)$	$a \barwedge (a \barwedge b) = (a \barwedge b) \barwedge a$
$a \barwedge (a \barwedge b) = a \barwedge (b \barwedge a)$	$(a \barwedge a) \barwedge b = (a \barwedge b) \barwedge b$	$a \barwedge (a \barwedge b) = a \barwedge (b \barwedge b)$	$a \barwedge (a \barwedge b) = (b \barwedge a) \barwedge a$
$(a \barwedge a) \barwedge b = b \barwedge (a \barwedge a)$	$(a \barwedge a) \barwedge b = (b \barwedge a) \barwedge b$	$(a \barwedge a) \barwedge b = b \barwedge (a \barwedge b)$	$a \barwedge (a \barwedge b) = (b \barwedge b) \barwedge a$
$(a \barwedge a) \barwedge b = b \barwedge (b \barwedge a)$	$(a \barwedge b) \barwedge a = a \barwedge (b \barwedge a)$	$(a \barwedge b) \barwedge a = a \barwedge (b \barwedge b)$	$a \barwedge (b \barwedge a) = a \barwedge (b \barwedge b)$
$(a \barwedge b) \barwedge a = (b \barwedge a) \barwedge a$	$a \barwedge (b \barwedge a) = (b \barwedge a) \barwedge a$	$(a \barwedge b) \barwedge a = (b \barwedge b) \barwedge a$	$a \barwedge (b \barwedge a) = (b \barwedge b) \barwedge a$
$a \barwedge (b \barwedge b) = (b \barwedge a) \barwedge a$	$(a \barwedge b) \barwedge b = b \barwedge (a \barwedge a)$	$(a \barwedge b) \barwedge b = (b \barwedge a) \barwedge b$	$(a \barwedge b) \barwedge b = b \barwedge (a \barwedge b)$
$a \barwedge (b \barwedge b) = (b \barwedge b) \barwedge a$	$(a \barwedge b) \barwedge b = b \barwedge (b \barwedge a)$	$a \barwedge (b \barwedge c) = a \barwedge (c \barwedge b)$	$(a \barwedge b) \barwedge c = (b \barwedge a) \barwedge c$
$a \barwedge (b \barwedge c) = (b \barwedge c) \barwedge a$	$(a \barwedge b) \barwedge c = c \barwedge (a \barwedge b)$	$a \barwedge (b \barwedge c) = (c \barwedge b) \barwedge a$	$(a \barwedge b) \barwedge c = c \barwedge (b \barwedge a)$
$(a \barwedge a) \barwedge (a \barwedge a) = (a \barwedge a) \barwedge (a \barwedge b)$	$(a \barwedge a) \barwedge (a \barwedge a) = (a \barwedge a) \barwedge (b \barwedge a)$	$(a \barwedge a) \barwedge (a \barwedge a) = (a \barwedge b) \barwedge (a \barwedge a)$	$(a \barwedge a) \barwedge (a \barwedge a) = (b \barwedge a) \barwedge (a \barwedge a)$
$(a \barwedge a) \barwedge (a \barwedge b) = (a \barwedge a) \barwedge (a \barwedge c)$	$(a \barwedge a) \barwedge (a \barwedge b) = (a \barwedge a) \barwedge (b \barwedge a)$	$(a \barwedge a) \barwedge (a \barwedge b) = (a \barwedge a) \barwedge (c \barwedge a)$	$(a \barwedge a) \barwedge (a \barwedge b) = (a \barwedge b) \barwedge (a \barwedge a)$

⋮ *118 lines*

$a \barwedge ((a \barwedge b) \barwedge b) = (c \barwedge (a \barwedge a)) \barwedge a$	$a \barwedge ((a \barwedge b) \barwedge b) = ((c \barwedge a) \barwedge c) \barwedge a$	$a \barwedge ((a \barwedge b) \barwedge b) = (c \barwedge (a \barwedge c)) \barwedge a$	$(a \barwedge (a \barwedge b)) \barwedge b = (c \barwedge (b \barwedge b)) \barwedge b$
$(a \barwedge (a \barwedge b)) \barwedge b = ((c \barwedge b) \barwedge c) \barwedge b$	$(a \barwedge (a \barwedge b)) \barwedge b = (c \barwedge (b \barwedge c)) \barwedge b$	$a \barwedge ((a \barwedge b) \barwedge b) = (c \barwedge (c \barwedge a)) \barwedge a$	$(a \barwedge (a \barwedge b)) \barwedge b = (c \barwedge (c \barwedge b)) \barwedge b$
$a \barwedge ((a \barwedge b) \barwedge b) = ((c \barwedge c) \barwedge c) \barwedge a$	$a \barwedge ((a \barwedge b) \barwedge b) = (c \barwedge (c \barwedge c)) \barwedge a$	$(a \barwedge (a \barwedge b)) \barwedge b = ((c \barwedge c) \barwedge c) \barwedge b$	$(a \barwedge (a \barwedge b)) \barwedge b = (c \barwedge (c \barwedge c)) \barwedge b$
$a \barwedge (a \barwedge (b \barwedge c)) = a \barwedge (a \barwedge (c \barwedge b))$	$(a \barwedge (a \barwedge b)) \barwedge c = ((a \barwedge b) \barwedge a) \barwedge c$	$(a \barwedge (a \barwedge b)) \barwedge c = (a \barwedge (b \barwedge a)) \barwedge c$	$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge ((b \barwedge a) \barwedge c)$
$((a \barwedge a) \barwedge b) \barwedge c = ((a \barwedge b) \barwedge b) \barwedge c$	$(a \barwedge (a \barwedge b)) \barwedge c = (a \barwedge (b \barwedge b)) \barwedge c$	$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge ((b \barwedge b) \barwedge c)$	$a \barwedge ((a \barwedge b) \barwedge c) = ((a \barwedge b) \barwedge c) \barwedge a$
$a \barwedge (a \barwedge (b \barwedge c)) = (a \barwedge (b \barwedge c)) \barwedge a$	$a \barwedge (a \barwedge (b \barwedge c)) = a \barwedge ((b \barwedge c) \barwedge a)$	$a \barwedge (a \barwedge (b \barwedge c)) = ((a \barwedge b) \barwedge c) \barwedge c$	$(a \barwedge (a \barwedge b)) \barwedge c = (a \barwedge (b \barwedge c)) \barwedge c$
$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge ((b \barwedge c) \barwedge c)$	$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge (c \barwedge (a \barwedge b))$	$a \barwedge (a \barwedge (b \barwedge c)) = (a \barwedge (c \barwedge b)) \barwedge a$	$a \barwedge (a \barwedge (b \barwedge c)) = a \barwedge ((c \barwedge b) \barwedge a)$
$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge (c \barwedge (b \barwedge a))$	$a \barwedge (a \barwedge (b \barwedge c)) = ((a \barwedge c) \barwedge b) \barwedge b$	$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge (c \barwedge (b \barwedge b))$	$((a \barwedge a) \barwedge b) \barwedge c = ((a \barwedge c) \barwedge b) \barwedge c$
$(a \barwedge (a \barwedge b)) \barwedge c = (a \barwedge (c \barwedge b)) \barwedge c$	$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge ((c \barwedge b) \barwedge c)$	$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge (c \barwedge (b \barwedge c))$	$a \barwedge ((a \barwedge b) \barwedge c) = a \barwedge (c \barwedge (c \barwedge b))$

⋮

The theorems of logic formulated in terms of NAND. Theorems which cannot be derived from ones earlier in the list are highlighted. The last highlighted theorem is 539th in the list. No later theorems would be highlighted since the ones shown form a complete axiom system from which any theorem of logic can be derived. The last highlighted theorem is however an example of one that follows from the axioms, but is hard to prove.

Now there is no particular historical tradition to rely on. But the criterion nevertheless still seems to agree rather well with judgements a human might make. And much as in the picture on page 817, what one sees is that right at the beginning of the list there are several theorems that are identified as interesting. But after these one has to go a long way before one finds other ones.

So if one were to go still further, would one eventually find yet more? It turns out that with the criterion we have used one would not. And the reason is that just the six theorems highlighted already happen to form an axiom system from which any possible theorem about NANDs can ultimately be derived.

And indeed, whenever one is dealing with theorems that can be derived from a finite axiom system the criterion implies that only a finite number of theorems should ever be considered interesting—ending as soon as one has in a sense got enough theorems to be able to reproduce some formulation of the axiom system.

But this is essentially like saying that once one knows the rules for a system nothing else about it should ever be considered interesting. Yet most of this book is concerned precisely with all the interesting behavior that can emerge even if one knows the rules for a system.

And the point is that if computational irreducibility is present, then there is in a sense all sorts of information about the behavior of a system that can only be found from its rules by doing an irreducibly large amount of computational work. And the analog of this in an axiom system is that there are theorems that can be reached only by proofs that are somehow irreducibly long.

So what this suggests is that a theorem might be considered interesting not only if it cannot be derived at all from simpler theorems but also if it cannot be derived from them except by some long proof. And indeed in basic logic the last theorem identified as interesting on page 817—the distributivity of OR—is an example of one that can in principle be derived from earlier theorems, but only by a proof that seems to be much longer than other theorems of comparable size.

In logic, however, all proofs are in effect ultimately of limited length. But in any axiom system where there is universality—and thus undecidability—this is no longer the case, and as I discussed above I suspect that it will actually be quite common for there to be all sorts of short theorems that have only extremely long proofs.

No doubt many such theorems are much too difficult ever to prove in practice. But even if they could be proved, would they be considered interesting? Certainly they would provide what is in essence new information, but my strong suspicion is that in mathematics as it is currently practiced they would only rarely be considered interesting.

And most often the stated reason for this would be that they do not seem to fit into any general framework of mathematical results, but instead just seem like isolated random mathematical facts.

In doing mathematics, it is common to use terms like difficult, powerful, surprising and deep to describe theorems. But what do these really mean? As I mentioned above, any field of mathematics can at some level be viewed as a giant network of statements in which the connections correspond to theorems. And my suspicion is that our intuitive characterizations of theorems are in effect just reflections of our perception of various features of the structure of this network.

And indeed I suspect that by looking at issues such as how easy a given theorem makes it to get from one part of a network to another it will be possible to formalize many intuitive notions about the practice of mathematics—much as earlier in this book we were able to formalize notions of everyday experience such as complexity and randomness.

Different fields of mathematics may well have networks with characteristically different features. And so, for example, what are usually viewed as more successful areas of pure mathematics may have more compact networks, while areas that seem to involve all sorts of isolated facts—like elementary number theory or theory of specific cellular automata—may have sparser networks with more tendrils.

And such differences will be reflected in proofs that can be given. For example, in a sparser network the proof of a particular theorem may not contain many pieces that can be used in proving other theorems. But in a more compact network there may be intermediate definitions and concepts that can be used in a whole range of different theorems.

Indeed, in an extreme case it might even be possible to do the analog of what has been done, say, in the computation of symbolic integrals, and to set up some kind of uniform procedure for finding a proof of essentially any short theorem.

And in general whenever there are enough repeated elements within a single proof or between different proofs this indicates the presence of computational reducibility. Yet while this means that there is in effect less new information in each theorem that is proved, it turns out that in most areas of mathematics these theorems are usually the ones that are considered interesting.

The presence of universality implies that there must at some level be computational irreducibility—and thus that there must be theorems that cannot be reached by any short procedure. But the point is that mathematics has tended to ignore these, and instead to concentrate just on what are in effect limited patches of computational reducibility in the network of all possible theorems.

Yet in a sense this is no different from what has happened, say, in physics, where the phenomena that have traditionally been studied are mostly just those ones that show enough computational reducibility to allow analysis by traditional methods of theoretical physics.

But whereas in physics one has only to look at the natural world to see that other more complex phenomena exist, the usual approaches to mathematics provide almost no hint of anything analogous.

Yet with the new approach based on explicit experimentation used in this book it now becomes quite clear that phenomena such as computational irreducibility occur in abstract mathematical systems.

And indeed the Principle of Computational Equivalence implies that such phenomena should be close at hand in almost every direction: it is merely that—despite its reputation for generality—mathematics has in the past implicitly tended to define itself to avoid them.

So what this means is that in the future, when the ideas and methods of this book have successfully been absorbed, the field of mathematics as it exists today will come to be seen as a small and surprisingly uncharacteristic sample of what is actually possible.

Intelligence in the Universe

Whether or not we as humans are the only examples of intelligence in the universe is one of the great unanswered questions of science.

Just how intelligence should be defined has never been quite clear. But in recent times it has usually been assumed that it has something to do with an ability to perform sophisticated computations.

And with traditional intuition it has always seemed perfectly reasonable that it should take a system as complicated as a human to exhibit such capabilities—and that the whole elaborate history of life on Earth should have been needed to generate such a system.

With the development of computer technology it became clear that many features of intelligence could be achieved in systems that are not biological. Yet our experience has still been that to build a computer requires sophisticated engineering that in a sense exists only because of human biological and cultural development.

But one of the central discoveries of this book is that in fact nothing so elaborate is needed to get sophisticated computation. And indeed the Principle of Computational Equivalence implies that a vast range of systems—even ones with very simple underlying rules—should be equivalent in the sophistication of the computations they perform.

So in as much as intelligence is associated with the ability to do sophisticated computations it should in no way require billions of years of biological evolution to produce—and indeed we should see it all over the place, in all sorts of systems, whether biological or otherwise.

And certainly some everyday turns of phrase might suggest that we do. For when we say that the weather has a mind of its own we are in effect attributing something like intelligence to the motion of a fluid. Yet surely, one might argue, there must be something fundamentally more to true intelligence of the kind that we as humans have.

So what then might this be?

Certainly one can identify all sorts of specific features of human intelligence: the ability to understand language, to do mathematics, solve puzzles, and so on. But the question is whether there are more general features that somehow capture the essence of true intelligence, independent of the particular details of human intelligence.

Perhaps it could be the ability to learn and remember. Or the ability to adapt to a wide range of different and complex situations. Or the ability to handle abstract general representations of data.

At first, all of these might seem like reasonable indicators of true intelligence. But as soon as one tries to think about them independent of the particular example of human intelligence, it becomes much less clear. And indeed, from the discoveries in this book I am now quite certain that any of them can actually be achieved in systems that we would normally never think of as showing anything like intelligence.

Learning and memory, for example, can effectively occur in any system that has structures that form in response to input, and that can persist for a long time and affect the behavior of the system. And this can happen even in simple cellular automata—or, say, in a physical system like a fluid that carves out a long-term pattern in a solid surface.

Adaptation to all sorts of complex situations also occurs in a great many systems. It is well recognized when natural selection is present. But at some level it can also be thought of as occurring whenever a constraint ends up getting satisfied—even say that a fluid flowing around a complex object minimizes the energy it dissipates.

Handling abstraction is also in a sense rather common. Indeed, as soon as one thinks of a system as performing computations one can immediately view features of those computations as being like abstract representations of input to the system.

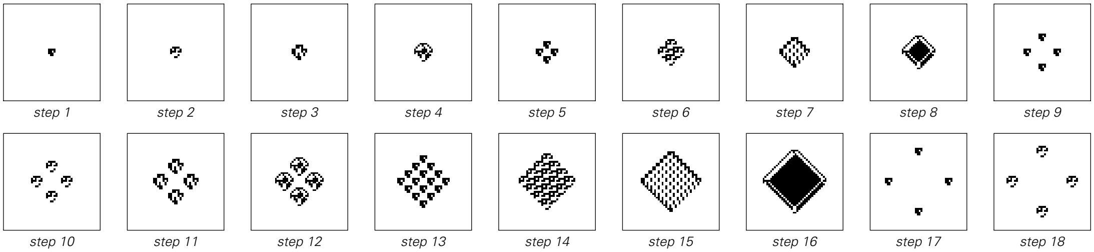
So given all of this is there any way to define a general notion of true intelligence? My guess is that ultimately there is not, and that in fact any workable definition of what we normally think of as intelligence will end up having to be tied to all sorts of seemingly rather specific details of human intelligence.

And as it turns out this is quite similar to what happens if one tries to define the seemingly much simpler notion of life.

There was a time when it was thought that practically any system that moves spontaneously and responds to stimuli must be alive. But with the development of machines having even the most primitive sensors it became clear that this was not correct.

Work in the field of thermodynamics led to the idea that perhaps living systems could be defined by their ability to take disorganized material and spontaneously organize it—usually to incorporate it into their own structure. Yet all sorts of non-living systems—from crystals to flames—also do this. And Chapter 6 showed that self-organization is actually extremely common even among systems with simple rules.

For a while it was thought that perhaps life might be defined by its ability for self-reproduction. But in the 1950s abstract computational systems were constructed that also had this ability. Yet it seemed that they needed highly complex rules—not unlike those found in actual living cells. But in fact no such complexity is really necessary. And as one might now expect from the intuition in this book, even systems like the one below with remarkably simple rules can still manage to show self-reproduction—despite the fact that they bear almost no other resemblance to ordinary living systems.



A two-dimensional cellular automaton that exhibits an almost trivial form of self-reproduction, in which multiple copies of any initial pattern appear every time the number of steps of evolution doubles. The rule used is additive, and takes a cell to be black whenever an odd number of its neighbors were black on the step before (outer totalistic code 204). The same basic self-reproduction phenomenon occurs in elementary rule 90, as well as in essentially any other additive rule, in any number of dimensions.

If one looks at typical living systems one of their most obvious features is great apparent complexity. And for a long time it has been thought that such complexity must somehow be unique to living systems—perhaps requiring billions of years of biological evolution to develop. But what I have shown in this book is that this is not the case, and that in fact a vast range of systems—including ones with very simple underlying rules—can generate at least as much complexity as we see in the components of typical living systems.

Yet despite all this, we do not in our everyday experience typically have much difficulty telling living systems from non-living ones. But the reason for this is that all living systems on Earth share an immense number of detailed structural and chemical features—reflecting their long common history of biological evolution.

So what about extraterrestrial life? To be able to recognize this we would need some kind of general definition of life, independent of the details of life on Earth. But just as in the case of intelligence, I believe that no reasonable definition of this kind can actually be given.

Indeed, following the discoveries in this book I have come to the conclusion that almost any general feature that one might think of as characterizing life will actually occur even in many systems with very simple rules. And I have little doubt that all sorts of such systems can be identified both terrestrially and extraterrestrially—and certainly require nothing like the elaborate history of life on Earth to produce.

But most likely we would not consider these systems even close to being real examples of life. And in fact I expect that in the end the only way we would unquestionably view a system as being an example of life is if we found that it shared many specific details with life on Earth—probably down, say, to being made of gelatinous materials and having components analogous to proteins, enzymes, cell membranes and so on—and perhaps even down to being based on specific chemical substances like water, sugars, ATP and DNA.

So what then of extraterrestrial intelligence? To what extent would it have to show the same details as human intelligence—and perhaps even the same kinds of knowledge—for us to recognize it as a valid example of intelligence?

Already just among humans it can in practice be somewhat difficult to recognize intelligence in the absence of shared education and culture. Indeed, in young children it remains almost completely unclear at what stage different aspects of intelligence become active.

And when it comes to other animals things become even more difficult. If one specifically tries to train an animal to solve mathematical puzzles or to communicate using human language then it is usually possible to recognize what intelligence it shows.

But if one just observes the normal activities of the animal it can be remarkably difficult to tell whether they involve intelligence. And so as a typical example it remains quite unclear whether there is intelligence associated with the songs of either birds or whales.

To us these songs may sound quite musical—and indeed they even seem to show some of the same principles of organization as human music. But do they really require intelligence to generate?

Particularly for birds it has increasingly been possible to trace the detailed processes by which songs are produced. And it seems that at least some of their elaborate elements are just direct consequences of the complex patterns of air flow that occur in the vocal tracts of birds.

But there is definitely also input from the brain of the bird. Yet within the brain some of the neural pathways responsible are known. And one might think that if all such pathways could be found then this would immediately show that no intelligence was involved.

Certainly if the pathways could somehow be seen to support only simple computations then this would be a reasonable conclusion. But just using definite pathways—or definite underlying rules—does not in any way preclude intelligence. And in fact if one looks inside a human brain—say in the process of generating speech—one will no doubt also see definite pathways and definite rules in use.

So how then can we judge whether something like a bird song, or a whale song—or, for that matter, an extraterrestrial signal—is a reflection of intelligence? The fundamental criterion we tend to use is whether it has a meaning—or whether it communicates anything.

Everyday experience shows us that it can often be very hard to tell. For even if we just hear a human language that we do not know it can be almost impossible for us to recognize whether what is being said is meaningful or not. And the same is true if we pick up data of any kind that is encoded in a format we do not know.

We might start by trying to use our powers of perception and analysis to find regularities in the data. And if we found too many regularities we might conclude that the data could not represent enough information to communicate anything significant—and indeed perhaps this is the case for at least some highly repetitive bird songs.

But what if we could find no particular regularities? Our everyday experience with human language might make us think that the data could then have no meaning. But there is nothing to say that it might not be a perfectly meaningful message—even one in human language—that just happens to have been encrypted or compressed to a point where it shows no detectable regularities.

And indeed it is sobering to notice that if one just listens even to bird songs and whale songs there is little that fundamentally seems to distinguish them from what can be generated by all sorts of processes in nature—say the motion of chimes blowing in the wind or of plasma in the Earth's magnetosphere.

One might imagine that one could find out whether a meaningful message had been communicated in a particular case by looking for correlations it induces between the actions of sender and receiver. But it is extremely common in all sorts of natural systems to see effects that propagate from one element to another. And when it comes even to whale songs it turns out that no clear correlations have ever in the end been identified between senders and receivers.

But what if one were to notice some event happen to the sender? If one were somehow to see a representation of this in what the sender produced, would it not be evidence for meaningful communication?

Once again, it need not be. For there are a great many cases in which systems generate signals that reflect what happens to them. And so, for example, a drum that is struck in a particular pattern will produce a sound that reflects—and in effect represents—that pattern.

Yet on the other hand even among humans different training or culture can lead to vastly different responses to a given event. And for animals there is the added problem of emphasis on different forms of perception. For presumably dogs can sense the detailed pattern of smell in their environment, and dolphins the detailed pattern of fluid motion around them. Yet we as humans would almost certainly not recognize descriptions presented in such terms.

So if we cannot identify intelligence by looking for meaningful communication, can we perhaps at least tell for a given object whether intelligence has been involved in producing it?

For certainly our everyday experience is that it is usually quite easy to tell whether something is an artifact created by humans.

But a large part of the reason for this is just that most artifacts we encounter in practice have specific elements that look rather similar. Yet presumably this is for the most part just a reflection of the historical development of engineering—and of the fact that the same basic geometrical and other forms have ended up being used over and over again.

So are there then more general ways to recognize artifacts?

A fairly good way in practice to guess whether something is an artifact is just to look and see whether it appears simple. For although there are exceptions—like crystals, bubbles and animal horns—the majority of objects that exist in nature have irregular and often very intricate forms that seem much more complex than typical artifacts.

And indeed this fact has often been taken to show that objects in nature must have been created by a deity whose capabilities go beyond human intelligence. For traditional intuition suggests that if one sees more complexity it must always in a sense have more complex origins.

But one of the main discoveries of this book is that in fact great complexity can arise even in systems with extremely simple underlying rules, so that in the end nothing with rules even as elaborate as human intelligence—let alone beyond it—is needed to explain the kind of complexity we see in nature.

But the question then remains why when human intelligence is involved it tends to create artifacts that look much simpler than objects that just appear in nature. And I believe the basic answer to this has to do with the fact that when we as humans set up artifacts we usually need to be able to foresee what they will do—for otherwise we have no way to tell whether they will achieve the purposes we want.

Yet nature presumably operates under no such constraint. And in fact I have argued that among systems that appear in nature a great many exhibit computational irreducibility—so that in a sense it becomes irreducibly difficult to foresee what they will do.

Yet at least with its traditional methodology engineering tends to rely on computational reducibility. For typically it operates by building systems up in such a way that the behavior of each element can always readily be predicted by something like a simple mathematical formula.

And the result of this is that most systems created by engineering are forced in some sense to seem simple—in mechanical cases for example typically being based only on simple repetitive motion.

But is simplicity a necessary feature of artifacts? Or might artifacts created by extraterrestrial intelligence—or by future human technology—seem to show no signs of simplicity?

As soon as we say that a system achieves a definite purpose this means that we can summarize at least some part of what the system does just by describing this purpose. So if we have a simple description of the purpose it follows that we must be able to give a simple summary of at least some part of what the system does.

But does this then mean that the whole behavior of the system must be simple? Traditional engineering might tend to make one think so. For typically our experience is that if we are able to get a particular kind of system to generate a particular outcome at all, then normally the behavior involved in doing so is quite simple.

But one of the results of this book is that in general things need not work like this. And so for example at the end of Chapter 5 we saw several systems in which a simple constraint of achieving a particular outcome could in effect only be satisfied with fairly complex behavior.

And as I will discuss in the next section I believe that in the effort to optimize things it is almost inevitable that even to achieve comparatively simple purposes more advanced forms of technology will make use of systems that have more and more complex behavior.

So this means that there is in the end no reason to think that artifacts with simple purposes will necessarily look simple.

And so if we are just presented with something, how then can we tell if it has a purpose? Even with things that we know were created by humans it can already be difficult. And so, for example, there are many archeological structures—such as Stonehenge—where it is at best unclear which features were intended to be purposeful.

And even in present-day situations, if we are exposed to objects or activities outside the areas of human endeavor with which we happen to be familiar, it can be very hard for us to tell which features are immediately purposeful, and which are unintentional—or have, say, primarily ornamental or ceremonial functions.

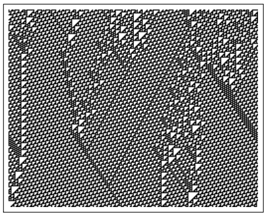
Indeed, even if we are told a purpose we will often not recognize it. And the only way we will normally become convinced of its validity is by understanding how some whole chain of consequences can lead to purposes that happen to fit into our own specific personal context.

So given this how then can we ever expect in general to recognize the presence of purpose—say as a sign of extraterrestrial intelligence?

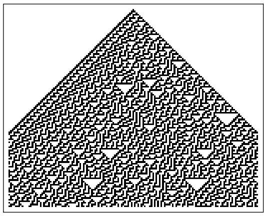
And as an example if we were to see a cellular automaton how would we be able to tell whether it was created for a purpose?

Of the cellular automata in this book—especially in Chapter 11—a few were specifically constructed to achieve particular purposes. But the vast majority originally just arose as part of my investigation of what happens with the simplest possible underlying rules.

And at first I did not think of most of them as achieving any particular purposes at all. But gradually as I built up the whole context of the science in this book I realized that many of them could in fact be thought of as achieving very definite purposes.



rule 110



rule 30

Cellular automata whose behavior does and does not give the impression of being purposeful.

Systems like rule 110 shown on the left have a kind of local coherence in their behavior that reminds one of the operation of traditional engineering systems—or of purposeful human activity. But the same is not true of systems like rule 30. For although one can see that such systems have a lot going on, one tends to assume that somehow none of it is coherent enough to achieve any definite purpose.

Yet in the context of the ideas in this book, a system like rule 30 can be viewed as achieving the purpose of performing a fairly sophisticated computation. And indeed we know that this computation is useful in practice for generating sequences that appear random.

But of course it is not necessary for us to talk about purpose when we describe the behavior of rule 30. We can perfectly well instead talk only about mechanism, and about the way in which the underlying rules for the cellular automaton lead to the behavior we see.

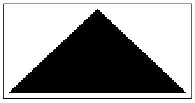
And indeed this is true of any system. But as a practical matter we often end up describing what systems do in terms of purpose when this seems to us simpler than describing it in terms of mechanism.

And so for example if we can identify some simple constraint that a system always tries to satisfy it is not uncommon for us to talk of this as being the purpose of the system. And in fact we do this even in cases like minimization of energy in physical systems or natural selection for fitness in biological systems where nothing that we ordinarily think of as intelligence is involved.

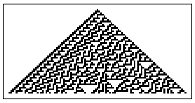
So the fact that we may be able to interpret a system as achieving some purpose does not necessarily mean that the system was really created with that purpose in mind. And indeed just looking at the system we will never ultimately be able to tell for sure that it was.

But we can still often manage to guess. And given a particular supposed purpose one potential criterion to use is that the system in a sense not appear to do too much that is extraneous to that purpose.

And so, for example, in looking at the pictures on the right it would normally seem much more plausible that rule 254 might have been set up for the purpose of generating a uniformly expanding pattern than that rule 30 might have been. For while rule 30 does generate such a pattern, it also does a lot else that appears irrelevant to this purpose.



rule 254



rule 30

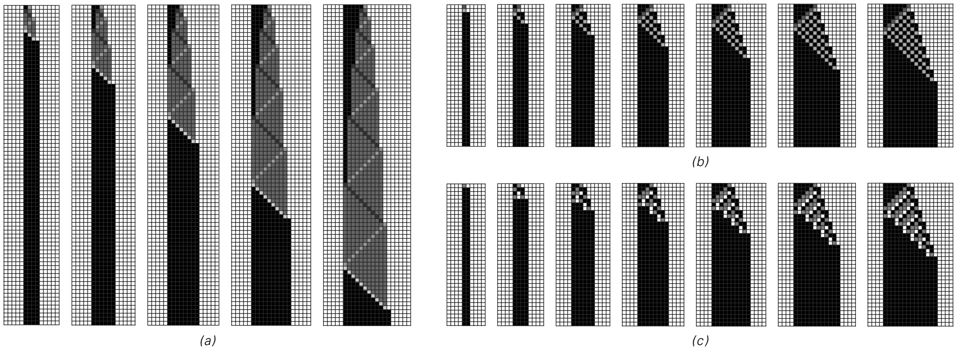
If the purpose is to generate a uniformly expanding pattern it seems more plausible that the top cellular automaton should have been the one created for this purpose.

So what this might suggest is that perhaps one could tell that a system was set up for a given purpose if the system turns out to be in a sense the minimal one that achieves that purpose.

But an immediate issue is that in traditional engineering we normally do not come even close to getting systems that are minimal. Yet it seems reasonable to suppose that as technology becomes more advanced it should become more common that the systems it sets up for a given purpose are ones that are minimal.

So as an example of all this consider cellular automata that achieve the purpose of doubling the width of the pattern given in their input. Case (a) in the picture on the next page is a cellular automaton one might construct for this purpose by using ideas from traditional engineering.

But while this cellular automaton seems to have little extraneous going on, it operates in a slow and sequential way, and its underlying



Examples of cellular automata that can be viewed as achieving the purpose of doubling the width of the pattern given in their input. Rule (a) involves 6 colors, and works sequentially, much as a typical traditional engineering system might. Rule (b) involves 4 colors, and works in parallel. Rule (c) was found by a large search, and involves only 3 colors. It takes the fewest steps of any 3-color rule to generate its result. Its rule number is 5407067979.

rules turn out to be far from minimal. For case (b) gets its results much more quickly—in effect by operating in parallel—and its rules involve four possible colors rather than six.

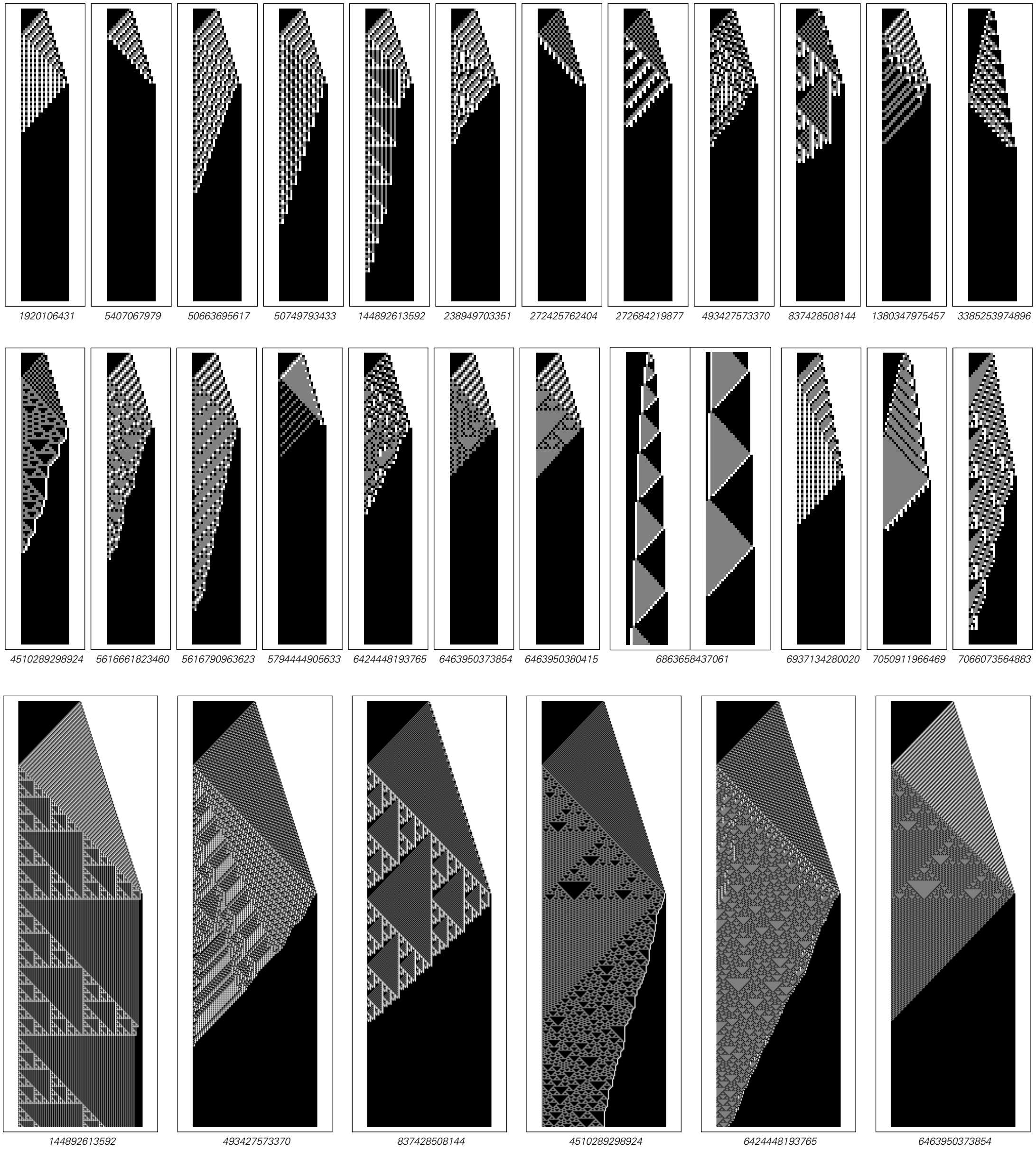
But is case (b) really the minimal cellular automaton that achieves the purpose of doubling its input? Just thinking about it, one might not be able to come up with anything better. But if one in effect explicitly searches all 8 trillion or so rules that involve less than four colors, it turns out that one can find 4277 three-color rules that work.

The pictures on the facing page show a few typical examples.

Each uses at least a slightly different scheme, but all achieve the same purpose of doubling their input. Yet often they operate in ways that seem considerably more complex than most familiar artifacts. And indeed some of the examples might look to us more like systems that just occur in nature than like artifacts.

But the point is that with sufficiently advanced technology one might expect that doubling of input would be implemented using a rule that is in some sense optimal. Different criteria for optimality could lead to different rules, but usually they will be rules like those on the facing page—and sometimes rules with quite complex behavior.

But now the question is if one were just to encounter such a rule, would one be able to guess that it was created for a purpose? After all,



Examples of rules with three colors that achieve the purpose of doubling the width of the pattern given in their input. These examples are taken from the 4277 found in effect by searching exhaustively all 7,625,597,484,987 possible rules with three colors. In most cases the number of steps to generate the final pattern increases roughly linearly with the width of the input—although in the case of the fourth-to-last rule on the second row it is $2(n^2 - n + 1)$ for width n.

there are all sorts of features in the behavior of these rules that could in principle represent a possible purpose. But what is special about rules like those on the previous page is that they are the minimal ones that exhibit the particular feature of doubling their input.

And in general if one sees some feature in the behavior of a system then finding out that the rule for the system is the minimal or optimal one for producing that feature may make it seem more likely that at least with sufficiently advanced technology the system might have specifically been created for the purpose of exhibiting that feature.

Computational irreducibility implies that it can be arbitrarily difficult to find minimal or optimal rules. Yet given any procedure for trying to do this it is certainly always possible that the procedure could just occur in nature without any purpose or intelligence being involved.

And in fact one might consider this not all that unlikely for the kind of fairly straightforward exhaustive searches that I ended up using to find the cellular automaton rules in the pictures on the previous page.

So what does all this mean for extraterrestrial intelligence?

Extrapolating from our own development we might expect that given sufficiently advanced technology it would be almost inevitable for artifacts to be constructed on an astronomical scale—perhaps for example giant machines with objects like stars as components.

Yet we do not believe that we have ever seen any such artifacts.

But how do we know for sure? For certainly our astronomical observations have revealed all sorts of phenomena for which we do not yet have any very satisfactory explanations. And indeed until just a few centuries ago most such unexplained phenomena would routinely have been attributed to some kind of divine intelligence.

But in more recent times it has become almost universally assumed that they must instead be the result of physical processes in which nothing like intelligence is involved.

Yet what the discoveries in this book have shown is that even such physical processes can often correspond to computations that are at least as sophisticated as any that we as humans perform.

But what we believe is that somehow none of the phenomena we see have any sense of purpose analogous to typical human artifacts.

Occasionally we do see evidence of simple geometrical shapes like those familiar from human artifacts—or visible on the Earth from space. But normally our explanations for these end up being short enough that they seem to leave no room for anything like intelligence. And when we see elaborate patterns, say in nebulas or galaxies, we assume that these can have no purpose—even though they may remind us to some extent of human art.

So if we do not recognize any objects that seem to be artifacts, what about signals that might correspond to messages?

If we looked at the Earth from far away the most obvious signs of human intelligence would probably be found in radio signals.

And in fact in the past it was often assumed that just to generate radio signals at all must require intelligence and technology. So when complex radio signals not of human origin were discovered in the early 1900s it was at first thought that they must be coming from extraterrestrial intelligence. But it was eventually realized that in fact the signals were just produced by effects in the Earth's magnetosphere.

And then again in the 1960s when the intense and highly regular signals of pulsars were discovered it was briefly thought that they too must come from extraterrestrial intelligence. But it was soon realized that these signals could actually be produced just by ordinary physical processes in the magnetospheres of rapidly rotating neutron stars.

So what might a real signal from extraterrestrial intelligence be like? Human radio signals currently tend to be characterized by the presence of sharply defined carrier frequencies, corresponding in effect to almost perfect small-scale repetition. But such regularity greatly reduces the rate at which information can be transmitted. And as technology advances less and less regularity needs to be present.

But in practice essentially all serious searches for extraterrestrial intelligence made so far have been based on using radio telescopes to look for signals with sharply defined frequencies. And indeed no such signals have been found. But as we saw in Chapter 10 even signals that are nested rather than purely repetitive cannot reliably be recognized just by looking for peaks in frequency spectra.

And there is certainly in general no lack of radio signals that we receive from around our galaxy and beyond. But the point is that these signals typically seem to us quite random. And normally this has made us assume that they must in effect just be some kind of radio noise that is being produced by one of several simple physical processes.

But could it be that some of these signals instead come from extraterrestrial intelligence—and are in fact meaningful messages?

Ongoing communications between extraterrestrials seem likely to be localized to regions of space where they are needed, and therefore presumably not accessible to us. And even if some signals involved in such communications are broadcast, my guess is that they will exhibit essentially no detectable regularities. For any such regularity represents in a sense a redundancy or inefficiency that can be removed by the sender and receiver both using appropriate data compression.

But if there are beacons that are intended to be noticed even if one does not already know that they are there, then the signals these produce must necessarily have recognizable distinguishing features, and thus regularities that can be detected, at least by their potential users.

So perhaps the problem is just that the methods of perception and analysis that we as humans have are not powerful enough. And perhaps if we could only find the appropriate new method it would suddenly be clear that some of what we thought was random radio noise is actually the output of beacons set up by extraterrestrial intelligence.

For as we saw in Chapter 10 most of the methods of perception and analysis that we currently use can in general do little more than recognize repetition—and sometimes nesting. Yet in the course of this book we have seen a great many examples where data that appears to us quite random can in fact be produced by very simple underlying rules.

And although I somewhat doubt it, one could certainly imagine that if one were to show data like the center column of rule 30 or the digit sequence of π to an extraterrestrial then they would immediately be able to deduce simple rules that can produce these.

But even if at some point we were to find that some of the seemingly random radio noise that we detect can be generated by simple rules, what would this mean about extraterrestrial intelligence?

In many respects, the simpler the rules, the more likely it might seem that they could be associated with ordinary physical processes, without anything like intelligence being involved.

Yet as we discussed above, if one could actually determine that the rules used in a given case were the simplest possible, then this might suggest that they were somehow set up on purpose. But in practice if one just receives a signal one normally has no way to tell which of all possible rules for producing it were in fact used.

So is there then any kind of signal that could be sent that would unambiguously communicate the presence of intelligence?

In the past, one might have thought that it would be enough for the production of the signal to involve sophisticated computation. But the discoveries in this book have made it clear that in fact such computation is quite common in all sorts of systems that do not show anything that we would normally consider intelligence.

And indeed it seems likely that for example an ordinary physical process like fluid turbulence in the gas around a star should rather quickly do more computation than has by most measures ever been done throughout the whole course of human intellectual history.

In discussions of extraterrestrial intelligence it is often claimed that mathematical constructs—such as the sequence of primes—somehow serve as universal signs of intelligence.

But from the results in this book it is clear that this is not correct.

For while in the past it might have seemed that the only way to generate primes was by using intelligence, we now know that the rather straightforward computations required can actually be carried out by a vast range of different systems—with no apparent need for intelligence.

One might nevertheless imagine that any sufficiently advanced intelligence would somehow at least consider the primes significant.

But here again I do not believe that this is correct. For very little even of current human technology depends on ideas about primes. And I am also fairly sure that not much can be deduced from the fact that primes happen to be popular in present-day human mathematics.

For despite its reputation for generality I argued at length in the previous section that the whole field of mathematics that we as humans have historically developed ultimately covers only a tiny fraction of what is possible—notably leaving out the vast majority of systems that I have studied in this book.

And if one identifies a feature—such as repetition or nesting—that is common to many possible systems, then it becomes inevitable that this feature will appear not only when intelligence or mathematics is involved, but also in all sorts of systems that just occur in nature.

So what about trying to set up a signal that gives evidence of somehow having been created for a purpose? I argued above that if the rules for a system are as simple as they can be, then this may suggest the presence of purpose. But such a criterion relies on seeing not only a signal but also the mechanism by which the signal was produced.

So what about a signal on its own? One might imagine that one could set something up—say the solution to a difficult mathematical problem—that was somehow easy to describe in terms of a constraint or purpose, but difficult to explain in terms of an explicit mechanism.

But in a sense such a thing cannot exist. For given a constraint, it is always in principle simple to set up an exhaustive search that provides a mechanism for finding what satisfies the constraint.

However, this may still take a lot of computational effort. But we cannot use that alone as a criterion. For as we have seen, many systems that just occur in nature actually end up doing more computation than typical systems that we explicitly set up for a purpose.

So even if we cannot find an abstract way to give evidence of purpose or intelligence, what about using the practical fact that both the sender and receiver of a signal exist in the same physical universe? Can one perhaps use a signal that is a representation of actual data in, say, astronomy, physics or chemistry?

As I discussed earlier, the more direct the representation the more easily an ordinary physical process can be expected to generate it, and the less there will be any indication of intelligence—just as, for example, something like a photograph can be produced essentially just by projecting light, while a diagram or a painting requires more.

But as soon as there is interpretation of data, it can become very difficult to recognize the results. For different forms of perception and different experiences and contexts can cause vastly different features to be emphasized. And thus, for example, the fact that we can readily recognize pictures of animals in cave paintings made by Stone Age humans depends greatly on the fact that our visual system still picks out the same specific features.

But what about more abstract art?

Although one has the feeling that this involves more human input, it rapidly becomes extremely difficult to tell what has been created on purpose. And so, for example, if one sees a splash of paint it is almost impossible to know without detailed cultural background and context whether it is intended to be purposeful art.

So what does all this mean about extraterrestrial intelligence?

My main conclusion is rather similar to my conclusion about artificial intelligence in Chapter 10: that the basic issue is not finding systems that perform sophisticated enough computations, but rather finding ones whose details happen to be similar enough to us as humans that we recognize what they do as showing intelligence.

And there is perhaps some analogy to recognizing the capability for sophisticated computation in the first place. For while this is undoubtedly very common say in cellular automata, the most immediate suggestions of it are in class 4 systems like rule 110 that in effect happen to do their computations in a way that looks at least somewhat similar to the way we as humans are used to doing them.

So should we expect that somehow recognizable extraterrestrial intelligence will occur at a level of a few percent—like class 4 systems?

There is clearly more to the phenomenon of intelligence than this. But if we require something that follows too many of the details of us as humans there is already evidence that it does not exist. For if such intelligence had ever arisen in the past, then extrapolating from our own history we would expect that some of it would long ago have colonized our galaxy—at least with signals, if not with physical objects.

But I suspect that if we generalize even quite modestly our definition of intelligence then there will be examples that can be found—at least with sufficiently powerful methods of perception and analysis. Yet it seems likely that they will behave in some ways that are as bizarrely different from human intelligence as many of the simple programs in this book are different from the systems that have traditionally been studied in human mathematics and science.

Implications for Technology

My main purpose in this book has been to build a new kind of basic science. But I expect that in time what I have done will also have many implications for technology. No doubt there will be all sorts of specific applications of particular results and ideas. But in the long run probably the most important consequence will be to introduce a vast new range of systems and processes that can be used for technology.

And indeed one of the things that emerges from this book is that traditional engineering has actually considered only a tiny and quite unrepresentative fraction of all the kinds of systems and processes that are in principle possible.

Presumably the reason—as I have mentioned several times in this book—is that its whole methodology has tended to be based on setting up systems whose behavior is simple enough that almost every aspect of them can always readily be predicted. But doing this has immediately excluded many of the systems that I have studied in this book—or for that matter that occur in nature. And no doubt this is why systems created by engineering have in the past usually ended up looking so much simpler than typical systems in nature.

And with traditional intuition it has normally been assumed that the only way to create systems that show a higher degree of complexity is somehow to build this complexity into their underlying rules.

But one of the central discoveries of this book is that this is not the case, and that in fact it is perfectly possible for systems even with extremely simple underlying rules to produce behavior that has immense complexity—and that looks like what one sees in nature.

And I believe that if one uses such systems it is almost inevitable that a vast amount of new technology will become possible.

There are some places where just the abstract ability to produce complexity from simple rules is already important. One example discussed in Chapter 10 is cryptography. Other examples include all sorts of practical processes in which bias or deadlock can be avoided by using randomness, or in which one wants to generate behavior that is somehow too complex for an adversary to predict.

Being able to produce complexity that is even roughly like what we see in nature also has immediate consequences—say in generating realistic textures and computer graphics or in producing artistic images that we abstractly perceive as having features familiar from nature.

The phenomenon of computational irreducibility implies that to find out what some specific system with complex behavior will do can require explicit simulation that involves an irreducible amount of computational work. But as a practical matter, if one can set up a model that is based on sufficiently simple rules then it becomes more likely that one will be able to make designs and build control devices that work even with some system in nature that shows complex behavior.

So what about computers? Although the components used have shifted from vacuum tubes to semiconductors the fundamental rules by which computers operate have changed very little in half a century.

But what the Principle of Computational Equivalence implies is that there are actually a vast range of very different kinds of rules that all lead to exactly the same computational capabilities—and so can all in principle be used as a basis for making computers.

Traditional intuition suggests that to be able to do sophisticated computations one would inevitably need a system with complicated underlying rules. But what I have shown in this book is that this is not the case, and that in fact even systems with extremely simple rules—like the rule 110 cellular automaton—can often be universal, and thus be capable of doing computations as sophisticated as any other system.

And the fact that the underlying rules can be so simple vastly expands the kinds of components that can realistically be used to implement them. For while it is quite implausible that some simple chemical process could successfully assemble a traditional computer out of atoms, it seems quite plausible that this could be done for something like a rule 110 cellular automaton.

Indeed, it seems likely that a system could be set up in which just one or a few atoms would correspond to a cell in a system like a cellular automaton. And one thing this would mean is that doing computations would then translate almost directly into building actual physical structures out of atoms.

In the past biology—with all its details of DNA, proteins, ribosomes and so on—has provided our only example of programmable construction on an atomic scale. But the discoveries in this book suggest that there are vastly simpler systems that could also be used.

And indeed my guess is that the essential features of all sorts of intricate structures that are seen in living systems can actually be reproduced with remarkably simple rules—making it for example possible to use technology to repair or replace a whole new range of functions of biological tissues and organs.

But given some form of perhaps complex behavior, how can one find rules that will manage to generate it? The traditional engineering approach—if it works at all—will almost inevitably give rules that are in effect at least as complicated as the behavior one is trying to get.

At first biology seems to do better by repeatedly making random modifications to genetic programs, and then applying natural selection. But while this process does quite often yield programs with complex behavior, I argued earlier in this book that it does not usually manage to mold anything but fairly simple aspects of this behavior.

So what then can one do? Occasionally some kind of iterative or directed search may work. But in my experience there are so many different and unexpected things that can happen with simple programs that ultimately the only way to find what one wants is essentially just to do an exhaustive search of all possibilities.

And with computers as they are today one can already often look at trillions of cases—as on page 833. But while this is enough to see a tremendous range of behavior, there is no guarantee that one will in fact run across whatever specific features one is looking for.

Yet in a sense this is a familiar problem. For especially early in their history many branches of technology have ended up searching the natural world for ingredients or systems that serve particular purposes—whether for making light bulb filaments or drugs. And in some sense the only difference here is that in the abstract world of simple programs doing a search becomes much more systematic.

But while traditional engineering has usually ended up finding ways to avoid searches for the limited kinds of systems it considers, the phenomenon of computational irreducibility makes it inevitable that if one considers all possible simple programs then finding particular forms of behavior can require doing searches that involve irreducibly large amounts of computational work.

And in a sense this means that if one tries directly to produce specific pieces of technology one can potentially always get stuck. So in practice a better approach will often be in effect just to do basic science—and much as I have done in this book to try to build up a body of abstract knowledge about how all sorts of simple programs behave.

In chemistry for example one might start by studying the basic science of how all sorts of different substances behave. But having developed a library of results one is then in a position to pick out substances that might be relevant for a specific technological purpose.

And I believe much the same will happen with simple programs. Indeed, in my experience it is remarkable just how often even elementary cellular automata like rule 90 and rule 30 can be applied in one way or another to technological situations.

In general one can think of technology as trying to take systems that exist in nature or elsewhere and harness them to achieve human purposes. But history suggests that it is often difficult even to imagine a purpose without having seen at least something that achieves it.

And indeed a vast quantity of current technology is in the end based on trying to set up our own systems to emulate features that we have noticed exist in ordinary biological or physical systems.

But inevitably we tend to notice only those features that somehow fit into the whole conceptual framework we use. And insofar as that framework is based even implicitly on traditional science it will tend to miss much of what I have discussed in this book.

So in the decades to come, when the science in this book has been absorbed, it is my expectation that it will not only suggest many new ways to achieve existing technological purposes but will also suggest many new purposes that technology can address.

Historical Perspectives

It would be most satisfying if science were to prove that we as humans are in some fundamental way special, and above everything else in the universe. But if one looks at the history of science many of its greatest advances have come precisely from identifying ways in which we are not special—for this is what allows science to make ever more general statements about the universe and the things in it.

Four centuries ago we learned for example that our planet does not lie at a special position in the universe. A century and a half ago we learned that there was nothing very special about the origin of our species. And over the past century we have learned that there is nothing special about our various physical, chemical and other constituents.

Yet in Western thought there is still a strong belief that there must be something fundamentally special about us. And nowadays the most common assumption is that it must have to do with the level of intelligence or complexity that we exhibit. But building on what I have discovered in this book, the Principle of Computational Equivalence now makes the fairly dramatic statement that even in these ways there is nothing fundamentally special about us.

For if one thinks in computational terms the issue is essentially whether we somehow show a specially high level of computational sophistication. Yet the Principle of Computational Equivalence asserts that almost any system whose behavior is not obviously simple will tend to be exactly equivalent in its computational sophistication.

So this means that there is in the end no difference between the level of computational sophistication that is achieved by humans and by all sorts of other systems in nature and elsewhere.

For my discoveries imply that whether the underlying system is a human brain, a turbulent fluid, or a cellular automaton, the behavior it exhibits will correspond to a computation of equivalent sophistication.

And while from the point of view of modern intellectual thinking this may come as quite a shock, it is perhaps not so surprising at the level of everyday experience. For there are certainly many systems in nature whose behavior is complex enough that we often describe it in human terms. And indeed in early human thinking it is very common to encounter the idea of animism: that systems with complex behavior in nature must be driven by the same kind of essential spirit as humans.

But for thousands of years this has been seen as naive and counter to progress in science. Yet now essentially this idea—viewed in computational terms through the discoveries in this book—emerges as crucial. For as I discussed earlier in this chapter, it is the computational equivalence of us as observers to the systems in nature that we observe that makes these systems seem to us so complex and unpredictable.

And while in the past it was often assumed that such complexity must somehow be special to systems in nature, what my discoveries and the Principle of Computational Equivalence now show is that in fact it is vastly more general. For what we have seen in this book is that even when their underlying rules are almost as simple as possible, abstract systems like cellular automata can achieve exactly the same level of computational sophistication as anything else.

It is perhaps a little humbling to discover that we as humans are in effect computationally no more capable than cellular automata with very simple rules. But the Principle of Computational Equivalence also implies that the same is ultimately true of our whole universe.

So while science has often made it seem that we as humans are somehow insignificant compared to the universe, the Principle of Computational Equivalence now shows that in a certain sense we are at the same level as it is. For the principle implies that what goes on inside us can ultimately achieve just the same level of computational sophistication as our whole universe.

But while science has in the past shown that in many ways there is nothing special about us as humans, the very success of science has tended to give us the idea that with our intelligence we are in some way above the universe. Yet now the Principle of Computational Equivalence implies that the computational sophistication of our intelligence should in a sense be shared by many parts of our universe—an idea that perhaps seems more familiar from religion than science.

Particularly with all the successes of science, there has been a great desire to capture the essence of the human condition in abstract scientific terms. And this has become all the more relevant as its replication with technology begins to seem realistic. But what the Principle of Computational Equivalence suggests is that abstract descriptions will never ultimately distinguish us from all sorts of other systems in nature and elsewhere. And what this means is that in a sense there can be no abstract basic science of the human condition—only something that involves all sorts of specific details of humans and their history.

So while we might have imagined that science would eventually show us how to rise above all our human details what we now see is that in fact these details are in effect the only important thing about us.

And indeed at some level it is the Principle of Computational Equivalence that allows these details to be significant. For this is what leads to the phenomenon of computational irreducibility. And this in turn is in effect what allows history to be significant—and what implies that something irreducible can be achieved by the evolution of a system.

Looking at the progress of science over the course of history one might assume that it would only be a matter of time before everything would somehow be predicted by science. But the Principle of Computational Equivalence—and the phenomenon of computational irreducibility—now shows that this will never happen.

There will always be details that can be reduced further—and that will allow science to continue to show progress. But we now know that there are some fundamental boundaries to science and knowledge.

And indeed in the end the Principle of Computational Equivalence encapsulates both the ultimate power and the ultimate weakness of science. For it implies that all the wonders of our universe can in effect be captured by simple rules, yet it shows that there can be no way to know all the consequences of these rules, except in effect just to watch and see how they unfold.

The Principle of Computational Equivalence

Basic Framework

▪ **All is computation.** The early history of science includes many examples of attempts to treat all aspects of the universe in a uniform way. Some were more successful than others. "All is fire" was never definite enough to lead to much, but "all is number" can be viewed as an antecedent to the whole application of mathematics to science, and "all is atoms" to the atomic theory of matter and quantum mechanics. My "all is computation" will, I believe, form the basis for a fruitful new direction in science. It should be pointed out, however, that it is wrong to think that once one has described everything as, say, computation, then there is nothing more to do. Indeed, the phenomenon of computational irreducibility discussed in this chapter specifically implies that in many cases irreducible work has to be done in order to find out how any particular system will behave.

Outline of the Principle

▪ **Note for mathematicians.** The way I discuss the Principle of Computational Equivalence is in a sense opposite to what would be typical in modern mathematics. For rather than starting with very specific definitions and then expanding from these, I start from general intuition and then use this to come up with more specific results. In the years to come there will no doubt be many attempts to formulate parts of the Principle of Computational Equivalence in ways that are closer to the traditions of modern mathematics. But at least at first, I suspect that huge simplifications will be made, with the result that all sorts of misleading conclusions will probably be reached, perhaps in some cases even seemingly contradicting the principle.

▪ **History.** As I discuss elsewhere, aspects of the Principle of Computational Equivalence have many antecedents. But the complete principle is presented for the first time in this book, and is the result of thinking I did in the late 1980s and early 1990s.

▪ **Page 717 · Church's Thesis.** The idea that any computation that can be done at all can be done by a universal system such as a universal Turing machine is often referred to as Church's Thesis. Following the introduction of so-called primitive recursive functions (see page 907) in the 1880s, there had by the 1920s emerged the idea that perhaps any reasonable function could be computed using the small set of operations on which primitive recursive functions are based. This notion was supported by the fact that certain modifications to these operations were found to allow only the exact same set of functions. But the discovery of the Ackermann function in the late 1920s (see page 906) showed that there are reasonable functions that are not primitive recursive. The proof of Gödel's Theorem in 1931 made use of so-called general recursive functions (see page 1121) as a way to represent possible functions in arithmetic. And in the early 1930s the two basic idealizations used in foundational studies of mathematical processes were then general recursive functions and lambda calculus (see page 1121). By 1934 these were known to be equivalent, and in 1935 Alonzo Church suggested that either of them could be used to do any mathematical calculation which could effectively be done. (It had been noted that many specific kinds of calculations could be done within such systems—and that processes like diagonalization led to operations of a seemingly rather different character.) In 1936 Alan Turing then introduced the idea of Turing machines, and argued that any mathematical process that could be carried out in practice, say by a person, could be carried out by a Turing machine. Turing proved that his machines were exactly equivalent in their computational capabilities to lambda calculus. By the 1940s Emil Post had shown that the string rewriting systems he had studied were also equivalent, and as electronic computers began to be developed it became quite firmly established that Turing machines provided an appropriate idealization for what computations could be done. From the 1940s to 1960s many different types of systems—almost all mentioned at some point or another in this book—were shown to be equivalent in their computational capabilities. (Starting in the 1970s, as discussed on page 1143, emphasis shifted to studies not of overall equivalence but instead equivalence with respect to classes of transformations such as polynomial time.)

When textbooks of computer science began to be written some confusion developed about the character of Church's Thesis: was it something that could somehow be deduced, or was it instead essentially just a definition of computability? Turing and Post seem to have thought of Church's Thesis as characterizing the "mathematicizing power" of humans, and Turing at least seems to have thought that it might not apply to continuous processes in physics. Kurt Gödel privately discussed the question of whether the universe could be viewed as following Church's Thesis and being "mechanical". And starting in the 1950s a few physicists, notably Richard Feynman, asked about fundamental comparisons between computational and physical processes. But it was not until the 1980s—perhaps particularly following some of my work—that it began to be more widely realized that Church's Thesis should best be considered a statement about nature and about the kinds of computations that can be done in our universe. The validity of Church's Thesis has long been taken more or less for granted by computer scientists, but among physicists there are still nagging doubts, mostly revolving around the perfect continua assumed in space and quantum mechanics in the traditional formalism of theoretical physics (see page 730). Such doubts will in the end only be put to rest by the explicit construction of a discrete fundamental theory along the lines I discuss in Chapter 9.

The Content of the Principle

▪ **Page 719 · Character of principles.** Examples of principles that can be viewed in several ways include the Principle of Entropy Increase (Second Law of Thermodynamics), the Principle of Relativity, Newton's Laws, the Uncertainty Principle and the Principle of Natural Selection. The Principle of Entropy Increase, for example, is partly a law of nature relating to properties of heat, partly an abstract fact about ensembles of dynamical systems, and partly a foundation for the definition of entropy. In this case and in others, however, the most important role of a principle is as a guide to intuition and understanding.

▪ **Page 720 · Oracles.** Following his introduction of Turing machines Alan Turing tried in 1937 to develop models that would somehow allow the ultimate result of absolutely every conceivable computation to be determined. And as a step towards this, he introduced the idea of oracles which would give results of computations that could not be found by any Turing machine in any limited number of steps. He then noted, for example, that if an oracle were set up that could answer the question for a particular universal system of whether that system would ever halt when given any specific input, then with an appropriate transformation of input this same oracle could also answer the question for any other system that can be emulated by the universal system. But it turns out that this is no longer true if one allows systems which themselves can access the oracle in the course of their evolution. Yet one can then imagine a higher-level oracle for these systems, and indeed a whole hierarchy of levels of oracles—as studied in the theory of degrees of unsolvability. (Note that for example to answer the question of whether or not a given Turing machine always halts can require a second-order oracle, since it is a Π_2 question in the sense of page 1139.)

▪ **Initial conditions.** Oracles are usually imagined as being included in the internal rules for a system. But if there are an infinite number of elements that can be specified in the initial condition—as in a cellular automaton—then a table for an oracle could also be given in the initial conditions.

▪ **Page 722 · Criteria for universality.** To be universal a system must in effect be able to emulate any feature of any system. So at some level any feature can be thought of as a criterion for universality. Some features—like the possibility of information transmission—may be more obvious than others, but despite occasional assertions to the contrary in the scientific literature none is ever the whole story. Since any given universal system must be able to emulate any other universal system it follows that within any such system it must in a sense be possible to find any known universal system. But inevitably the encoding will sometimes be very complicated. And in practice if there are many simple rules that are universal they cannot all be related by simple encodings. (See also the end of Chapter 11.)

▪ **Page 722 · Encodings.** One can prevent an encoding from itself introducing universality by insisting, for example, that it be primitive recursive (see page 907) or always involve only a bounded number of steps. One can also do this—as in the rule 110 proof in the previous chapter—by having programs and data be encoded separately, and appear, say, as distinct parts of the initial conditions for the system one is studying. (See also page 1118.)

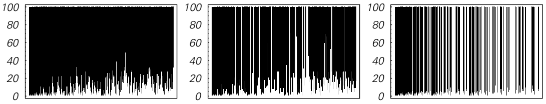
▪ **Density of universal systems.** One might imagine that it would be possible to make estimates of the overall density of universal systems, perhaps using arguments like those for the density of primes, or for the density of algorithmically random sequences. But as it turns out I know of no way to make any such estimates. If one has shown that various simple rules are universal, then it follows that rules which generalize these must also be universal. But even from this I do not know, for example, how to prove that the density of universal rules cannot decrease when rules become more complicated.

■ **Page 723 · Proving universality.** The question of whether a system is universal is in general undecidable. Using a specific mathematical axiom system such as Peano arithmetic or set theory it may also be that there is no proof that can be given. (It is straightforward to construct complicated examples where this is the case.) In practice it seems to get more difficult to prove universality when the structure of a system gets simpler. Current proofs of universality all work by showing how to emulate a known universal system. Some level of checking can be done by tracing the emulation of random initial conditions for the universal system. In the future it seems likely that automated theorem-proving methods should help in finding proofs of universality.

■ **Page 724 · History.** There are various precedents in philosophy and mysticism for the idea of encoding all possible knowledge of some kind in a single object. An example in computation theory is the concept emphasized by Gregory Chaitin of a number whose n^{th} digit specifies whether a computation with initial condition n in a particular system will ever halt. This particular number is far from being computable (see page 1128), as a result of the undecidability of the halting problem (see page 754). But a finite version in which one looks at results after a limited number of steps is similar to my concept of a universal object. (See also page 1067.)

■ **Page 725 · Universal objects.** A more direct way to create a universal object is to set up, say, a 4D array in which two of the dimensions range respectively over possible 1D cellular automaton rules and over possible initial conditions, while the other two dimensions correspond to space and time in the evolution of each cellular automaton from each initial condition. (Compare the parameter space sets of page 1006.)

■ **Page 725 · Block occurrences.** The pictures below show at which step each successive block of length up to 8 first appears in evolution according to various cellular automaton rules starting from a single black cell. For rule 30, the numbers of steps needed for each block of lengths 1 through 10 to appear at least once is *{1, 2, 4, 12, 22, 24, 33, 59, 69, 113}*. (See also page 871.)



rule 30 rule 110 rule 150

The Validity of the Principle

■ **Page 729 · Continuum and cardinality.** Some notion of a distinction between continuous and discrete systems has existed since antiquity. But in the 1870s the distinction became more precise with Georg Cantor's characterization of the total numbers of possible objects of various types in terms of different orders of infinity (see page 1162). The total number of possible integers corresponds to the smallest level of infinity, usually denoted $\aleph_0$. The total number of possible lists of integers of given finite length—and thus the number of possible rational numbers—turns out also to be $\aleph_0$. The reason is that it is always possible to encode any finite list of integers as a single integer, as discussed on page 1120. (A way to do this for pairs of non-negative integers is to use *σ[{x_, y_}] := 1/2 (x + y) (x + y + 1) + x*.) But for real numbers the story is different. Any real number x can be represented as a set of integers using for example

```
Rest[FoldList[Plus, 1, ContinuedFraction[x]]]
```

but except when x is rational this list is not finite. Since the number of possible subsets of a set with k elements is 2^k, the number of possible real numbers is $2^{\aleph_0}$. And using Cantor's diagonal argument (see note below) one can then show that this must be larger than $\aleph_0$. (The claim that there are no sets intermediate in size between $\aleph_0$ and $2^{\aleph_0}$ is the so-called continuum hypothesis, which is known to be independent of the standard axioms of set theory, as discussed on page 1155.) Much as for integers, finite lists of real numbers can be encoded as single real numbers—using for example roughly *FromDigits[Flatten[Transpose[RealDigits[list]]]]*—so that the number of such lists is $2^{\aleph_0}$. (Space-filling curves yield a more continuous version of such an encoding.) But unlike for integers the same turns out to be true even for infinite lists of real numbers. (The function σ above can for example be used to specify the order in which to sample elements in *RealDigits[list]*). The total number of possible functions of real numbers is $2^{2^{\aleph_0}}$; the number of continuous such functions (which can always be represented by a list of coefficients for a series) is however only $2^{\aleph_0}$.

In systems like cellular automata, finite arrangements of black cells on a background of white cells can readily be specified by single integers, so the number of them is $\aleph_0$. But infinite configurations of cells are like digit sequences of real numbers (as discussed on page 869 they correspond more precisely to elements in a Cantor set), so the number of them is $2^{\aleph_0}$. Continuous cellular automata (see page 155) also have $2^{\aleph_0}$ possible states.

■ **Computable reals.** The stated purpose of Alan Turing's original 1936 paper on computation was to introduce the notion of computable real numbers, whose n^{th} digit for any n could be found by a Turing machine in a finite number of steps. Real numbers used in any explicit way in traditional mathematics are always computable in this sense. But as Turing pointed out, the overwhelming majority of all possible real numbers are not computable. For certainly there can be no more computable real numbers than there are possible Turing machines. But with his discovery of universality, Turing established that any Turing machine can be emulated by a single universal Turing machine with suitable initial conditions. And the point is that any such initial conditions can always be encoded as an integer.

As examples of non-computable reals that can readily be defined, Turing considered numbers whose successive digits are determined by the eventual behavior after an infinitely long time of a universal system with successive possible initial conditions (compare page 964). With two possible forms of behavior $h[i] = 0$ or 1 for initial condition i, an example of such a number is $Sum[2^{-i} h[i], \{i, \infty\}]$. Closely related is the total probability for each form of behavior, given for example by $Sum[2\hat{}(-Ceiling[Log[2, i]]) h[i], \{i, \infty\}]$. I suspect that many limiting properties of systems like cellular automata in general correspond to non-computable reals. An example is the average density of black cells after an arbitrarily long time. For many rules, this converges rapidly to a definite value; but for some rules it will wiggle forever as more and more initial conditions are included in the average.

■ **Diagonal arguments.** Similar arguments were used by Georg Cantor in 1891 to show that there must be more real numbers than integers and by Alan Turing in 1936 to show that the problem of enumerating computable real numbers is unsolvable. One might imagine that it should be possible to set up a function $f[i, n]$ which if given successive integers i would give the n^{th} base 2 digit in every possible real number. But what about the number whose n^{th} digit is $1 - f[n, n]$? This is still a real number, yet it cannot be generated by $f[i, n]$ for any i—thus showing that there are more real numbers than integers. Analogously, one might imagine that it should be possible to have a function $f[i, n]$ which enumerates all possible programs that always halt, and specifies a digit in their output when given input n. But what about the program with output $1 - f[n, n]$? This program always halts, yet it does not correspond to any possible value of i—even though universality implies that any program should be encodable by a single integer i. And the only possible conclusion from this is that $f[i, n]$ cannot in fact be implemented as a program that always halts—thus demonstrating that the computable real numbers cannot explicitly be enumerated. (Closely related is the undecidability of the problem discussed on page 1137 of whether a system halts given any particular input.) (See also pages 907 and 1162.)

■ **Continuous computation.** Various models of computation that involve continuous elements have been proposed since the 1930s, and unlike those with discrete elements they have often not proved ultimately equivalent. One general class of models based on the work of Alan Turing in 1936 follow the operation of standard digital computers, and involve looking at real numbers in terms of digits, and using discrete processes to generate these digits. Such models inevitably handle only computable reals (in the sense defined above), and can never do computations beyond those possible in ordinary discrete systems. Functions are usually considered computable in such models if one can take the procedure for finding the digits of x and get a procedure for finding the digits of $f[x]$. And with this definition all standard mathematical functions are computable—even those from chaos theory that excavate digits rapidly. (It seems possible however to construct functions computable in this sense whose derivatives are not computable.) The same basic approach can be used whenever numbers are represented by constructs with discrete elements (see page 143), including for example symbolic formulas.

Several times since the 1940s it has been suggested that models of computation should be closer to traditional continuous mathematics, and should look at real numbers as a whole, not in terms of their digit or other representations. In a typical case, what is done is to generalize the register machines of page 97 to have registers that hold arbitrary real numbers. It is then usually assumed, however, that the primitive operations performed on these registers are just those of ordinary arithmetic, with the result that only a very limited set of functions (not including for example the exponential function) can be computed in a finite number of steps. Introducing other standard mathematical functions as primitives does not usually help much, unless one somehow gives the system the capability to solve any equation immediately (see below). (Other appropriate primitives may conceivably be related to the solubility of Hilbert's Thirteenth Problem and the fact that any continuous function with any number of arguments can be written as a one-argument function of a sum of a handful of fixed one-argument functions applied to the arguments of the original function.)

Most of the types of programs that I have discussed in this book can be generalized to allow continuous data, often just by having a continuous range of values for their elements (see e.g. page 155). But the programs themselves normally remain discrete, typically involving discrete choices made at discrete steps. If one has a table of choices, one can imagine generalizing this to a function of a real number. But to specify this function one normally has no choice but to use some type of finite formula. And to set up any kind of continuous evolution, the most obvious approach is to use traditional mathematical ideas of calculus and differential equations (see page 161). This leads to models in which possible computations are assumed, say, to correspond to combinations of differential equations—as in Claude Shannon's 1941 general-purpose analog computer. And if one assumes—as is usually implicitly done in traditional mathematics—that any solutions that exist to these equations can somehow always be found then at least in principle this allows computations impossible for discrete systems to be done.

■ **Initial conditions.** Traditional mathematics tends to assume that real numbers with absolutely any digit sequence can be set up. And if this were the case, then the digits of an initial condition could for example be the table for an oracle of the kind discussed on page 1126—and even a simple shift mapping could then yield output that is computationally more sophisticated than any standard discrete system. But just as in my discussion of chaos theory in Chapter 7, any reasonably complete theory must address how such an initial condition could have been constructed. And presumably the only way is to have another system that already violates the Principle of Computational Equivalence.

■ **Constructible reals.** Instead of finding successive digits using systems like Turing machines, one can imagine constructing complete real numbers using idealizations of mechanical processes. An example studied since antiquity involves finding lengths or angles using a ruler and compass (i.e. as intersections between lines and circles). However, as was shown in the 1800s, this method can yield only numbers formed by operating on rationals with combinations of *Plus*, *Times* and *Sqrt*. (Thus it is impossible with ruler and compass to construct π and "square the circle" but it is possible to construct 17-gons or other n-gons for which $FunctionExpand[Sin[\pi/n]]$ contains only *Plus*, *Times* and *Sqrt*.) Linkages consisting of rods of integer lengths always trace out algebraic curves (or algebraic surfaces in 3D) and in general allow any algebraic number (as represented by *Root*) to be constructed. (Linkages were used by the late 1800s not only in machines such as steam engines, but also in devices for analog computation. More recently they have appeared in robotics.) Note that above degree 4, algebraic numbers cannot in general be expressed in radicals involving only *Plus*, *Times* and *Power* (see page 945).

■ **Page 732 · Equations.** For any purely algebraic equation involving real numbers it is possible to find a bound on the size of any isolated solutions it has, and then to home in on their actual values. But as discussed on page 786, nothing similar is true for equations involving only integers, and in this case finding solutions can in effect require following the evolution of a system like a cellular automaton for infinitely many steps. If one allows trigonometric functions, any equation for integers can be converted to one for real numbers; for example $x^2 + y^2 == z^2$ for integers is equivalent to $Sin[\pi x]^2 + Sin[\pi y]^2 + Sin[\pi z]^2 + (x^2 + y^2 - z^2)^2 == 0$ for real numbers.

■ **Page 732 · ODEs.** The method of compressing time using algebraic transformations works not only in partial but also in ordinary differential equations.

■ **Emulating discrete systems.** Despite it often being assumed that continuous systems are computationally more sophisticated than discrete ones, it has in practice proved surprisingly difficult to make continuous systems emulate discrete ones. Some integer functions can readily be obtained by supplying integer arguments to continuous functions, so that for example $Mod[x, 2]$ corresponds to $Sin[\pi x/2]^2$ or $(1 - Cos[\pi x])/2$,

$$Mod[x, 3] \longleftrightarrow 1 + 2/3\,(Cos[2/3\,\pi\,(x - 2)] - Cos[2\,\pi\,x/3])$$

$$Mod[x, 4] \longleftrightarrow (3 - 2\,Cos[\pi\,x/2] - Cos[\pi\,x] - 2\,Sin[\pi\,x/2])/2$$

$$Mod[x, n] \longleftrightarrow Sum[j\,Product[(Sin[\pi\,(x - i - j)/n]/ \\ Sin[\pi\,i/n])^2, \{i, n - 1\}], \{j, n - 1\}]$$

(As another example, $If[x > 0, 1, 0]$ corresponds to $1 - 1/Gamma[1 - x]$.) And in this way the discrete system $x \to If[EvenQ[x], 3x/2, 3(x + 1)/2]$ from page 122 can be emulated by the continuous iterated map $x \to (3 + 6x - 3\,Cos[\pi x])/4$. This approach can then be applied to the universal arithmetic system on page 673, establishing that continuous iterated maps can in principle emulate discrete universal systems. A similar result presumably holds for ordinary and therefore also partial differential equations (PDEs). One might expect, however, that it should be possible to construct a PDE that quite directly emulates a system like a cellular automaton. And to do this approximately is not difficult. For as suggested by the bottom row of pictures on page 732 one can imagine having localized structures whose interactions emulate the rules of the cellular automaton. And one can set things up so that these structures exhibit the analog of attractors, and evolve towards one of a few discrete states. But the problem is that in finite time one cannot expect that they will precisely reach such states. (This is somewhat analogous to the issue of asymptotic particle states in the foundations of quantum field theory.) And this means that the overall state of the system will not be properly prepared for the next step of cellular automaton evolution.

Generating repetitive patterns with continuous systems is straightforward, but generating even nested ones is not. Page 147 showed how $Sin[x] + Sin[\sqrt{2}\ x]$ has nested features, and these are reflected in the distribution of eigenvalues for ODEs containing such functions. Strange attractors for many continuous systems also show various forms of Cantor sets and nesting.

■ **Page 732 · Time and gravity.** General relativity implies that time can be affected by gravitational fields—and that for example a process in a lower gravitational field will seem to be going faster if it is looked at by an observer in a higher gravitational field. (Related phenomena associated with motion in special relativity are more difficult to interpret in a static way.) But presumably there are effects that prevent infinite speedups. For if, say, energy were coming from a process at a constant rate, then an infinite speedup would lead to infinite energy density, and thus presumably to infinite gravitational fields that would change the system.

At least formally, general relativity does nevertheless suggest infinite transformations of time in various cases. For example, to a distant observer, an object falling into a black hole will seem to take an infinite time to cross the event horizon—even though to the object itself only a finite time will seem to have passed. One might have thought that this would imply in reverse that to an observer moving with the object the whole infinite future of the outside universe would in effect seem to go by in a finite time. But in the simplest case of a non-rotating black hole (Schwarzschild metric), it turns out that an object will always hit the singularity at the center before this can happen. In a rotating but perfectly spherical black hole (Kerr metric), the situation is nevertheless different, and in this case the whole infinite future of the outside universe can indeed in principle be seen in the finite time between crossing the outer and inner event horizons. But for the reasons mentioned above, this very fact presumably implies instability, and the whole effect disappears if there is any deviation from perfect spherical symmetry.

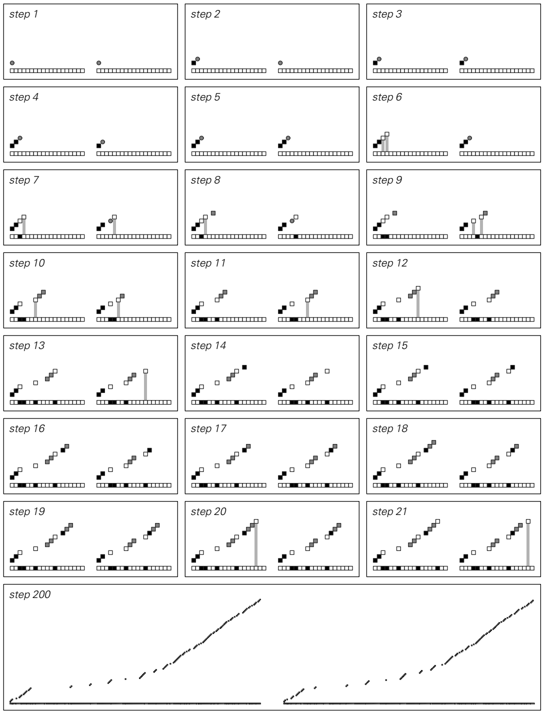
Even without general relativity there are already issues with time and gravity. For example, it was shown in 1990 that close encounters in a system of 5 idealized point masses can lead to infinite accelerations which cause one mass to be able to go infinitely far in a finite time.

■ **Page 733 · Human thinking.** The discovery in this book that even extremely simple programs can give rise to behavior vastly more complex than expected casts suspicion on any claim that programs are fundamentally unable to reproduce features of human thinking. But complete evidence that human thinking follows the Principle of Computational Equivalence will presumably come only gradually as practical computer systems manage to emulate more and more aspects of human thinking. (See page 628.)

■ **Page 734 · Intermediate degrees.** As discussed on page 753, an important indication of computational sophistication in a system is for its ultimate behavior to be undecidable, in the sense that a limited number of steps in a standard universal system cannot determine in general what the system will do after an infinite number of steps, and whether, for example, it will ever in some sense halt. Such undecidability is inevitable in any system that is universal. But what about other systems? So long as one only ever looks at the original input and final output it turns out that one can construct a system that exhibits undecidability but is not universal. One trivial way to do so is to take a universal system but modify it so that if it ever halts its output is discarded and, say, replaced by its original input. The lack of meaningful output prevents such a system from being universal, but the question of whether the system halts is still undecidable. Nevertheless, the pattern of this undecidability is just the same as for the underlying universal system. So one can then ask whether it is possible to have a system which exhibits undecidability, but with a pattern that does not correspond to that of any universal system.

As I discuss on page 1137, almost all known proofs of undecidability in practice work by reduction to the halting problem for some universal system—this is, by showing that if one could resolve whatever is supposed to be undecidable then one could also solve the halting problem for a universal system. But in 1956 Richard Friedberg and Albert Muchnik both gave an intricate and abstract construction of a system that has a halting problem which is undecidable but is not reducible to the halting problem of any universal system.

The pictures at the top of the facing page show successive steps in the evolution of an analog of their system. The input is an integer that gives a position in either of the two rows of cells at the bottom of each picture. All these cells are initially white, but some eventually become black—and the system is considered to halt for a particular input if the corresponding cell ever becomes black.



The rules for the system are quite complicated, and in essence work by progressively implementing a generalization of a diagonal argument of the kind discussed on page 1128. Note first that the configuration of cells in the rows at the bottom of each picture can be thought of as successive finite approximations to tables for an oracle (see page 1126) which gives the solution to the halting problem for each possible input to the system. To set up the generalized diagonal argument one needs a way to list all possible programs. Any type of program that supports universality can be used for this purpose; the pictures shown use essentially the register machines from page 97. Each row above the bottom one corresponds in effect to a successive register machine—and shows, if relevant, its output when given as input the integer corresponding to that position in the row, together with the complete bottom row of cells found so far. (A dot indicates that the register machine does not halt.) The way the system works is to put down new black cells in the bottom row in just such a way as to arrange that for any register machine at least the output shown will ultimately not agree with the cells in the bottom row. As indicated by vertical gray lines, there is sometimes temporary agreement, but this is always removed within a finite number of steps.

The fact that no register machine can ever ultimately give output that agrees everywhere with the bottom row of cells then demonstrates that the halting problem for the system—whose results appear in the bottom row—must be undecidable. Yet if this halting problem were reducible to a halting problem for a universal system, then by using its results one should ultimately be able to solve the halting problem for any system. However, even using the complete bottom row of cells on the left it turns out that the construction is such that no register machine can ever yield results after any finite number of steps that agree everywhere with the row of cells on the right—thus demonstrating that the halting problem for the system is not reducible to the halting problem for a universal system.

Note however that this result is extremely specific to looking only at what is considered output from the system, and that inside the system there are all sorts of components that are definitely universal.

Explaining the Phenomenon of Complexity

▪ **Definition of complexity.** See page 557.

▪ **Ingredients for complexity.** With its emphasis on breaking systems down to find their underlying elements traditional science tends to make one think that any important overall property of a system must be a consequence of some specific feature of its underlying construction. But the results of this section imply that for complexity this is not the case. For as discussed on page 1126 there is no direct structural criterion for sophisticated computation and universality. And indeed most ways of ensuring that these do not occur are in essence equivalent just to saying that the overall behavior exhibits some specific regularity and is therefore not complex.

▪ **Relativism and equivalence.** Although the notion has been discussed since antiquity, it has become particularly common in the academic humanities in the past few decades to believe that there can be no valid absolute conclusions about the world—only statements made relative to particular cultural contexts. My emphasis of the importance of perception and analysis might seem to support this view, and to some extent it does. But the Principle of Computational Equivalence implies that in the end essentially any method of perception and analysis that can actually be implemented in our universe must have a certain computational equivalence, and must therefore at least in some respects come to the same absolute conclusions.

Computational Irreducibility

▪ **History.** The notion that there could be fundamental limits to knowledge or predictability has been discussed repeatedly since antiquity. But most often it has been assumed that the origin of this must be inadequacy in models, not difficulty in working out their consequences. And indeed already in the 1500s with the introduction of symbolic algebra and the discovery of formulas for solving cubic and quartic equations the expectation began to develop that with sufficient cleverness it should be possible to derive a formula for the solution to any purely mathematical problem. Infinitesimals were sometimes thought to get in the way of finite understanding—but this was believed to be overcome by calculus. And when mathematical models for natural systems became widespread in the late 1600s it was generally assumed that their basic consequences could always be found in terms of formulas or geometrical theorems, perhaps with fairly straightforward numerical calculations required for connection to practical situations. In discussing gravitational interactions between many planets Isaac Newton did however comment in 1684 that "to define these motions by exact laws admitting of easy calculation exceeds, if I am not mistaken, the force of any human mind". But in the course of the 1700s and 1800s formulas were successfully found for solutions to a great many problems in mathematical physics (see note below)—at least when suitable special functions (see page 1091) were introduced. The three-body problem (see page 972) nevertheless continued to resist efforts at general solution. In the 1820s it was shown that quintic equations cannot in general be solved in terms of radicals (see page 1137), and by the 1890s it was known that degree 7 equations cannot in general be solved even if elliptic functions are allowed. Around 1890 it was then shown that the three-body problem could not be solved in general in terms of ordinary algebraic functions and integrals (see page 972). However, perhaps in part because of a shift towards probabilistic theories such as quantum and statistical mechanics there remained the conviction that for relevant aspects of behavior formulas should still exist. The difficulty for example of finding more than a few exact solutions to the equations of general relativity was noted—but a steady stream of results (see note below) maintained the belief that with sufficient cleverness a formula could be found for behavior according to any model.

In the 1950s computers began to be used to work out numerical solutions to equations—but this was seen mostly as a convenience for applications, not as a reflection of any basic necessity. A few computer experiments were done on systems with simple underlying rules, but partly because Monte Carlo methods were sometimes used, it was typically assumed that their results were just approximations to what could in principle be represented by exact formulas. And this view was strengthened in the 1960s when solitons given by simple formulas were found in some of these systems.

The difficulty of solving equations for numerical weather prediction was noted even in the 1920s. And by the 1950s and 1960s the question of whether computer calculations would be able to outrun actual weather was often discussed. But it was normally assumed that the issue was just getting a better approximation to the underlying equations—or better initial measurements—not something more fundamental.

Particularly in the context of game theory and cybernetics the idea had developed in the 1940s that it should be possible to make mathematical predictions even about complex human situations. And for example starting in the early 1950s government control of economies based on predictions from linear models became common. By the early 1970s, however, such approaches were generally seen as unsuccessful, but it was usually assumed that the reason was not fundamental, but was just that there were too many disparate elements to handle in practice.

The notions of universality and undecidability that underlie computational irreducibility emerged in the 1930s, but they were not seen as relevant to questions arising in natural science. Starting in the 1940s they were presumably the basis for a few arguments made about free will and fundamental unpredictability of human behavior (see page 1135), particularly in the context of economics. And in the late 1950s there was brief interest among philosophers in connecting results like Gödel's Theorem to questions of determinism—though mostly there was just confusion centered around the difficulty of finding countable proofs for statements about the continuous processes assumed to occur in physics.

The development of algorithmic information theory in the 1960s led to discussion of objects whose information content cannot be compressed or derived from anything shorter. But as indicated on page 1067 this is rather different from what I call computational irreducibility. In the 1970s computational complexity theory began to address questions about overall resources needed to perform computations, but concentrated on computations that perform fairly specific known practical tasks. At the beginning of the 1980s, however, it was noted that certain problems about models of spin glasses were NP-complete. But there was no immediate realization that this was connected to any underlying general phenomenon.

Starting in the late 1970s there was increasing interest in issues of predictability in models of physical systems. And it was emphasized that when the equations in such models are nonlinear it often becomes difficult to find their solutions. But usually this was at some level assumed to be associated with sensitive dependence on initial conditions and the chaos phenomenon—even though as we saw on page 1098 this alone does not even prevent there from being formulas.

By the early 1980s it had become popular to use computers to study various models of natural systems. Sometimes the idea was to simulate a large collection of disparate elements, say as involved in a nuclear explosion. Sometimes instead the idea was to get a numerical approximation to some fairly simple partial differential equation, say for fluid flow. Sometimes the idea was to use randomized methods to get a statistical approximation to properties say of spin systems or lattice gauge theories. And sometimes the idea was to work out terms in a symbolic perturbation series approximation, say in quantum field theory or celestial mechanics. With any of these approaches huge amounts of computer time were often used. But it was almost always implicitly assumed that this was necessary in order to overcome the approximations being used, and not for some more fundamental reason.

Particularly in physics, there has been some awareness of examples such as quark confinement in QCD where it seems especially difficult to deduce the consequences of a theory—but no general significance has been attached to this.

When I started studying cellular automata in the early 1980s I was quickly struck by the difficulty of finding formulas for their behavior. In traditional models based for example on continuous numbers or approximations to them there was usually no obvious correspondence between a model and computations that might be done about it. But the evolution of a cellular automaton was immediately reminiscent of other computational processes—leading me by 1984 to formulate explicitly the concept of computational irreducibility.

No doubt an important reason computational irreducibility was not identified before is that for more than two centuries students had been led to think that basic theoretical science could somehow always be done with convenient formulas. For almost all textbooks tend to discuss only those cases that happen to come out this way. Starting in earnest in the 1990s, however, the influence of *Mathematica* has gradually led to broader ranges of examples. But there still remains a very widespread belief that if a theoretical result about the behavior of a system is truly fundamental then it must be possible to state it in terms of a simple mathematical formula.

■ **Exact solutions.** Some notable cases where closed-form analytical results have been found in terms of standard mathematical functions include: quadratic equations (~2000 BC) (*Sqrt*); cubic, quartic equations (1530s) ($x^{1/n}$); 2-body problem (1687) (*Cos*); catenary (1690) (*Cosh*); brachistochrone (1696) (*Sin*); spinning top (1849; 1888; 1888) (*JacobiSN*; *WeierstrassP*; hyperelliptic functions); quintic equations (1858) (*EllipticTheta*); half-plane diffraction (1896) (*FresnelC*); Mie scattering (1908) (*BesselJ*, *BesselY*, *LegendreP*); Einstein equations (Schwarzschild (1916), Reissner-Nordström (1916), Kerr (1963) solutions) (rational and trigonometric functions); quantum hydrogen atom and harmonic oscillator (1927) (*LaguerreL*, *HermiteH*); 2D Ising model (1944) (*Sinh*, *EllipticK*); various Feynman diagrams (1960s–1980s) (*PolyLog*); KdV equation (1967) (*Sech* etc.); Toda lattice (1967) (*Sech*); six-vertex spin model (1967) (*Sinh* integrals); Calogero-Moser model (1971) (*Hypergeometric1F1*); Yang-Mills instantons (1975) (rational functions); hard-hexagon spin model (1979) (*EllipticTheta*); additive cellular automata (1984) (*MultiplicativeOrder*); Seiberg-Witten supersymmetric theory (1994) (*Hypergeometric2F1*). When problems are originally stated as differential equations, results in terms of integrals ("quadrature") are sometimes considered exact solutions—as occasionally are convergent series. When one exact solution is found, there often end up being a whole family—with much investigation going into the symmetries that relate them. It is notable that when many of the examples above were discovered they were at first expected to have broad significance in their fields. But the fact that few actually did can be seen as further evidence of how narrow the scope of computational reducibility usually is. Notable examples of systems that have been much investigated, but where no exact solutions have been found include the 3D Ising model, quantum anharmonic oscillator and quantum helium atom.

■ **Amount of computation.** Computational irreducibility suggests that it might be possible to define "amount of computation" as an independently meaningful quantity—perhaps vaguely like entropy or amount of information. And such a quantity might satisfy laws vaguely analogous to the laws of thermodynamics that would for example determine what processes are possible and what are not. If one knew the fundamental rules for the universe then one way in principle to define the amount of computation associated with a given process would be to find the minimum number of applications of the rules for the universe that are needed to reproduce the process at some level of description.

■ **Page 743 · More complicated rules.** The standard rule for a cellular automaton specifies how every possible block of cells of a certain size should be updated at every step. One can imagine finding the outcome of evolution more efficiently by adding rules that specify what happens to larger blocks of cells after more steps. And as a practical matter, one can look up different blocks using a method like hashing. But much as one would expect from data compression this will only in the end work more efficiently if there are some large blocks that are sufficiently common. Note that dealing with blocks of different sizes requires going beyond an ordinary cellular automaton rule. But in a sequential substitution system—and especially in a multiway system (see page 776)—this can be done just as part of an ordinary rule.

▪ **Page 744 · Reducible systems.** The color of a cell at step *t* and position *x* can be found by starting with initial condition

```
Flatten[With[{w = Max[Ceiling[Log[2, {t, x}]]]},
     {2 Reverse[IntegerDigits[t, 2, w]] + 1,
       5, 2 IntegerDigits[x, 2, w] + 2}]]
```

then for rule 188 running the cellular automaton with rule

```
{{a : (1 | 3), 1 | 3, _} → a, {_, 2 | 4, a : (2 | 4)} → a,
  {3, 5 | 10, 2} → 6, {1, 5 | 7, 4} → 0, {3, 5, 4} → 7,
  {1, 6, 2} → 10, {1, 6 | 11, 4} → 8, {3, 6 | 8 | 10 | 11, 4} → 9,
  {3, 7 | 9, 2} → 11, {1, 8 | 11, 2} → 9, {3, 11, 2} → 8,
  {1, 9 | 10, 4} → 11, {_, a_ /; a > 4, _} → a, {_, _, _} → 0}
```

and for rule 60 running the cellular automaton with rule

```
{{a : (1 | 3), 1 | 3, _} → a, {_, 2 | 4, a : (2 | 4)} → a,
  {1, 5, 4} → 0, {_, 5, _} → 5, {_, _, _} → 0}
```

▪ **Speed-up theorems.** That there exist computations that are arbitrarily computationally reducible was noted in work on the theory of computation in the mid-1960s.

▪ **Page 745 · Mathematical functions.** The number of bit operations needed to add two *n*-digit numbers is of order *n*. The number of operations *m[n]* needed to multiply them increases just slightly more rapidly than *n* (see page 1093). (Even if one can do operations on all digits in parallel it still takes of order *n* steps in a system like a cellular automaton for the effects of different digits to mix together—though see also page 1149.) The number of operations to evaluate *Mod[a, b]* is of order *n* if *a* has *n* digits and *b* is small. Many standard continuous mathematical functions just increase or decrease smoothly at large *x* (see page 917). The main issue in evaluating those that exhibit regular oscillations at large *x* is to find their oscillation period with sufficient precision. Thus for example if *x* is an integer with *n* digits then evaluating *Sin[x]* or *FractionalPart[x c]* requires respectively finding π or *c* to *n*-digit precision. It is known how to evaluate π (see page 912) and all standard elementary functions to *n*-digit precision using about *Log[n] m[n]* operations. (This can be done by repeatedly making use of functional relations such as *Exp[2 x] == Exp[x]*^{2} which express *f[2 x]* as a polynomial in *f[x]*; such an approach is known to work for elementary, elliptic, modular and other functions associated with *ArithmeticGeometricMean* and for example *DedekindEta*.) Known methods for high-precision evaluation of special functions—usually based in the end on series representations—typically require of order $n^{1/s}$ *m[n]* operations, where *s* is often 2 or 3. (Examples of more difficult cases include *HypergeometricPFQ[a, b, 1]* and *StieltjesGamma[k]*, where logarithmic series can require an exponential number of terms. Evaluation of *BernoulliB[x]* is also difficult.) Any iterative procedure (such as *FindRoot*) that yields a constant multiple more digits at each step will take about *Log[n]* steps to get *n* digits. Roots of polynomials can thus almost always be found with *NSolve* in about *Log[n] m[n]* operations. If one evaluates *NIntegrate* or *NDSolve* by effectively fitting functions to order *s* polynomials the difficulty of getting results with *n*-digit precision typically increases like $2^{n/s}$. An adaptive algorithm such as Romberg integration reduces this to about *2* ^ $\sqrt{n}$. The best-known algorithms for evaluating *Zeta[1/2 + ⅈ x]* (see page 918) to fixed precision take roughly $\sqrt{x}$ operations—or $2^{n/2}$ operations if *x* is an *n*-digit integer. (The evaluation is based on the Riemann-Siegel formula, which involves sums of about $\sqrt{x}$ cosines.) Unlike for continuous mathematical functions, known algorithms for number theoretical functions such as *FactorInteger[x]* or *MoebiusMu[x]* typically seem to require a number of operations that grows faster with the number of digits *n* in *x* than any power of *n* (see page 1090).

▪ **Formulas.** It is always in principle possible to build up some kind of formula for the outcome of any process of evolution, say of a cellular automaton (see page 618). But for there to be computational reducibility this formula needs to be simple and easy to evaluate—as it is if it consists just of a few standard mathematical functions (see note above; page 1098).

▪ **Page 747 · Short computations.** Some properties include:

(a) The regions are bounded by the hyperbolas *x y == Exp[n/2]* for successive integers *n*.

(d) There is approximate repetition associated with rational approximations to π (for example with period 22), but never precise repetition.

(e) The pattern essentially shows which *x* are divisors of *y*, just as on pages 132 and 909.

(h) *Mod[Quotient[s,* 2^n*], 2]* extracts the digit associated with 2^n in the base 2 digit sequence of *s*.

(i) Like (e), except that colors at neighboring positions alternate.

(l) See page 613.

(m) The pattern can be generated by a 2D substitution system with rule *{1 -> {{0, 0}, {0, 1}}, 0 -> {{1, 1}, {1, 0}}}* (see page 583).

(See also page 870.)

Even though standard mathematical functions are used, few of the pictures can readily be generalized to continuous values of x and y.

▪ **Intrinsic limits in science.** Before computational irreducibility other sources of limits to science that have been discussed include: measurement in quantum mechanics, prediction in chaos theory and singularities in gravitation theory. As it happens, in each of these cases I suspect that the supposed limits are actually just associated with a lack of correct analysis of all elements of the relevant systems. In mathematics, however, more valid intrinsic limits—much closer to computational irreducibility—follow for example from Gödel's Theorem.

The Phenomenon of Free Will

▪ **History.** Early in history it seems to have generally been assumed that everything about humans must ultimately be determined by unchangeable fate—which it was sometimes thought could be foretold by astrology or other forms of divination. Most Greek philosophers seem to have believed that their various mechanical or moral theories implied rigid determination of human actions. But especially with the advent of the Christian religion the notion that humans can at some level make free choices—particularly about whether to do good or not—emerged as a foundational idea. (The idea had also arisen in Persian and Hebrew religions and legal systems, and was supported by Roman lawyers such as Cicero.) How this could be consistent with God having infinite power was not clear, although around 420 AD Augustine suggested that while God might have infinite knowledge of the future we as humans could not—yielding what can be viewed as a very rough analog of my explanation for free will. In the 1500s some early Protestants made theological arguments against free will—and indeed issues of free will remain a feature of controversy between Christian denominations even today.

In the mid-1600s philosophers such as Thomas Hobbes asserted that minds operate according to definite mechanisms and therefore cannot exhibit free will. In the late 1700s philosophers such as Immanuel Kant—agreeing with earlier work by Gottfried Leibniz—claimed instead that at least some parts of our minds are free and not determined by definite laws. But soon thereafter scientists like Pierre-Simon Laplace began to argue for determinism throughout the universe based on mathematical laws. And with the increasing success of science in the 1800s it came to be widely believed that there must be definite laws for all human actions—providing a foundation for the development of psychology and the social sciences.

In the early 1900s historians and economists emphasized that there were at least not simple laws for various aspects of human behavior. But it was nevertheless typically assumed that methods based on physics would eventually yield deterministic laws for human behavior—and this was for example part of the inspiration for the behaviorist movement in psychology in the mid-1900s. The advent of quantum mechanics in the 1920s, however, showed that even physics might not be entirely deterministic—and by the 1940s the possibility that this might lead to human free will was being discussed by physicists, philosophers and historians. Around this time Karl Popper used both quantum mechanics and sensitive dependence on initial conditions (see also page 971) to argue for fundamental indeterminism. And also around this time Friedrich Hayek (following ideas of Ludwig Mises in the early 1900s) suggested—presumably influenced by work in mathematical logic—that human behavior might be fundamentally unpredictable because in effect brains can explain only systems simpler than themselves, and can thus never explain their own operation. But while this has some similarity to the ideas of computational irreducibility in this book it appears never to have been widely studied.

Questions of free will and responsibility have been widely discussed in criminal and other law since at least the 1800s (see note below). In the 1960s and 1970s ideas from popular psychology tended to diminish the importance of free will relative to physiology or environment and experiences. In the 1980s, however, free will was increasingly attributed to animals other than humans. Free will for computers and robots was discussed in the 1950s in science fiction and to some extent in the field of cybernetics. But following lack of success in artificial intelligence it has for the most part not been seriously studied. Sometimes it is claimed that Gödel's Theorem shows that humans cannot follow definite rules—but I argue on page 1158 that this is not correct.

▪ **Determinism in brains.** Early investigations of internal functioning in the brain tended to suggest considerable randomness—say in the sequence of electrical pulses from a nerve cell. But in recent years, with more extensive measurement methods, there has been increasing evidence for precise deterministic underlying rules. (See pages 976 and 1011.)

▪ **Amounts of free will.** In my theory the amount of free will associated with a particular decision is in effect related to the amount of computation required to arrive at it. In conscious thinking we can to some extent scrutinize the processes we use, and assess how much computation they involve. But in unconscious thinking we cannot. And probably often these just involve memory lookups with rather little computation. But other unconscious abilities like intuition presumably involve more sophisticated computation.

■ **Responsibility.** It is often assumed that if there are definite underlying rules for our brains then it cannot be meaningful to say that we have any ultimate moral or legal responsibility for our actions. For traditional ideas lead to the notion that in this case all our actions must somehow be thought of as the direct result of whatever external causes (over which we have no control) are responsible for the underlying rules in our brains and the environment in which we find ourselves. But if the processes in our brains are computationally irreducible then as discussed in the main text their outcome can seem in many respects free of underlying rules, making it reasonable to view the processes themselves as what is really responsible for our actions. And since these processes are intrinsic to us, it makes sense to treat us as responsible for their effects.

Several different theories are used in practical legal systems. The theory popular from the behavioral sciences tends to assume that human actions can be understood from underlying rules for the brain, and that people should be dealt with according to the rules they have—which can perhaps be modified by some form of treatment. But computational irreducibility can make it essentially impossible to find what general behavior will arise from particular rules—making it difficult to apply this theory. The alternative pragmatic theory popular in rational philosophy and economics suggests that behavior in legal matters is determined through calculations based on laws and the deterrents they provide. But here there is the issue that computational irreducibility can make it impossible to foresee what consequences a given law will have. Western systems of law tend to be dominated by the moral theory that people should somehow get what they deserve for choices they made with free will—and my explanation now makes this consistent with the existence of definite underlying rules for the brain.

Young children, animals and the insane are typically held less responsible for their actions. And in a moral theory of law this can be understood in my approach as a consequence of the computations they do being less sophisticated—so that their outcome is less free of the environment and of their underlying rules. (In a pragmatic theory the explanation would presumably be that less sophisticated computations would not be up to the task of handling the elaborate system of incentives that laws had defined.)

■ **Will and purpose.** Things that are too predictable do not normally seem free. But things that are too random also do not normally seem to be associated with the exercise of a will. Thus for example continual random twitching in our muscles is not normally thought to be a matter of human will, even though some of it is the result of signals from our brains. For typically one imagines that if something is to be a genuine reflection of human will then there must be some purpose to it. In general it is very difficult to assess whether something has a purpose (see page 829). But in capturing the most obvious aspects of human will what seems to be most important is at least short-term coherence and consistency of action—as often exists in class 4, but not class 3, systems.

■ **Source of will.** Damage to a human brain can lead to apparent disappearance of the will to act, and there is some evidence that one small part of the brain is what is crucial.

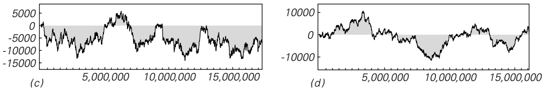
Undecidability and Intractability

■ **History.** In the early 1900s, particularly in the context of the ideas of David Hilbert, it was commonly believed that there should be a finite procedure to decide the truth of any mathematical statement. That this is not the case in the standard theory of arithmetic was in effect established by Kurt Gödel in 1931 (see page 1158). Alonzo Church gave the first explicit example of an undecidable problem in 1935 when he showed that no finite procedure in lambda calculus could guarantee to determine the equivalence of two lambda expressions. (A corollary to Gödel's proof had in fact already supplied another explicit undecidable problem by implying that no finite procedure based on recursive functions could decide whether a given primitive recursive function is identically 0.) In 1936 Alan Turing then showed that the halting problem for Turing machines could not be solved in general in a finite number of steps by any Turing machine. Some similar issues had already been considered by Emil Post in the context of tag and multiway systems starting in the 1920s, and in 1947 Post and Andrei Markov were able to establish that an existing mathematical question—the word problem for semigroups (see page 1141)—was undecidable. By the 1960s undecidability was being found in all sorts of systems, but most of the examples were too complicated to seem of much relevance in practical mathematics or computing. And apart from a few vague mentions in fields like psychology, undecidability was viewed mainly as a highly abstract curiosity of no importance to ordinary science. But in the early 1980s my experiments on cellular automata convinced me that undecidability is vastly more common than had been assumed, and in my 1984 paper

"Undecidability and intractability in theoretical physics" I argued that it should be important in many issues in physics and elsewhere.

■ **Mathematical impossibilities.** It is sometimes said that in the 1800s problems such as trisecting angles, squaring the circle, solving quintics, and integrating functions like $Exp[x^2]$ were proved mathematically impossible. But what was actually done was just to show that these problems could not be solved in terms of particular levels of mathematical constructs—say square roots (as in ruler and compass constructions discussed on page 1129), arbitrary roots, or elementary transcendental functions. And in each case higher mathematical constructs that seem in some sense no less implementable immediately allow the problems to be solved. Yet with undecidability one believes that there is absolutely no construct that can explicitly exist in our universe that allows the problem to be solved in any finite way. And unlike traditional mathematical impossibilities, undecidability is normally formulated purely in terms of ordinary integers—making it in a sense necessary to collapse basic distinctions between finite and infinite quantities if any higher-level constructs are to be included.

■ **Page 755 · Code 1004600.** In cases (c) and (d) steady growth at about 0.035 and 0.039 cells per step (of which 28% on average are non-white) is seen up to at least 20 million steps, though there continue to be fluctuations as shown below.



■ **Halting problems.** A classic example of a problem that is known in general to be undecidable is whether a given Turing machine will ever halt when started from a given initial condition. Halting is usually defined by the head of the Turing machine reaching a special halt state. But other criteria can equally well be used—say the head reaching a particular position (see page 759), or a certain pattern of colors being formed on the tape. And in a system like a cellular automaton a halting problem can be set up by asking whether a cell at a particular position ever turns a particular color, or whether, more globally, the complete state of the system ever reaches a fixed point and no longer changes.

In practical computing, one usually thinks of computational programs as being set up much like the register machines of page 896 and halting when they have finished executing their instructions. User interface and operating system programs are not normally intended to halt in an explicit sense, although without external input they often reach states that do not change. *Mathematica* works by taking its input and repeatedly applying transformation rules—a process which normally reaches a fixed point that is returned as the answer, but with definitions like $x = x + 1$ (x having no value) formally does not.

■ **Proofs of undecidability.** Essentially the same argument due to Alan Turing used on page 1128 to show that most numbers cannot be computable can also be used to show that most problems cannot be decidable. For a problem can be thought of as an infinite list of solutions for successive possible inputs. But this is analogous to a digit sequence of a real number. And since any program for a universal system can be specified by an integer it follows that there must be many problems for which no such program can be given.

To show that a particular problem like the halting problem is undecidable one typically argues by contradiction, setting up analogs of self-referential logic paradoxes such as "this statement is false". Suppose that one had a Turing machine m that could solve the halting problem, in the sense that it itself would always halt after a finite number of steps, but it would determine whether any Turing machine whose description it was given as input would ever halt. One way to see that this is not possible is to imagine modifying m to make a machine m' that halts if its input corresponds to a machine that does not halt, but otherwise goes into an infinite loop and does not itself halt. For if one considers feeding m' as input to itself there is immediately no consistent answer to the question of whether m' halts—leading to the conclusion that in fact no machine m could ever exist in the first place. (To make the proof rigorous one must add another level of self-reference, say setting up m' to ask m whether a Turing machine will halt when fed its own description as input.) In the main text I argued that undecidability is a consequence of universality. In the proof above universality is what guarantees that any Turing machine can successfully be described in a way that can be fed as input to another Turing machine.

■ **Page 756 · Examples of undecidability.** Once universality exists in a system it is known from Gordon Rice's 1953 theorem and its generalizations that most questions about ultimate behavior will be undecidable unless their answers are always trivially the same. Undecidability has been demonstrated in various seemingly rather different types of systems, most often by reduction to halting (termination) problems for multiway systems.

In formal language theory, questions about regular languages are always decidable, but ones about context-free languages (see page 1103) are already often not. It is decidable whether such a language is finite, but not whether it contains every possible string, is regular, is unambiguous, or is equivalent to a language with a different grammar.

In mathematical logic, it can be undecidable whether statements are provable from a given axiom system—say predicate logic or Peano arithmetic (see page 782). It is also undecidable whether one axiom system is equivalent to another—even for basic logic (see page 1170).

In algebra and related areas of mathematics problems of equivalence between objects built up from elements that satisfy relations are often in general undecidable. Examples are word problems for groups and semigroups (see page 1141), and equivalence of finitely specified 4D manifolds (see page 1051). (Equivalence for 3D manifolds is thought to be decidable.) A related undecidable problem is whether two integer matrices can be multiplied together in some sequence to yield the zero matrix. It is also undecidable whether two sets of relations specify the same group or semigroup.

In combinatorics it is known in general to be undecidable whether a given set of tiles can cover the plane (see page 1139). And from this follows the undecidability of various problems about 2D cellular automata (see note below) and spin systems. Also undecidable are many questions about whether strings exist that satisfy particular constraints (see below).

In number theory it is known to be undecidable whether Diophantine equations have solutions (i.e. whether algebraic equations have integer solutions) (see page 786). And this means for example that it is in general undecidable whether expressions that involve both algebraic and trigonometric functions can be zero for real values of variables, or what the values of integrals are in which such expressions appear as denominators (compare page 916).

In computer science, general problems about verifying the possible behavior of programs tend to be undecidable, usually being directly related to halting problems. It is also for example undecidable whether a given program is the shortest one that produces particular output (see page 1067).

It is in general undecidable whether a given system exhibits universality—or undecidability.

▪ **Undecidability in cellular automata.** For 1D cellular automata, almost all questions about ultimate limiting behavior are undecidable, even ones that ask about average properties such as density and entropy. (This results in undecidability in classification schemes, as mentioned on page 948.) Questions about behavior after a finite number of steps, even with infinite initial conditions, tend to be decidable for 1D cellular automata, and related to regular languages (see page 957). In 2D cellular automata, however, even questions about a single step are often undecidable. Examples include whether any configurations are invariant under the cellular automaton evolution (see page 942), and, as established by Jarkko Kari in the late 1980s, whether the evolution is reversible, or can generate every possible configuration (see page 959).

▪ **Natural systems.** Undecidable questions arise even in some traditional classes of models for natural systems. For example, in a generalized Ising model (see page 944) for a spin system the undecidability of the tiling problem implies that it is undecidable whether a given energy function leads to a phase transition in the infinite size limit. Somewhat similarly, the undecidability of equivalence of 4-manifolds implies undecidability of questions about quantum gravity models. In models based both on equations and other kinds of rules the existence of formulas for conserved quantities is in general undecidable. In models that involve continuous quantities it can be more difficult to formulate undecidability. But I strongly suspect that with appropriate definitions there is often undecidability in for example the three-body problem, so that the questions such as whether one of the bodies in a particular scattering process will ever escape to infinity are in general undecidable. In biology formal models for neural processes often involve undecidability, so that in principle it can be undecidable whether, say, there is any particular stimulus that will lead to a given response. Formal models for morphogenesis can also involve undecidability, so that for example it can in principle be undecidable whether a particular organism will ever stop growing, or whether a given structure can ever be formed in some class of organisms. (Compare page 407.)

▪ **Undecidability in *Mathematica*.** In choosing functions to build into *Mathematica* I tried to avoid ones that would often encounter undecidability. And this is why for example there is no built-in function in *Mathematica* that tries to predict whether a given program will terminate. But inevitably functions like *FixedPoint*, *ReplaceRepeated* and *FullSimplify* can run into undecidability—so that ultimately they have to be limited by constructs such as *$IterationLimit* and *TimeConstraint*.

▪ **Undecidability and sets.** Functions that can be computed in finite time by systems like Turing machines are often called recursive (or effectively computable). Sets are called recursive if there is a recursive function that can test whether or not any given element is in them. Sets are called recursively enumerable if there is a recursive function that can eventually generate any element in them.

The set of initial conditions for which a given Turing machine halts is thus not recursive. But it turns out that this set is recursively enumerable. And the reason is that one can generate the elements in it by effectively maintaining a copy of the Turing machine for each possible initial condition, then following a procedure where for example at step *n* one updates the one for initial condition *IntegerExponent[n, 2]*, and watches to see if it halts. Note that while the complement of a recursive set is always recursive, the complement of a recursively enumerable set may not be recursively enumerable. (An example is the set of initial conditions for which a Turing machines does not halt.) Recursively enumerable sets are characteristically associated with so-called Σ_1 statements of the form $\exists_t\ \phi[t]$ (where ϕ is recursive). (Asking whether a system ever halts is equivalent to asking whether there exists a number of steps *t* at which the system can be determined to be in its halting state.) Complements of recursively enumerable sets are characteristically associated with Π_1 statements of the form $\forall_t\ \phi[t]$—an example being whether a given system never halts. (Π_1 and Σ_1 statements are such that if they can be shown to be undecidable, then respectively they must be true or false, as discussed on page 1167.) If a statement in minimal form involves *n* alternations of $\exists$ and $\forall$ it is Σ_{n+1} if it starts with $\exists$ and Π_{n+1} if it starts with $\forall$. The Π_n and Σ_n form the so-called arithmetic hierarchy in which statements with larger *n* can be constructed by allowing ϕ to access an oracle for statements with smaller *n* (see page 1126). (Showing that a statement with $n \geq 1$ is undecidable does not establish that it is always true or always false.)

■ **Undecidability in tiling problems.** The question of whether a particular set of constraints like those on page 220 can be satisfied over the whole 2D plane is in general undecidable. For much as on page 943, one can imagine setting up a 1D cellular automaton with the property that, say, the absence of a particular color of cell throughout the 2D pattern formed by its evolution signifies satisfaction of the constraints. But even starting from a fixed line of cells, the question of whether a given color will ever occur in the evolution of a 1D cellular automaton is in general undecidable, as discussed in the main text. And although it is somewhat more difficult to show, this question remains undecidable even if one allows any possible configuration of cells on the starting line. (There are several different detailed formulations; the first explicit proof of undecidability in a tiling problem was given by Hao Wang in 1960; the version with no fixed cells by Robert Berger in 1966 by setting up an elaborate emulation of a register machine.) (See also page 943.)

■ **Page 757 · Correspondence systems.** Given a list of pairs *p* with *{u, v} = Transpose[p]* the constraint to be satisfied is

StringJoin[u⟦s⟧] == StringJoin[v⟦s⟧]

Thus for example *p = {{"ABB", "B"}, {"B", "BA"}, {"A", "B"}}* has shortest solution *s = {2, 3, 2, 2, 3, 2, 1, 1}*. (One can have lists instead of strings, replacing *StringJoin* by *Flatten*.)

Correspondence systems were introduced by Emil Post in 1945 to give simple examples of undecidability; he showed that the so-called Post Correspondence Problem (PCP) of satisfying their constraints is in general undecidable (see below). With 2 string pairs PCP was shown to be decidable in 1981. It is known to be undecidable when 9 pairs are used, but I strongly suspect that it is also undecidable with just 3 pairs. The undecidability of PCP has been used to establish undecidability of many problems related to groups, context-free languages, and other objects defined by relations (see page 1141). Finding PCP solutions shorter than a given length is known to be an NP-complete problem.

With *r* string pairs and *n = StringLength[StringJoin[p]]* there are 2^n *Binomial[n - 1, 2 r - 1]* possible constraints (assuming no strings of zero length), each being related to at most *8 r!* others by straightforward symmetries (or altogether 4^{n-1} for given *n*). The number of constraints which yield solutions of specified lengths *Length[s]* for *r = 2* and *r = 3* are as follows (the boxes at the end give the number of cases with no solution):

r = 2, n = 4	1:12	4							
r = 2, n = 5	1:64	64							
r = 2, n = 6	1:208	2:28	404						
r = 2, n = 7	1:640	3:32	1888						
r = 2, n = 8	1:1680	2:176	4:48	7056					
r = 2, n = 9	1:4352	3:112	5:56	24152					
r = 2, n = 10	1:10496	2:744	3:80	4:168	6:64	74464			
r = 3, n = 6	1:56	8							
r = 3, n = 7	1:576	192							
r = 3, n = 8	1:3312	2:168	3:84	1812					
r = 3, n = 9	1:14592	2:1140	3:192	4:288	5:96	8:48	30:48	44:48	12220
r = 3, n = 10	1:55296	2:4752	3:2712	4:372	5:492	6:264	7:216		
	12:24	18:48	24:48	36:48	75:48	78:48	64656		

With *r = 2*, as *n* increases an exponentially decreasing fraction of possible constraints have solutions; with *r = 3* it appears that a fraction more than 1/4 continue to do so. With *r = 2*, it appears that if a solution exists, it must have length *n + 4* or less. With *r ≤ 3*, the longest minimal solution lengths for *n ≤ 10* are given above. (Allowing *r > 3* yields no greater lengths for these values of *n*.) With *n = 11*, example (l) yields a solution of length 112. The only possible longer *n = 11* case is *{{"AAB", "B"}, {"B", "A"}, {"A", "AABB"}}*, for which any possible solution must be longer than 200. With n = 12, {{"AABAAB", "B"}, {"B", "A"}, {"A", "AB"}} has minimal solution length 120 and {{"A", "AABB"}, {"AAB", "B"}, {"B", "AA"}} has minimal solution length 132.

A given constraint can fail to have a solution either because the colors of cells at some point cannot be made to match, or because the two strings can never have the same finite length (as in {{"A", "AA"}}). To know that a solution exists in a particular case, it is sufficient just to exhibit it. To know that no solution is possible of any length, one must in effect have a proof.

In general, one condition for a solution to exist is that integer numbers of pairs can yield strings of the same length, so that given the length differences d = Map[StringLength, p, {2}] . {1, -1} there is a vector v of non-negative integers such that v . d == 0. If only one color of element ever appears this is the complete condition for a solution—and for r = 2 solutions exist if Apply[Times, d] < 0 and are then of length at least Apply[Plus[##]/GCD[##] &, Abs[d]]. With two colors of elements additional conditions can be constructed involving counting elements of each color, or various blocks of elements.

The undecidability of PCP can be seen to follow from the undecidability of the halting problem through the fact that the question of whether a tag system of the kind on page 93 with initial sequence s ever reaches a halting state (where none of its rules apply) is equivalent to the question of whether there is a way to satisfy the PCP constraint

```
TSToPCP[{n_, rule_}, s_] :=
 Map[Flatten[IntegerDigits[#, 2, 2]] &, Module[{f}, f[u_] :=
    Flatten[Map[{1, #} &, 3 u]]; Join[Map[{f[Last[#]],
       RotateLeft[f[First[#]]]} &, rule], {{f[s], {1}}}, Flatten[
     Table[{{1, 2}, Append[RotateLeft[f[IntegerDigits[j, 2,
          i]]], 2]}, {i, 0, n - 1}, {j, 0, 2^i - 1}], 1]]], {2}]
```

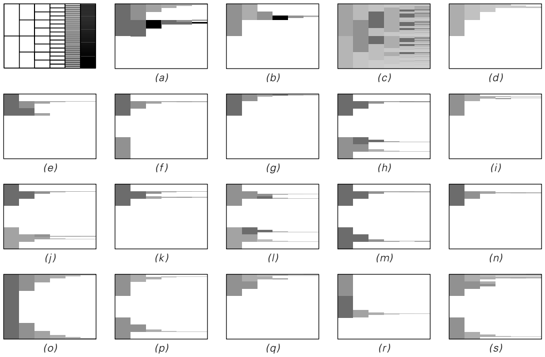
Any PCP constraint can also immediately be related to the evolution of a multiway tag system of the kind discussed in the note below. Assuming that the upper string is never shorter than the lower one, the rules for the relevant tag system are given simply by

```
Apply[Append[#2, s___] → Prepend[#1, s] &, p, {1}]
```

In the case of example (e) the existence of a solution of length 24 can then be seen to follow from the fact that MWTSEvolve[rule, {{"B"}}, 22] contains {"B", "A"}.

This correspondence with tag systems can be used in practice to search for PCP solutions, though it is usually most efficient to run tag systems that correspond both to moving forward and backward in the string, and to see whether their results ever agree. (In most PCP systems, including all the examples shown except (a) and (g), one string is always systematically longer than the other.) The tag system approach is normally limited by the number of intermediate strings that may need to be kept.

The pictures below show which possible sequences of up to 6 blocks yield upper and lower strings that agree in each of the PCP systems in the main text. As indicated in the first picture for the case of two blocks, each possible successively longer sequence corresponds to a rectangle in the picture (compare page 594). When a sequence of blocks leads to upper and lower strings that disagree, the rectangle is left white. If the strings agree so far, then the rectangle is colored with a gray that is darker if the strings are closer in length. Rectangles that are black (as visible in cases (a) and (b)) correspond to actual PCP solutions where the strings are the same length. Note that in case (c) the presence of only one color in either block means that strings will always agree so far. In cases (m) through (s) there is ultimately no solution, but as the pictures indicate, in these specific PCP systems there are always strings that agree as far as they have gone—it is just that they never end up the same length.



As one example of how one proves that a PCP constraint cannot be satisfied, consider case (s). From looking at the structure of the individual pairs one can see that if there is a solution it must begin with pair 1 or pair 3, and end with pair 1. But in fact it cannot begin with pair 1 because this would mean that the upper string would have to start off being longer, then at some point cross over to being shorter. However, the only way that such a crossover can occur is by pair 3 appearing with its upper A aligned with its second lower A. Yet starting with pair 1, the upper string is longer by 2 As, and the pairs are such that the length difference must always remain even—preventing the crossover from occurring. This means that any solution must begin with pair 3. But this pair must then be followed by another pair 3, which leaves BAAB sticking out on the bottom. So how can this *BAAB* be removed? The only way is to use the sequence of pairs 2, 3, 3, 2—yet doing this will just produce another *BAAB* further on. And thus one concludes that there is no way to satisfy these particular PCP constraints.

One can generalize PCP to allow any number of colors, and to require correspondence among any number of strings—though it is fairly easy to translate any such generalization to the 2-string 2-color case.

■ **Multiway tag systems.** As an extension of ordinary multiway systems one can generalize tag systems from page 93 to allow a list of strings at each step. Representing the strings by lists, one can write rules in the form

```
{{1, 1, s___} → {s, 1, 0}, {1, s___} → {s, 1, 0, 1}}
```

so that the evolution is given by

```
MWTSEvolve[rule_, list_, t_] :=
  Nest[Flatten[Map[ReplaceList[#, rule] &, #], 1] &, list, t]
```

■ **Word problems.** The question of whether a particular string can be generated in a given multiway system is an example of a so-called word problem. An original more specialized version of this was posed by Max Dehn in 1911 for groups and by Axel Thue in 1914 for semigroups. As discussed on page 938 a finitely presented group or semigroup can be viewed as a special case of a multiway system, in which the rules of the multiway system are obtained from relations between strings consisting of products of generators. The word problem then asks if a given product of such generators is equal to the identity element. Following work by Alan Turing in the mid-1930s, it was shown in 1947 by Emil Post from the undecidability of PCP that the word problem for semigroups is in general undecidable. Andrei Markov gave a specific example of this for a semigroup with 13 generators and 33 relations, and by 1966 Gennadií Makanin had found the simpler example

```
{"CCBB" ↔ "BBCC", "BCCCBB" ↔ "CBBBCC", "ACCBB" ↔ "BBA",
  "ABCCCBB" ↔ "CBBA", "BBCCBBBBCC" ↔ "BBCCBBBBCCA"}
```

Using these relations as rules for a multiway system most initial strings yield behavior that either dies out or becomes repetitive. The shortest initial strings that give unbounded growth are *"BBBBABB"* and *"BBBBBBA"*—though both of these still eventually yield just exponentially increasing numbers of distinct strings. In 1967 Yuri Matiyasevich constructed a semigroup with 3 complicated relations that has an undecidable word problem. It is not yet known whether undecidability can occur in a semigroup with a single relation. The word problem is known to be decidable for commutative semigroups.

The word problem for groups was shown to be undecidable in the mid-1950s by Petr Novikov and William Boone. There are however various classes of groups for which it is decidable. Abelian groups are one example. Another are so-called automatic groups, studied particularly in the 1980s, in which equivalence of words can be recognized by a finite automaton. (Such groups turn out to have definite geometrical properties, and are associated with spaces of negative curvature.) Even if a group ultimately has only a finite number of distinct elements, its word problem (with elements specified as products of generators) may still be undecidable. Constructions of groups with undecidable word problems have been based on setting up relations that correspond to the rules in a universal Turing machine. With the simplest such machine known in the past (see page 706) one gets a group with 32 generators and 142 relations. But with the universal Turing machine from page 707 one gets a group with 14 generators and 52 relations. (In general $s k + 4$ generators and $5 s k + 2$ relations are needed.) From the results in this book it seems likely that there are still much simpler examples—some of which could perhaps be found by setting up groups to emulate rule 110. Note that groups with just one relation were shown always to have decidable word problems by Wilhelm Magnus in 1932.

For ordinary multiway (semi-Thue) systems, an example with an undecidable word problem is known with 2 types of elements and 5 very complicated rules—but I am quite certain that much simpler examples are possible. (1-rule multiway systems always have decidable word problems.)

■ **Sequence equations.** One can ask whether by replacing variables by sequences one can satisfy so-called word or string equations such as

```
Flatten[{x, 0, x, 0, y}] == Flatten[{y, x, 0, y, 1, 0, 1, 0, 0}]
```

(with shortest solution *x = {1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0}*, *y = {1, 0, 1, 0, 0, 1, 0, 1, 0, 0}*). Knowing about PCP and Diophantine equations one might expect that in general this would be undecidable. But in 1977 Gennadií Makanin gave a complicated algorithm that solves the problem completely in a finite number of steps (though in general triple exponential in the length of the equation).

■ **Fast algorithms.** Most of the fast algorithms now known seem to fall into a few general classes. The most common are ones based on repetition or iteration, classic examples being Euclid's algorithm for *GCD* (page 915), Newton's method for *FindRoot* and the Gaussian elimination method for *LinearSolve*. Starting in the 1960s it began to be realized that fast algorithms could be based on nested or recursive processes, and such algorithms became increasingly popular in the 1980s. In most cases, the idea is recursively to divide data into parts, then to do operations on these parts, and finally reassemble the results. An example is the algorithm of Anatolii Karatsuba from 1961 for finding products of n-digit numbers (with $n = 2^s$) by operating on their digits in the nested pattern of page 608 (see also page 1093) according to

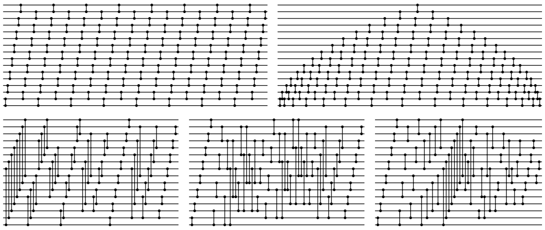
```
First[f[IntegerDigits[x, 2, n], IntegerDigits[y, 2, n], n/2]]
f[x_, y_, n_] :=
  If[n < 1, x y, g[Partition[x, n], Partition[y, n], n]]
g[{x1_, x0_}, {y1_, y0_}, n_] :=
  With[{z1 = f[x1, y1, n/2], z0 = f[x0, y0, n/2]},
    z1 2^(2 n) + (f[x0 + x1, y0 + y1, n/2] - z1 - z0) 2^n + z0]
```

Other examples include the fast Fourier transform (page 1074) and related algorithms for *ListConvolve*, the quicksort algorithm for *Sort*, and many algorithms in fields such as computational geometry. Starting in the 1980s fast algorithms based on randomized methods (see page 1192) have also become popular. But particularly from the discoveries in this book, it seems likely that the very fastest algorithms for many kinds of problems will not in the end have the type of regular structure that characterizes almost all algorithms currently used.

▪ **Sorting networks.** Any list can be sorted using *Fold[PairSort, list, pairs]* by doing a fixed sequence of comparisons of pairs

```
PairSort[a_, p : {_, _}] := Block[{t = a}, t[[p]] = Sort[t[[p]]]; t]
```

(Different comparisons often do not interfere and so can be done in parallel.) The pictures below show a few sequences of pair comparisons that sort lists of length $n = 16$.



The top two (both with 120 comparisons) have a repetitive structure and correspond to standard sorting algorithms: transposition sort and insertion sort. (Quicksort does not use a fixed sequence of comparisons.) The first one on the bottom (with 63 comparisons) has a nested structure and uses the method invented by Kenneth Batcher in 1964:

```
Flatten[Reverse[Flatten[With[{m = Ceiling[Log[2, n]] - 1},
    Table[With[{d = If[i == m, 2^t, 2^(i+1) - 2^t]}, Map[
        {0, d} + # &, Select[Range[n - d], BitAnd[# - 1, 2^t] ==
            If[i == m, 0, 2^t] &]]], {t, 0, m}, {i, t, m}]], 1]], 1]
```

The second one on the bottom also uses 63 comparisons, while the last one is the smallest known for $n = 16$: it uses 60 comparisons and was invented by Milton Green in 1969. For $n \leq 16$ the smallest numbers of comparisons known to work are *{0, 1, 3, 5, 9, 12, 16, 19, 25, 29, 35, 39, 45, 51, 56, 60}*. (In general all lists will be sorted correctly if lists of just 0's and 1's are sorted correctly; allowing even just one of these 2^n cases to be wrong greatly reduces the number of comparisons needed.) For $n \leq 8$ the Batcher method is known to give minimal length sequences of comparisons (for $n \leq 5$ the total numbers of minimal sequences that work are *{1, 6, 3, 13866}*). The Batcher method in general requires about $n\,Log[n]^2$ comparisons; it is known that in principle $n\,Log[n]$ are sufficient. Various structures such as de Bruijn and Cayley graphs can be used as the basis for sorting networks, though it is my guess that typically the smallest networks for given n will have no obvious regularity. (See also page 832.)

▪ **Page 758 · Computational complexity theory.** Despite its rather general name, computational complexity theory has for the most part been concerned with the quite specific issue of characterizing how the computational resources needed to solve problems grow with input size. From knowing explicit algorithms many problems can be assigned to such classes as:

- NC: can be solved in a number of steps that increases like a polynomial in the logarithm of the input size if processing is done in parallel on a number of arbitrarily connected processors that increases like a polynomial in the input size. (Examples include addition and multiplication.)
- P (polynomial time): can be solved (with one processor) in a number of steps that increases like a polynomial in the input size. (Examples include evaluating standard mathematical functions and simulating the evolution of cellular automata and Turing machines.)
- NP (non-deterministic polynomial time): solutions can be checked in polynomial time. (Examples include many problems based on constraints as well as simulating the evolution of multiway systems and finding initial conditions that lead to given behavior in a cellular automaton.)
- PSPACE (polynomial space): can be solved with an amount of memory that increases like a polynomial in the input size. (Examples include finding repetition periods in systems of limited size.)

Central to computational complexity theory are a collection of hypotheses that imply that NC, P, NP and PSPACE form a strict hierarchy. At each level there are many problems known that are complete at that level in the sense that all other problems at that level can be translated to instances of that problem using only computations at a lower level. (Thus, for example, all problems in NP can be translated to instances of any given NP-complete problem using computations in P.)

▪ **History.** Ideas of characterizing problems by growth rates in the computational resources needed to solve them were discussed in the 1950s, notably in the context of operation counts for numerical calculations, sizes of circuits for switching and other applications, and theoretical lengths of proofs. In the 1960s such ideas were increasingly formalized, particularly for execution times on Turing machines, and in 1965 the suggestion was made that one should consider computations feasible if they take times that grow like polynomials in their input size. NP completeness (see below) was introduced by Stephen Cook in 1971 and Leonid Levin around the same time. And over the course of the 1970s a great many well-known problems were shown to be NP-complete. A variety of additional classes of computations—notably ones like NC with various kinds of parallelism, ones based on circuits and ones based on algebraic operations—were defined in the 1970s and 1980s, and many detailed results about them were found. In the 1980s much work was also done on the average difficulty of solving NP-complete problems—both exactly and approximately (see page 985). When computational complexity theory was at its height in the early 1980s it was widely believed that if a problem could be shown, for example, to be NP-complete then there was little chance of being able to work with it in a practical situation. But increasingly it became clear that general asymptotic results are often quite irrelevant in typical problems of reasonable size. And certainly pattern matching with _ in *Mathematica*, as well as polynomial manipulation functions like *GroebnerBasis*, routinely deal with problems that are formally NP-complete.

▪ **Lower bounds.** If one could prove for example that P ≠ NP then one would immediately have lower bounds on all NP-complete problems. But absent such a result most of the general lower bounds known so far are based on fairly straightforward information content arguments. One cannot for example sort n objects in less than about n steps since one must at least look at each object, and one cannot multiply two n-digit numbers in less than about n steps since one must at least look at each digit. (As it happens the fastest known algorithms for these problems require very close to n steps.) And if the output from a computation can be of size 2^n then this will normally take at least 2^n steps to generate. Subtleties in defining how big the input to a computation really is can lead to at least apparently exponential lower bounds. An example is testing whether one can match all possible sequences with a regular expression that involves s-fold repetitions. It is fairly clear that this cannot be done in less than about s steps. But this seems exponentially large if s is specified by its digit sequence in the original input regular expression. Similar issues arise in the problem of determining truth or falsity in Presburger arithmetic (see page 1152).

Diagonalization arguments analogous to those on pages 1128 and 1162 show that in principle there must exist functions that can be evaluated only by computations that exceed any given bound. But actually finding examples of such functions that can readily be described as having some useful purpose has in the past seemed essentially impossible.

If one sufficiently restricts the form of the underlying system then it sometimes becomes possible to establish meaningful lower bounds. For example, with deterministic finite automata (see page 957), there are sequences that can be recognized, but only by having exponentially many states. And with DNF Boolean expressions (see page 1096) functions like *Xor* are known to require exponentially many terms, even—as discovered in the 1980s—if any limited number of levels are allowed (see page 1096).

▪ **Algorithmic complexity theory.** Ordinary computational complexity theory asks about the resources needed to run programs that perform a given computation. But algorithmic complexity theory (compare page 1067) asks instead about how large the programs themselves need to be. The results of this book indicate however that even programs that are very small—and thus have low algorithmic complexity—can nevertheless perform all sorts of complex computations.

▪ **Turing machines.** The Turing machines used here in effect have tapes that extend only to the left, and have no explicit halt states. (They thus differ from the Turing machines which Marvin Minsky and Daniel Bobrow studied in 1961 in the $s = 2$, $k = 2$ case and concluded all had simple behavior.) One can think of each Turing machine as computing a function $f[x]$ of the number x given as its input. The function is total (i.e. defined for all x) if the Turing machine always halts; otherwise it is partial (and undefined for at least some x). Turing machines can be numbered according to the scheme on page 888. The number of steps before a machine with given rule halts can be computed from (see page 888)

```
Module[{s = 1, a, i = 1, d}, a[_] = 0; MapIndexed[a[#2[[1]]] =
    #1 &, Reverse[IntegerDigits[x, 2]]]; Do[{s, a[i], d} =
   {s, a[i]} /. rule; i -= d; If[i == 0, Return[t]], {t, tmax}]]
```

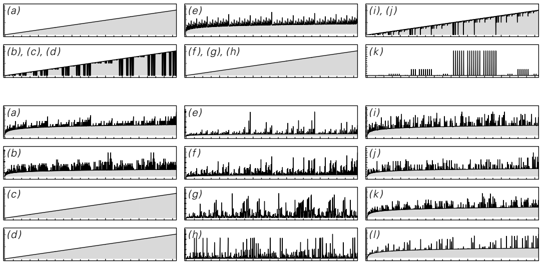
Of the 4096 Turing machines with $s = 2$, $k = 2$, 748 never halt, 3348 sometimes halt and 1683 always halt. (The most rarely halting are ones like machine 3112 that halt only when $x = 4j - 1$.) The number of distinct functions $f[x]$ that can be computed by such machines is 351, of which 149 are total. 17 machines compute $x + 1$; none compute $x + 2$; 17 compute $x - 1$ and do not halt when $x = 0$—an example being 2575. Most machines compute functions that involve digit manipulations without traditional interpretations as mathematical functions. It is quite common to find machines that compute almost the same function: 1507 and 1511 disagree (where 1507 halts) only for $x \geq 35$. If $t[x]$ is the number of steps to compute $f[x]$ then the number of distinct pairs $\{f[x], t[x]\}$ is 492, or 230 for total $f[x]$. In 164 $t[x]$ does not increase with the number of digits n in x, in 295 it increases linearly, in 27 quadratically, and in 6 exponentially. For total $f[x]$ the corresponding numbers are 84, 136, 7, 3; the 3 machines with exponential growth are 378 (example (f) on page 761), 1953 and 2289; all compute trivial functions. Machine 1447 (example (e)) computes the function which takes the digit sequence of x and replaces its first $3 + \mathit{IntegerExponent}[x+1, 2]$ 0's by 1's.

Among the 2,985,984 Turing machines with $s = 3$, $k = 2$, at least 2,550,972 sometimes halt, and about 1,271,304 always do. The number of distinct functions that can be computed is about 36,392 (or 75,726 for $\{f[x], t[x]\}$ pairs). 8934 machines compute $x + 1$ (by 25 different methods, including ones like machine 164850 that take exponential steps), 14 compute $x + 2$, and none compute $x + 3$. Those machines that take times that grow precisely like 2^n all tend to compute very straightforward functions which can be computed much faster by other machines.

Among the 2,985,984 Turing machines with $s = 2$, $k = 3$, at least 2,760,721 sometimes halt, and about 974,595 always halt. The number of distinct functions that can be computed is about 315,959 (or 457,508 for $\{f[x], t[x]\}$ pairs). (The fact that there are far fewer distinct functions in the $s = 3$, $k = 2$ case is a consequence of equivalences between states but not colors.)

Among the 2^{32} Turing machines with $s = 4$, $k = 2$ about 80% at least sometimes halt, and about 16% always do. Still none compute $x + 3$. And no Turing machine of any size can directly compute a function like x^2, $2x$ or $\mathit{Mod}[x, 2]$ that involves manipulating all digits in x.

■ **Functions.** The plots below show the values of the functions $f[x]$ for x from 0 to 1023 computed by the Turing machines on pages 761 and 763. Many of the plots use logarithmic scales. Rarely are the values close to their absolute maximum $t[x]$.



■ **Machine 1507.** This machine shows in some ways the most complicated behavior of any $s = 2$, $k = 2$ Turing machine. As suggested by picture (k) it fails to halt if and only if its configuration at some step matches $\{(0)\ldots, \{1, 1\}, 1, \_\_\_\}$ (in the alternative form of page 888). For any input x one can test whether the machine will ever halt using

```
u[{Reverse[IntegerDigits[x, 2]], 0}]

u[list_] := v[Split[Flatten[list]]]

v[{a_, b_ : {}, c_ : {}, d_ : {}, e_ : {}, f_ : {}, g___}] :=
  Which[a == {1} || First[a] == 0, True, c == {}, False,
    EvenQ[Length[b]], u[{a, 1 - b, c, d, e, f, g}],
    EvenQ[Length[c]], u[{a, 1 - b, c, 1, Rest[d], e, f, g, 0}],
    e == {} || Length[d] ≥ Length[b] + Length[a] - 2,
    True, EvenQ[Length[e]], u[{a, b, c, d, f, g}],
    True, u[{a, 1 - b, c, 1 - d, e, 1, Rest[f], g, 0}]]
```

This test takes at most $n/3$ recursive steps, even though the original machine can take of order n^2 steps to halt. Among $s = 3$, $k = 2$ machines there are 314 machines that do the same computation as 1507, but none any faster.

■ **Page 763 · Properties.** The maximum numbers of steps increase with input size according to:

(a) $14\, 2^{\wedge}\mathit{Floor}[n/2] - 11 + 2\,\mathit{Mod}[n, 2]$

(b) (does not halt for $x = 1$)

(c) $2^n - 1$

(d) $(7\,(1 + \mathit{Mod}[n, 2])\, 4^{\wedge}\mathit{Floor}[n/2] + 2\,\mathit{Mod}[n, 2] - 7)/3$

(h) (see note below)

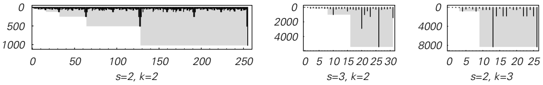
(i) (does not halt for various $x > 53$)

(j) (does not halt for various $x > 39$)

(k) (does not halt for $x = 1$)

(l) $5\,(2^{n-2} - 1)$

■ **Longest halting times.** The pictures below show the largest numbers of steps $t[x]$ that it takes any machine of a particular type to halt when given successive inputs x. For $s = 2$, $k = 2$ the largest results for all inputs of sizes 0 to 4 are $\{7, 17, 31, 49, 71\}$, all obtained with machine 1447. For $n > 4$ the largest results are $2^{n+2} - 3$, achieved for $x = 2^n - 1$ with machines 378 and 1351. For $s = 3$, $k = 2$ the largest results for successive sizes are $\{25, 53, 159, 179, 1021, 5419\}$ (often achieved by machine 600720; see below) and for $s = 2$, $k = 3$ $\{35, 83, 843, 8335\}$ (often achieved by machine 840971). Note the similarity to the busy beaver problem discussed on page 889.



▪ **Growth rates.** Some Turing machine can always be found that has halting times that grow at any specified rate. (See page 103 for a symbolic system with halting times that grow like $\mathit{Nest}[2^{\#}\ \&, 0, n]$.) As discussed on page 1162, if the growth rate is too high then it may not be possible to prove that the machines halt using, say, the standard axioms of arithmetic. The maximum halting times above increase faster than the halting times for any specific Turing machine, and are therefore ultimately not computable by any single Turing machine.

▪ **Machine 600720.** (Case (h) of page 763.) The maximum halting times for the first few sizes *n* are

```
{5, 159, 161, 1021, 5419, 315391,
  1978213883, 1978213885, 3018415453261}
```

These occur for inputs *{1, 2, 5, 10, 26, 34, 106, 213, 426}* and correspond to outputs (each themselves maximal for given *n*)

```
2^{3, 23, 24, 63, 148, 1148, 91148, 91149, 3560523} - 1
```

Such maxima often seem to occur when the input *x* has the form $(20\,4^s - 2)/3$ (and so has digits *{1, 1, 0, 1, 0, …, 1, 0}*). The output *f[x]* in such cases is always $2^u - 1$ where

```
u = Nest[(13 + (6 # + 8) (5/2)^
                IntegerExponent[6 # + 8, 2])/6 &, 1, s + 1]
```

One then finds that *6 u + 8* has the form *Nest[If[EvenQ[#], 5 #/2, # + 21] &, 14, m]* for some *m*, suggesting a connection with the number theory systems of page 122. The corresponding halting time *t[x]* is *Last[Nest[h, {8, 4 s + 24}, s]] - 1* with

$$\mathit{h}[\{i\_, j\_\}] := \mathit{With}[\{e = \mathit{IntegerExponent}[3 i + 4, 2]\}, \{13/6 + (i + 4/3)(5/2)^{e+1}, ((154 + 75 (i + 4/3)(5/2)^{e})^2 - 16321 - 7860 i - 900 i^2 + 3360 e)/3780 + j\}]$$

For $s > 3$ it then turns out that *f[x]* is extremely close to $3560523 (5/2)^r$, and *t[x]* to $18865098979373 (5/2)^{2 r}$, for some integer *r*.

It is very difficult in general to find traditional formulas for *f[x]* and *t[x]*. But if *IntegerDigits[x, 2]* involves no consecutive 0's then for example *f[x]* can be obtained from

```
2^(b[Join[{1, 1}, #], Length[#]] &)[IntegerDigits[x, 2]] - 1

a[{l_, _}, r_] := ({l + (5 r - 3 #)/2, #} &)[Mod[r, 2]]

a[{l_, 0}, 0] := {l + 1, 0}

a[{l_, 1}, 0] :=
  ({(13 + # (5/2)^IntegerExponent[#, 2])/6, 0} &)[6 l + 2]

b[list_, i_] := First[Fold[a, {Apply[Plus, Drop[list, -i]], 0},
      Apply[Plus, Split[Take[list, -i], #1 == #2 ≠ 0 &], 1]]]
```

(The corresponding expression for *t[x]* is more complicated.) A few special cases are:

$$f[4 s] = 4 s + 3$$

$$f[4 s + 1] = 2 f[2 s] + 1$$

$$f[2^s - 1] = 2^{(10 s + 5 + 3 (-1)^s)/4} - 1$$

How the halting times behave for large *n* is not clear. It is certainly possible that they could increase like $\mathit{NestList}[\#^2\ \&, 2, n]$, or 2^{2^n}, although for $x = (20\,4^s - 2)/3$ a better fit for $n \lesssim 200$ is just $2^{2.6 n}$, with outputs increasing like $2^{2^{1.3 n}}$.

▪ **Page 766 · NP completeness.** Among the hundreds of problems known to be NP-complete are:

- Can a non-deterministic Turing machine reach a certain state in a given number of steps?
- Can a multiway system generate a certain string in a given number of steps?
- Is there an assignment of truth values to variables that makes a given Boolean expression true? (Satisfiability; related to minimal Boolean expressions of page 1095.)
- Will a given sequence of pair comparisons correctly sort any list (see page 1142)?
- Will a given pattern of origami folds yield an object that can be made flat?
- Does a network have any parts that match a given subnetwork (see page 1038)?
- Is there a path shorter than some given length that visits all of some set of points in the plane? (Travelling salesman; related to the network layout problem of page 1031.)
- Is there a solution of a certain size to an integer linear programming problem?
- Is there any $x < a$ such that $\mathit{Mod}[x^2, b] == c$? (See page 1090.)
- Does a matrix have a permanent of given value?
- Is there a way to satisfy tiling constraints in a finite region? (See page 984.)
- Is there a string of some limited length that solves a correspondence problem?
- Is there an initial condition to a cellular automaton that yields particular behavior after a given number of steps?

(In cases where numbers are involved, it is usually crucial that these be represented by base 2 digit sequences, and not, say, in unary.) Many NP-complete problems at first seem quite unrelated. But often their equivalence becomes clear just by straightforward identification of terms. And so for example the equivalence of satisfiability to problems about networks can be seen by identifying variables and clauses in Boolean expressions respectively with connections and nodes in networks.

One can get an idea of the threshold of NP completeness by looking at seemingly similar problems where one is NP-complete but the other is in P. Examples include:

- Finding a Hamiltonian circuit that visits once every connection in a given network is NP-complete, but finding an Euler circuit that visits once every node is in P.
- Finding the longest path between two nodes in a network is NP-complete, but finding the shortest path is in P.
- Determining satisfiability for a Boolean expression with 3 variables in each clause is NP-complete, but for one with 2 variables is in P. (The latter is like a network with only 2 connections at each node.)
- Solving quadratic Diophantine equations $a x^2 + b y == c$ is NP-complete, but solving linear ones $a x + b y == c$ is in P.
- Finding a minimum energy configuration for a 2D Ising spin glass in a magnetic field is NP-complete, but is in P if there is no magnetic field.
- Finding the permanent of a matrix is NP-complete, but finding its determinant is in P.

It is not known whether problems such as integer factoring or equivalence of networks under relabelling of nodes (graph isomorphism) are NP-complete. It is known that in principle there exist NP problems that are not in P, yet are not NP-complete.

▪ **Natural systems.** Finding minimum energy configurations is formally NP-complete in standard models of natural systems such as folding protein and DNA molecules (see page 1003), collections of charges on a sphere (compare page 987), and finite regions of spin glasses (see page 944). As discussed on page 351, however, it seems likely that in nature true minima are very rare, and that instead what is usually seen are just the results of actual dynamical processes of evolution.

In quantum field theory and to a lesser extent quantum mechanics and celestial mechanics, approximation schemes based on perturbation series seem to require computations that grow very rapidly with order. But exactly what this implies about the underlying physical processes is not clear.

▪ **P versus NP questions.** Most programs that are explicitly constructed to solve specific problems tend at some level to have rather simple behavior—often just repetitive or nested, so long as appropriate number representations are used. And it is this that makes it realistic to estimate asymptotic growth rates using traditional mathematics, and to determine whether the programs operate in polynomial time. But as the pictures on page 761 suggest, arbitrary computational systems—even Turing machines with very simple rules—can exhibit much more complicated behavior with no clear asymptotic growth rate. And indeed the question of whether the halting times for a system grow only like a power of input size is in general undecidable. And if one tries to prove a result about halting times using, say, standard axioms of arithmetic or set theory, one may find that the result is independent of those axioms. So this makes it far from clear that the general P = NP question has a definite answer within standard axiom systems of mathematics. If one day someone were to find a provably polynomial time algorithm that solves an NP-complete problem then this would establish that P = NP. But it could well be that the fastest programs for NP-complete problems behave in ways that are too complicated to prove much about using the standard axioms of mathematics.

▪ **Non-deterministic Turing machines.** Generalizing rules from page 888 by making each right-hand side a list of possible outcomes, the list of configurations that can be reached after *t* steps is given by

```
NTMEvolve[rule_, inits_, t_Integer] := Nest[
   Union[Flatten[Map[NTMStep[rule, #] &, #], 1]] &, inits, t]
NTMStep[rule_List, {s_, a_, n_}] /; 1 ≤ n ≤ Length[a] :=
  Apply[{#1, ReplacePart[a, #2, n], n + #3} &,
    Replace[{s, a[[n]]}, rule], {1}]
```

▪ **Page 767 · Implementation.** Given a non-deterministic Turing machine with rules in the form above, the rules for a cellular automaton which emulates it can be obtained from

```
NDTMToCA[tm_] := Flatten[{{_, h, _} → h, {s, _c, _} → e, {s,
  _, _} → s, {_, s, c[i_]} → s[i], {_, s, x_} → x, {a[_, _], _s, _} → s,
  {_, a[x_, y_], s[i_]} → a[x, y, i], {x_, _s, _} → x, {_, _, s[i_]} →
  s[i], Map[Table[With[{b = (#[[Min[Length[#], z]]] &)[
   {x, #} /. tm]}, If[Last[b] == -1, {{a[_], a[x, #, z], e} → h, {a[
   _], a[x, #, z], s} → a[x, #, z], {a[_], a[x, #, z], _} → a[b[[2]]],
   {a[x, #, z], a[w_], _} → a[b[[1]], w], {_, a[w_], a[x, #, z]} →
   a[w]}, {{a[_], a[x, #, z], _} → a[b[[2]]], {a[x, #, z], a[w_],
   _} → a[w], {_, a[w_], a[x, #, z]} → a[b[[1]], w]}]], {x,
   Max[Map[#[[1, 1]] &, tm]]}, {z, Max[Map[Length[#[[2]]] &,
   tm]]}] &, Union[Map[#[[1, 2]] &, tm]]], {_, x_, _} → x}]
```

▪ **Page 768 · Satisfiability.** Given variables *𝕤[t, s]*, *𝕒[t, x, a]*, *𝕟[t, n]* representing whether at step *t* a non-deterministic Turing machine is in state *s*, the tape square at position *x* has color *a*, and the head is at position *n*, the following CNF expression represents the assertion that a Turing machine with *stot* states and *ktot* possible colors follows the specified rules and halts after at most *t* steps:

```
NDTMToCNF[rules_, {s_, a_, n_}, t_] :=
 {Table[Apply[Or, Table[𝕤[i, j], {j, stot}]], {i, t - 1}],
  Table[! 𝕤[i, j] || ! 𝕤[i, k], {i, 0, t - 1}, {j, stot}, {k, j + 1, stot}],
  Table[Apply[Or, Table[𝕟[i, j], {j, n + i, Max[0, n - i], -2}]],
   {i, 0, t}], Table[! 𝕟[i, j] || ! 𝕟[i, k], {i, 0, t}, {j, n + i, Max[0,
    n - i], -2}, {k, j + 2, n + i}], Table[Apply[Or, Table[𝕒[i, j, k],
   {k, 0, ktot - 1}]], {i, 0, t - 1}, {j, Max[1, n - i], n + i}],
  Table[! 𝕒[i, j, k] || ! 𝕒[i, j, m], {i, 0, t - 1}, {j, Max[1, n - i],
   n + i}, {k, 0, ktot - 1}, {m, k + 1, ktot - 1}], 𝕤[0, s],
Cases[MapIndexed[𝕒[Abs[n - First[#2]], First[#2], #1] &,
 a], 𝕒[x_, _, _] /; x < t], Table[𝕒[Abs[n - i], i, 0],
 {i, Length[a] + 1, n + t - 1}], Table[! 𝕒[i, j, k] ||
  If [EvenQ[n + i - j], 𝕞[i, j], False] || 𝕒[i + 1, j, k], {i, 0, t - 2},
 {j, Max[1, n - i], n + i}, {k, 0, ktot - 1}], Table[Map[Function[
  z, Outer[! 𝕞[i, j] || ! 𝕤[i, z[[1, 1]]] || ! 𝕒[i, j, z[[1, 2]]] || ## &,
   Apply[Sequence, Map[If [i < t - 1, {𝕤[i + 1, #[[1]]], 𝕞[
      i + 1, j - #[[3]]], 𝕒[i + 1, j, #[[2]]]}, {𝕞[i + 1, j - #[[3]]]}] &,
    z[[2]]]]]], rules], {i, 0, t - 1}, {j, n + i, Max[1, n - i], -2}],
Apply[Or, Table[𝕞[i, 0], {i, n, t, 2}]]]} /. List → And
```

■ **Density of difficult problems.** There are arguments that in an asymptotic sense most instances chosen at random of problems like limited-size PCP or tiling will be difficult to solve. In a problem like satisfiability, however, difficult instances tend to occur only on the boundary between cases where the density of black or white squares implies that there is usually satisfaction or usually not satisfaction. If one looks at simple instances of problems (say PCP with short strings) then my experience is that many are easy to solve. But just as some fraction of cellular automata with very simple rules show immensely complex behavior, so similarly it seems that some fraction of even simple instances of many NP-complete problems also tend to be difficult to solve.

■ **Page 770 · Rule 30 inversion.** The total numbers of sequences for *t* from 1 to 15 not yielding stripes of heights 1 and 2 are respectively

```
{1, 2, 2, 3, 3, 6, 6, 10, 16, 31, 52, 99, 165, 260}
```

```
{2, 5, 8, 14, 23, 40, 66, 111, 182,
  316, 540, 921, 1530, 2543, 4122}
```

The sideways evolution of rule 30 discussed on page 601 implies that if one fills cells from the left rather than the right then some sequence of length $t + 1$ will always yield any given stripe of height t.

If the evolution of rule 30 can be set up as on page 704 to emulate any Boolean function then the problem considered here is immediately equivalent to satisfiability.

■ **Systems of limited size.** In the system *x → Mod[x + m, n]* from page 255 the repetition period *n/GCD[m, n]* can be computed using Euclid's algorithm in at most about *Log[GoldenRatio, n]* steps. In the system *x → Mod[2 x, n]* from page 257, the repetition period *MultiplicativeOrder[2, n]* probably cannot always be computed in any polynomial of *Log[n]* steps, since otherwise *FactorInteger[n]* could also be computed in about this number of steps. (But see note below.) In a cellular automaton with n cells, the problem of finding the repetition period is in general PSPACE-complete—as follows from the possibility of universality in the underlying cellular automaton. And even in a case like rule 30 I suspect that the period cannot be found much faster than by tracing nearly 2^n steps of evolution. (I know of no way for example to break the computation into parts that can be done in parallel.) With sufficiently simple behavior, a cellular automaton repetition period can readily be determined in some power of *Log[n]* steps. But even with an additive rule and nested behavior, the period depends on quantities like *MultiplicativeOrder[2, n]*, which probably take more like n steps to evaluate. (But see note below.)

■ **Page 771 · Quantum computers.** In an ordinary classical setup one typically describes the state of something like a 2-color cellular automaton with n cells just by giving a list of n color values. But the standard formalism of quantum theory (see page 1058) implies that for an analogous quantum system—like a line of n quantum spins each either up or down—one instead has to give a whole vector of probability amplitudes for each of the 2^n possible complete underlying spin configurations. And these amplitudes a_i are assumed to be complex numbers with a continuous range of possible values, subject only to the conventional constraint of unit total probability *Sum[Abs[*a_i*]*^{2}*, {i,* 2^n*}] == 1*. The evolution of such a quantum system can then formally be represented by successive multiplication of the vector of amplitudes by appropriate $2^n \times 2^n$ unitary matrices.

In a classical system like a cellular automaton with n cells a probabilistic ensemble of states can similarly be described by a vector of 2^n probabilities p_i—now satisfying *Sum[*p_i*, {i,* 2^n*}] == 1*, and evolving by multiplication with $2^n \times 2^n$ matrices having a single *1* in each row. (If the system is reversible—as in the quantum case—then the matrices are invertible.) But even if one assumes that all 2^n states in the ensemble somehow manage to evolve in parallel, it is still fairly clear that to do reliable computations takes essentially as much effort as evolving single instances of the underlying system. For even though the vector of probabilities can formally give outcomes for 2^n different initial conditions, any specific individual outcome could have probability as small as 2^{-n}—and so would take 2^n trials on average to detect.

The idea of setting up quantum analogs of systems like Turing machines and cellular automata began to be pursued in the early 1980s by a number of people, including myself. At first it was not clear what idealizations to make, but by the late 1980s—especially through the work of David Deutsch—the concept had emerged that a quantum computer should be described in terms of a network of basic quantum gates. The idea was to have say n quantum spins (each representing a so-called qubit), then to do computations much like in the reversible logic systems of page 1097 or the sorting networks of page 1142 by applying some appropriate sequence of elementary operations. It was found to be sufficient to do operations on just one and two spins at a time, and in fact it was shown that any $2^n \times 2^n$ unitary matrix can be approximated arbitrarily closely by a suitable sequence of for example underlying 2-spin *{x, y} → {x, Mod[x + y, 2]}* operations (assuming values 0 and 1), together with 1-spin arbitrary phase change operations. Such phase changes can be produced by repeatedly applying a single irrational rotation, and using the fact that *Mod[h s, 2 π]* will eventually for some *s* come close to any given phase (see page 903). From the involvement of continuous numbers, one might at first imagine that it should be possible to do fundamentally more computations than can be done say in ordinary discrete cellular automata. But all the evidence is that—just as discussed on page 1128—this will not in fact be possible if one makes the assumption that at some level discrete must be used to set up the initial values of probability amplitudes.

From the fact that the basic evolution of an *n*-spin quantum system in effect involves superpositions of 2^n spin configurations one might however still imagine that in finite computations exponential speedups should be possible. And as a potential example, consider setting up a quantum computer that evaluates a given Boolean function—with its initial configurations of spins encoding possible inputs to the function, and the final configuration of a particular spin representing the output from the function. One might imagine that with such a computer it would be easy to solve the NP-complete problem of satisfiability from page 768: one would just start off with a superposition in which all 2^n possible inputs have equal amplitude, then look at whether the spin representing the output from the function has any amplitude to be in a particular configuration. But in an actual physical system one does not expect to be able to find values of amplitudes directly. For according to the standard formalism of quantum theory all amplitudes do is to determine probabilities for particular outcomes of measurements. And with the setup described, even if a particular function is ultimately satisfiable the probability for a single output spin to be measured say as up can be as little as 2^{-n}—requiring on average 2^n trials to distinguish from *0*, just as in the classical probabilistic case.

With a more elaborate setup, however, it appears sometimes to be possible to spread out quantum amplitudes so as to make different outcomes correspond to much larger probability differences. And indeed in 1994 Peter Shor found a way to do this so as to get quantum computers at least formally to factor integers of size *n* using resources only polynomial in *n*. As mentioned in the note above, it becomes straightforward to factor *m* if one can get the values of *MultiplicativeOrder[a, m]*. But these correspond to periodicities in the list *Mod[a ^ Range[m], m]*. Given *n* spins one can imagine using their 2^n possible configurations to represent each element of *Range[m]*. But now if one sets up a superposition of all these configurations, one can compute *Mod[a^#, m] &*, then essentially use *Fourier* to find periodicities—all with a polynomial number of quantum gates. And depending on *FactorInteger[m]* the resulting amplitudes show fairly large differences which can then be detected in the probabilities for different outcomes of measurements.

In the mid-1990s it was thought that quantum computers might perhaps give polynomial solutions to all NP problems. But in fact only a very few other examples were found—all ultimately based on very much the same ideas as factoring. And indeed it now seems decreasingly likely that quantum computers will give polynomial solutions to NP-complete problems. (Factoring is not known to be NP-complete.)

And even in the case of factoring there are questions about the idealizations used. It does appear that only modest precision is needed for the initial amplitudes. And it seems that perturbations from the environment can be overcome using versions of error-correcting codes. But it remains unclear just what might be needed actually to perform for example the final measurements required.

Simple physical versions of individual quantum gates have been built using particles localized for example in ion traps. But even modestly larger setups have been possible only in NMR and optical systems—which show formal similarities to quantum systems (and for example exhibit interference) but presumably do not have any unique quantum advantage. (There are other approaches to quantum computation that involve for example topology of 4D quantum fields. But it is difficult to see just what idealizations are realistic for these.)

▪ **Circuit complexity.** Any function with a fixed size of input can be computed by a circuit of the kind shown on page 619. How the minimal size or depth of circuit needed grows with input size then gives a measure of the difficulty of the computation, with circuit depth growing roughly like number of steps for a Turing machine. Note that much as on page 662 one can construct universal circuits that can be arranged by appropriate choice of parts of their input to compute any function of a given input size. (Compare page 703.)

▪ **Page 771 · Finding outcomes.** If one sets up a function to compute the outcome after *t* steps of evolution from some fixed initial condition—say a single black cell in a cellular automaton—then the input to this function need contain only *Log[2, t]* digits. But if the evolution is computationally irreducible then to find its outcome will involve explicitly following each of its *t* steps—thereby effectively finding results for each of the *2 ^ Log[2, t]* possible arrangements of digits corresponding to numbers less than *t*. Note that the computation that is involved is not necessarily in either NP or PSPACE.

▪ **P completeness.** If one allows arbitrary initial conditions in a cellular automaton with nearest-neighbor rules, then to compute the color of a particular cell after *t* steps in general requires specifying as input the colors of all *2 t + 1* initial cells up to distance *t* away (see page 960). And if one always does computations using systems that have only nearest-neighbor rules then just combining *2 t + 1* bits of information can take up to *t* steps—even if the bits are combined in a way that is not computationally irreducible. So to avoid this one can consider systems that are more like circuits in which any element can get data from any other. And given *t* elements operating in parallel one can consider the class NC studied by Nicholas Pippenger in 1978 of computations that can be done in a number of steps that is at most some power of *Log[t]*. Among such computations are *Plus*, *Times*, *Divide*, *Det* and *LinearSolve* for integers, as well as determining outcomes in additive cellular automata (see page 609). But I strongly suspect that computational irreducibility prevents outcomes in systems like rule 30 and rule 110 from being found by computations that are in NC—implying in effect that allowing arbitrary connections does not help much in computing the evolution of such systems. There is no way yet known to establish this for certain, but just as with NP and P one can consider showing that a computation is P-complete with respect to transformations in NC. It turns out that finding the outcome of evolution in any standard universal Turing machine or cellular automaton is P-complete in this sense, since the process of emulating any such system by any other one is in NC. Results from the mid-1970s established that finding the output from an arbitrary circuit with *And* or *Or* gates is P-complete, and this has made it possible to show that finding the outcome of evolution in various systems not yet known to be universal is P-complete. A notable example due to Cristopher Moore from 1996 is the 3D majority cellular automaton with rule *UnitStep[a + AxesTotal[a, 3] - 4]* (see page 927); another example is the Ising model cellular automaton from page 982.

Implications for Mathematics and Its Foundations

▪ **History.** Babylonian and Egyptian mathematics emphasized arithmetic and the idea of explicit calculation. But Greek mathematics tended to focus on geometry, and increasingly relied on getting results by formal deduction. For being unable to draw geometrical figures with infinite accuracy this seemed the only way to establish anything with certainty. And when Euclid around 330 BC did his work on geometry he started from 10 axioms (5 “common notions” and 5 “postulates”) and derived 465 theorems. Euclid’s work was widely studied for more than two millennia and viewed as a quintessential example of deductive thinking. But in arithmetic and algebra—which in effect dealt mostly with discrete entities—a largely calculational approach was still used. In the 1600s and 1700s, however, the development of calculus and notions of continuous functions made use of more deductive methods. Often the basic concepts were somewhat vague, and by the mid-1800s, as mathematics became more elaborate and abstract, it became clear that to get systematically correct results a more rigid formal structure would be needed.

The introduction of non-Euclidean geometry in the 1820s, followed by various forms of abstract algebra in the mid-1800s, and transfinite numbers in the 1880s, indicated that mathematics could be done with abstract structures that had no obvious connection to everyday intuition. Set theory and predicate logic were proposed as ultimate foundations for all of mathematics (see note below). But at the very end of the 1800s paradoxes were discovered in these approaches. And there followed an increasing effort—notably by David Hilbert—to show that everything in mathematics could consistently be derived just by starting from axioms and then using formal processes of proof.

Gödel’s Theorem showed in 1931 that at some level this approach was flawed. But by the 1930s pure mathematics had already firmly defined itself to be based on the notion of doing proofs—and indeed for the most part continues to do so even today (see page 859). In recent years, however, the increasing use of explicit computation has made proof less important, at least in most applications of mathematics.

▪ **Models of mathematics.** Gottfried Leibniz’s notion in the late 1600s of a “universal language” in which arguments in mathematics and elsewhere could be checked with logic can be viewed as an early idealization of mathematics. Starting in 1879 with his “formula language” (*Begriffsschrift*) Gottlob Frege followed a somewhat similar direction, suggesting that arithmetic and from there all of mathematics could be built up from predicate logic, and later an analog of set theory. In the 1890s Giuseppe Peano in his *Formulario* project organized a large body of mathematics into an axiomatic framework involving logic and set theory. Then starting in 1910 Alfred Whitehead and Bertrand Russell in their *Principia Mathematica* attempted to derive many areas of mathematics from foundations of logic and set theory. And although its methods were flawed and its notation obscure this work did much to establish the idea that mathematics could be built up in a uniform way.

Starting in the late 1800s, particularly with the work of Gottlob Frege and David Hilbert, there was increasing interest in so-called metamathematics, and in trying to treat mathematical proofs like other objects in mathematics. This led in the 1920s and 1930s to the introduction of various idealizations for mathematics—notably recursive functions, combinators, lambda calculus, string rewriting systems and Turing machines. All of these were ultimately shown to be universal (see page 784) and thus in a sense capable of reproducing any mathematical system. String rewriting systems—as studied particularly by Emil Post—are close to the multiway systems that I use in this section (see page 938).

Largely independent of mathematical logic the success of abstract algebra led by the end of the 1800s to the notion that any mathematical system could be represented in algebraic terms—much as in the operator systems of this section. Alfred Whitehead to some extent captured this in his 1898 *Universal Algebra*, but it was not until the 1930s that the theory of structures emphasized commonality in the axioms for different fields of mathematics—an idea taken further in the 1940s by category theory (and later by topos theory). And following the work of the Bourbaki group beginning at the end of the 1930s it has become almost universally accepted that structures together with set theory are the appropriate framework for all of pure mathematics.

But in fact the *Mathematica* language released in 1988 is now finally a serious alternative. For while it emphasizes calculation rather than proof its symbolic expressions and transformation rules provide an extremely general way to represent mathematical objects and operations—as for example the notes to this book illustrate.

(See also page 1176.)

■ **Page 773 · Axiom systems.** In the main text I argue that there are many consequences of axiom systems that are quite independent of their details. But in giving the specific axiom systems that have been used in traditional mathematics one needs to take account of all sorts of fairly complicated details.

As indicated by the tabs in the picture, there is a hierarchy to axiom systems in traditional mathematics, with those for basic and predicate logic, for example, being included in all others. (Contrary to usual belief my results strongly suggest however that the presence of logic is not in fact essential to many overall properties of axiom systems.)

As discussed in the main text (see also page 1155) one can think of axioms as giving rules for transforming symbolic expressions—much like rules in *Mathematica*. And at a fundamental level all that matters for such transformations is the structure of expressions. So notation like $a + b$ and $a \times b$, while convenient for interpretation, could equally well be replaced by more generic forms such as $f[a, b]$ or $g[a, b]$ without affecting any of the actual operation of the axioms.

My presentation of axiom systems generally follows the conventions of standard mathematical literature. But by making various details explicit I have been able to put all axiom systems in forms that can be used almost directly in *Mathematica*. Several steps are still necessary though to get the actual rules corresponding to each axiom system. First, the definitions at the top of page 774 must be used to expand out various pieces of notation. In basic logic I use the notation $u = v$ to stand for the pair of rules $u \to v$ and $v \to u$. (Note that $=$ has the precedence of $\to$ not $==$.) In predicate logic the tab at the top specifies how to construct rules (which in this case are often called rules of inference, as discussed on page 1155). $x\_ \land y\_ \to x\_$ is the *modus ponens* or detachment rule (see page 1155). $x\_ \to \forall_{y\_}\, x\_$ is the generalization rule. $x\_ \to x\_ \land \#\, \&$ is applied to the axioms given to get a list of rules. Note that while $=$ in basic logic is used in the underlying construction of rules, $==$ in predicate logic is just an abstract operator with properties defined by the last two axioms given.

As is typical in mathematical logic, there are some subtleties associated with variables. In the axioms of basic logic literal variables like a must be replaced with patterns like $a\_$ that can stand for any expression. A rule like $a\_ \land (b\_ \lor \lnot b\_) \to a$ can then immediately be applied to part of an expression using $Replace$. But to apply a rule like $a\_ \to a \land (b \lor \lnot b)$ requires in effect choosing some new expression for b (see page 1155). And one way to represent this process is just to have the pattern $a\_ \to a\_ \land (b\_ \lor \lnot b\_)$ and then to say that any actual rule that can be used must match this pattern. The rules given in the tab for predicate logic work the same way. Note, however, that in predicate logic the expressions that appear on each side of any rule are required to be so-called well-formed formulas (WFFs) consisting of variables (such as a) and constants (such as 0 or $\emptyset$) inside any number of layers of functions (such as $+$, $\cdot$, or Δ) inside a layer of predicates (such as $==$ or $\in$) inside any number of layers of logical connectives (such as $\land$ or $\Rightarrow$) or quantifiers (such as $\forall$ or $\exists$). (This setup is reflected in the grammar of the *Mathematica* language, where the operator precedences for functions are higher than for predicates, which are in turn higher than for quantifiers and logical connectives—thus yielding for example few parentheses in the presentation of axiom systems here.)

In basic logic any rule can be applied to any part of any expression. But in predicate logic rules can be applied only to whole expressions, always in effect using *Replace[expr, rules]*. The axioms below (devised by Matthew Szudzik as part of the development of this book) set up basic logic in this way.

$(a \vee b) \vee c = (b \vee a) \vee c$	$a \vee ((b \wedge c) \wedge d) = a \vee ((c \wedge b) \wedge d)$
$((a \vee b) \vee c) \vee d = (a \vee (b \vee c)) \vee d$	$a \vee (((b \wedge c) \wedge d) \wedge e) = a \vee ((b \wedge (c \wedge d)) \wedge e)$
$a \vee (b \wedge (c \wedge \neg c)) = a$	$a \vee (b \wedge (c \vee \neg c)) = a \vee b$
$a \vee (b \vee (c \wedge d)) = a \vee ((b \vee c) \wedge (b \vee d))$	$a \vee (b \wedge (c \vee d)) = a \vee ((b \wedge c) \vee (b \wedge d))$
$a \vee (b \wedge \neg (c \vee d)) = a \vee (b \wedge (\neg c \wedge \neg d))$	$a \vee (b \wedge \neg (c \wedge d)) = a \vee (b \wedge (\neg c \vee \neg d))$
$a \vee (b \wedge c) = a \vee (b \wedge \neg \neg c)$	

■ **Basic logic.** The formal study of logic began in antiquity (see page 1099), with verbal descriptions of many templates for valid arguments—corresponding to theorems of logic—being widely known by medieval times. Following ideas of abstract algebra from the early 1800s, the work of George Boole around 1847 introduced the notion of representing logic in a purely symbolic and algebraic way. (Related notions had been considered by Gottfried Leibniz in the 1680s.) Boole identified *1* with *True* and *0* with *False*, then noted that theorems in logic could be stated as equations in which *Or* is roughly *Plus* and *And* is *Times*—and that such equations can be manipulated by algebraic means. Boole's work was progressively clarified and simplified, notably by Ernst Schröder, and by around 1900, explicit axiom systems for Boolean algebra were being given. Often they included most of the 14 highlighted theorems of page 817, but slight simplifications led for example to the "standard version" of page 773. (Note that the duality between *And* and *Or* is no longer explicit here.) The "Huntington version" of page 773 was given by Edward Huntington in 1933, along with

$$\{(\neg \neg a) = a,\ (a \vee \neg (b \vee \neg b)) = a,\\ (\neg (\neg (a \vee \neg b) \vee \neg (a \vee \neg c))) = (a \vee \neg (b \vee c))\}$$

The "Robbins version" was suggested by Herbert Robbins shortly thereafter, but only finally proved correct in 1996 by William McCune using automated theorem proving (see page 1157). The "Sheffer version" based on *Nand* (see page 1173) was given by Henry Sheffer in 1913. The shorter version was devised by David Hillman as part of the development of this book. The shortest version is discussed on page 808. (See also page 1175.)

In the main text each axiom defines an equivalence between expressions. The tradition in philosophy and mathematical logic has more been to take axioms to be true statements from which others can be deduced by the *modus ponens* inference rule $\{x, x \Rightarrow y\} \rightarrow y$ (see page 1155). In 1879 Gottlob Frege used his diagrammatic notation to set up a symbolic representation for logic on the basis of the axioms

$$\{a \Rightarrow (b \Rightarrow a),\ (a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c)),\\ (a \Rightarrow (b \Rightarrow c)) \Rightarrow (b \Rightarrow (a \Rightarrow c)),\\ (a \Rightarrow b) \Rightarrow ((\neg b) \Rightarrow (\neg a)),\ (\neg \neg a) \Rightarrow a,\ a \Rightarrow (\neg \neg a)\}$$

Charles Peirce did something similar at almost the same time, and by 1900 this approach to so-called propositional or sentential calculus was well established. (Alfred Whitehead and Bertrand Russell used an axiom system based on *Or* and *Not* in their original 1910 edition of *Principia Mathematica*.) In 1948 Jan Łukasiewicz found the single axiom version

$$\{((a \Rightarrow (b \Rightarrow a)) \Rightarrow ((((\neg c) \Rightarrow (d \Rightarrow (\neg e))) \Rightarrow ((c \Rightarrow (d \Rightarrow f)) \Rightarrow\\ ((e \Rightarrow d) \Rightarrow (e \Rightarrow f)))) \Rightarrow g)) \Rightarrow (h \Rightarrow g)\}$$

equivalent for example to

$$\{((\neg a) \Rightarrow (b \Rightarrow (\neg c))) \Rightarrow\\ ((a \Rightarrow (b \Rightarrow d)) \Rightarrow ((c \Rightarrow b) \Rightarrow (c \Rightarrow d))),\ a \Rightarrow (b \Rightarrow a)\}$$

It turns out to be possible to convert any axiom system that works with *modus ponens* (and supports the properties of ⇒) into a so-called equational one that works with equivalences between expressions by using

```
Module[{a}, Join[Thread[axioms == a ⇒ a],
    {((a ⇒ a) ⇒ b) == b, ((a ⇒ b) ⇒ b) == (b ⇒ a) ⇒ a}]]
```

An analog of *modus ponens* for *Nand* is $\{x, x \barwedge (y \barwedge z)\} \rightarrow z$, and with this Jean Nicod found in 1917 the single axiom

$$\{(a \barwedge (b \barwedge c)) \barwedge ((e \barwedge (e \barwedge e)) \barwedge ((d \barwedge b) \barwedge ((a \barwedge d) \barwedge (a \barwedge d))))\}$$

which was highlighted in the 1925 edition of *Principia Mathematica*. In 1931 Mordechaj Wajsberg found the slightly simpler

$$\{(a \barwedge (b \barwedge c)) \barwedge (((d \barwedge c) \barwedge ((a \barwedge d) \barwedge (a \barwedge d))) \barwedge (a \barwedge (a \barwedge b)))\}$$

Such an axiom system can be converted to an equational one using

```
Module[{a}, With[{t = a ⊼ (a ⊼ a), i = #1 ⊼ (#2 ⊼ #2) &},
    Join[Thread[axioms == t], {i[t ⊼ (b ⊼ c), c] == t,
        i[t, b] == b, i[i[a, b], b] == i[i[b, a], a]}]]]
```

but then involves 4 axioms.

The question of whether any particular statement in basic logic is true or false is always formally decidable, although in general it is NP-complete (see page 768).

■ **Predicate logic.** Basic logic in effect concerns itself with whole statements (or "propositions") that are each either *True* or *False*. Predicate logic on the other hand takes into account how such statements are built up from other constructs—like those in mathematics. A simple statement in predicate logic is $\forall_x (\forall_y\, x == y) \vee \forall_x (\exists_y (\neg x == y))$, where $\forall$ is "for all" and $\exists$ is "there exists" (defined in terms of $\forall$ on page 774)—and this particular statement can be proved *True* from the axioms. In general statements in predicate logic can contain arbitrary so-called predicates, say *p[x]* or *r[x, y]*, that are each either *True* or *False* for given *x* and *y*. When predicate logic is used as part of other axiom systems, there are typically axioms which define properties of the predicates. (In real algebra, for example, the predicate $>$ satisfies $a > b \Rightarrow a \neq b$.) But in pure predicate logic the predicates are not assumed to have any particular properties.

Notions of quantifiers like $\forall$ and $\exists$ were already discussed in antiquity, particularly in the context of syllogisms. The first explicit formulation of predicate logic was given by Gottlob Frege in 1879, and by the 1920s predicate logic had become widely accepted as a basis for mathematical axiom systems. (Predicate logic has sometimes also been used as a model for general reasoning—and particularly in the 1980s was the basis for several initiatives in artificial intelligence. But for the most part it has turned out to be too rigid to capture directly typical everyday reasoning processes.)

Monadic pure predicate logic—in which predicates always take only a single argument—reduces in effect to basic logic and is not universal. But as soon as there is even one arbitrary predicate with two arguments the system becomes universal (see page 784). And indeed this is the case even if one considers only statements with quantifiers $\forall\,\exists\,\forall$. (The system is also universal with one two-argument function or two one-argument functions.)

In basic logic any statement that is true for all possible assignments of truth values to variables can always be proved from the axioms of basic logic. In 1930 Kurt Gödel showed a similar result for pure predicate logic: that any statement that is true for all possible explicit values of variables and all possible forms of predicates can always be proved from the axioms of predicate logic. (This is often called Gödel's Completeness Theorem, but is not related to completeness of the kind I discuss on page 782 and elsewhere in this section.)

In discussions of predicate logic there is often much said about scoping of variables. A typical issue is that in, say, $\forall_x(\exists_y(\neg x == y))$, x and y are dummy variables whose specific names are not supposed to be significant; yet the names become significant if, say, x is replaced by y. In *Mathematica* most such issues are handled automatically. The axioms for predicate logic given here follow the work of Alfred Tarski in 1962 and use properties of $==$ to minimize issues of variable scoping.

(See also higher-order logics on page 1167.)

▪ **Arithmetic.** Most of the Peano axioms are straightforward statements of elementary facts about arithmetic. The last axiom is a schema (see page 1156) that states the principle of mathematical induction: that if a statement is valid for $a = 0$, and its validity for $a = b$ implies its validity for $a = b + 1$, then it follows that the statement must be valid for all a. Induction was to some extent already used in antiquity—for example in Euclid's proof that there are always larger primes. It began to be used in more generality in the 1600s. In effect it expresses the idea that the integers form a single ordered sequence, and it provides a basis for the notion of recursion.

In the early history of mathematics arithmetic with integers did not seem to need formal axioms, for facts like $x + y == y + x$ appeared to be self-evident. But in 1861 Hermann Grassmann showed that such facts could be deduced from more basic ones about successors and induction. And in 1891 Giuseppe Peano gave essentially the Peano axioms listed here (they were also given slightly less formally by Richard Dedekind in 1888)—which have been used unchanged ever since. (Note that in second-order logic—and effectively set theory— $+$ and $\times$ can be defined just in terms of Δ; see page 1160. In addition, as noted by Julia Robinson in 1948 it is possible to remove explicit mention of $+$ even in the ordinary Peano axioms, using the fact that if $c == a + b$ then $(\Delta a \times c) \times (\Delta b \times c) == \Delta(c \times c) \times (\Delta a \times b)$. Axioms 3, 4 and 6 can then be replaced by $a \times b == b \times a$, $a \times (b \times c) == (a \times b) \times c$ and $(\Delta a) \times (\Delta a \times b) == \Delta a \times (\Delta b \times (\Delta a))$. See also page 1163.)

The proof of Gödel's Theorem in 1931 (see page 1158) demonstrated the universality of the Peano axioms. It was shown by Raphael Robinson in 1950 that universality is also achieved by the Robinson axioms for reduced arithmetic (usually called Q) in which induction—which cannot be reduced to a finite set of ordinary axioms (see page 1156)—is replaced by a single weaker axiom. Statements like $x + y == y + x$ can no longer be proved in the resulting system (see pages 800 and 1169).

If any single one of the axioms given for reduced arithmetic is removed, universality is lost. It is not clear however exactly what minimal set of axioms is needed, for example, for the existence of solutions to integer equations to be undecidable (see page 787). (It is known, however, that essentially nothing is lost even from full Peano arithmetic if for example one drops axioms of logic such as $\neg\neg a = a$.)

A form of arithmetic in which one allows induction but removes multiplication was considered by Mojzesz Presburger in 1929. It is not universal, although it makes statements of size n potentially take as many as about 2^{2^n} steps to prove (though see page 1143).

The Peano axioms for arithmetic seem sufficient to support most of the whole field of number theory. But if as I believe there are fairly simple results that are unprovable from these axioms it may in fact be necessary to extend the Peano axioms to make certain kinds of progress even in practical number theory. (See also page 1166.)

■ **Algebraic axioms.** Axioms like $a \circ (b \circ c) = (a \circ b) \circ c$ can be used in at least three ways. First, as equations which can be manipulated—like the axioms of basic logic—to establish whether expressions are equal. Second, as on page 773, as statements to be added to the axioms of predicate logic to yield results that hold for every possible system described by the axioms (say every possible semigroup). And third, as definitions of sets whose properties can be studied—and compared—using set theory. High-school algebra typically treats axioms as equations. More advanced algebra often uses predicate logic, but implicitly uses set theory whenever it addresses for example mappings between objects. Note that as discussed on page 1159 how one uses algebraic axioms can affect issues of universality and undecidability. (See also page 1169.)

■ **Groups.** Groups have been used implicitly in the context of geometrical symmetries since antiquity. In the late 1700s specific groups began to be studied explicitly, mainly in the context of permutations of roots of polynomials, and notably by Evariste Galois in 1831. General groups were defined by Arthur Cayley around 1850 and their standard axioms became established by the end of the 1800s. The alternate axioms given in the main text are the shortest known. The first for ordinary groups was found by Graham Higman and Bernhard Neumann in 1952; the second by William McCune (using automated theorem proving) in 1992. For commutative (Abelian) groups the first alternate axioms were found by Alfred Tarski in 1938; the second by William McCune (using automated theorem proving) in 1992. In this case it is known that no shorter axioms are possible. (See page 806.) Note that in terms of the $\bar{\circ}$ operator $1 == a \,\bar{\circ}\, a$, $\overline{b} == (a \,\bar{\circ}\, a) \,\bar{\circ}\, b$, and $a \cdot b == a \,\bar{\circ}\, ((a \,\bar{\circ}\, a) \,\bar{\circ}\, b)$. Ordinary group theory is universal; commutative group theory is not (see page 1159).

■ **Semigroups.** Despite their simpler definition, semigroups have been much less studied than groups, and there have for example been about 7 times fewer mathematical publications about them (and another 7 times fewer about monoids). Semigroups were defined by Jean-Armand de Séguier in 1904, and beginning in the late 1920s a variety of algebraic results about them were found. Since the 1940s they have showed up sporadically in various areas of mathematics—notably in connection with evolution processes, finite automata and category theory.

■ **Fields.** With $\oplus$ being $+$ and $\otimes$ being $\times$ rational, real and complex numbers are all examples of fields. Ordinary integers lack inverses under $\times$, but reduction modulo a prime p gives a finite field. Since the 1700s many examples of fields have arisen, particularly in algebra and number theory. The general axioms for fields as given here emerged around the end of the 1800s. Shorter versions can undoubtedly be found. (See page 1168.)

■ **Rings.** The axioms given are for commutative rings. With $\oplus$ being $+$ and $\otimes$ being $\times$ the integers are an example. Several examples of rings arose in the 1800s in number theory and algebraic geometry. The study of rings as general algebraic structures became popular in the 1920s. (Note that from the axioms of ring theory one can only expect to prove results that hold for any ring; to get most results in number theory, for example, one needs to use the axioms of arithmetic, which are intended to be specific to ordinary integers.) For non-commutative rings the last axiom given is replaced by $(a \oplus b) \otimes c == a \otimes c \oplus b \otimes c$. Non-commutative rings already studied in the 1800s include quaternions and square matrices.

■ **Other algebraic systems.** Of algebraic systems studied in traditional mathematics the vast majority are special cases of either groups, rings or fields. Probably the most common other examples are those based on lattice theory. Standard axioms for lattice theory are ($\wedge$ is usually called meet, and $\vee$ join)

$$\{a \wedge b == b \wedge a,\ a \vee b == b \vee a,\ (a \wedge b) \wedge c == a \wedge (b \wedge c),$$
$$(a \vee b) \vee c == a \vee (b \vee c),\ a \wedge (a \vee b) == a,\ a \vee a \wedge b == a\}$$

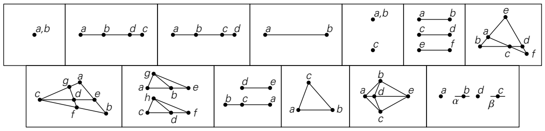
Boolean algebra (basic logic) is a special case of lattice theory, as is the theory of partially ordered sets (of which the causal networks in Chapter 9 are an example). The shortest single axiom currently known for lattice theory has *LeafCount* 79 and involves 7 variables. But I suspect that in fact a *LeafCount* less than about 20 is enough.

(See also page 1171.)

■ **Real algebra.** A notion of real numbers as measures of space or quantity has existed since antiquity. The development of basic algebra gave a formal way to represent operations on such numbers. In the late 1800s there were efforts—notably by Richard Dedekind and Georg Cantor—to set up a general theory of real numbers relying only on basic concepts about integers—and these efforts led to set theory. For purely algebraic questions of the kind that might arise in high-school algebra, however, one can use just the axioms given here. These add to field theory several axioms for ordering, as well as the axiom at the bottom expressing a basic form of continuity (specifically that any polynomial which changes sign must have a zero). With these axioms one can prove results about real polynomials, but not about arbitrary mathematical functions, or integers. The axioms were shown to be complete by Alfred Tarski in the 1930s. The proof was based on setting up a procedure that could in principle resolve any set of real polynomial equations or inequalities. This is now in practice done by *Simplify* and other functions in *Mathematica* using methods of cylindrical algebraic decomposition invented in the 1970s—which work roughly by finding a succession of points of change using *Resultant*. (Note that with n variables the number of steps needed can increase like 2^{2^n}.) (See the note about real analysis below.)

■ **Geometry.** Euclid gave axioms for basic geometry around 300 BC which were used with fairly little modification for more than 2000 years. In the 1830s, however, it was realized that the system would remain consistent even if the so-called parallel postulate was modified to allow space to be curved. Noting the vagueness of Euclid's original axioms there was then increasing interest in setting up more formal axiom systems for geometry. The best-known system was given by David Hilbert in 1899—and by describing geometrical figures using algebraic equations he showed that it was as consistent as the underlying axioms for numbers.

The axioms given here are illustrated below. They were developed by Alfred Tarski and others in the 1940s and 1950s. (Unlike Hilbert's axioms they require only first-order predicate logic.) The first six give basic properties of betweenness of points and congruence of line segments. The second- and third-to-last axioms specify that space has two dimensions; they can be modified for other dimensions. The last axiom is a schema that asserts the continuity of space. (The system is not finitely axiomatizable.)



The axioms given can prove most of the results in an elementary geometry textbook—indeed all results that are about geometrical figures such as triangles and circles specified by a fixed finite number of points, but which do not involve concepts like area. The axioms are complete and consistent—and thus not universal. They can however be made universal if axioms from set theory are added.

■ **Category theory.** Developed in the 1940s as a way to organize constructs in algebraic topology, category theory works at the level of whole mathematical objects rather than their elements. In the basic axioms given here the variables represent morphisms that correspond to mappings between objects. (Often morphisms are shown as arrows in diagrams, and objects as nodes.) The axioms specify that when morphisms are composed their domains and codomains must have appropriately matching types. Some of the methodology of category theory has become widely used in mathematics, but until recently the basic theory itself was not extensively studied—and its axiomatic status remains unclear. Category theory can be viewed as a formalization of operations on abstract data types in computer languages—though unlike in *Mathematica* it normally requires that functions take a single bundle of data as an argument.

■ **Set theory.** Basic notions of finite set theory have been used since antiquity—though became widespread only after their introduction into elementary mathematics education in the 1960s. Detailed ideas about infinite sets emerged in the 1880s through the work of Georg Cantor, who found it useful in studying trigonometric series to define sets of transfinite numbers of points. Several paradoxes associated with infinite sets were quickly noted—a 1901 example due to Bertrand Russell being to ask whether a set containing all sets that do not contain themselves in fact contains itself. To avoid such paradoxes Ernst Zermelo in 1908 suggested formalizing set theory using the first seven axioms given in the main text. (The axiom of infinity, for example, was included to establish that an infinite set such as the integers exists.) In 1922 Abraham Fraenkel noted that Zermelo's axioms did not support certain operations that seemed appropriate in a theory of sets, leading to the addition of Thoralf Skolem's axiom of replacement, and to what is usually called Zermelo-Fraenkel set theory (ZF). (The replacement axiom formally makes the subset axiom redundant.) The axiom of choice was first explicitly formulated by Zermelo in 1904 to capture the idea that in a set all elements can be ordered, so that the process of transfinite induction is possible (see page 1160). The non-constructive character of the axiom of choice has made it always remain somewhat controversial. It has arisen in many different guises and been useful in proving theorems in many areas of mathematics, but it has seemingly peculiar consequences such as the Banach-Tarski result that a solid sphere can be divided into six pieces (each a non-measurable set) that can be reassembled into a solid sphere twice the size. (The nine axioms with the axiom of choice are usually known as ZFC.) The axiom of regularity (or axiom of foundation) formulated by John von Neumann in 1929 explicitly forbids sets which for example can be elements of themselves. But while this axiom is convenient in simplifying work in set theory it has not been found generally useful in mathematics, and is normally considered optional at best.

A few additional axioms have also arisen as potentially useful. Most notable is the Continuum Hypothesis discussed on page 1127, which was proved independent of ZFC by Paul Cohen in 1963. (See also page 1166.)

Note that by using more complicated axioms the only construct beyond predicate logic needed to formulate set theory is $\in$. As discussed on page 1176, however, one cannot avoid axiom schemas in the formulation of set theory given here. (The von Neumann-Bernays-Gödel formulation does avoid these, but at the cost of introducing additional objects more general than sets.)

(See also page 1160.)

▪ **General topology.** The axioms given define properties of open sets of points in spaces—and in effect allow issues like connectivity and continuity to be discussed in terms of set theory without introducing any explicit distance function.

▪ **Real analysis.** The axiom given is Dedekind's axiom of continuity, which expresses the connectedness of the set of real numbers. Together with set theory it allows standard results about calculus to be derived. But as well as ordinary real numbers, these axioms allow non-standard analysis with constructs such as explicit infinitesimals (see page 1172).

▪ **Axiom systems for programs.** (See pages 794 and 1168.)

▪ **Page 775 · Implementation.** Given the axioms in the form

```
s[1] = (a_ ⊼ a_) ⊼ (a_ ⊼ b_) → a;
s[2, x_] := b_ → (b ⊼ b) ⊼ (b ⊼ x); s[3] =
  a_ ⊼ (a_ ⊼ b_) → a ⊼ (b ⊼ b); s[4] = a_ ⊼ (b_ ⊼ b_) → a ⊼ (a ⊼ b);
s[5] = a_ ⊼ (a_ ⊼ (b_ ⊼ c_)) → b ⊼ (b ⊼ (a ⊼ c));
```

the proof shown here can be represented by

```
{{s[2, b], {2}}, {s[4], {}}, {s[2, (b ⊼ b) ⊼ ((a ⊼ a) ⊼ (b ⊼ b))],
    {2, 2}}, {s[1], {2, 2, 1}}, {s[2, b ⊼ b], {2, 2, 2, 2, 2}},
  {s[5], {2, 2, 2}}, {s[2, b ⊼ b], {2, 2, 2, 2, 1}},
  {s[1], {2, 2, 2, 2, 2}}, {s[3], {2, 2, 2}},
  {s[1], {2, 2, 2, 2}}, {s[4], {2, 2, 2}}, {s[5], {}},
  {s[2, a], {2, 2, 1}}, {s[1], {2, 2}}, {s[3], {}}, {s[1], {2}}}
```

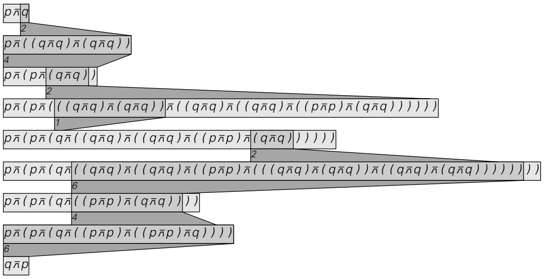
and applied using

```
FoldList[Function[{u, v},
    MapAt[Replace[#, v[[1]]] &, u, {v[[2]]}]], a ⊼ b, proof]
```

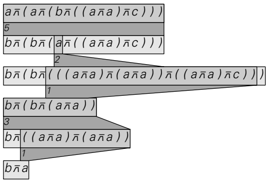
▪ **Page 776 · Proof structures.** The proof shown is in a sense based on very low-level steps, each consisting of applying a single axiom from the original axiom system. But in practical mathematics it is usual for proofs to be built up in a more hierarchical fashion using intermediate results or lemmas. In the way I set things up lemmas can in effect be introduced as new axioms which can be applied repeatedly during a proof. And in the case shown here if one first proves the lemma

$$(a \barwedge (a \barwedge (b \barwedge ((a \barwedge a) \barwedge c)))) = (b \barwedge a)$$

and treats it as rule 6, then the main proof can be shortened:



p⊼q
2
p⊼((q⊼q)⊼(q⊼q))
4
p⊼(p⊼(q⊼q))
2
p⊼(p⊼(((q⊼q)⊼(q⊼q))⊼((q⊼q)⊼((q⊼q)⊼((p⊼p)⊼(q⊼q))))))
1
p⊼(p⊼(q⊼((q⊼q)⊼((q⊼q)⊼((p⊼p)⊼(q⊼q))))))
2
p⊼(p⊼(q⊼((q⊼q)⊼((q⊼q)⊼((p⊼p)⊼(((q⊼q)⊼(q⊼q))⊼((q⊼q)⊼(q⊼q))))))))
6
p⊼(p⊼(q⊼((p⊼p)⊼(q⊼q))))
4
p⊼(p⊼(q⊼((p⊼p)⊼((p⊼p)⊼q))))
6
q⊼p



When one just applies axioms from the original axiom system one is in effect following a single line of steps. But when one proves a lemma one is in effect on a separate branch, which only merges with the main proof when one uses the lemma. And if one has nested lemmas one can end up with a proof that is in effect like a tree. (Repeated use of a single lemma can also lead to cycles.) Allowing lemmas can in extreme cases probably make proofs as much as exponentially shorter. (Note that lemmas can also be used in multiway systems.)

In the way I have set things up one always gets from one step in a proof to the next by taking an expression and applying some transformation rule to it. But while this is familiar from algebraic mathematics and from the operation of *Mathematica* it is not the model of proofs that has traditionally been used in mainstream mathematical logic. For there one tends to think not so much about transforming expressions as about taking collections of true statements (such as equations $u == v$), and using so-called rules of inference to deduce other ones. Most often there are two basic rules of inference: *modus ponens* or detachment which uses the logic result $(x \wedge x \Rightarrow y) \Rightarrow y$ to deduce the statement y from statements x and $x \Rightarrow y$, and substitution, which takes statements x and y and deduces $x /. p \to y$, where p is a logical variable in x (see page 1151). And with this approach axioms enter merely as initial true statements, leaving rules of inference to generate successive steps in proofs. And instead of being mainly linear sequences of results, proofs instead become networks in which pairs of results are always combined when *modus ponens* is used. But it is still always in principle possible to convert any proof to a purely sequential one—though perhaps at the cost of having exponentially many more steps.

▪ **Substitution strategies.** With the setup I am using each step in a proof involves transforming an expression like $u = v$ using an expression like $s = t$. And for this to happen s or t must match some part w of u or v. The simplest way this can be achieved is for s or t to reproduce w when its variables are replaced by appropriate expressions. But in general one can make replacements not only for variables in s and t, but also for ones in w. And in practice this often makes many more matches possible. Thus for example the axiom $a \circ a = a$ cannot be applied directly to $(p \circ q) \circ (p \circ r) = q \circ r$. But after the replacement $r \to q$, $a \circ a$ matches $(p \circ q) \circ (p \circ r)$ with $a \to p \circ q$, yielding the new theorem $p \circ q = q \circ q$. These kinds of substitutions are used in the proof on page 810. One approach to finding them is so-called paramodulation, which was introduced around 1970 in the context of automated theorem-proving systems, and has been used in many such systems (see page 1157). (Such substitutions are not directly relevant to *Mathematica*, since it transforms expressions rather than theorems or equations. But when I built SMP in 1981, its semantic pattern matching mechanism did use essentially such substitutions.)

▪ **One-way transformations.** As formulated in the main text, axioms define two-way transformations. One can also set up axiom systems based on one-way transformations (as in multiway systems). For basic logic, examples of this were studied in the mid-1900s, and with the transformations thought of as rules of inference they were sometimes known as "axiomless formulations".

▪ **Axiom schemas.** An axiom like $a + 0 == a$ is a single well-formed formula in the sense of page 1150. But sometimes one needs infinite collections of such individual axioms, and in the main text these are represented by axiom schemas given as *Mathematica* patterns involving objects like $x\_$. Such schemas are taken to stand for all individual axioms that match the patterns and are well-formed formulas. The induction axiom in arithmetic is an example of a schema. (See the note on finite axiomatizability on page 1176.) Note that as mentioned on page 1150 all the axioms given for basic logic should really be thought of as schemas.

▪ **Reducing axiom details.** Traditional axiom systems have many details not seen in the basic structure of multiway systems. But in most cases these details can be avoided—and in the end the universality of multiway systems implies that they can always be made to emulate any axiom system.

Traditional axiom systems tend to be based on operator systems (see page 801) involving general expressions, not just strings. But any expression can always be written as a string using something like *Mathematica* FullForm. (See also page 1169.) Traditional axiom systems also involve symbolic variables, not just literal string elements. But by using methods like those for combinators on page 1121 explicit mention of variables can always be eliminated.

▪ **Proofs in practice.** At some level the purpose of a proof is to establish that something is true. But in the practice of modern mathematics proofs have taken on a broader role; indeed they have become the primary framework for the vast majority of mathematical thinking and discourse. And from this perspective the kinds of proofs given on pages 810 and 811—or typically generated by automated theorem proving—are quite unsatisfactory. For while they make it easy at a formal level to check that certain statements are true, they do little at a more conceptual level to illuminate why this might be so. And indeed the kinds of proofs normally considered most mathematically valuable are ones that get built up in terms of concepts and constructs that are somehow expected to be as generally applicable as possible. But such proofs are inevitably difficult to study in a uniform and systematic way (though see page 1176). And as I argue in the main text, it is in fact only for the rather limited kinds of mathematics that have historically been pursued that such proofs can be expected to be sufficient. For in general proofs can be arbitrarily long, and can be quite devoid of what might be considered meaningful structure.

Among practical proofs that show signs of this (and whose mathematical value is thus often considered controversial) most have been done with aid of computers. Examples include the Four-Color Theorem (coloring of maps), the optimality of the Kepler packing (see page 986), the completeness of the Robbins axiom system (see page 1151) and the universality of rule 110 (see page 678).

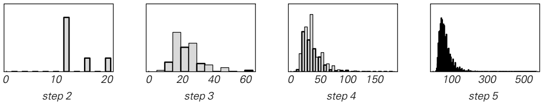
In the past it was sometimes claimed that using computers is somehow fundamentally incompatible with developing mathematical understanding. But particularly as the use of *Mathematica* has become more widespread there has been increasing recognition that computers can provide crucial raw material for mathematical intuition—a point made rather forcefully by the discoveries in this book. Less well recognized is the fact that formulating mathematical ideas in a *Mathematica* program is at least as effective a way to produce clarity of thinking and understanding as formulating a traditional proof.

▪ **Page 778 · Properties.** The second rule shown has the property that black elements always appear before white, so that strings can be specified just by the number of elements of each color that they contain—making the rule one of the sorted type discussed on page 937, based on the difference vector *{{2, -1}, {-1, 3}, {-4, -1}}*. The question of whether a given string can be generated is then analogous to finding whether there is a solution with certain positivity properties to a set of linear Diophantine equations.

■ **Page 781 · NAND tautologies.** At each step every possible transformation rule in the axioms is applied wherever it can. New expressions are also created by replacing each possible variable with $x \barwedge y$, where x and y are new variables, and by setting every possible pair of variables equal in turn. The longest tautology at step t is

$$Nest[(\# \barwedge \#) \barwedge (\# \barwedge p_t)\ \&,\ p \barwedge (p \barwedge p),\ t-1]$$

whose *LeafCount* grows like 3^t. The distribution of sizes of statements generated at each step is shown below.



Even with the same underlying axioms the tautologies are generated in a somewhat different order if one uses a different strategy—say one based on paramodulation (see page 1156). Pages 818 and 1175 discuss the sequence of all NAND theorems listed in order of increasing complexity.

■ **Proof searching.** To find a proof of some statement $p = q$ in a multiway system one can always in principle just start from p, evolve the system until it first generates q, then pick out the sequence of strings on the path from p to q. But doing this will usually involve building up a vast network of strings. And although at some level computational irreducibility and NP completeness (see page 766) imply that in general only a limited amount of this computational work can be saved, there are in practice quite often important optimizations that can be made. For finding a proof of $p = q$ is like searching for a path satisfying the constraint of going from p to q. And just like in the systems based on constraints in Chapter 5 one can usually do at least somewhat better than just to look at every possible path in turn.

For a start, in generating the network of paths one only ever need keep a single path that leads to any particular string; just like in many of my pictures of multiway systems one can in effect always drop any duplicate strings that occur. One might at first imagine that if p and q are both short strings then one could also drop any very long strings that are produced. But as we have seen, it is perfectly possible for long intermediate strings to be needed to get from p to q. Still, it is often reasonable to weight things so that at least at first one looks at paths that involve only shorter strings.

In the most direct approach, one takes a string and at each step just applies the underlying rules or axioms of the multiway system. But as soon as one knows that there is a path from a string u to a string v, one can also imagine applying the rule $u \to v$ to any string—in effect like a lemma. And one can choose which lemmas to try first by looking for example at which involve the shortest or commonest strings.

It is often important to minimize the number of lemmas one has to keep. Sometimes one can do this by reducing every lemma—and possibly every string—to some at least partially canonical form. One can also use the fact that in a multiway system if $u \to v$ and $r \to s$ then $u <> r \to v <> s$.

If one wants to get from p to q the most efficient thing is to use properties of q to avoid taking wrong turns. But except in systems with rather simple structure this is usually difficult to achieve. Nevertheless, one can for example always in effect work forwards from p, and backwards from q, seeing whether there is any overlap in the sets of strings one gets.

■ **Automated theorem proving.** Since the 1950s a fair amount of work has been done on trying to set up computer systems that can prove theorems automatically. But unlike systems such as *Mathematica* that emphasize explicit computation none of these efforts have ever achieved widespread success in mathematics. And indeed given my ideas in this section this now seems not particularly surprising.

The first attempt at a general system for automated theorem proving was the 1956 Logic Theory Machine of Allen Newell and Herbert Simon—a program which tried to find proofs in basic logic by applying chains of possible axioms. But while the system was successful with a few simple theorems the searches it had to do rapidly became far too slow. And as the field of artificial intelligence developed over the next few years it became widely believed that what would be needed was a general system for imitating heuristics used in human thinking. Some work was nevertheless still done on applying results in mathematical logic to speed up the search process. And in 1963 Alan Robinson suggested the idea of resolution theorem proving, in which one constructs $\neg\, theorem \vee axioms$, then typically writes this in conjunctive normal form and repeatedly applies rules like $(\neg p \vee q) \wedge (p \vee q) \to q$ to try to reduce it to *False*, thereby proving given *axioms* that *theorem* is *True*. But after early enthusiasm it became clear that this approach could not be expected to make theorem proving easy—a point emphasized by the discovery of NP completeness in the early 1970s. Nevertheless, the approach was used with some success, particularly in proving that various mechanical and other engineering systems would behave as intended—although by the mid-1980s such verification was more often done by systematic Boolean function methods (see page 1097). In the 1970s simple versions of the resolution method were incorporated into logic programming languages such as Prolog, but little in the way of mathematical theorem proving was done with them. A notable system under development since the 1970s is the Boyer-Moore theorem prover Nqthm, which uses resolution together with methods related to induction to try to find proofs of statements in a version of LISP. Another family of systems under development at Argonne National Laboratory since the 1960s are intended to find proofs in pure operator (equational) systems (predicate logic with equations). Typical of this effort was the Otter system started in the mid-1980s, which uses the resolution method, together with a variety of ad hoc strategies that are mostly versions of the general ones for multiway systems in the previous note. The development of so-called unfailing completion algorithms (see page 1037) in the late 1980s made possible much more systematic automated theorem provers for pure operator systems—with a notable example being the Waldmeister system developed around 1996 by Arnim Buch and Thomas Hillenbrand.

Ever since the 1970s I at various times investigated using automated theorem-proving systems. But it always seemed that extensive human input—typically from the creators of the system—was needed to make such systems actually find non-trivial proofs. In the late 1990s, however, I decided to try the latest systems and was surprised to find that some of them could routinely produce proofs hundreds of steps long with little or no guidance. Almost any proof that was easy to do by hand almost always seemed to come out automatically in just a few steps. And the overall ability to do proofs—at least in pure operator systems—seemed vastly to exceed that of any human. But as page 810 illustrates, long proofs produced in this way tend to be difficult to read—in large part because they lack the higher-level constructs that are typical in proofs created by humans. As I discuss on page 821, such lack of structure is in some respects inevitable. But at least for specific kinds of theorems in specific areas of mathematics it seems likely that more accessible proofs can be created if each step is allowed to involve sophisticated computations, say as done by *Mathematica*.

▪ **Proofs in *Mathematica*.** Most of the individual built-in functions of *Mathematica* I designed to be as predictable as possible—applying transformations in definite ways and using algorithms that are never of fundamentally unknown difficulty. But as their names suggest *Simplify* and *FullSimplify* were intended to be less predictable—and just to do what they can and then return a result. And in many cases these functions end up trying to prove theorems; so for example *FullSimplify[(a + b)/2 ≥ Sqrt[a b], a > 0 && b > 0]* must in effect prove a theorem to get the result *True*.

▪ **Page 781 · Truth and falsity.** The notion that statements can always be classified as either true or false has been a common idealization in logic since antiquity. But in everyday language, computer languages and mathematics there are many ways in which this idealization can fail. An example is $x + y == z$, which cannot reasonably be considered either true or false unless one knows what x, y and z are. Predicate logic avoids this particular kind of case by implicitly assuming that what is meant is a general statement about all values of any variable—and avoids cases like the expression $x + y$ by requiring all statements to be well-formed formulas (see page 1150). In *Mathematica* functions like *TrueQ* and *IntegerQ* are set up always to yield *True* or *False*—but just by looking at the explicit structure of a symbolic expression.

Note that although the notion of negation seems fairly straightforward in everyday language it can be difficult to implement in computational or mathematical settings. And thus for example even though it may be possible to establish by a finite computation that a particular system halts, it will often be impossible to do the same for the negation of this statement. The same basic issue arises in the intuitionistic approach to mathematics, in which one assumes that any object one handles must be found by a finite construction. And in such cases one can set up an analog of logic in which one no longer takes $\neg\neg a = a$.

It is also possible to assume a specific number $k > 2$ of truth values, as on page 1175, or to use so-called modal logics.

(See also page 1167.)

▪ **Page 782 · Gödel's Theorem.** What is normally known as "Gödel's Theorem" (or "Gödel's First Incompleteness Theorem") is the centerpiece of the paper "On Undecidable Propositions of *Principia Mathematica* and Related Systems" published by Kurt Gödel in 1931. What the theorem shows is that there are statements that can be formulated within the standard axiom system for arithmetic but which cannot be proved true or false within that system. Gödel's paper does this first for the statement "this statement is unprovable", and much of the paper is concerned with showing how such a statement can be encoded within arithmetic. Gödel in effect does this by first converting the statement to one about recursive functions and then—by using tricks of number theory such as the beta function of page 1120—to one purely about arithmetic. (Gödel's main achievement is sometimes characterized as the "arithmetization of metamathematics": the discovery that concepts such as provability related to the processes of mathematics can be represented purely as statements in arithmetic.) (See page 784.)

Gödel originally based his theorem on Peano arithmetic (as discussed in the context of *Principia Mathematica*), but expected that it would in fact apply to any reasonable formal system for mathematics—and in later years considered that this had been established by thinking about Turing machines. He suggested that his results could be avoided if some form of transfinite hierarchy of formalisms could be used, and appears to have thought that at some level humans and mathematics do this (compare page 1167).

Gödel's 1931 paper came as a great surprise, although the issues it addressed were already widely discussed in the field of mathematical logic. And while the paper is at a technical level rather clear, it has never been easy for typical mathematicians to read. Beginning in the late 1950s its results began to be widely known outside of mathematics, and by the late 1970s Gödel's Theorem and various misstatements of it were often assigned an almost mystical significance. Self-reference was commonly seen as its central feature, and connections with universality and computation were usually missed. And with the belief that humans must somehow have intrinsic access to all truths in mathematics, Gödel's Theorem has been used to argue for example that computers can fundamentally never emulate human thinking.

The picture on page 786 can be viewed as a modern proof of Gödel's Theorem based on Diophantine equations.

In addition to what is usually called Gödel's Theorem, Kurt Gödel established a second incompleteness theorem: that the statement that the axioms of arithmetic are consistent cannot be proved by using those axioms (see page 1168). He also established what is often called the Completeness Theorem for predicate logic (see page 1152)—though here "completeness" is used in a different sense.

■ **Page 783 · Properties.** The first multiway system here generates all strings that end in □■; the third all strings that end in ■. The second system generates all strings where the second-to-last element is white, or the string ends with a run of black elements delimited by white ones.

■ **Page 783 · Essential incompleteness.** If a consistent axiom system is complete this means that any statement in the system can be proved true or false using its axioms, and the question of whether a statement is true can always be decided by a finite procedure. If an axiom system is incomplete then this means that there are statements that cannot be proved true or false using its axioms—and which must therefore be considered independent of those axioms. But even given this it is still possible that a finite procedure can exist which decides whether a given statement is true, and indeed this happens in the theory of commutative groups (see note below). But often an axiom system will not only be incomplete, but will also be what is called essentially incomplete. And what this means is that there is no finite set of axioms that can consistently be added to make the system complete. A consequence of this is that there can be no finite procedure that always decides whether a given statement is true—making the system what is known as essentially undecidable. (When I use the term "undecidable" I normally mean "essentially undecidable". Early work on mathematical logic sometimes referred to statements that are independent as being undecidable.)

One might think that adding rules to a system could never reduce its computational sophistication. And this is correct if with suitable input one can always avoid the new rules. But often these rules will allow transformations that in effect short-circuit any sophisticated computation. And in the context of axiom systems, adding axioms can be thought of as putting more constraints on a system—thus potentially in effect forcing it to be simpler. The result of all this is that an axiom system that is universal can stop being universal when more axioms are added to it. And indeed this happens when one goes from ordinary group theory to commutative group theory, and from general field theory to real algebra.

■ **Page 784 · Predicate logic.** The universality of predicate logic with a single two-argument function follows immediately from the result on page 1156 that it can be used to emulate any two-way multiway system.

■ **Page 784 · Algebraic axioms.** How universality works with algebraic axioms depends on how those axioms are being used (compare page 1153). What is said in the main text here assumes that they are being used as on page 773—with each variable in effect standing for any object (compare page 1169), and with the axioms being added to predicate logic. The first of these points means that one is concerned with so-called pure group theory—and with finding results valid for all possible groups. The second means that the statements one considers need not just be of the form $\ldots == \ldots$, but can explicitly involve logic; an example is Cayley's theorem

$$a \cdot x == a \cdot y \Rightarrow (x == y \wedge \exists_z\, a \cdot z == x) \wedge \\ a \cdot x == b \cdot x \Rightarrow a == b \wedge (a \cdot b) \cdot x == a \cdot (b \cdot x)$$

With this setup, Alfred Tarski showed in 1946 that any statement in Peano arithmetic can be encoded as a statement in group theory—thus demonstrating that group theory is universal, and that questions about it can be undecidable. This then also immediately follows for semigroup theory and monoid theory. It was shown for ring theory and field theory by Julia Robinson in 1949. But for commutative group theory it is not the case, as shown by Wanda Szmielew around 1950. And indeed there is a procedure based on quantifier elimination for determining in a finite number of steps whether any statement in commutative group theory can be proved. (Commutative group theory is thus a decidable theory. But as mentioned in the note above, it is not complete—since for example it cannot establish the theorem $a == b$ which states that a group has just one element. It is nevertheless not essentially incomplete—and for example adding the axiom $a == b$ makes it complete.) Real algebra is also not universal (see page 1153), and the same is for example true for finite fields—but not for arbitrary fields.

As discussed on page 1141, word problems for systems such as groups are undecidable. But to set up a word problem in general formally requires going beyond predicate logic, and including axioms from set theory. For a word problem relates not, say, to groups in general, but to a particular group, specified by relations between generators. Within predicate logic one can give the relations as statements, but in effect one cannot specify that no other relations hold. It turns out, however, that undecidability for word problems occurs in essentially the same places as universality for axioms with predicate logic. Thus, for example, the word problem is undecidable for groups and semigroups, but is decidable for commutative groups.

One can also consider using algebraic axioms without predicate logic—as in basic logic or in the operator systems of page 801. And one can now ask whether there is then universality. In the case of semigroup theory there is not. But certainly systems of this type can be universal—since for example they can be set up to emulate any multiway system. And it seems likely that the axioms of ordinary group theory are sufficient to achieve universality.

■ **Page 784 · Set theory.** Any integer n can be encoded as a set using for example *Nest[Union[#, {#}] &, { }, n]*. And from this a statement s in Peano arithmetic (with each variable explicitly quantified) can be translated to a statement in set theory by using

$$\text{Replace}[s, \{\forall_{a\_}\ b\_ \to \forall_a\ (a \in \mathbb{N} \Rightarrow b),$$
$$\exists_{a\_}\ b\_ \to \exists_a\ (a \in \mathbb{N} \wedge b)\}, \{0, \infty\}]$$

and then adding the statements below to provide definitions ($\mathbb{N}$ is the set of non-negative integers, $\langle x, y, z\rangle$ is an ordered triple, and $\Updownarrow a$ determines whether each triple in a set a is of the form $\langle x, y, f[x, y]\rangle$ specifying a single-valued function).

$a = \mathbb{N} \Leftrightarrow \forall_b\ ((\emptyset \in b \wedge \forall_c\ (c \in b \Rightarrow \cup\{c, \{c\}\} \in b)) \Rightarrow a \subseteq b)$
$a = \Delta\, b \Leftrightarrow a = \cup\{b, \{b\}\}$
$a = \langle b, c, d\rangle \Leftrightarrow a = \{\{\{\{b, c\}, \{c\}\}, d\}, \{d\}\}$
$\Updownarrow a \Leftrightarrow (\forall_b\ \forall_c\ \forall_d\ (\langle b, c, d\rangle \in a \Rightarrow \forall_e\ (\langle b, c, e\rangle \in a \Rightarrow d = e)) \wedge$ $\forall_b\ \forall_c\ ((b \in \mathbb{N} \wedge c \in \mathbb{N}) \Rightarrow \exists_d\ (d \in \mathbb{N} \wedge \langle b, c, d\rangle \in a)))$
$a = b + c \Leftrightarrow \forall_d\ ((\Updownarrow d \wedge \forall_e\ \forall_f\ \forall_g\ (\langle \Delta\, e, f, g\rangle \in d \Rightarrow \langle e, f, \Delta\, g\rangle \in d) \wedge$ $\forall_f\ \forall_g\ (\langle \emptyset, f, g\rangle \in d \Rightarrow g = f)) \Rightarrow \langle b, c, a\rangle \in d)$
$a = b \times c \Leftrightarrow \forall_d\ ((\Updownarrow d \wedge \forall_e\ \forall_f\ \forall_g\ (\langle \Delta\, e, f, g\rangle \in d \Rightarrow \langle e, f, f + g\rangle \in d) \wedge$ $\forall_f\ \forall_g\ (\langle \emptyset, f, g\rangle \in d \Rightarrow g = \emptyset)) \Rightarrow \langle b, c, a\rangle \in d)$

This means that set theory can be used to prove any statement that can be proved in Peano arithmetic. But it can also prove other statements—such as Goodstein's result (see note below), and the consistency of arithmetic (see page 1168). An important reason for this is that set theory allows not just ordinary induction over sequences of integers but also transfinite induction over arbitrary ordered sets (see below).

■ **Page 786 · Universal Diophantine equation.** The equation is built up from ones whose solutions are set up to be integers that satisfy particular relations. So for example the equation $a^2 + b^2 == 0$ has solutions that are exactly those integers that satisfy the relation $a == 0 \wedge b == 0$. Similarly, assuming as in the rest of this note that all variables are non-negative, $b == a + c + 1$ has solutions that are exactly those integers that satisfy $a < b$, with c having some allowed value. From various number-theoretical results many relations can readily be encoded as integer equations:

$$(a == 0 \vee b == 0) \longleftrightarrow a\, b == 0$$

$$(a == 0 \wedge b == 0) \longleftrightarrow a + b == 0$$

$$a < b \longleftrightarrow b == a + c + 1$$

$$a == \text{Mod}[b, c] \longleftrightarrow (b == a + c\, d \wedge a < c)$$

$$a == \text{Quotient}[b, c] \longleftrightarrow (b == a\, c + d \wedge d < c)$$

$$a == \text{Binomial}[b, c] \longleftrightarrow \text{With}[\{n = 2^b + 1\},$$
$$(n + 1)^b == n^c\, (a + d\, n) + e \wedge e < n^c \wedge a < n]$$

$$a == b\,! \longleftrightarrow a == \text{Quotient}[c^b, \text{Binomial}[c, b]]$$

$$a == \text{GCD}[b, c] \longleftrightarrow (b\, c > 0 \wedge a\, d == b \wedge a\, e == c \wedge a + c\, f == b\, g)$$

$$a == \text{Floor}[b/c] \longleftrightarrow (a\, c + d == b \wedge d < c)$$

$$\text{PrimeQ}[a] \longleftrightarrow (\text{GCD}[(a - 1)\,!, a] == 1 \wedge a > 1)$$

$$a == \text{BitAnd}[c, d] \wedge b == \text{BitOr}[c, d] \longleftrightarrow$$
$$(\sigma[c, a] \wedge \sigma[d, a] \wedge \sigma[b, c] \wedge \sigma[b, d] \wedge a + b == c + d)\ /.$$
$$\sigma[x\_, y\_] \to \text{Mod}[\text{Binomial}[x, y], 2] == 1$$

where the last encoding uses the result on page 608. (Note that any variable a can be forced to be non-negative by including an equation $a == w^2 + x^2 + y^2 + z^2$, as on page 910.)

Given an integer a for which *IntegerDigits[a, 2]* gives the cell values for a cellular automaton, a single step of evolution according say to rule 30 is given by

$$\text{BitXor}[a, 2\, \text{BitOr}[a, 2\, a]]$$

where (see page 871)

```
BitXor[x, y] == BitOr[x, y] - BitAnd[x, y]
```

and a is assumed to be padded with 0's at each end. The corresponding form for rule 110 is

```
BitXor[BitAnd[a, 2 a, 4 a], BitOr[2 a, 4 a]]
```

The final equation is then obtained from

$$
\begin{aligned}
&\{1 + x_4 + x_{12} == 2^{(1+x_3)(x_1+2 x_3)},\ x_2 + x_{13} == 2^{x_1},\\
&\quad 1 + x_5 + x_{14} == 2^{x_1},\ 2^{x_3} x_5 + 2^{x_1+2 x_3} x_6 + 2^{x_1+x_3} x_{15} + x_{16} == x_4,\\
&\quad 1 + x_{15} + x_{17} == 2^{x_3},\ 1 + x_{16} + x_{18} == 2^{x_3},\\
&\quad 2^{1+x_3 (1+x_1+2 x_3)} (-1 + x_2) - x_{10} + x_{11} == 2 x_4,\\
&\quad x_7 == \mathit{BitAnd}[x_6, 2 x_6] \land x_8 == \mathit{BitOr}[x_6, 2 x_6],\\
&\quad x_9 == \mathit{BitAnd}[x_6, 2 x_7] \land x_{19} == \mathit{BitOr}[x_6, 2 x_7],\\
&\quad x_{10} == \mathit{BitAnd}[x_9, 2 x_8] \land x_{11} == \mathit{BitOr}[x_9, 2 x_8]\}
\end{aligned}
$$

where x_1 through x_4 have the meanings indicated in the main text, and satisfy $x_i \geq 0$. Non-overlapping subsidiary variables are introduced for *BitOr* and *BitAnd*, yielding a total of 79 variables.

Note that it is potentially somewhat easier to construct Diophantine equations to emulate register machines—or arithmetic systems from page 673—than to emulate cellular automata, but exactly the same basic methods can be used.

In the universal equation in the main text variables appear in exponents. One can reduce such an exponential equation to a pure polynomial equation by encoding powers using integer equations. The simplest known way of doing this (see note below) involves a degree 8 equation with 60 variables:

$$
\begin{aligned}
&a == b^c \longleftrightarrow \alpha[d, 4 + b e, 1 + z] \land \alpha[f, e, 1 + z] \land\\
&\quad a == \mathit{Quotient}[d, f] \land \alpha[g, 4 + b, 1 + z] \land e == 16 g (1 + z)\\
&\lambda[a\_, b\_, c\_] := \mathit{Module}[\{x\},\\
&\qquad 2 a + x_1 == c \land (\mathit{Mod}[b - a, c] == 0 \lor \mathit{Mod}[b + a, c] == 0)]\\
&\alpha[a\_, b\_, c\_] := \mathit{Module}[\{x\}, {x_1}^2 - b x_1 x_2 + {x_2}^2 == 1 \land\\
&\quad {x_3}^2 - b x_3 x_4 + {x_4}^2 == 1 \land 1 + x_4 + x_5 == x_3 \land \mathit{Mod}[x_3, {x_1}^2] ==\\
&\quad 0 \land 2 x_4 + x_7 == b x_3 \land \mathit{Mod}[-b + x_8, x_7] == 0 \land\\
&\quad \mathit{Mod}[-2 + x_8, x_1] == 0 \land x_8 - x_{11} == 3 \land {x_{12}}^2 - x_8 x_{12} x_{13} +\\
&\quad {x_{13}}^2 == 1 \land 1 + 2 a + x_{14} == x_1 \land \lambda[a, x_{12}, x_7] \land \lambda[c, x_{12}, x_1]]
\end{aligned}
$$

(This roughly uses the idea that solutions to Pell equations grow exponentially, so that for example $x^2 == 2 y^2 + 1$ has solutions $\mathit{With}[\{u = 3 + 2 \sqrt{2}\}, (u^n + u^{-n})/2]$.) From this representation of *Power* the universal equation can be converted to a purely polynomial equation with 2154 variables—which when expanded has 1683150 terms, total degree 16 (average per term 6.8), maximum coefficient 17827424 and *LeafCount* 16540206.

Note that the existence of universal Diophantine equations implies that any problem of mathematics—even, say, the Riemann Hypothesis—can in principle be formulated as a question about the existence of solutions to a Diophantine equation. It also means that given any specific enumeration of polynomials, there must be some universal polynomial u which if fed the enumeration number of a polynomial p, together with an encoding of the values of its variables, will yield the corresponding value of p as a solution to $u == 0$.

▪ **Hilbert's Tenth Problem.** Beginning in antiquity various procedures were developed for solving particular kinds of Diophantine equations (see page 1164). In 1900, as one of his list of 23 important mathematical problems, David Hilbert posed the problem of finding a single finite procedure that could systematically determine whether a solution exists to any specified Diophantine equation. The original proof of Gödel's Theorem from 1931 in effect involves showing that certain logical and other operations can be represented by Diophantine equations—and in the end Gödel's Theorem can be viewed as saying that certain statements about Diophantine equations are unprovable. The notion that there might be universal Diophantine equations for which Hilbert's Tenth Problem would be fundamentally unsolvable emerged in work by Martin Davis in 1953. And by 1961 Davis, Hilary Putnam and Julia Robinson had established that there are exponential Diophantine equations that are universal. Extending this to show that Hilbert's original problem about ordinary polynomial Diophantine equations is unsolvable required proving that exponentiation can be represented by a Diophantine equation, and this was finally done by Yuri Matiyasevich in 1969 (see note above).

By the mid-1970s, Matiyasevich had given a construction for a universal Diophantine equation with 9 variables—though with a degree of about 10^{45}. It had been known since the 1930s that any Diophantine equation can be reduced to one with degree 4—and in 1980 James Jones showed that a universal Diophantine equation with degree 4 could be constructed with 58 variables. In 1979 Matiyasevich also showed that universality could be achieved with an exponential Diophantine equation with many terms, but with only 3 variables. As discussed in the main text I believe that vastly simpler Diophantine equations can also be universal. It is even conceivable that a Diophantine equation with 2 variables could be universal: with one variable essentially being used to represent the program and input, and the other the execution history of the program—with no finite solution existing if the program does not halt.

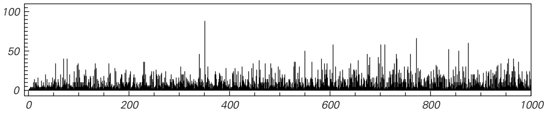
▪ **Polynomial value sets.** Closely related to issues of solving Diophantine equations is the question of what set of positive values a polynomial can achieve when fed all possible positive integer values for its variables. A polynomial with a single variable must always yield either be a finite set, or a simple polynomial progression of values. But already the sequence of values for $x^2 y - x y^3$ or even $x (y^2 + 1)$ seem quite complicated. And for example from the fact that $x^2 == y^2 + (x y \pm 1)$ has solutions $\mathit{Fibonacci}[n]$ it follows that the positive values of $(2-(x^2-y^2-x\,y)^2)\,x$ are just *Fibonacci[n]* (achieved when *{x, y}* is *Fibonacci[{n, n - 1}]*). This is the simplest polynomial giving *Fibonacci[n]*, and there are for example no polynomials with 2 variables, up to 4 terms, total degree less than 4, and integer coefficients between -2 and +2, that give any of 2^n, 3^n or *Prime[n]*. Nevertheless, from the representation for *PrimeQ* in the note above it has been shown that the positive values of a particular polynomial with 26 variables, 891 terms and total degree 97 are exactly the primes. (Polynomials with 42 variables and degree 5, and 10 variables and degree 10^{45}, are also known to work, while it is known that one with 2 variables cannot.) And in general the existence of a universal Diophantine equation implies that any set obtained by any finite computation must correspond to the positive values of some polynomial. The analog of doing a long computation to find a result is having to go to large values of variables to find a positive polynomial value. Note that one can imagine, say, emulating the evolution of a cellular automaton by having the t^{th} positive value of a polynomial represent the t^{th} step of evolution. That universality can be achieved just in the positive values of a polynomial is already remarkable. But I suspect that in the end it will take only a surprisingly simple polynomial, perhaps with just three variables and fairly low degree.

(See also page 1165.)

▪ **Statements in Peano arithmetic.** Examples include:

- $\sqrt{2}$ is irrational:
  $\neg\,\exists_a\,(\exists_b\,(b \neq 0 \land a \times a == (\triangle\triangle 0) \times (b \times b)))$
- There are infinitely many primes of the form $n^2 + 1$:
  $\neg\,\exists_n\,(\forall_c\,(\exists_a\,(\exists_b\,(n+c) \times (n+c) + \triangle 0 == (\triangle\triangle a) \times (\triangle\triangle b))))$
- Every even number (greater than 2) is the sum of two primes (Goldbach's Conjecture; see page 135):
  $\forall_a\,(\exists_b\,(\exists_c\,((\triangle\triangle 0) \times (\triangle\triangle a) == b + c \land \forall_d\,(\forall_e\,(\forall_f\,((f == (\triangle\triangle d) \times (\triangle\triangle e) \lor f == \triangle 0) \Rightarrow (f \neq b \land f \neq c)))))))$

The last two statements have never been proved true or false, and remain unsolved problems of number theory. The picture shows spacings between n for which $n^2 + 1$ is prime.



▪ **Transfinite numbers.** For most mathematical purposes it is quite adequate just to have a single notion of infinity, usually denoted ∞. But as Georg Cantor began to emphasize in the 1870s, it is possible to distinguish different levels of infinity. Most of the details of this have not been widely used in typical mathematics, but they can be helpful in studying foundational issues. Cantor's theory of ordinal numbers is based on the idea that every integer must have a successor. The next integer after all of the ordinary ones—the first infinite integer—is given the name ω. In Cantor's theory $\omega + 1$ is still larger (though $1 + \omega$ is not), as are $2\,\omega$, ω^2 and ω^ω. Any arithmetic expression involving ω specifies an ordinal number—and can be thought of as corresponding to a set containing all integers up to that number. The ordinary axioms of arithmetic do not apply, but there are still fairly straightforward rules for manipulating such expressions. In general there are many different expressions that correspond to a given number, though there is always a unique Cantor normal form—essentially a finite sequence of digits giving coefficients of descending powers of ω. However, not all infinite integers can be represented in this way. The first one that cannot is ϵ_0, given by the limit $\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$, or effectively *Nest[ω^# &, ω, ω]*. ϵ_0 is the smallest solution to $\omega^\epsilon == \epsilon$. Subsequent solutions ($\epsilon_1$, ..., ϵ_ω, ..., ϵ_{ϵ_0}, ...) define larger ordinals, and one can go on until one reaches the limit $\epsilon_{\epsilon_{\epsilon_{\cdot_{\cdot_{\cdot}}}}}$, which is the first solution to $\epsilon_\alpha == \alpha$. Giving this ordinal a name, one can then go on again, until eventually one reaches another limit. And it turns out that in general one in effect has to introduce an infinite sequence of names in order to be able to specify all transfinite integers. (Naming a single largest or "absolutely infinite" integer is never consistent, since one can always then talk about its successor.) As Cantor noted, however, even this only allows one to reach the lowest class of transfinite numbers—in effect those corresponding to sets whose size corresponds to the cardinal number $\aleph_0$. Yet as discussed on page 1127, one can also consider larger cardinal numbers, such as $\aleph_1$, considered in connection with the number of real numbers, and so on. And at least for a while the ordinary axioms of set theory can be used to study the sets that arise.

▪ **Growth rates.** One can characterize most functions by their ultimate rates of growth. In basic mathematics these might be n, $2\,n$, $3\,n$, ... or n^2, n^3, ..., or 2^n, 3^n, ..., or 2^n, 2^{2^n}, 2^{2^n}, ... To go further one begins by defining an analog to the Ackermann function of page 906:

```
f[1][n_] = 2 n; f[s_][n_] := Nest[f[s - 1], 1, n]
```

f[2][n] is then 2^n, *f[3]* is iterated power, and so on. Given this one can now form the "diagonal" function

```
f[ω][n_] := f[n][n]
```

and this has a higher growth rate than any of the *f[s][n]* with finite s. This higher growth rate is indicated by the transfinite index ω. And in direct analogy to the transfinite numbers discussed above one can then in principle form a hierarchy of functions using operations like

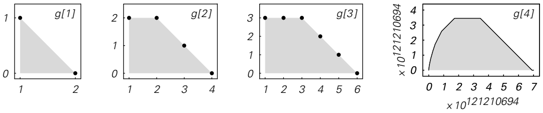
```
f[ω + s][n_] := Nest[f[ω + s - 1], 1, n]
```

together with diagonalization at limit ordinals. In practice, however, it gets more and more difficult to determine that the functions defined in this way actually in a sense halt and yield definite values—and indeed for $f[\epsilon_0]$ this can no longer be proved using the ordinary axioms of arithmetic (see below). Yet it is still possible to define functions with even more rapid rates of growth. An example is the so-called busy beaver function (see page 1144) that gives the maximum number of steps that it takes for any Turing machine of size *n* to halt when started from a blank tape. In general this function must grow faster than any computable function, and is not itself computable.

■ **Page 787 · Unprovable statements.** After the appearance of Gödel’s Theorem a variety of statements more or less directly related to provability were shown to be unprovable in Peano arithmetic and certain other axiom systems. Starting in the 1960s the so-called method of forcing allowed certain kinds of statements in strong axiom systems—like the Continuum Hypothesis in set theory (see page 1155)—to be shown to be unprovable. Then in 1977 Jeffrey Paris and Leo Harrington showed that a variant of Ramsey’s Theorem (see page 1068)—a statement that is much more directly mathematical—is also unprovable in Peano arithmetic. The approach they used was in essence based on thinking about growth rates—and since the 1970s almost all new examples of unprovability have been based on similar ideas. Probably the simplest is a statement shown to be unprovable in Peano arithmetic by Laurence Kirby and Jeff Paris in 1982: that certain sequences *g[n]* defined by Reuben Goodstein in 1944 are of limited length for all *n*, where

```
g[n_] := Map[First, NestWhileList[
        {f[#] - 1, Last[#] + 1} &, {n, 3}, First[#] > 0 &]]
f[{0, _}] = 0; f[{n_, k_}] := Apply[Plus, MapIndexed[#1
     k^f[{#2[[1]] - 1, k}] &, Reverse[IntegerDigits[n, k - 1]]]]
```

As in the pictures below, *g[1]* is *{1, 0}*, *g[2]* is *{2, 2, 1, 0}* and *g[3]* is *{3, 3, 3, 2, 1, 0}*. *g[4]* increases quadratically for a long time, with only element $3 \times 2^{402653211} - 2$ finally being 0. And the point is that in a sense *Length[g[n]]* grows too quickly for its finiteness to be provable in general in Peano arithmetic.



The argument for this as usually presented involves rather technical results from several fields. But the basic idea is roughly just to set up a correspondence between elements of *g[n]* and possible proofs in Peano arithmetic—then to use the fact that if one knew that *g[n]* always terminated this would establish the validity of all these proofs, which would in turn prove the consistency of arithmetic—a result which is known to be unprovable from within arithmetic.

Every possible proof in Peano arithmetic can in principle be encoded as an ordinary integer. But in the late 1930s Gerhard Gentzen showed that if proofs are instead encoded as ordinal numbers (see note above) then any proof can validly be reduced to a preceding one just by operations in logic. To cover all possible proofs, however, requires going up to the ordinal ϵ_0. And from the unprovability of consistency one can conclude that this must be impossible using the ordinary operation of induction in Peano arithmetic. (Set theory, however, allows transfinite induction—essentially induction on arbitrary sets—letting one reach such ordinals and thus prove the consistency of arithmetic.) In constructing *g[n]* the integer *n* is in effect treated like an ordinal number in Cantor normal form, and a sequence of numbers that should precede it are found. That this sequence terminates for all *n* is then provable in set theory, but not Peano arithmetic—and in effect *Length[g[n]]* must grow like $f[\epsilon_0][n]$.)

In general one can imagine characterizing the power of any axiom system by giving a transfinite number κ which specifies the first function $f[\kappa]$ (see note above) whose termination cannot be proved in that axiom system (or similarly how rapidly the first example of *y* must grow with *x* to prevent $\exists_y\, p[x, y]$ from being provable). But while it is known that in Peano arithmetic $\kappa = \epsilon_0$, quite how to describe the value of κ for, say, set theory remains unknown. And in general I suspect that there are a vast number of functions with simple definitions whose termination cannot be proved not just because they grow too quickly but instead for the more fundamental reason that their behavior is in a sense too complicated.

Whenever a general statement about a system like a Turing machine or a cellular automaton is undecidable, at least some instances of that statement encoded in an axiom system must be unprovable. But normally these tend to be complicated and not at all typical of what arise in ordinary mathematics. (See page 1167.)

■ **Encodings of arithmetic.** Statements in arithmetic are normally written in terms of $+$, $\times$ and Δ (and logical operations). But it turns out also to be possible to encode such statements in terms of other basic operations. This was for example done by Julia Robinson in 1949 with Δ (or $a + 1$) and *Mod[a, b] == 0*. And in the 1990s Ivan Korec and others showed that it could be done just with *Mod[Binomial[a + b, a], k]* with $k = 6$ or any product of primes—and that it could not be done with k a prime or prime power. These operations can be thought of as finding elements in nested Pascal's triangle patterns produced by k-color additive cellular automata. Korec showed that finding elements in the nested pattern produced by the $k = 3$ cellular automaton with rule *{{1, 1, 3}, {2, 2, 1}, {3, 3, 2}}[[#1, #2]] &* (compare page 886) was also enough.

■ **Page 788 · Infinity.** See page 1162.

■ **Page 789 · Diophantine equations.** If variables appear only linearly, then it is possible to use *ExtendedGCD* (see page 944) to find all solutions to any system of Diophantine equations—or to show that none exist. Particularly from the work of Carl Friedrich Gauss around 1800 there emerged a procedure to find solutions to any quadratic Diophantine equation in two variables—in effect by reduction to the Pell equation $x^2 == a\,y^2 + 1$ (see page 944), and then computing $\mathit{ContinuedFraction}[\sqrt{a}\,]$. The minimal solutions can be large; the largest ones for successive coefficient sizes are given below. (With size s coefficients it is for example known that the solutions must always be less than $(14\,s)^{5\,s}$.).

1	$1 + x + x^2 + y - x\,y == 0$	$\{x == 2, y == 7\}$
2	$1 + 2\,x + 2\,x^2 + 2\,y + x\,y - 2\,y^2 == 0$	$\{x == 687, y == 881\}$
3	$2 + 2\,x + 3\,x^2 + 3\,y + x\,y - y^2 == 0$	$\{x == 545759, y == 1256763\}$
4	$-4 - x + 4\,x^2 - y - 3\,x\,y - 4\,y^2 == 0$	$\{x == 251996202018, y == 174633485974\}$

There is a fairly complete theory of homogeneous quadratic Diophantine equations with three variables, and on the basis of results from the early and mid-1900s a finite procedure should in principle be able to handle quadratic Diophantine equations with any number of variables. (The same is not true of simultaneous quadratic Diophantine equations, and indeed with a vector x of just a few variables, a system $m\,.\,x^2 == a$ of such equations could quite possibly show undecidability.)

Ever since antiquity there have been an increasing number of scattered results about Diophantine equations involving higher powers. In 1909 Axel Thue showed that any equation of the form $p[x, y] == a$, where $p[x, y]$ is a homogeneous irreducible polynomial of degree at least 3 (such as $x^3 + x\,y^2 + y^3$) can have only a finite number of integer solutions. (He did this by formally factoring $p[x, y]$ into terms $x - \alpha_i\,y$, then looking at rational approximations to the algebraic numbers α_i.) In 1966 Alan Baker then proved an explicit upper bound on such solutions, thereby establishing that in principle they can be found by a finite search procedure. (The proof is based on having bounds for how close to zero $\mathit{Sum}[\alpha_i\,\mathit{Log}[\alpha_j], i, j]$ can be for independent algebraic numbers α_k.) His bound was roughly $\mathit{Exp}[(c\,s)^{10^5}]$—but later work in essence reduced this, and by the 1990s practical algorithms were being developed. (Even with a bound of 10^{100}, rational approximations to real number results can quickly give the candidates that need to be tested.)

Starting in the late 1800s and continuing ever since a series of progressively more sophisticated geometric and algebraic views of Diophantine equations have developed. These have led for example to the 1993 proof of Fermat's Last Theorem and to the 1983 Faltings theorem (Mordell conjecture) that the topology of the algebraic surface formed by allowing variables to take on complex values determines whether a Diophantine equation has only a finite number of rational solutions—and shows for example that this is the case for any equation of the form $x^n == a\,y^n + 1$ with $n > 3$. Extensive work has been done since the early 1900s on so-called elliptic curve equations such as $x^2 == a\,y^3 + b$ whose corresponding algebraic surface has a single hole (genus 1). (A crucial feature is that given any two rational solutions to such equations, a third can always be found by a simple geometrical construction.) By the 1990s explicit algorithms for such equations were being developed—with bounds on solutions being found by Baker's method (see above). In the late 1990s similar methods were applied to superelliptic (e.g. $x^n == p[y]$) and hyperelliptic (e.g. $x^2 == p[y]$) equations involving higher powers, and it now at least definitely seems possible to handle any two-variable cubic Diophantine equation with a finite procedure. Knowing whether Baker's method can be made to work for any particular class of equations involves, however, seeing whether certain rather elaborate algebraic constructions can be done—and this may perhaps in general be undecidable. Most likely there are already equations of degree 4 where Baker's method cannot be used—perhaps ones like $x^3 == y^4 + x\,y + a$. But in recent years there have begun to be results by other methods about two-variable Diophantine equations, giving, for example, general upper bounds on the number of possible solutions. And although this has now led to the assumption that all two-variable Diophantine equations will eventually be resolved, based on the results of this book I would not be surprised if in fact undecidability and universality appeared in such equations—even perhaps at degree 4 with fairly small coefficients.

The vast majority of work on Diophantine equations has been for the case of two variables (or three for some homogeneous equations). No clear analog of Baker's method is known beyond two variables, and my suspicion is that with three variables undecidability and universality may already be present even in cubic equations.

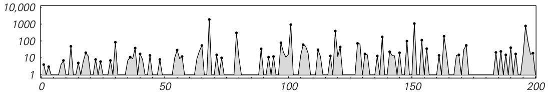
As mentioned in the main text, proving that even simple specific Diophantine equations have no solutions can be very difficult. Obvious methods involve for example showing that no solutions exist for real variables, or for variables reduced modulo some n. (For quadratic equations Hasse's Principle implies that if no solutions exist for any n then there are no solutions for ordinary integers—but a cubic like $3 x^3 + 4 y^3 + 5 z^3 == 0$ is a counterexample.) If one can find a bound on solutions—say by Baker's method—then one can also try to show that no values below this bound are actually solutions. Over the history of number theory the sophistication of equations for which proofs of no solutions can be given has gradually increased—though even now it is state of the art to show say that $x == y == 1$ is the only solution to $x^2 == 3 y^4 - 2$.

Just as for all sorts of other systems with complex behavior, some idea of overall properties of Diophantine equations can be found on the basis of an approximation of perfect randomness. Writing equations in the form $p[x_1, x_2, \ldots, x_n] == 0$ the distribution of values of p will in general be complicated (see page 1161), but as a first approximation one can try taking it to be purely random. (Versions of this for large numbers of variables are validated by the so-called circle method from the early 1900s.) If p has total degree d then with $x_i < x$ the values of $Abs[p]$ will range up to about x^d. But with n variables the number of different cases sampled for $x_i < x$ will be x^n. The assumption of perfect randomness then suggests that for $d < n$, more and more cases with $p == 0$ will be seen as x increases, so that the equation will have an infinite number of solutions. For $d > n$, on the other hand, it suggests that there will often be no solutions, and that any solutions that exist will usually be small. In the boundary case $d == n$ it suggests that even for arbitrarily large x an average of about one solution should exist—suggesting that the smallest solution may be very large, and presumably explaining the presence of so many large solutions in the $n = d = 2$ and $n = d = 3$ examples in the main text. Note that even though large solutions may be rare when $d > n$ they must always exist in at least some cases whenever there is undecidability and universality in a class of equations. (See also page 1161.)

If one wants to enumerate all possible Diophantine equations there are many ways to do this, assigning different weights to numbers of variables, and sizes of coefficients and of exponents. But with several ways I have tried, it seems that of the first few million equations, the vast majority have no solutions—and this can in most cases be established by fairly elementary methods that are presumably within Peano arithmetic. When solutions do exist, most are fairly small. But as one continues the enumeration there are increasingly a few equations that seem more and more difficult to handle.

■ **Page 790 · Properties.** (All variables are assumed positive.)

- $2 x + 3 y == a$. There are $Ceiling[a/2] + Ceiling[2 a/3] - (a + 1)$ solutions, the one with smallest x being $\{Mod[2 a + 2, 3] + 1, 2 Floor[(2 a + 2)/3] - (a + 2)\}$. Linear equations like this were already studied in antiquity. (Compare page 915.)
- $x^2 == y^2 + a$. Writing a in terms of distinct factors as $r s$, $\{r + s, r - s\}/2$ gives a solution if it yields integers—which happens when $Abs[a] > 4$ and $Mod[a, 4] \neq 2$.
- $x^2 == a y^2 + 1$ (Pell equation). As discussed on page 944, whenever a is not a perfect square, there are always an infinite number of solutions given in terms of $ContinuedFraction[\sqrt{a}]$. Note that even when the smallest solution is not very large, subsequent solutions can rapidly get large. Thus for example when $a = 13$, the second solution is already $\{842401, 233640\}$.
- $x^2 == y^3 + a$ (Mordell equation). First studied in the 1600s, a complete theory of this so-called elliptic curve equation was only developed in the late 1900s—using fairly sophisticated algebraic number theory. The picture below shows as a function of a the minimum x that solves the equation. For $a = 68$, the only solution is $x = 1874$; for $a = 1090$, it is $x = 149651610621$. The density of cases with solutions gradually thins out as a increases (for $0 < a \leq 10000$ there are 2468 such cases). There are always only a finite number of solutions (for $0 < a \leq 10000$ the maximum is 12, achieved for $a = 8900$).



- $x^2 == a y^3 + 1$. Also an elliptic curve equation.
- $x^3 == y^4 + x y + a$. For most values of a (including specifically $a = 1$) the continuous version of this equation defines a surface of genus 3, so there are at most a finite number of integer solutions. (An equation of degree d generically defines a surface of genus $1/2 (d - 1) (d - 2)$.) Note that $x^3 == y^4 + a$ is equivalent to $x^3 == z^2 + a$ by a simple substitution.
- $x^2 == y^5 + a y + 3$. The second smallest solution to $x^2 == y^5 + 5 y + 3$ is $\{45531, 73\}$. As for the equations above, there are always at most a finite number of integer solutions.
- $x^3 + y^3 == z^2 + a$. For the homogenous case $a = 0$ the complete solution was found by Leonhard Euler in 1756.

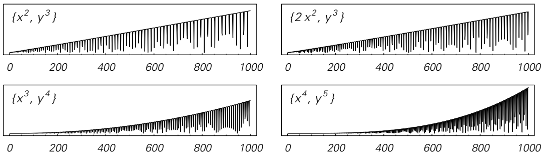
- $x^3 + y^3 == z^3 + a$. No solutions exist when $a = 9n \pm 4$; for $a = n^3$ or $2n^3$ infinite families of solutions are known. Particularly in its less strict form $x^3 + y^3 + z^3 == a$ with x, y, z positive or negative the equation was mentioned in the 1800s and again in the mid-1900s; computer searches for solutions were begun in the 1960s, and by the mid-1990s solutions such as $\{283059965, 2218888517, 2220422932\}$ for the case $a = -30$ had been found. Any solution to the difficult case $x^3 + y^3 == z^3 - 3$ must have $Mod[x, 9] == Mod[y, 9] == Mod[z, 9]$. (Note that $x^2 + y^2 + z^2 == a$ always has solutions except when $a = 4^s (8n + 7)$, as mentioned on page 135.)

■ **Large solutions.** A few other 2-variable equations with fairly large smallest solutions are:

- $x^3 == 3y^3 - xy + 63$: $\{7149, 4957\}$
- $x^4 == y^3 + 2xy - 2y + 81$: $\{19674, 531117\}$
- $x^4 == 5y^3 + xy + y - 8x$: $\{69126, 1659072\}$

The equation $x^x y^y == z^z$ is known to have smallest non-trivial solution $\{2985984, 1679616, 4478976\}$.

■ **Nearby powers.** One can potentially find integer equations with large solutions but small coefficients by looking say for pairs of integer powers close in value. The pictures below show what happens if one computes x^m and y^n for many x and y, sorts these values, then plots successive differences. The differences are trivially zero when $x = s^n$, $y = s^m$. Often they are large, but surprisingly small ones can sometimes occur (despite various suggestions from the so-called ABC conjecture). Thus, for example, $5853886516781223^3 - 1641843$ is a perfect square, as found by Noam Elkies in 1998. (Another example is $55^5 - 22434^2 == 19$.)



■ **Page 791 · Unsolved problems.** Problems in number theory that are simple to state (say in the notation of Peano arithmetic) but that so far remain unsolved include:

- Is there any odd number equal to the sum of its divisors? (Odd perfect number; 4th century BC) (See page 911.)
- Are there infinitely many primes that differ by 2? (Twin Prime Conjecture; 1700s?) (See page 909.)
- Is there a cuboid in which all edges and all diagonals are of integer length? (Perfect cuboid; 1719)
- Is there any even number which is not the sum of two primes? (Goldbach's Conjecture; 1742) (See page 135.)
- Are there infinitely many primes of the form $n^2 + 1$? (Quadratic primes; 1840s?) (See page 1162.)
- Are there infinitely many primes of the form $2^{2^n} + 1$? (Fermat primes; 1844)
- Are there no solutions to $x^m - y^n == 1$ other than $3^2 - 2^3 == 1$? (Catalan's Conjecture; 1844)
- Can every integer not of the form $9n \pm 4$ be written as $a^3 \pm b^3 \pm c^3$? (See note above.)
- How few n^{th} powers need be added to get any given integer? (Waring's Problem; 1770)

(See also Riemann Hypothesis on page 918.)

■ **Page 791 · Fermat's Last Theorem.** That $x^n + y^n == z^n$ has no integer solutions for $n > 2$ was suggested by Pierre Fermat around 1665. Fermat proved this for $n = 4$ around 1660; Leonhard Euler for $n = 3$ around 1750. It was proved for $n = 5$ and $n = 7$ in the early 1800s. Then in 1847 Ernst Kummer used ideas of factoring with algebraic integers to prove it for all $n < 37$. Extensions of this method gradually allowed more cases to be covered, and by the 1990s computers had effectively given proofs for all n up to several million. Meanwhile, many connections had been found between the general case and other areas of mathematics—notably the theory of elliptic curves. And finally around 1995, building on extensive work in number theory, Andrew Wiles managed to give a complete proof of the result. His proof is long and complicated, and relies on sophisticated ideas from many areas of mathematics. But while the statement of the proof makes extensive use of concepts from areas like set theory, it seems quite likely that in the end a version of it could be given purely in terms of Peano arithmetic. (By the 1970s it had for example been shown that many classic proofs with a similar character in analytic number theory could at least in principle be carried out purely in Peano arithmetic.)

■ **Page 791 · More powerful axioms.** If one looks for example at progressively more complicated Diophantine equations then one can expect that one will find examples where more and more powerful axiom systems are needed to prove statements about them. But my guess is that almost as soon as one reaches cases that cannot be handled by Peano arithmetic one will also reach cases that cannot be handled by set theory or even by still more powerful axiom systems.

Any statement that one can show is independent of the Peano axioms and at least not inconsistent with them one can potentially consider adding as a new axiom. Presumably it is best to add axioms that allow the widest range of new statements to be proved. But I strongly suspect that the set of statements that cannot be proved is somehow sufficiently fragmented that adding a few new axioms will actually make very little difference.

In set theory (see page 1155) a whole sequence of new axioms have historically been added to allow particular kinds of statements to be proved. And for several decades additional so-called large cardinal axioms have been discussed, that in effect state that sets exist larger than any that can be reached with the current axioms of set theory. (As discussed on page 816 any axiom system that is universal must in principle be able to prove any statement that can be proved in any axiom system—but not with the kinds of encodings normally considered in mathematical logic.)

It is notable, however, that if one looks at classic theorems in mathematics many can actually be derived from remarkably weak axioms. And indeed the minimal axioms needed to obtain most of mathematics as it is now practiced are probably much weaker than those on pages 773 and 774.

(If one considers for example theorems about computational issues such as whether Turing machines halt, then it becomes inevitable that to cover more Turing machines one needs more axioms—and to cover all possible machines one needs an infinite set of axioms, that cannot even be generated by any finite set of rules.)

■ **Higher-order logics.** In ordinary predicate—or so-called first-order—logic the objects x that $\forall_x$ and $\exists_x$ range over are variables of the kind used as arguments to functions (or predicates) such as `f[x]`. To set up second-order logic, however, one imagines also being able to use $\forall_f$ and $\exists_f$ where f is a function (say the head of `f[x]`). And then in third-order logic one imagines using $\forall_g$ and $\exists_g$ where g appears in `g[f][x]`.

Early formulations of axiom systems for mathematics made little distinction between first- and second-order logic. The theory of types used in *Principia Mathematica* introduced some distinction, and following the proof of Gödel's Completeness Theorem for first-order logic in 1930 (see page 1152) standard axiom systems for mathematics (as given on pages 773 and 774) began to be reformulated in first-order form, with set theory taking over many of the roles of second-order logic.

In current mathematics, second-order logic is sometimes used at the level of notation, but almost never in its full form beyond. And in fact with any standard computational system it can never be implemented in any explicit way. For even to enumerate theorems in second-order logic is in general impossible for a system like a Turing machine unless one assumes that an oracle can be added. (Note however that this is possible in Henkin versions of higher-order logic that allow only limited function domains.)

■ **Truth and incompleteness.** In discussions of the foundations of mathematics in the early 1900s it was normally assumed that truth and provability were in a sense equivalent—so that all true statements could in principle be reached by formal processes of proof from fixed axioms (see page 782). Gödel's Theorem showed that there are statements that can never be proved from given axioms. Yet often it seemed inevitable just from the syntactic structure of statements (say as well-formed formulas) that each of them must at some level be either true or false. And this led to the widespread claim that Gödel's Theorem implies the existence of mathematical statements that are true but unprovable—with their negations being false but unprovable. Over the years this often came to be assigned a kind of mystical significance, mainly because it was implicitly assumed that somehow it must still ultimately be possible to know whether any given statement is true or false. But the Principle of Computational Equivalence implies that in fact there are all sorts of statements that simply cannot be decided by any computational process in our universe. So for example, it must in some sense be either true or false that a given Turing machine halts with given input—but according to the Principle of Computational Equivalence there is no finite procedure in our universe through which we can guarantee to know which of these alternatives is correct.

In some cases statements can in effect have default truth values—so that showing that they are unprovable immediately implies, say, that they must be true. An example in arithmetic is whether some integer equation has no solution. For if there were a solution, then given the solution it would be straightforward to give a proof that it is correct. So if it is unprovable that there is no solution, then it follows that there must in fact be no solution. And similarly, if it could be shown for example that Goldbach's Conjecture is unprovable then it would follow that it must be true, for if it were false then there would have to be a specific number which violates it, and this could be proved. Not all statements in mathematics have this kind of default truth value. And thus for example the Continuum Hypothesis in set theory is unprovable but could be either of true or false: it is just independent of the axioms of set theory. In computational systems, showing that it is unprovable that a given Turing machine halts with given input immediately implies that in fact it must not halt. But showing that it is unprovable whether a Turing machine halts with every input (a Π_2 statement in the notation of page 1139) does not immediately imply anything about whether this is in fact true or false.

■ **Page 793 · Generalization in mathematics.** Systems that have evolved from the basic notion of numbers provide a characteristic example of the process of progressive generalization in mathematics. The main such systems and their dates of earliest known reasonably formalized use have been (see also page 901): positive integers (before 10,000 BC), rationals (3000 BC), square roots (2000 BC), other roots (1800 BC), all integers (600 AD, 1600s), decimals (950 AD), complex numbers (1500s, 1800s), polynomials (1591), infinitesimals (1635), algebraic numbers (1744), quaternions (1843), Grassmann algebra (1844), ideals (1844, 1871), octonions (~1845), Boolean algebra (1847), fields (1850s, 1871), matrices (1858), associative algebras (1870), axiomatic real numbers (1872), vectors (1881), transfinite ordinals (1883), transfinite cardinals (1883), operator calculus (1880s), Boolean algebras (1890), algebraic number fields (1893), rings (1897), *p*-adic numbers (1897), non-Archimedean fields (1899), *q*-numbers (1926), non-standard integers (1930s), non-standard reals (hyperreals) (1960), interval arithmetic (1968), fuzzy arithmetic (1970s), surreal numbers (1970s). New systems have usually been introduced in connection with extending the domains of particular existing operations. But in almost all cases the systems are set up so as to preserve as many theorems as possible—a notion that was for example made explicit in the Principle of Permanence discussed by George Peacock in 1830 and extended by Hermann Hankel in 1869.

■ **Page 794 · Cellular automaton axioms.** The first 4 axioms are general to one-dimensional cellular automata. The next 8 are specific to rule 110. The final 3 work whenever patterns are embedded in a background of white cells. The universality of rule 110 presumably implies that the axiom system given is universal. (A complete proof would require handling various issues about boundary conditions.)

If the last 2 axioms are dropped any statement can readily be proved true or false essentially just by running rule 110 for a finite number of steps equal to the number of nested ↓ plus ⟨…⟩ in the statement. In practice, a large number of steps can however be required. As an example the statement

$$\begin{array}{l} O\blacksquare \Rightarrow (\exists_a (\exists_b\, O\, a \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\square \diamond \\ \quad (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\square \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\square \diamond \\ \qquad (\square \diamond (\square \diamond (\blacksquare \diamond (\square \diamond (\square \diamond (\blacksquare \diamond (\blacksquare \diamond (\square \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond (\blacksquare \diamond \\ \qquad\quad (\square \diamond (\square \diamond (\square \diamond (\blacksquare \diamond (\square \diamond b))))))))))))))))))))))))))))))))))))) \end{array}$$

asserts that a particular localized structure occurs in the evolution of rule 110 from a single black cell. But page 38 shows that this happens for the first time after 2867 steps. (A proof of this without lemmas would probably have to be of length at least 32,910,300.)

The axioms as they are stated apply to any rule 110 evolution, regardless of initial conditions. One can establish that the statement at the bottom on the right cannot be proved either true or false from the axioms by showing that it is true for some initial conditions and false for others. Note from page 279 that the sequence □■□■□ cannot occur in rule 110 evolution except as an initial condition. So this means that the statement is false if the initial condition is ■ and true if the initial condition is □■□■□.

■ **Practical programs.** Any equivalence between programs in a programming language can be thought of as a theorem. Simple examples in *Mathematica* include:

```
First[Prepend[p, q]] === q
Join[Join[p, q], r] === Join[p, Join[q, r]]
Partition[Union[p], 1] === Split[Union[p]]
```

One can set up axiom systems say by combining definitions of programming languages with predicate logic (as done by John McCarthy for LISP in 1963). And for programs whose structure is simple enough it has sometimes been possible to prove theorems useful for optimization or verification. But in the vast majority of cases this has been essentially impossible in practice. And I suspect that this is a reflection of widespread fundamental unprovability. In setting up programs with specific purposes there is inevitably some computational reducibility (see page 828). But I suspect that enough computational irreducibility usually remains to make unprovability common when one asks about all possible forms of behavior of the program.

■ **Page 796 · Rules.** The examples shown here (roughly in order of increasing complexity) correspond respectively to cases (a), (k), (b), (q), (p), (r), (o), (d) on page 798.

■ **Page 797 · Consistency.** Any axiom system that is universal can represent the statement that the system is consistent. But normally such a statement cannot be proved true or false within the system itself. And thus for example Kurt Gödel showed this in 1931 for Peano arithmetic (in his so-called second incompleteness theorem). In 1936, however, Gerhard Gentzen showed that the axioms of set theory imply the consistency of Peano arithmetic (see page 1160). In practical mathematics set theory is always taken to be consistent, but to set up a proof of this would require axioms beyond set theory.

■ **Page 798 · Properties.** For most of the rules shown, there ultimately turn out to be quite easy characterizations of what strings can be produced.

- (a) At step *t*, the only new string produced is the one containing *t* black elements.

- (b) All strings of length n containing exactly one black cell are produced—after at most $2n-1$ steps.
- (c) All strings containing even-length runs of white cells are produced.
- (d) The set of strings produced is complicated. The last length 4 string produced is □□□■, after 16 steps; the last length 6 one is □□□□□■, after 26 steps.
- (e) All strings that begin with a black element are produced.
- (f) All strings that end with a white element but contain at least one black element, or consist of all white elements ending with black, are produced. Strings of length n take n steps to produce.
- (g) The same strings as in (f) are produced, but now a string of length n with m black elements takes $n+m-1$ steps.
- (h) All strings appear in which the first run of black elements is of length 1; a string of length n with m black elements appears after $n+m-1$ steps.
- (i) All strings containing an odd number of black elements are produced; a string of length n with m black cells occurs at step $n+m-1$.
- (j) All strings that end with a black element are produced.
- (k) Above length 1, the strings produced are exactly those starting with a white element. Those of length n appear after at most $3n-3$ steps.
- (l) The same strings as in (k) are produced, taking now at most $2n+1$ steps.
- (m) All strings beginning with a black element are produced, after at most $3n+1$ steps.
- (n) The set of strings produced is complicated, and seems to include many but not all that do not end with ■□.
- (o) All strings that do not end in ■□ are produced.
- (p) All strings are produced, except ones in which every element after the first is white. ■□□■ takes 14 steps.
- (q) All strings are produced, with a string of length n with m white elements taking $n+2m$ steps.
- (r) All strings are ultimately produced—which is inevitable after the lemmas ■ → ■■ and ■ → □ appear at steps 12 and 13. (See the first rule on page 778.)

▪ **Page 800 · Non-standard arithmetic.** Goodstein's result from page 1163 is true for all ordinary integers. But since it is independent of the axioms of arithmetic there must be objects that still satisfy the axioms but for which it is false. It turns out however that any such objects must in effect be infinite. For any set of objects that satisfy the axioms of arithmetic must include all finite ordinary integers, since each of these can be reached just by using Δ repeatedly. And the axioms then turn out to imply that any additional objects must be larger than all these integers—and must therefore be infinite. But for any such truly infinite objects operations like $+$ and $\times$ cannot be computed by finite procedures, making it difficult to describe such objects in an explicit way. Ever since the work of Thoralf Skolem in 1933 non-standard models of arithmetic have been discussed, particularly in the context of ultrafilters and constructs like infinite trees. (See also page 1172.)

▪ **Page 800 · Reduced arithmetic.** (See page 1152.) Statements that can be proved with induction but are not provable only with Robinson's axioms are: $x \neq \Delta x$; $x+y == y+x$; $x+(y+z) == (x+y)+z$; $0+x == x$; $\exists_x (\Delta x + y == z \Rightarrow y \neq z)$; $x \times y == y \times x$; $x \times (y \times z) == (x \times y) \times z$; $x \times (y+z) == x \times y + x \times z$.

▪ **Page 800 · Generators and relations.** In the axiom systems of page 773, a single variable can stand for any element—much like a *Mathematica* pattern object such as $x\_$. In studying specific instances of objects like groups one often represents elements as products of constants or generators, and then for example specifies the group by giving relations between these products. In traditional mathematical notation such relations normally look just like ordinary axioms, but in fact the variables that appear in them are now assumed to be literal objects—like x in *Mathematica*—that are generically taken to be unequal. (Compare page 1159.)

▪ **Page 801 · Comparison to multiway systems.** Operator systems are normally based on equations, while multiway systems are based on one-way transformations. But for multiway systems where each rule $p \to q$ is accompanied by its reverse $q \to p$, and such pairs are represented say by "AAB" ↔ "BBAA", an equivalent operator system can immediately be obtained either from

```
Apply[Equal,
  Map[Fold[#2[#1] &, x, Characters[#]] &, rules, {2}], {1}]
```

or from (compare page 1172)

```
Append[Apply[Equal,
    Map[(Fold[f, First[#], Rest[#]] &)[Characters[#]] &,
      rules, {2}], {1}], f[f[a, b], c] == f[a, f[b, c]]]
```

where now objects like "A" and "B" are treated as constants—essentially functions with zero arguments. With slightly more effort multiway systems with ordinary one-way rules can also be converted to operator systems. Converting from operator systems to multiway systems is more difficult, though ultimately always possible (see page 1156).

As discussed on page 898, one can set up operator evolution systems similar to symbolic systems (see page 103) that have essentially the same relationship to operator systems as sequential substitution systems do to multiway systems. (See also page 1172.)

■ **Page 802 · Operator systems.** One can represent the possible values of expressions like *f[f[p, q], p]* by rule numbers analogous to those used for cellular automata. Specifying an operator *f* (taken in general to have *n* arguments with *k* possible values) by giving the rule number *u* for *f[p, q, …]*, the rule number for an expression with variables *vars* can be obtained from

```
With[{m = Length[vars]}, FromDigits[
 Block[{f = Reverse[IntegerDigits[u, k, k^n]][[FromDigits[
    {##}, k] + 1]] &}, Apply[Function[Evaluate[vars], expr],
  Reverse[Array[IntegerDigits[# - 1, k, m] &, k^m]], {1}]], k]]
```

■ **Truth tables.** The method of finding results in logic by enumerating all possible combinations of truth values seems to have been rediscovered many times since antiquity. It began to appear regularly in the late 1800s, and became widely known after its use by Emil Post and Ludwig Wittgenstein in the early 1920s.

■ **Page 803 · Proofs of axiom systems.** One way to prove that an axiom system can reproduce all equivalences for a given operator is to show that its axioms can be used to transform any expression to and from a unique standard form. For then one can start with an expression, convert it to standard form, then convert back to any expression that is equivalent. We saw on page 616 that in ordinary logic there is a unique DNF representation in terms of *And*, *Or* and *Not* for any expression, and in 1921 Emil Post used essentially this to give the first proof that an axiom system like the first one on page 773 can completely reproduce all theorems of logic. A standard form in terms of *Nand* can be constructed essentially by direct translation of DNF; other methods can be used for the various other operators shown. (See also page 1175.)

Given a particular axiom system that one knows reproduces all equivalences for a given operator one can tell whether a new axiom system will also work by seeing whether it can be used to derive the axioms in the original system. But often the derivations needed may be very long—as on page 810. And in fact in 1948 Samuel Linial and Emil Post showed that in general the problem is undecidable. They did this in effect by arguing (much as on page 1169) that any multiway system can be emulated by an axiom system of the form on page 803, then knowing that in general it is undecidable whether a multiway system will ever reach some given result. (Note that if an axiom system does manage to reproduce logic in full then as indicated on page 814 its consequences can always be derived by proofs of limited length, if nothing else by using truth tables.)

Since before 1920 it has been known that one way to disprove the validity of a particular axiom system is to show that with $k > 2$ truth values it allows additional operators (see page 805). (Note that even if it works for all finite k this does not establish its validity.) Another way to do this is to look for invariants that should not be present—seeing if there are features that differ between equivalent expressions, yet are always left the same by transformations in the axiom system. (Examples for logic are axiom systems which never change the size of an expression, or which are of the form *{expr == a}* where *Flatten[expr]* begins or ends with *a*.)

■ **Junctional calculus.** Expressions are equivalent when *Union[Level[expr, {-1}]]* is the same, and this canonical form can be obtained from the axiom system of page 803 by flattening using $(a \circ b) \circ c == a \circ (b \circ c)$, sorting using $a \circ b == b \circ a$, and removing repeats using $a \circ a == a$. The operator can be either *And* or *Or* (8 or 14). With $k = 3$ there are 9 operators that yield the same results:

```
{13203, 15633, 15663, 16401,
  17139, 18063, 19539, 19569, 19599}
```

With $k = 4$ there are 3944 such operators (see below). No single axiom can reproduce all equivalences, since such an axiom must of the form *expr == a*, yet *expr* cannot contain variables other than a, and so cannot for example reproduce $a \circ b = b \circ a$.

■ **Equivalential calculus.** Expressions with variables *vars* are equivalent if they give the same results for

```
Mod[Map[Count[expr, #, {-1}] &, vars], 2]
```

With n variables, there are thus 2^n equivalence classes of expressions (compared to 2^{2^n} for ordinary logic). The operator can be either *Xor* or *Equal* (6 or 9). With $k = 3$ there are no operators that yield the same results; with $k = 4$ *{458142180, 1310450865, 2984516430, 3836825115}* work (see below). The shortest axiom system that works up to $k = 2$ is $\{(a \circ b) \circ a = b\}$. With *modus ponens* as the rule of inference, the shortest single-axiom system that works is known to be $\{(a \circ b) \circ ((c \circ b) \circ (a \circ c))\}$. Note that equivalential calculus describes the smallest non-trivial group, and can be viewed as an extremely minimal model of algebra.

■ **Implicational calculus.** With $k = 2$ the operator can be either 2 or 11 (*Implies*), with $k = 3$ *{2694, 9337, 15980}*, and with $k = 4$ any of 16 possibilities. (Operators exist for any k.) No single axiom, at least with up to 7 operators and 4 variables, reproduces all equivalences. With *modus ponens* as the rule of inference, the shortest single-axiom system that works is known to be $\{((a \circ b) \circ c) \circ ((c \circ a) \circ (d \circ a))\}$. Using the method of page 1151 this can be converted to the equational form

$$\{((a \circ b) \circ c) \circ ((c \circ a) \circ (d \circ a)) = d \circ d,\\ (a \circ a) \circ b = b,\ (a \circ b) \circ b = (b \circ a) \circ a\}$$

from which the validity of the axiom system in the main text can be established.

■ **Page 803 · Operators on sets.** There is always more than one operator that yields a given collection of equivalences. So for ordinary logic both *Nand* and *Nor* work. And with $k = 4$ any of the 12 operators

```
{1116699, 169585339, 290790239, 459258879,
  1090522958, 1309671358, 1430343258, 1515110058,
  2184380593, 2575151445, 2863760025, 2986292093}
```

also turn out to work. One can see why this happens by considering the analogy between operations in logic and operations on sets. As reflected in their traditional notations—and emphasized by Venn diagrams—*And* ($\wedge$), *Or* ($\vee$) and *Not* correspond directly to *Intersection* ($\cap$), *Union* ($\cup$) and *Complement*. If one starts from the single-element set *{1}* then applying *Union*, *Intersection* and *Complement* one always gets either *{}* or *{1}*. And applying *Complement[s, Intersection[a, b]]* to these two elements gives the same results and same equivalences as $a \bar{\wedge} b$ applied to *True* and *False*. But if one uses instead $s = \{1, 2\}$ then starts with *{1}* and *{2}* one gets any of *{{}, {1}, {2}, {1, 2}}* and in general with *s = Range[n]* one gets any of the 2^n elements in the powerset

```
Distribute[Map[{{}, {#}} &, s], List, List, List, Join]
```

But applying *Complement[s, Intersection[a, b]]* to these elements still always produces the same equivalences as with $a \bar{\wedge} b$. Yet now $k = 2^n$. And so one therefore has a representation of Boolean algebra of size 2^n. For ordinary logic based on *Nand* it turns out that there are no other finite representations (though there are other infinite ones). But if one works, say, with *Implies* then there are for example representations of size 3 (see above). And the reason for this is that with $s = \{1, 2\}$ the function *Union[Complement[s, a], b]* corresponding to $a \Rightarrow b$ only ever gets to the 3 elements *{{1}, {2}, {1, 2}}*. Indeed, in general with operators *Implies*, *And* and *Or* one gets to $2^n - 1$ elements, while with operators *Xor* and *Equal* one gets to *2^(2 Floor[n/2])* elements.

(One might think that one could force there only ever to be two elements by adding an axiom like $a == b \vee b == c \vee c == a$. But all this actually does is to force there to be only two objects analogous to *True* and *False*.)

■ **Page 805 · Implementation.** Given an axiom system in the form $\{f[a, f[a, a]] = a, f[a, b] = f[b, a]\}$ one can find rule numbers for the operators $f[x, y]$ with k values for each variable that are consistent with the axiom system by using

```
Module[{c, v}, c = Apply[Function,
  {v = Union[Level[axioms, {-1}]], Apply[And, axioms]}];
 Select[Range[0, k^(k^2) - 1], With[{u = IntegerDigits[#, k, k^2]},
   Block[{f}, f[x_, y_] := u[[-1 - k x - y]];
    Array[c, Table[k, {Length[v]}], 0, And]]] &]]
```

For $k = 4$ this involves checking nearly 16^4 or 4 billion cases, though many of these can often be avoided, for example by using analogs of the so-called Davis-Putnam rules. (In searching for an axiom system for a given operator it is in practice often convenient first to test whether each candidate axiom holds for the operator one wants.)

■ **Page 805 · Properties.** There are k^{k^2} possible forms for binary operators with k possible values for each argument. There is always at least some operator that satisfies the constraints of any given axiom system—though in a case like $a = b$ it has $k = 1$. Of the 274,499 axiom systems of the form $\{\ldots = a\}$ where $\ldots$ involves $\circ$ up to 6 times, 32,004 allow only operators *{6, 9}*, while 964 allow only *{1, 7}*. The only cases of 2 or less operators that appear with $k = 2$ are *{{}, {10}, {12}, {1, 7}, {3, 12}, {5, 10}, {6, 9}, {10, 12}}*. (See page 1174.)

■ **Page 806 · Algebraic systems.** Operator systems can be viewed as algebraic systems of the kind studied in universal algebra (see page 1150). With a single two-argument operator (such as $\circ$) what one has is in general known as a groupoid (though this term means something different in topology and category theory); with two such operators a ringoid. Given a particular algebraic system, it is sometimes possible—as we saw on page 773—to reduce the number of operators it involves. But the number of systems that have traditionally been studied in mathematics and that are known to require only one 2-argument operator are fairly limited. In addition to basic logic, semigroups and groups, there are essentially only the rather obscure examples of semilattices, with axioms $\{a \circ (b \circ c) = (a \circ b) \circ c, a \circ b = b \circ a, a \circ a = a\}$, central groupoids, with axioms $\{(b \circ a) \circ (a \circ c) = b\}$, and squags (quasigroup representations of Steiner triple systems), with axioms $\{a \circ b = b \circ a, a \circ a = a, a \circ (a \circ b) = b\}$ or equivalently $\{a \circ ((b \circ (b \circ (((c \circ c) \circ d) \circ c))) \circ a) = d\}$. (Ordinary quasigroups are defined by $\{a \circ c = b, d \circ a = b\}$ with c, d unique for given a, b—so that their table is a Latin square; their axioms can be set up with 3 operators as $\{a \backslash a \circ b = b, a \circ b / b = a, a \circ (a \backslash b) = b, (a / b) \circ b = a\}$.)

Pages 773 and 774 indicate that most axiom systems in mathematics involve operators with at most 2 arguments (there are exceptions in geometry). (Constants such as *1* or $\emptyset$ can be viewed as 0-argument operators.) One can nevertheless generalize say to polyadic groups, with 3-argument composition and analogs of associativity such as

$$f[f[a, b, c], d, e] = f[a, f[b, c, d], e] = f[a, b, f[c, d, e]]$$

Another example is the cellular automaton axiom system of page 794; see also page 886. (A perhaps important generalization is to have expressions that are arbitrary networks rather than just trees.)

▪ **Symbolic systems.** By introducing constants (0-argument operators) and interpreting $\circ$ as function application one can turn any symbolic system such as `e[x][y] → x[x[y]]` from page 103 into an algebraic system such as $(e \circ a) \circ b = a \circ (a \circ b)$. Doing this for the combinator system from page 711 yields the so-called combinatory algebra $\{((s \circ a) \circ b) \circ c = (a \circ c) \circ (b \circ c), (k \circ a) \circ b = a\}$.

▪ **Page 806 · Groups and semigroups.** With k possible values for each variable, the forms of operators allowed by axiom systems for group theory and semigroup theory correspond to multiplication tables for groups and semigroups with k elements. Note that the first group that is not commutative (Abelian) is the group S3 with $k = 6$ elements. The total number of commutative groups with k elements is just

```
Apply[Times,
  Map[PartitionsP[Last[#]] &, FactorInteger[k]]]
```

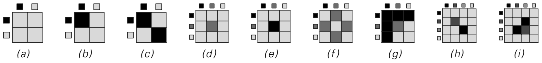
(Relabelling of elements makes the number of possible operator forms up to $k!$ times larger.) (See also pages 945, 1153 and 1173.)

▪ **Forcing of operators.** Given a particular set of forms for operators one can ask whether an axiom system can be found that will allow these but no others. As discussed in the note on operators on sets on page 1171 some straightforwardly equivalent forms will always be allowed. And unless one limits the number of elements k it is in general undecidable whether a given axiom system will allow no more than a given set of forms. But even with fixed k it is also often not possible to force a particular set of forms. And as an example of this one can consider commutative group theory. The basic axioms for this allow forms of operators corresponding to multiplication tables for all possible commutative groups (see note above). So to force particular forms of operators would require setting up axioms satisfied only by specific commutative groups. But it turns out that given the basic axioms for commutative group theory any non-trivial set of additional axioms can always be reduced to a single axiom of the form $a^n == 1$ (where exponentiation is repeated application of $\circ$). Yet even given a particular number of elements k, there can be several distinct groups satisfying $a^n == 1$ for a given exponent n. (The groups can be written as products of cyclic ones whose orders correspond to the possible factors of n.) (Something similar is also known in principle to be true for general groups, though the hierarchy of axioms in this case is much more complicated.)

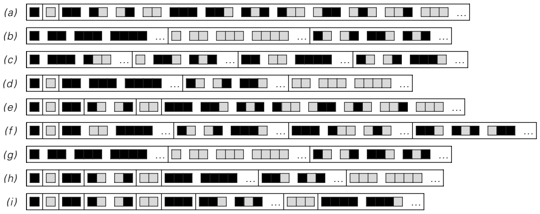
▪ **Model theory.** In model theory each form of operator that satisfies the constraints of a given axiom system is called a model of that axiom system. If there is only one inequivalent model the axiom system is said to be categorical—a notion discussed for example by Richard Dedekind in 1887. The Löwenheim-Skolem theorem from 1915 implies that any axiom system must always have a countable model. (For an operator system such a model can have elements which are simply equivalence classes of expressions equal according to the axioms.) So this means that even if one tries to set up an axiom system to describe an uncountable set—such as real numbers—there will inevitably always be extra countable models. Any axiom system that is incomplete must always allow more than one model. The model intended when the axiom system was originally set up is usually called the standard model; others are called non-standard. In arithmetic non-standard models are obscure, as discussed on page 1169. In analysis, however, Abraham Robinson pointed out in 1960 that one can potentially have useful non-standard models, in which there are explicit infinitesimals—much along the lines suggested in the original work on calculus in the late 1600s.

▪ **Pure equational logic.** Proofs in operator systems always rely on certain underlying rules about equality, such as the equivalence of `u == v` and `v == u`, and of `u == v` and `u == v /. a → b`. And as Garrett Birkhoff showed in 1935, any equivalence between expressions that holds for all possible forms of operator must have a finite proof using just these rules. (This is the analog of Gödel's Completeness Theorem from page 1152 for pure predicate logic.) But as soon as one introduces actual axioms that constrain the operators this is no longer true—and in general it can be undecidable whether or not a particular equivalence holds.

▪ **Multiway systems.** One can use ideas from operator systems to work out equivalences in multiway systems (compare page 1169). One can think of concatenation of strings as being an operator, in terms of which a string like `"ABB"` can be written `f[f[a, b], b]`. (The arguments to $\circ$ should strictly be distinct constants, but no equivalences are lost by allowing them to be general variables.) Assuming that the rules for a multiway system come in pairs `p → q`, `q → p`, like `"AB" → "AAA"`, `"AAA" → "AB"`, these can be written as statements about operators, like `f[a, b] == f[f[a, a], a]`. The basic properties of concatenation then also imply that `f[f[a, b], c] == f[a, f[b, c]]`. And this means that the possible forms for the operator $\circ$ correspond to possible semigroups. Given a particular such semigroup satisfying axioms derived from a multiway system, one can see whether the operator representations of particular strings are equal—and if they are not, then it follows that the strings can never be reached from each other through evolution of the multiway system. (Such operator representations are a rough analog for multiway systems of truth tables.) As an example, with the multiway system *"AB"* ↔ *"BA"* some possible forms of operators are shown below. (In this case these are the commutative semigroups. With $k = 2$, elements 6 out of the total of 8 possible semigroups appear; with $k = 3$, 63 out of 113, and with $k = 4$, 1140 out of 3492—all as shown on page 805.) (See also page 952.)



Taking ∘ to be each of these operators, one can work out a representation for any given string like *"ABAA"* by for example constructing the expression *f[f[f[a, b], a], a]* and finding its value for each of the k^2 possible pairs of values of a and b. Then for each successive operator, the sets of strings where the arrays of values are the same are as shown below.



Ultimately the sets of strings equivalent under the multiway system are exactly those containing particular numbers of black and white elements. But as the pictures above suggest, only some of the distinctions between sets of strings are ever captured when any specific form for the operator is used.

Just as for operator systems, any bidirectional multiway system will allow a certain set of operators. (When there are multiple rules in the multiway system, tighter constraints are obtained by combining them with *And*.) And the pattern of results for simple multiway systems is roughly similar to those on page 805 for operator systems—although, for example, the associativity of concatenation makes it impossible for example to get the operators for *Nand* and basic logic.

■ **Page 806 · Logic in languages.** Human languages always seem to have single words for AND, OR and NOT. A few have distinct words for OR and XOR: examples are Latin with *vel* and *aut* and Finnish with *vai* and *tai*. NOR is somewhat rare, though Dutch has *noch* and Old English *ne*. (Modern English has only the compound form *neither ... nor*.) But remarkably enough it appears that no ordinary language has a single word for NAND. The reason is not clear. Most people seem to find it difficult to think in terms of NAND (NAND is for example not associative, but then neither is NOR). And NAND on the face of it rarely seems useful in everyday situations. But perhaps these are just reflections of the historical fact that NAND has never been familiar from ordinary languages.

Essentially all computer languages support AND, OR and NOT as ways to combine logical statements; many support AND, OR and XOR as bitwise operations. Circuit design languages like Verilog and VHDL also support NAND, NOR and XNOR. (NAND is the operation easiest to implement with CMOS FETs—the transistors used in most current chips; it was also implemented by pentode vacuum tubes.) Circuit designers sometimes use the linguistic construct "*p* nand *q*".

The Laws of Form presented by George Spencer Brown in 1969 introduce a compact symbolic notation for NAND with any number of arguments and in effect try to develop a way of discussing NAND and reasoning directly in terms of it. (The axioms normally used are essentially the Sheffer ones from page 773.)

■ **Page 806 · Properties.** Page 813 lists theorems satisfied by each function. *{0, 1, 6, 7, 8, 9, 14, 15}* are commutative (orderless) so that $a \circ b = b \circ a$, while *{0, 6, 8, 9, 10, 12, 14, 15}* are associative (flat), so that $a \circ (b \circ c) = (a \circ b) \circ c$. (Compare page 886.)

■ **Notations.** Among those in current use are (highlighted ones are supported directly in *Mathematica*):

True	T	1	⊤	⋎					
False	F	0	⊥	⋏					
Not[p]	¬ p	$\bar{p}$	~p	p′	-p	! p	¬ p	N p	(negation)
And[p, q]	p ∧ q	p & q	p · q	p q	p && q	K p q	(conjunction)		
Or[p, q]	p ∨ q	p + q	p ‖ q	A p q	(disjunction)				
Xor[p, q]	p ⊻ q	p ⊕ q	p ≢ q	p ∨̇ q	J p q	(inequivalence)			
Implies[p, q]	p ⇒ q	p ⊃ q	p → q	If[p, q]	C p q	(material implication)			
Equal[p, q]	p = q	p ⇔ q	p ≡ q	p ↔ q	p ~ q	E p q	(material equivalence; xnor)		
Nand[p, q]	p ⊼ q	p | q	p ⌐ q	(p q)	D p q	(Sheffer stroke; alternative denial)			
Nor[p, q]	p ⊽ q	p↓q	X p q	(joint denial)					

The grouping of terms is normally inferred from precedence of operators (typically ordered ==, ¬, ⊼, ∧, ⊻, ⊽, ∨, ⇒), or explicitly indicated by parentheses, function brackets, or sometimes nested underbars or dots. So-called Polish notation given second-to-last above avoids all explicit delimiters (see page 896).

■ **Page 807 · Universal logical functions.** The fact that combinations of *Nand* or *Nor* are adequate to reproduce any logical function was noted by Charles Peirce around 1880, and became widely known after the work of Henry Sheffer in 1913. (See also page 1096.) *Nand* and *Nor* are the only 2-input functions universal in this sense. (*{Equal}* can for example reproduce only functions $\{9, 10, 12, 15\}$, $\{Implies\}$ only functions $\{10, 11, 12, 13, 14, 15\}$, and $\{Equal, Implies\}$ only functions $\{8, 9, 10, 11, 12, 13, 14, 15\}$.) For 3-input functions, corresponding to elementary cellular automaton rules, 56 of the 256 possibilities turn out to be universal. Of these, 6 are straightforward generalizations of *Nand* and *Nor*. Other universal functions include rules 1, 45 and 202 (*If[a == 1, b, c]*), but not 30, 60 or 110. For large n roughly 1/4 of all n-input functions are universal. (See also page 1175.)

■ **Page 808 · Searching for logic.** For axiom systems of the form $\{\ldots = a\}$ one finds:

	2 variables					3 variables				
number of ∘	2	3	4	5	6	2	3	4	5	6
total systems	4	16	80	448	2688	54	405	3402	30618	288684
allow ⊼	0	5	44	168	1532	0	9	124	744	8764
allow only ⊼ etc. for $k=2$	0	0	2	12	76	0	0	12	84	868
allow only ⊼ etc. for $k \le 3$	0	0	0	0	0	0	0	8	16	296
allow only ⊼ etc. for $k \le 4$	0	0	0	0	0	0	0	0	0	100

$\{((b\circ b)\circ a)\circ(a\circ b) = a\}$ allows the $k = 3$ operator 15552 for which the NAND theorem $(p\circ p)\circ q = (p\circ q)\circ q$ is not true. $\{(((b\circ a)\circ c)\circ a)\circ(a\circ c) = a\}$ allows the $k = 4$ operator 95356335 for which even $p\circ q = q\circ p$ is not true. Of the 100 cases that remain when $k = 4$, the 25 inequivalent under renaming of variables and reversing arguments of $\circ$ are

$$\begin{gathered}\{(b\circ(b\circ(a\circ a)))\circ(a\circ(b\circ c)),\\ (b\circ(b\circ(a\circ a)))\circ(a\circ(c\circ b)),\ (b\circ(b\circ(a\circ b)))\circ(a\circ(b\circ c)),\\ (b\circ(b\circ(a\circ b)))\circ(a\circ(c\circ b)),\ (b\circ(b\circ(a\circ c)))\circ(a\circ(c\circ b)),\\ (b\circ(b\circ(b\circ a)))\circ(a\circ(b\circ c)),\ (b\circ(b\circ(b\circ a)))\circ(a\circ(c\circ b)),\\ (b\circ(b\circ(c\circ a)))\circ(a\circ(b\circ c)),\ (b\circ((a\circ b)\circ b))\circ(a\circ(b\circ c)),\\ (b\circ((a\circ b)\circ b))\circ(a\circ(c\circ b)),\ (b\circ((a\circ c)\circ b))\circ(a\circ(c\circ b)),\\ ((b\circ c)\circ a)\circ(b\circ(b\circ(a\circ b))),\ ((b\circ c)\circ a)\circ(b\circ(b\circ(a\circ c))),\\ ((b\circ c)\circ a)\circ(b\circ((a\circ a)\circ b)),\ ((b\circ c)\circ a)\circ(b\circ((a\circ b)\circ b)),\\ ((b\circ c)\circ a)\circ(b\circ((a\circ c)\circ b)),\ ((b\circ c)\circ a)\circ(b\circ((b\circ a)\circ b)),\\ ((b\circ c)\circ a)\circ(b\circ((c\circ a)\circ b)),\ ((b\circ c)\circ a)\circ(c\circ(c\circ(a\circ b))),\\ ((b\circ c)\circ a)\circ(c\circ(c\circ(a\circ c))),\ ((b\circ c)\circ a)\circ(c\circ((a\circ a)\circ c)),\\ ((b\circ c)\circ a)\circ(c\circ((a\circ b)\circ c)),\ ((b\circ c)\circ a)\circ(c\circ((a\circ c)\circ c)),\\ ((b\circ c)\circ a)\circ(c\circ((b\circ a)\circ c)),\ ((b\circ c)\circ a)\circ(c\circ((c\circ a)\circ c))\}\end{gathered}$$

Of these I was able in 2000—using automated theorem proving—to show that the ones given as (g) and (h) in the main text are indeed axiom systems for logic. (My proof essentially as found by Waldmeister is given on page 810.)

If one adds $a\circ b = b\circ a$ to any of the other 23 axioms above then in all cases the resulting axiom system can be shown to reproduce logic. But from any of the 23 axioms on their own I have never managed to derive $p\circ q = q\circ p$. Indeed, it seems difficult to derive much at all from them—though for example I have found a fairly short proof of $(p\circ p)\circ(p\circ q) = p$ from $\{(b\circ(b\circ(b\circ a)))\circ(a\circ(b\circ c)) = a\}$.

It turns out that the first of the 25 axioms allows the $k = 6$ operator 1885760537655023865453442036 and so cannot be logic. Axioms 3, 19 and 23 allow similar operators, leaving 19 systems as candidate axioms for logic.

It has been known since the 1940s that any axiom system for logic must have at least one axiom that involves more than 2 variables. The results above now show that 3 variables suffice. And adding more variables does not seem to help. The smallest axiom systems with more than 3 variables that work up to $k = 2$ are of the form $\{(((b\circ c)\circ d)\circ a)\circ(a\circ d) = a\}$. All turn out also to work at $k = 3$, but fail at $k = 4$. And with 6 NANDs (as in (g) and (h)) no system of the form $\{\ldots = a\}$ works even up to $k = 4$.

For axiom systems of the form $\{\ldots = a,\ a\circ b = b\circ a\}$:

	2 variables					3 variables				
number of ∘	4	5	6	7	8	4	5	6	7	8
total systems	4	16	80	448	2688	54	405	3402	30618	288684
allow ⊼	0	5	44	168	1532	0	9	124	744	8764
allow only ⊼ etc. for $k=2$	0	4	20	160	748	0	8	80	736	6248
allow only ⊼ etc. for $k \le 3$	0	0	0	64	16	0	0	32	416	2752
allow only ⊼ etc. for $k \le 4$	0	0	0	48	16	0	0	32	384	2368

With 2 variables the inequivalent cases that remain are

$$\begin{gathered}\{(a\circ b)\circ(a\circ(b\circ(a\circ b))),\\ (a\circ b)\circ(a\circ(b\circ(b\circ b))),\ (a\circ(b\circ b))\circ(a\circ(b\circ(b\circ b)))\}\end{gathered}$$

but all of these allow the $k = 6$ operator

$$1885760537655125429738480884$$

and so cannot correspond to basic logic. With 3 variables, all 32 cases with 6 NANDs are equivalent to $(a\circ b)\circ(a\circ(b\circ c))$, which is axiom system (f) in the main text. With 7 NANDs there are 8 inequivalent cases:

$$\begin{gathered}\{(a\circ a)\circ(b\circ(b\circ(a\circ c))),\ (a\circ b)\circ(a\circ(b\circ(a\circ b))),\ (a\circ b)\circ(a\circ(b\circ(a\circ c))),\\ (a\circ b)\circ(a\circ(b\circ(b\circ b))),\ (a\circ b)\circ(a\circ(b\circ(b\circ c))),\\ (a\circ b)\circ(a\circ(b\circ(c\circ c))),\ (a\circ b)\circ(a\circ(c\circ(a\circ c))),\ (a\circ b)\circ(a\circ(c\circ(c\circ c)))\}\end{gathered}$$

and of these at least 5 and 6 can readily be proved to be axioms for logic.

Any axiom system must consist of equivalences valid for the operator it describes. But the fact that there are fairly few short such equivalences for *Nand* (see page 818) implies that there can be no axiom system for *Nand* with 6 or less NANDs except the ones discussed above.

■ **Two-operator logic.** If one allows two operators then one can get standard logic if one of these operators is forced to be *Not* and the other is forced to be *And*, *Or* or *Implies*—or in fact any of operators 1, 2, 4, 7, 8, 11, 13, 14 from page 806.

A simple example that allows *Not* and either *And* or *Or* is the Robbins axiom system from page 773. Given the first two axioms (commutativity and associativity) it turns out that no shorter third axiom will work in this case (though ones such as $f[g[f[a, g[f[a, b]]]], g[g[b]]] = b$ of the same size do work).

Much as in the single-operator case, to reproduce logic two pairs of operators must be allowed for $k = 2$, none for $k = 3$, 12 for $k = 4$, and so on. Among single axioms, the shortest that works up to $k = 2$ is $(\neg(\neg(\neg b \vee a) \vee \neg(a \vee b))) = a$. The shortest that works up to $k = 3$ is $(\neg(\neg(a \vee b) \vee \neg b) \vee \neg(\neg a \vee a)) = b$. It is known, however, that at least 3 variables must appear in order to reproduce logic, and an example of a single axiom with 4 variables that has been found recently to work is $\{(\neg(\neg(c \vee b) \vee \neg a) \vee \neg(\neg(\neg d \vee d) \vee \neg a \vee c)) = a\}$.

■ **Page 808 · History.** (See page 1151.) (c) was found by Henry Sheffer in 1913; (e) by Carew Meredith in 1967. Until this book, very little work appears to have been done on finding short axioms for logic in terms of *Nand*. Around 1949 Meredith found the axiom system

$$\{(a \circ (b \circ c)) \circ (a \circ (b \circ c)) = \\ ((c \circ a) \circ a) \circ ((b \circ a) \circ a),\ (a \circ a) \circ (b \circ a) = a\}$$

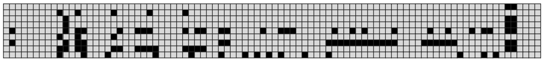
In 1967 George Spencer Brown found (see page 1173)

$$\{(a \circ a) \circ ((b \circ b) \circ b) = a, \\ a \circ (b \circ c) = (((c \circ c) \circ a) \circ ((b \circ b) \circ a)) \circ (((c \circ c) \circ a) \circ ((b \circ b) \circ a))\}$$

and in 1969 Meredith also gave the system

$$\{a \circ (b \circ (a \circ c)) = a \circ (b \circ (b \circ c)),\ (a \circ a) \circ (b \circ a) = a,\ a \circ b = b \circ a\}$$

■ **Page 812 · Theorem distributions.** The picture below shows which of the possible theorems from page 812 hold for each of the numbered standard mathematical theories from page 805. The theorem close to the right-hand end valid in many cases is $(p \circ p) \circ p = p \circ (p \circ p)$. The lack of regularity in this picture can be viewed as a sign that it is difficult to tell which theorems hold, and thus in effect to do mathematics.

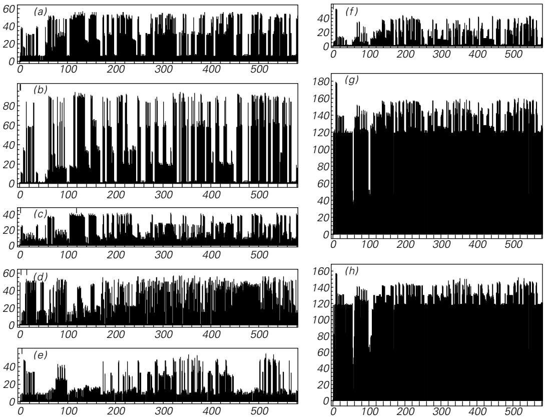


■ **Page 814 · Multivalued logic.** As noted by Jan Łukasiewicz and Emil Post in the early 1920s, it is possible to generalize ordinary logic to allow k values $Range[0, 1, 1/(k-1)]$, say with 0 being *False*, and 1 being *True*. Standard operations in logic can be generalized as $Not[a\_] = 1 - a$, $And[a\_, b\_] = Min[a, b]$, $Or[a\_, b\_] = Max[a, b]$, $Xor[a\_, b\_] = Abs[a - b]$, $Equal[a\_, b\_] = 1 - Abs[a - b]$, $Implies[a\_, b\_] = 1 - UnitStep[a - b]\,(a - b)$. An alternative generalization for *Not* is $Not[a\_] := Mod[(k-1)\,a + 1, k]/(k-1)$. The function $Nand[a\_, b\_] := Not[And[a, b]]$ used in the main text turns out to be universal for any k. Axiom systems can be set up for multivalued logic, but they are presumably more complicated than for ordinary $k = 2$ logic. (Compare page 1171.)

The idea of intermediate truth values has been discussed intermittently ever since antiquity. Often—as in the work of George Boole in 1847—a continuum of values between 0 and 1 are taken to represent probabilities of events, and this is the basis for the field of fuzzy logic popular since the 1980s.

■ **Page 814 · Proof lengths in logic.** As discussed on page 1170 equivalence between expressions can always be proved by transforming to and from canonical form. But with n variables a DNF-type canonical form can be of size 2^n—and can take up to at least 2^n proof steps to reach. And indeed if logic proofs could in general be done in a number of steps that increases only like a polynomial in n this would mean that the NP-complete problem of satisfiability could also be solved in this number of steps, which seems very unlikely (see page 768).

In practice it is usually extremely difficult to find the absolute shortest proof of a given logic theorem—and the exact length will depend on what axiom system is used, and what kinds of steps are allowed. In fact, as mentioned on page 1155, if one does not allow lemmas some proofs perhaps have to become exponentially longer. The picture below shows in each of the axiom systems from page 808 the lengths of the shortest proofs found by a version of Waldmeister (see page 1158) for all 582 equivalences (see page 818) that involve two variables and up to 3 NANDs on either side.



The longest of these are respectively $\{57, 94, 42, 57, 55, 53, 179, 157\}$ and occur for theorems

$$\{(((a \barwedge a) \barwedge b) \barwedge b) = (((a \barwedge b) \barwedge a) \barwedge a), \\ (a \barwedge (a \barwedge (a \barwedge a))) = (a \barwedge ((a \barwedge b) \barwedge b)),\ (((a \barwedge a) \barwedge a) \barwedge a) = \\ (((a \barwedge a) \barwedge b) \barwedge a),\ (((a \barwedge a) \barwedge b) \barwedge b) = (((a \barwedge b) \barwedge a) \barwedge a), \\ (a \barwedge ((b \barwedge b) \barwedge a)) = (b \barwedge ((a \barwedge a) \barwedge b)),\ ((a \barwedge a) \barwedge a) = ((b \barwedge b) \barwedge b), \\ ((a \barwedge a) \barwedge a) = ((b \barwedge b) \barwedge b),\ ((a \barwedge a) \barwedge a) = ((b \barwedge b) \barwedge b)\}$$

Note that for systems that do not already have it as an axiom, most theorems use the lemma $(a \barwedge b) = (b \barwedge a)$ which takes respectively $\{6, 1, 8, 49, 8, 1, 119, 118\}$ steps to prove.

■ **Page 818 · NAND theorems.** The total number of expressions with n NANDs and s variables is: $Binomial[2\,n, n]\,s^{n+1}/(n+1)$ (see page 897). With $s = 2$ and n from 0 to 7 the number of these *True* for all values of variables is $\{0, 0, 4, 0, 80, 108, 2592, 7296\}$, with the first few distinct ones being (see page 781)

$$\{(p \barwedge p) \barwedge p,\ (((p \barwedge p) \barwedge p) \barwedge p) \barwedge p,\ (((p \barwedge p) \barwedge p) \barwedge q) \barwedge q\}$$

The number of unequal expressions obtained is *{2, 3, 3, 7, 10, 15, 12, 16}* (compare page 1096), with the first few distinct ones being

$$\{p, p \barwedge p, p \barwedge q, (p \barwedge p) \barwedge p, (p \barwedge q) \barwedge p, (p \barwedge p) \barwedge q\}$$

Most of the axioms from page 808 are too long to appear early in the list of theorems. But those of system (d) appear at positions *{3, 15, 568}* and those of (e) at *{855, 4}*.

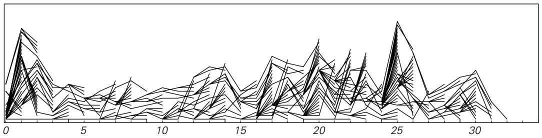
(See also page 1096.)

▪ **Page 819 · Finite axiomatizability.** It is known that the axiom systems (such as Peano arithmetic and set theory) given with axiom schemas on pages 773 and 774 can be set up only with an infinite number of individual axioms. But because such axioms can be described by schemas they must all have similar forms, so that even though the definition in the main text suggests that each corresponds to an interesting theorem these theorems are not in a sense independently interesting. (Note that for example the theory of specifically finite groups cannot be set up with a finite number even of schemas—or with any finite procedure for checking whether a given candidate axiom should be included.)

▪ **Page 820 · Empirical metamathematics.** One can imagine a network representing some field of mathematics, with nodes corresponding to theorems and connections corresponding to proofs, gradually getting filled in as the field develops. Typical rough characterizations of mathematical results—independent of their detailed history—might then end up being something like the following:

- lemma: short theorem appearing at an intermediate stage in a proof
- corollary: theorem connected by a short proof to an existing theorem
- easy theorem: theorem having a short proof
- difficult theorem: theorem having a long proof
- elegant theorem: theorem whose statement is short and somewhat unique
- interesting theorem (see page 817): theorem that cannot readily be deduced from earlier theorems, but is well connected
- boring theorem: theorem for which there are many others very much like it
- useful theorem: theorem that leads to many new ones
- powerful theorem: theorem that substantially reduces the lengths of proofs needed for many others
- surprising theorem: theorem that appears in an otherwise sparse part of the network of theorems
- deep theorem: theorem that connects components of the network of theorems that are otherwise far away
- important theorem: theorem that allows a broad new area of the network to be reached.

The picture below shows the network of theorems associated with Euclid's *Elements*. Each stated theorem is represented by a node connected to the theorems used in its stated proof. (Only the shortest connection from each theorem is shown explicitly.) The axioms (postulates and common notions) are given in the first column on the left, and successive columns then show theorems with progressively longer proofs. (Explicit annotations giving theorems used in proofs were apparently added to editions of Euclid only in the past few centuries; the picture below extends the usual annotations in a few cases.) The theorem with the longest proof is the one that states that there are only five Platonic solids.



▪ **Speedups in other systems.** Multiway systems are almost unique in being able to be sped up just by adding "results" already derived in the multiway system. In other systems, there is no such direct way to insert such results into the rules for the system.

▪ **Character of mathematics.** Since at least the early 1900s several major schools of thought have existed:

- *Formalism* (e.g. David Hilbert): Mathematics studies formal rules that have no intrinsic meaning, but are relevant because of their applications or history.
- *Platonism* (e.g. Kurt Gödel): Mathematics involves trying to discover the properties of a world of ideal mathematical forms, of which we in effect perceive only shadows.
- *Logicism* (e.g. Gottlob Frege, Bertrand Russell): Mathematics is an elaborate application of logic, which is itself fundamental.
- *Intuitionism* (e.g. Luitzen Brouwer): Mathematics is a precise way of handling ideas that are intuitive to the human mind.

The results in this book establish a new point of view somewhere between all of these.

▪ **Invention versus discovery in mathematics.** One generally considers things invented if they are created in a somewhat arbitrary way, and discovered if they are identified by what seems like a more inexorable process. The results of this section thus strongly suggest that the basic directions taken by mathematics as currently practiced should mostly be considered invented, not discovered. The new kind of science that I describe in this book, however, tends to suggest forms of mathematics that involve discovery rather than invention.

▪ **Ordering of constructs.** One can deduce some kind of ordering among standard mathematical constructs by seeing how difficult they are to implement in various systems—such as cellular automata, Turing machines and Diophantine equations. My experience has usually been that addition is easiest, followed by multiplication, powers, Fibonacci numbers, perfect numbers and then primes. And perhaps this is similar to the order in which these constructs appeared in the early history of mathematics. (Compare page 640.)

▪ **Mathematics and the brain.** A possible reason for some constructs to be more common in mathematics is that they are somehow easier for human brains to manipulate. Typical human experience makes small positive integers and simple shapes familiar—so that all human brains are at least well adapted to such constructs. Yet of the limited set of people exposed to higher mathematics, different ones often seem to think in bizarrely different ways. Some think symbolically, presumably applying linguistic capabilities to algebraic or other representations. Some think more visually, using mechanical experience or visual memory. Others seem to think in terms of abstract patterns, perhaps sometimes with implicit analogies to musical harmony. And still others—including some of the purest mathematicians—seem to think directly in terms of constraints, perhaps using some kind of abstraction of everyday geometrical reasoning.

In the history of mathematics there are many concepts that seem to have taken surprisingly long to emerge. And sometimes these are ones that people still find hard to grasp. But they often later seem quite simple and obvious—as with many examples in this book.

It is sometimes thought that people understand concepts in mathematics most easily if they are presented in the order in which they arose historically. But for example the basic notion of programmability seems at some level quite easy even for young children to grasp—even though historically it appeared only recently.

In designing *Mathematica* one of my challenges was to use constructs that are at least ultimately easy for people to understand. Important criteria for this in my experience include specifying processes directly rather than through constraints, the explicitness in the representation of input and output, and the existence of small, memorable, examples. Typically it seems more difficult for people to understand processes in which larger numbers of different things happen in parallel. (Notably, *FoldList* normally seems more difficult to understand than *NestList*.) Tree structures such as *Mathematica* expressions are fairly easy to understand. But I have never found a way to make general networks similarly easy, and I am beginning to suspect that they may be fundamentally difficult for brains to handle.

▪ **Page 821 · Frameworks.** Symbolic integration was in the past done by a collection of ad hoc methods like substitution, partial fractions, integration by parts, and parametric differentiation. But in *Mathematica Integrate* is now almost completely systematic, being based on structure theorems for finding general forms of integrals, and on general representations in terms of *MeijerG* and other functions. (In recognizing, for example, whether an expression involving a parameter can have a pole undecidable questions can in principle come up, but they seem rare in practice.) Proofs are essentially always still done in an ad hoc way—with a few minor frameworks like enumeration of cases, induction, and proof by contradiction (reductio ad absurdum) occasionally being used. (More detailed frameworks are used in specific areas; an example are ϵ-δ arguments in calculus.) But although still almost unknown in mainstream mathematics, methods from automated theorem proving (see page 1157) are beginning to allow proofs of many statements that can be formulated in terms of operator systems to be found in a largely systematic way (e.g. page 810). (In the case of Euclidean geometry—which is a complete axiom system—algebraic methods have allowed complete systematization.) In general, the more systematic the proofs in a particular area become, the less relevant they will typically seem compared to the theorems that they establish as true.

Intelligence in the Universe

▪ **Page 822 · Animism.** Attributing abstract human qualities such as intelligence to systems in nature is a central part of the idea of animism, discussed on page 1195.

▪ **Page 822 · The weather.** Almost all the intricate variations of atmospheric temperature, pressure, velocity and humidity that define the weather we see are in the end determined by fairly simple underlying rules for fluid behavior. (Details of phase changes in water are also important, as are features of topography, ocean currents, solar radiation levels and presumably land use.) Our everyday personal attempts to predict the weather tend to be based just on looking at local conditions and then recalling what happened when we saw these conditions before. But ever since the mid-1800s synoptic weather maps of large areas have been available that summarize conditions in terms of features like fronts and cyclones. And predictions made by looking at simple trends in these features tend at least in some situations to work fairly well. Starting in the 1940s more systematic efforts to predict weather were made by using computers to run approximations to fluid equations. The approximations have improved as larger computers have become available. But even though millions of discrete samples are now used, each one typically still represents something much larger then for example a single cloud. Yet ever since the 1970s, the approach has had at least some success in making predictions up to several days in advance. But although there has been gradual improvement it is usually assumed that—like in the Lorenz equations—the phenomenon of chaos must make forecasts that are based on even slightly different initial measurements eventually diverge exponentially (see page 972). Almost certainly this does indeed happen in at least some critical situations. But it seems that over most of a typical weather map there is no such sensitivity—so that in the end the difficulties of weather prediction are probably much more a result of computational irreducibility and of the sophisticated kinds of computations that the Principle of Computational Equivalence implies should often occur even in simple fluids.

▪ **Page 822 · Defining intelligence.** The problem of defining intelligence independent of specific education and culture has been considered important for human intelligence testing since the beginning of the 1900s. Charles Spearman suggested in 1904 that there might be a general intelligence factor (usually called *g*) associated with all intellectual tasks. Its nature was never very clear, but it was thought that its value could be inferred from performance on puzzles involving numbers, words and pictures. By the 1980s, however, there was increasing emphasis on the idea that different types of human tasks require different types of intelligence. But throughout the 1900s psychologists occasionally tried to give general definitions of intelligence—initially usually in terms of learning or problem-solving capabilities; later more often in terms of adaptation to complex environments.

Particularly starting at the end of the 1800s there was great interest in whether animals other than humans could be considered intelligent. The most common general criterion used was the ability to show behavior based on conceptual or abstract thinking rather than just simple instincts. More specific criteria also included ability to use tools, plan actions, use language, solve logical problems and do arithmetic. But by the mid-1900s it became increasingly clear that it was very difficult to interpret actual observations—and that unrecognized cues could for example often account for the behavior seen.

When the field of artificial intelligence began in the mid-1900s there was some discussion of appropriate definitions of intelligence (see page 1099). Most focused on mathematical or other problem solving, though some—such as the Turing test—emphasized everyday conversation with humans.

▪ **Page 823 · Mimesis.** The notion of inanimate analogs of memory—such as impressions in wax—was discussed for example by Plato in antiquity.

▪ **Page 823 · Defining life.** Greek philosophers such as Aristotle defined life by the presence of some form of soul, and the idea that there must be a single unique feature associated with life has always remained popular. In the 1800s the notion of a "life force" was discussed—and thought to be associated perhaps with chemical properties of protoplasm, and perhaps with electricity. The discovery from the mid-1800s to the mid-1900s of all sorts of elaborate chemical processes in living systems led biologists often to view life as defined by its ability to maintain fixed overall structure while achieving chemical functions such as metabolism. When the Second Law of Thermodynamics was formulated in the mid-1800s living systems were usually explicitly excluded (see page 1021), and by the 1930s physicists often considered local entropy decrease a defining feature of life. Among geneticists and soon mathematicians self-reproduction was usually viewed as the defining feature of life, and following the discovery of the structure of DNA in 1953 it came to be widely believed that the presence of self-replicating elements must be fundamental to life. But the recognition that just copying information is fairly easy led in the 1960s to definitions of life based on the large amounts of information encoded in its genetic material, and later to ones based on the apparent difficulty of deriving this information (see page 1069). And perhaps in part reacting to my discoveries about cellular automata it became popular in the 1980s to mention adaptation and essential interdependence of large numbers of different kinds of parts as further necessary characteristics of life. Yet in the end every single general definition that has been given both includes systems that are not normally considered alive, and excludes ones that are. (Self-reproduction, for example, suggests that flames are alive, but mules are not.)

One of the features that defines life on Earth is the presence of DNA, or at least RNA. But as one looks at smaller molecules they become less specific to living systems. It is sometimes thought significant that living systems perpetuate the use of only one chirality of molecules, but actually this can quite easily be achieved by various forms of non-chemical input without life.

The Viking spacecraft that landed on Mars in the 1970s tried specific tests for life on soil samples—essentially whether gases were generated when nutrients were added, whether this behavior changed if the samples were first heated, and whether molecules common in terrestrial life were present. The tests gave confusing results, presumably having to do not with life, but rather with details of martian soil chemistry

▪ **Origin of life.** Fossil traces of living cells have been found going back more than 3.8 billion years—to perhaps as little as 700 million years after the formation of the Earth. There were presumably simpler forms of life that preceded the advent of recognizable cells, and even if life arose more than once it is unlikely that evidence of this would remain. (One sees many branches in the fossil record—such as organisms with dominant symmetries other than fivefold—but all seem to have the same ancestry.)

From antiquity until the 1700s it was widely believed that smaller living organisms arise spontaneously in substances like mud, and this was not finally disproved until the 1860s. Controversy surrounding the theory of biological evolution in the late 1800s dissuaded investigation of non-biological origins for life, and at the end of the 1800s it was for example suggested that life on Earth might have arisen from spores of extraterrestrial origin. In the 1920s the idea developed that electrical storms in the atmosphere of the early Earth could lead to production of molecules seen in living systems—and this was confirmed by the experiment of Stanley Miller and Harold Urey in 1953. The molecules obtained were nevertheless still fairly simple—and as it turns out most of them have now also been found in interstellar space. Starting in the 1960s suggestions were made for the chemical and other roles of constituents of the crust as well as atmosphere. Schemes for early forms of self-replication were invented based on molecules such as RNA and on patterns in clay-like materials. (The smallest known system that independently replicates itself is a mycoplasma bacterium with about 580,000 base pairs and perhaps 470 genes. Viroids can be as small as 10,000 atoms but require a host for replication.) In the 1970s it then became popular to investigate complicated cycles of chemical reactions that seemed analogous to ones found in living systems. But with the advent of widespread computer simulations in the 1980s it became clear that all sorts of features normally associated with life were actually rather easy to obtain. (See note above.)

▪ **Page 824 · Self-reproduction.** That one can for example make a mold that will produce copies of a shape has been known since antiquity (see note above). The cybernetics movement highlighted the question of what it takes for self-reproduction to occur autonomously, and in 1952 John von Neumann designed an elaborate 2D cellular automaton that would automatically make a copy of its initial configuration of cells (see page 876). In the course of the 1950s suggestions of several increasingly simple mechanical systems capable of self-reproduction were made—notably by Lionel Penrose. The phenomenon in the main text was noticed around 1961 by Edward Fredkin (see page 877). But while it shows some of the essence of self-reproduction, it lacks many of the more elaborate features common in biological self-reproduction. In the 1980s, however, such features were nevertheless surprisingly often present in computer viruses and worms. (See also page 1092.)

▪ **Page 825 · Extraterrestrial life.** Conditions thermally and chemically similar to those on Earth have presumably existed on other bodies in the solar system. Venus, Mars, Europa (a moon of Jupiter) and Titan (a moon of Saturn) have for example all probably had liquid water at some time. But there is so far no evidence for life now or in the past on any of these. Yet if life had arisen one might expect it to have become widespread, since at least on Earth it has managed to spread to many extremes of temperature, pressure and chemical composition. On several occasions structures have been found in extraterrestrial rocks that look somewhat like small versions of microorganism fossils (most notably in 1996 in a meteorite from Mars discovered in Antarctica). But almost certainly these structures have nothing to do with life, and are instead formed by ordinary precipitation of minerals. And although even up to the 1970s it was thought that life might well be found on Mars, it now seems likely that there is nothing quite like terrestrial life anywhere else in our solar system. (Even if life is found elsewhere it might still have originally come from Earth, say via meteorites, since dormant forms such as spores can apparently survive for long periods in space.)

Away from our solar system there is increasing evidence that most stars have planets with a distribution of sizes—so presumably conditions similar to Earth are fairly common. But thus far it has not been possible to see—say in planetary atmospheres—whether there are for example molecules similar to ones characteristically found in life on Earth.

▪ **Forms of living systems.** This book has shown that even with underlying rules of some fixed type a vast range of different forms can often be produced. And this makes it reasonable to expect that with appropriate genetic programs the chemical building blocks of life on Earth should in principle allow a vast range of forms. But the comparative weakness of natural selection (see page 391) has meant that only a limited set of such forms have actually been explored. And from the experience of this book I suspect that what others might even be nearby is effectively impossible to foresee. The appearance in engineering of forms somewhat like those in living systems should not be taken to imply that other forms are fundamentally difficult to produce; instead I suspect that it is more a reflection of the copying of living systems for engineering purposes. The overall morphology of living systems on Earth seems to be greatly affected by their basically gelatinous character. So even systems based on solids or gases would likely not be recognized by us as life.

■ **Page 825 · History.** Although Greek philosophers such as Democritus believed that there must be an infinite number of worlds all with inhabitants like us, the prevailing view in antiquity—later supported by theological arguments—was that the Earth is special, and the only abode of life. However, with the development of Copernican ideas in the 1600s it came to be widely though not universally believed, even in theological circles, that other planets—as well as the Moon—must have inhabitants like us. Many astronomers attributed features they saw on the Moon to life if not intelligence, but in the late 1800s, after it was found that the Moon has no atmosphere, belief in life there began to wane. Starting in the 1870s, however, there began to be great interest in life on Mars, and it was thought—perhaps following the emphasis on terrestrial canal-building at the time—that a vast network of canals on Mars had been observed. And although in 1911 the apparent building of new canals on Mars was still being soberly reported by newspapers, there was by the 1920s increasing skepticism. The idea that lichens might exist on Mars and be responsible for seasonal changes in color nevertheless became popular, especially after the discovery of atmospheric carbon dioxide in 1947. Particularly in the 1920s there had been occasional claims of extraterrestrial radio signals (see page 1188), but by the 1950s interest in extraterrestrial intelligence had largely transferred to science fiction (see page 1190). Starting in the late 1940s many sightings were reported of UFOs believed to be alien spacecraft, but by the 1960s these were increasingly discredited. It had been known since the mid-1800s that many other stars are much like the Sun, but it was not until the 1950s that evidence of planets around other stars began to accumulate. Following a certain amount of discussion in the physics community in the 1950s, the first explicit search for extraterrestrial intelligence with a radio telescope was done in 1960 (see page 1189). In the 1960s landings of spacecraft on the Moon confirmed the absence of life there—though returning Apollo astronauts were still quarantined to guard against possible lunar microbes. And despite substantial expectations to the contrary, when spacecraft landed on Mars in 1976 they found no evidence of life there. Some searches for extraterrestrial signals have continued in the radio astronomy community, but perhaps because of its association with science fiction, the topic of extraterrestrial intelligence has generally not been popular with professional scientists. With the rise of amateur science on the web and the availability of low-cost radio telescope components the late 1990s may however have seen a renewal of serious interest.

■ **Page 826 · Bird songs.** Essentially all birds produce calls of some kind, but complex songs are mainly produced by male songbirds, usually in breeding season. Their general form is inherited, but specifics are often learned through imitation during a fixed period of infancy, leading birds in local areas to have distinctive songs. The songs sometimes seem to be associated with attracting mates, and sometimes with defining territory—but often their function is unclear, even when one bird seems to sing in response to another. (There are claims, however, that parrots can learn to have meaningful conversations with humans.) The syrinxes of songbirds have two membranes, which can vibrate independently, in a potentially complex way. A specific region in bird brains appears to coordinate singing; the region contains a few tens of thousands of nerve cells, and is larger in species with more complex songs.

Famous motifs from human music are heard in bird songs probably more often than would be expected by chance. It may be that some common neural mechanism makes the motifs seem pleasing to both birds and humans. Or it could be that humans find them pleasing because they are familiar from bird songs.

■ **Page 826 · Whale songs.** Male whales can produce complex songs lasting tens of minutes during breeding season. The songs often include rhyme-like repeating elements. At a given time all whales in a group typically sing almost the same song, which gradually changes. The function of the song is quite unclear. It has been claimed that its frequencies are optimized for long-range transmission in the ocean, but this appears not to be the case. In dolphins, it is known that one dolphin can produce patterns of sound that are repeated by a specific other dolphin.

■ **Page 826 · Animal communication.** Most animals that live in groups have the capability to produce at least a few specific auditory, visual (e.g. gestures and displays), chemical (e.g. pheromones) or other signals in response to particular situations such as danger. Some animals have also been found to produce much more complex and varied signals.

For example it was discovered in the 1980s that elephants can generate elaborate patterns of sounds—but at frequencies below human hearing. Animals such as octopuses and particularly cuttlefish can show complex and changing patterns of pigmentation. But despite a fair amount of investigation it remains unclear whether these represent more than just simple responses to the environment.

■ **Page 826 · Theories of communication.** Over the course of time the question of what the essential features of communication are has been discussed from many different angles. It appears to have always been a common view that communication somehow involves transferring thoughts from one mind to another. Even in antiquity it was nevertheless recognized that all sorts of aspects of language are purely matters of convention, so that shared conventions are necessary for verbal communication to be possible. In the 1600s the philosophical idea that the only way to get information with certainty is from the senses led to emphasis on observable aspects of communication, and to the conclusion that there is no way to tell whether an accurate transfer of abstract thoughts has occurred between one mind and another. In the late 1600s Gottfried Leibniz nevertheless suggested that perhaps a universal language—modelled on mathematics—could be created that would represent all truths in an objective way accessible to any mind (compare page 1149). But by the late 1800s philosophers like Charles Peirce had developed the idea that communication must be understood purely in terms of its observable features and effects. Three levels of so-called semiotics were then discussed. The first was syntax: the grammatical or other structure of a sequence of verbal or other elements. The second was semantics: the standardized meaning or meanings of the sequence of elements. And the third was pragmatics: the observable effect on those involved in the communication. In the early 1900s, the logical positivism movement suggested that perhaps a universal language or formalism based on logic could be developed that would allow at least scientific truths to be communicated in an unambiguous way not affected by issues of pragmatics—and that anything that could not be communicated like this was somehow meaningless. But by the 1940s it came to be believed—notably by Ludwig Wittgenstein—that ordinary language, with its pragmatic context, could in the end communicate fundamentally more than any formalized logical system, albeit more ambiguously.

Ever since antiquity work has been done to formalize grammatical and other rules of individual human languages. In the early 1900s—notably with the work of Ferdinand de Saussure—there began to be more emphasis on the general question of how languages really operate, and the point was made that the verbal elements or signs in a language should be viewed as somehow intermediate between tangible entities like sounds and abstract thoughts and concepts. The properties of any given sign were recognized as arbitrary, but what was then thought to be essential about a language is the structure of the network of relations between signs—with the ultimate meaning of any given sign inevitably depending on the meanings of signs related to it (as later emphasized in deconstructionism). By the 1950s anthropological studies of various languages—notably by Benjamin Whorf—had encouraged the idea that concepts that did not appear to fit in certain languages simply could not enter the thinking of users of those languages. Evidence to the contrary (notably about past and future among Hopi speakers) eroded this strong form of the so-called Sapir-Whorf hypothesis, so that by the 1970s it was generally believed just that language can have an influence on thinking—a phenomenon definitely seen with mathematical notation and computer languages. Starting in the 1950s, especially with the work of Noam Chomsky, there were claims of universal features in human languages—independent of historical or cultural context (see page 1103). But at least among linguists these are generally assumed just to reflect common aspects of verbal processing in the human brain, not features that must necessarily appear in any conceivable language. (And it remains unclear, for example, to what extent non-verbal forms of communication such as music, gestures and visual ornament show the same grammatical features as ordinary languages.)

The rise of communications technology in the early 1900s led to work on quantitative theories of communication, and for example in 1928 Ralph Hartley suggested that an objective measure of the information content of a message with n possible forms is `Log[n]`. (Similar ideas arose around the same time in statistics, and in fact there had already been work on probabilistic models of written language by Andrei Markov in the 1910s.) In 1948 Claude Shannon suggested using a measure of information based on p `Log[p]`, and there quickly developed the notion that this could be used to find the fundamental redundancy of any sequence of data, independent of its possible meaning (compare page 1071). Human languages were found on this basis to have substantial redundancy (see page 1086), and it has sometimes been suggested that this is important to their operation—allowing errors to be corrected and details of different users to be ignored. (There are also obvious features which reduce redundancy—for example that in most languages common words tend to be short. One can also imagine models of the historical development of languages which will tend to lead to redundancy at the level of Shannon information.)

■ **Mathematical notation.** While it is usually recognized that ordinary human languages depend greatly on history and context, it is sometimes believed that mathematical notation is somehow more universal. But although it so happens that essentially the same mathematical notation is in practice used all around the world by speakers of every ordinary language, I do not believe that it is in any way unique or inevitable, and in fact I think it shows most of the same issues of dependence on history and context as any ordinary language.

As a first example, consider the case of numbers. One can always just use *n* copies of the same symbol to represent an integer *n*—and indeed this idea seems historically to have arisen independently quite a few times. But as soon as one tries to set up a more compact notation there inevitably seem to be many possibilities. And so for example the Greek and Roman number systems were quite different from current Hindu-Arabic base-10 positional notation. Particularly from working with computers it is often now assumed that base-2 positional notation is somehow the most natural and fundamental. But as pages 560 and 916 show, there are many other quite different ways to represent numbers, each with different levels of convenience for different purposes. And it is fairly easy to see how a different historical progression might have ended up making another one of these seem the most natural.

The idea of labelling entities in geometrical diagrams by letters existed in Babylonian and Greek times. But perhaps because until after the 1200s numbers were usually also represented by letters, algebraic notation with letters for variables did not arise until the late 1500s. The idea of having notation for operators emerged in the early 1600s, and by the end of the 1600s, notably with the work of Gottfried Leibniz, essentially all the basic notation now used in algebra and calculus had been established. Most of it was ultimately based on shortenings and idealizations of ordinary language, an important early motivation just being to avoid dependence on particular ordinary languages. Notation for mathematical logic began to emerge in the 1880s, notably with the work of Giuseppe Peano, and by the 1930s it was widely used as the basis for notation in pure mathematics.

In its basic structure of operators, operands, and so on, mathematical notation has always been fairly systematic—and is close to being a context-free language. (In many ways it is like a simple idealization of ordinary language, with operators being like verbs, operands nouns, and so on.) And while traditional mathematical notation suffers from some inconsistencies and ambiguities, it was possible in developing *Mathematica* StandardForm to set up something very close that can be interpreted uniquely in all cases. Mathematical notation works well for things like ordinary formulas that involve a comparatively small number of basic operations. But there has been no direct generalization for more general constructs and computations. And indeed my goal in designing *Mathematica* was precisely to provide a uniform notation for these (see page 852). Yet to make this work I had to use names derived from ordinary language to specify the primitives I defined.

■ **Computer communication.** Most protocols for exchanging data between computers have in the end traditionally had rather simple structures—with different pieces of information usually being placed at fixed positions, or at least being arranged in predefined sequences—or sometimes being given in name-value pairs. A more general approach, however, is to use tree-structured symbolic expressions of the kind that form the basis for *Mathematica*—and now in essence appear in XML. In the most general case one can imagine directly exchanging a representation of a program, that is run on the computer that receives it, and induces whatever effect one wants. A simple example from 1984 is *PostScript*, which can specify a picture by giving a program for constructing it; a more sophisticated example from the late 1990s is client-side Java. (Advanced forms of data compression can also be thought of as working by sending simple programs.) But a practical problem in exchanging arbitrary programs is the difficulty of guarding against malicious elements like viruses. And although at some level most communications between present-day computers are very regular and structured, this is often obscured by compression or encryption.

When a program is sent between computers it is usually encoded in a syntactically very straightforward way. But computer languages intended for direct use by humans almost always have more elaborate syntax that is a simple idealization of ordinary human language (see page 1103). There are in practice many similarities between different computer languages. Yet except in very low-level languages few of these are necessary consequences of features or limitations of actual computers. And instead most of them must be viewed just as the results of shared history—and the need to fit in with human cognitive abilities.

■ **Meaning in programs.** Many issues about meaning arise for computer languages in more defined versions of the ways they arise for ordinary languages. Input to a computer language will immediately fail to be meaningful if it does not conform to a certain definite syntax. Before the input can have a chance of specifying meaningful action there are often all sorts of issues about whether variables in it refer to entities that can be considered to exist. And even if this is resolved, one can still get something that is in effect nonsense and does not usefully run. In most traditional computer languages it is usually the case that most programs chosen at random will just crash if run, often as a result of trying to write to memory outside the arrays they have allocated. In *Mathematica*, there is almost no similar issue, and programs chosen at random tend instead just to return unchanged. (Compare page 101.)

For the kinds of systems like cellular automata that I have discussed in this book programs chosen at random do very often produce some sort of non-trivial behavior. But as discussed in the main text there is still an issue of when this behavior can reasonably be considered meaningful.

For some purposes a more direct analog of messages is not programs or rules for systems like cellular automata but instead initial conditions. And one might imagine that the very process of running such initial conditions in a system with appropriate underlying rules would somehow be what corresponds to their meaning. But if one was just given a collection of initial conditions without any underlying rules one would then need to find out what underlying rules one was supposed to use in order to determine their meaning. Yet the system will always do something, whatever rules one uses. So then one is back to defining criteria for what counts as meaningful behavior in order to determine—by a kind of generalization of cryptanalysis—what rules one is supposed to use.

■ **Meaning and regularity.** If one considers something to show regularity one may or may not consider it meaningful. But if one considers something random then usually one will also consider it meaningless. For to say that something is meaningful normally implies that one somehow comes to a conclusion from it. And this typically implies that one can find some summary of some aspect of it—and thus some regularity. Yet there are still cases where things that are presumably quite random are considered meaningful—prices in financial markets being one example.

■ **Page 828 · Forms of artifacts.** Much as in biological evolution, once a particular engineering construct has been found to work it normally continues to be used. Examples with characteristic forms include (in rough order of their earliest known use): arrowheads, boomerangs, saws, boxes, stairs, fishhooks, wheels, arches, forks, balls, kites, lenses, springs, catenaries, cogs, screws, chains, trusses, cams, linkages, propellers, clocksprings, parabolic reflectors, airfoils, corrugation, zippers, and geodesic domes. It is notable that not even nested shapes are common, though they appear in cross-sections of rope (see page 874), as well as in address decoder trees on chips—and have recently been used in broadband antennas. (Some self-similarity is also present in standard log-periodic antennas.) When several distinct components are involved, more complicated structures are not uncommon—as in escapements, and many bearings and joints. More complex shapes for single elements sometimes arise when an analog of area maximization is desired—as with tire treads or fins in devices such as heat exchangers. Quadratic residue sequences *Mod[Range[n]^2, n]* (see page 1081) are used to give profiles for acoustic diffusers that operate uniformly over a range of frequencies. Musical instruments can have fairly complicated shapes maintained for historical reasons to considerable precision. Some knots can also be thought of as objects with complex forms. It is notable that elaborate types of mechanical motion (and sometimes surprising phenomena in general) are often first implemented in toys. Examples are early mechanical automata and model airplanes, and modern executive toys claiming to illustrate chaos theory through linkages, magnets or fluid systems. Complex trajectories (compare page 972) have sometimes been proposed or used for spacecraft. (See also notes on ornamental art on page 872.)

■ **Page 828 · Recognizing artifacts.** Various situations require picking out artifacts automatically. One example is finding buildings or machines from aerial reconnaissance images; another is finding boat or airplane wreckage on an ocean floor from sonar data. In both these cases the most common approach is to look for straight edges. Outdoor security systems also often need ways to distinguish animals and wind-induced motion from intentional human activity—and tend to have fairly simple procedures for doing this.

To recognize a regular crystal as not being a carefully cut artifact can take specific knowledge. The same can be true of patterns produced by wind on sand or rocks. Lenticular clouds are sometimes mistaken for UFOs on account of their regular shape. The exact cuboid form of the monolith in the movie *2001* was intended to suggest that it was an artifact.

Recognizing artifacts can be a central—and controversial—issue in prehistoric archeology. Sometimes human bones are found nearby. And sometimes chemical analysis suggests controlled fire—as with charcoal or baked clay. But to tell whether for example a piece of rock was formed naturally or was carefully made to be a stone tool can in general be very difficult. And a large part of the way this has been done in practice is just through comparison with known examples that fit into an overall pattern of gradual historical change. In recent decades there has been increasing emphasis on trying to understand and reproduce the whole process of making and using artifacts. And in the field of lithic analysis there are beginning to be fairly systematic ways to recognize for example the effects of the hundreds of orderly impacts needed to make a typical flint arrowhead by knapping. (Sometimes it is also possible to recognize microscopic features characteristic of particular kinds of use or wear—and it is conceivable that in the future analysis of trillions of atomic-scale features could reveal all sorts of details of the history of an object.)

To tell whether or not some arrangement of soil or rocks is an artifact can be extremely difficult—and there are many notorious cases of continuing controversy. Beyond looking for similarities to known examples, a typical approach is just to look for correlations with topographic or other features that might reveal some possible purpose.

▪ **Artifacts in data.** In fields like accounting and experimental science it is usually a sign of fraud if primary data is being created for a purpose, rather than merely being reported. If a large amount of numerical data has been made up by a person this can be detectable through statistical deviations from expected randomness—particularly in structural details such as frequency of digits. (So-called artifacts can also be the unintentional result of details of methods used to obtain or process data.)

In numerical computations effects are often called artifacts if they are believed not to be genuine features of an underlying mathematical system, but merely to reflect the computational scheme used. Such effects are usually first noticed through unexpected regularities in some detail of output. But in cases like chaos theory it remains unclear to what extent complex behavior seen in computations is an artifact (see page 920).

▪ **Animal artifacts.** Structures like mollusc shells, radiolarian skeletons and to some extent coral are formed through processes of growth like those discussed in Chapter 8. Structures like spider webs, wasp nests, termite mounds, bird nests and beaver dams rely on behavior determined by animal brains. (Even spider webs end up looking quite different if psychoactive drugs are administered to the spider.) And much like human artifacts, many of these structures tend to be distinguished by their comparative geometrical simplicity. In a few cases—particularly with insects—somewhat complicated forms are seen, but it seems likely that these are actually produced by rather simple local rules like those in aggregation systems (see page 1011).

▪ **Molecular biology.** DNA sequences of organisms can be thought of as artifacts created by biological evolution, and current data suggests that they contain some long-range correlations not present in typical random sequences. Most likely, however, these have fairly simple origins, perhaps being associated with iterative splicing of subsequences. And in the few thousand proteins currently known, standard statistical tests reveal no significant overall regularities in their sequences of up to a few thousand amino acids. (Some of the 20 standard amino acids do however occur more frequently than others.) Nevertheless, if one looks at overall shapes into which these proteins fold, there is some evidence that the same patterns of behavior are often seen. But probably such patterns would also occur in purely random proteins—at least if their folding happened in the same cellular apparatus. (See page 1003.) Note that the antibodies of the immune system are much like short random proteins—whose range of shapes must be sufficient to match any antigen. (See also page 1194.)

▪ **Messages in DNA.** Science fiction has sometimes suggested that an extraterrestrial source of life might have left some form of message in the DNA sequences of all terrestrial organisms, but to get evidence of this would seem to require extensive other knowledge of the source. (See also page 1190.)

▪ **Decompilers.** Trying to reverse engineer source code in a programming language like C from machine code output by compilers involves in effect trying to deduce higher-level purposes from lower-level computational steps. And normally this can be done with any reliability only when the machine code represents a fairly direct translation that has not been extensively rearranged or optimized.

▪ **Page 828 · Complexity and theology.** See page 861.

▪ **Page 829 · Purpose in archeology.** Ideas about the purpose of archeological objects most often ultimately tend to come from comparisons to similar-looking objects in use today. But great differences in typical beliefs and ways of life can make comparisons difficult. And certainly it is now very hard for us to imagine just what range of purposes the first known stone tools from 2.6 million years ago might have been put to—or what purpose the arrays of dots or handprints in cave paintings from 30,000 BC might have had. And even when it comes to early buildings from perhaps 10,000 BC it is still difficult to know just how they were used. Stone circles like Stonehenge from perhaps 3000 BC presumably served some community purpose, but beyond that little can convincingly be said. Definite geographical or astronomical alignments can be identified for many large prehistoric structures, but whether these were actually intentional is almost never clear. After the development of writing starting around 4000 BC, purposes can often be deduced from inscriptions and other written material. But still to work out for example the purpose of the Antikythera device from around 100 BC is very difficult, and depends on being able to trace a long historical tradition of astronomical clocks and orreries.

▪ **Dead languages.** Particularly over the past century or so, most of the known written human languages from every point in history have successfully been decoded. But to do this has essentially always required finding a case where there is explicit overlap with a known language—say a

Rosetta stone with the same text in multiple languages, or at least words or proper names that are transliterated. As in cryptanalysis, it is sometimes remarkable how small an amount of text is needed to find a decoding scheme. But usually what is done relies critically on the slowness with which human languages change, and the comparatively limited number of different basic ways in which they work.

▪ **Teleology.** There is a common tendency to project human purposes onto natural objects and events—and this is for example almost universally done by young children. Ancient beliefs often held that things in nature are set up by a variety of gods for a variety of purposes. By 400 BC, following ideas of Anaxagoras, Socrates and Plato discussed the notion that things in nature might in effect be optimally designed for coherent purposes by a single mind. Around 350 BC Aristotle claimed that a full explanation of anything should include its purpose (or so-called final cause, or telos)—but said that for systems in nature this is often just to make the final forms of these systems (their so-called formal cause). The rise of monotheistic religions led to the widespread belief that the universe and everything in it was created for definite purposes by a single god. But the development of mathematical science in the 1600s—and its focus on mechanisms ("efficient causes")—led away from ascribing explicit purposes to physical systems. In the mid-1700s David Hume then claimed on philosophical grounds that we fundamentally have no basis for ascribing purposes to any kind of natural system—though in the 1790s Immanuel Kant argued that even though we cannot know whether there really are such purposes, it is still often necessary for us to think in terms of them. And in fact the notion that systems in biology are so complex that they must have been intelligently designed for a purpose remained common. In the late 1800s Darwinian evolution nevertheless suggested that no such purposeful design was necessary—though in a sense it again introduced a notion of purpose associated with optimization of fitness. Ever since the 1700s economics had been discussed in terms of purposeful activities of rational agents. In the early 1900s there were however general attempts to develop mechanistic explanations in the social sciences, but by the mid-1900s purpose was again widely discussed, especially in economics. And in fact, even in physics, a notion of purpose had actually always been quite common. For whenever a physical system satisfies any kind of implicit equation, this defines a constraint that can be viewed as corresponding to some kind of purpose. (See page 940.) That something like a notion of purpose is being used has been more widely recognized for variational principles like the Principle of Least Action in mechanics from the mid-1700s. Results in the late 1900s in astrophysics and cosmology seemed to suggest that for us to exist our universe must satisfy all sorts of constraints—and to avoid explaining this in terms of purpose the Anthropic Principle was introduced (see page 1026). What I do in this book goes significantly further than traditional science in getting rid of notions of purpose from investigations of nature. For I essentially always consider systems that are based on explicit evolution rules rather than implicit constraints. And in fact I argue that simple programs constructed without known purposes are what one needs to study to find the kinds of complex behavior we see.

▪ **Possible purposes.** As part of asking whether the rules for a system are somehow minimal for a given purpose, one can ask what properties the system has that could reasonably be considered a purpose at all. In general one tends to talk of purpose only when doing so allows one to give a simpler description of some aspect of behavior than just describing the behavior directly. But whether one can give a simple description can depend greatly on the framework in which one is operating. And so, for example, while the digits of π have a simple description in terms of traditional mathematics, the results in Chapter 4 suggest that outside of this framework they normally do not. And what this means is that if one saw a system that had the property of generating the digits of π one would be unlikely to think that this could represent a meaningful purpose—unless one happened to be operating in traditional mathematics. And so similarly, one would be unlikely to think that generating the center column from rule 30 could represent any sort of meaningful purpose—unless one was operating within the framework that I have developed in this book.

▪ **Page 830 · Purposeful computation.** See page 638.

▪ **Page 832 · Doubling rules.** Rule (a) is

```
{{0, 2, _}→ 5, {5, 3, _}→ 5, {5, _, _}→ 1,
  {_, 5, _}→ 1, {_, 2, _}→ 3, {_, 3, 2}→ 2, {_, 1, 2}→ 4,
  {_, 4, _}→ 3, {4, 3, _}→ 4, {4, 0, _}→ 2, {_, x_, _}→ x}
```

and takes $2n^2+n$ steps to yield *Table[1, {2 n}]* given input *Append[Table[1, {n - 1}], 2]*. Rule (b) is

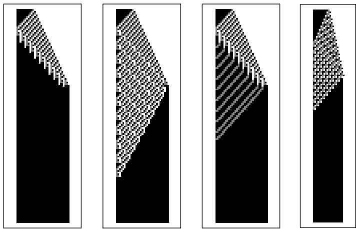
```
{{_, 2, _}→ 3, {_, 1, 2}→ 2, {3, 0, _}→ 1,
  {3, _, _}→ 3, {_, 3, _}→ 1, {_, x_, _}→ x}
```

and takes $3n$ steps. Rule (c) is $k = 3$, $r = 1$ rule *5407067979* and takes $3n-1$ steps.

▪ **Page 833 · Searching.** No symmetric $k = 3$, $r = 1$ rule yields doubling. General rules can show subtle bugs; rule *1340716537107* for example first fails at $n = 24$. The total number of $k = 3$, $r = 1$ rules that need to be searched can easily be reduced from 3^{27} to 3^{21}. Several different rules that work can behave identically, since up to 6 of the 27 cases in each rule are not sampled with the initial conditions used. In rules that work, between 8 and 19 cases lead to a change in the color of a cell, with 14 cases being the most common.

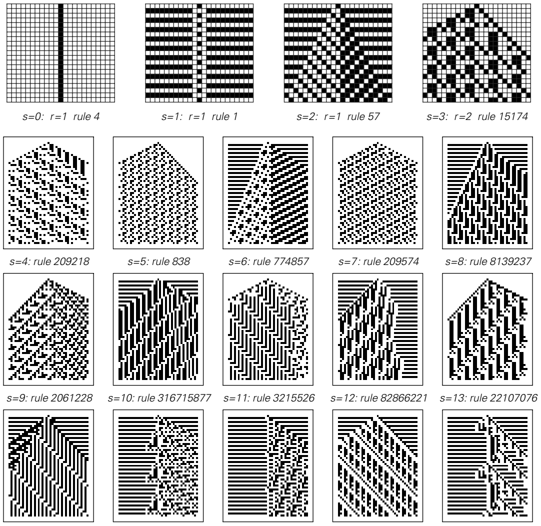
■ **Page 833 · Properties.** The number of steps increases irregularly but roughly quadratically with n in rule (a), and roughly linearly in (d) and (e). Rule (b) in the end repeats every 128 steps. The center of the complex pattern in both (d) and (e) emulates $k = 2$ rule 90.

■ **Other functions.** The first three pictures below show rules that yield $3n$ (no $k = 3$ rules yield $4n$, $5n$ or n^2), and the last picture $2n-2$ (corresponding to doubling with initial conditions analogous to page 639).



304911688608 308527554123 1183925790477 2672802162657

■ **Page 834 · Minimal cellular automata for sequences.** Given any particular sequence of black and white cells one can look for the simplest cellular automaton which generates that sequence as its center column when evolving from a single black cell (compare page 956). The pictures below show the lowest-numbered cellular automaton rules that manage to generate repetitive sequences containing black cells with successively greater separations s.



s=0: r=1 rule 4 s=1: r=1 rule 1 s=2: r=1 rule 57 s=3: r=2 rule 15174

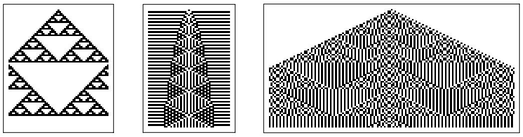
s=4: rule 209218 s=5: rule 838 s=6: rule 774857 s=7: rule 209574 s=8: rule 8139237

s=9: rule 2061228 s=10: rule 316715877 s=11: rule 3215526 s=12: rule 82866221 s=13: rule 22107076

14: rule 1006223533 15: rule 37788005 17: rule 72844195 19: rule 273781569 23: rule 22047073

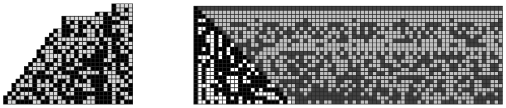
Elementary ($k = 2$, $r = 1$) cellular automata can be found only up to separations $s = 2$. But $k = 2$, $r = 2$ cellular automata can be found for all separations up to 15, as well as 17, 19 and 23. (Note that for example in the $s = 15$ case the lowest-numbered rule exhibits a complex 350-step transient away from the center column.)

The pictures below show the lowest-numbered cellular automata that generate respectively powers of two, squares and the nested Thue-Morse sequence of page 83 (compare rule 150). Of the 4 billion $k = 2$, $r = 2$ cellular automata none turn out to be able to produce for example sequences corresponding to the cubes, powers of 3, Fibonacci numbers, primes, digits of $\sqrt{2}$, or concatenation sequences.



r=1, rule 126 rule 1069090987 rule 69540422

If one looks not just at specific sequences, but instead at all 2^n possible sequences of length n, one can ask how many cellular automaton rules (say with $k = 2$, $r = 2$) one has to go through in order to generate every one of these. The pictures below show on the left the last rules needed to generate any sequence of each successive length—and on the right the form of the sequence (as well as its continuation after length n). Since some different rules generate the same sequences (see page 956) one needs to go through somewhat more than 2^n rules to get every sequence of length n. The sequences shown below can be thought of as being in a sense the ones of each length that are the most difficult to generate—or have the highest algorithmic information content. (Note that the sequence ■■■■□ is the first one that cannot be generated by any of the 256 elementary cellular automata; the first sequence that cannot be generated by any $k = 2$, $r = 2$ cellular automata is probably of length 26.)



rule numbers sequences

■ **Other examples.** Minimal systems achieving particular purposes are shown on page 619 for Boolean functions evaluated with NANDs, pages 759 and 889 for Turing machines, page 1142 for sorting networks, and page 1035 for firing squad synchronization.

■ **Page 834 · Minimal theories.** Particularly in fundamental physics it has been found that the correct theory is often the minimal one consistent with basic observations. Yet barring supernatural intervention, the laws of physics embodied in such a theory presumably cannot be considered to have been created for any particular purpose. (See page 1025.)

▪ **Page 835 · Earth from space.** Human activity has led to a few large simple geometrical structures that are visually noticeable from space. One is the almost-straight 30-mile railroad causeway built in 1959 that divides halves of the Great Salt Lake in Utah where the water is colored blue and orange. Another is the almost-circular 12-mile-diameter national park created in 1900 that encloses ungrazed vegetation on the Egmont Volcano in New Zealand. On the scale of a few miles, there is also rectilinear arrangement of fields in the U.S. Midwest, as well as straight-line political boundaries with different agriculture on each side. Large geometrical patterns of logging were for example briefly visible after snow in 1961 near Cochrane, Canada—as captured by an early weather satellite. Perfectly straight sections of roads (such as the 90-mile Balladonia-Caiguna road in Australia), as well as the 4-mile-diameter perfectly circular Fermilab accelerator ring are not so easy to see. The Great Wall of China from 200 BC follows local topography and so is not straight.

Some of the most dramatic geometrical structures—such as the dendritic fossil drainage pattern in south Yemen or the bilaterally symmetric coral reefs around islands like Bora Bora—are not artifacts. The same is true of fields of parallel sand dunes, as well as of almost-circular structures such as the 40-mile-diameter impact crater in Manicouagan, Canada (highlighted by an annular lake) and the 30-mile-diameter Richat structure in the Sahara desert of Mauritania. On the Moon, the 50-mile-diameter crater Tycho is also almost circular—and has 1000-mile almost-straight rays coming out from it.

At night, lights of cities are obvious—notably hugging the coast of the Mediterranean—as are fire plumes from oil rigs. In addition, in some areas, sodium streetlamps make the light almost monochromatic. But it would seem difficult to be sure that these were artifacts without more information. In western Kansas there is however a 200-mile square region with light produced by a strikingly regular grid of towns—many at the centers of square counties laid out around 1870 in connection with land grants for railroad development. In addition, there is an isolated 1000-mile straight railroad built in the late 1800s across Kazakhstan between Aktyubinsk and Tashkent, with many towns visible at night along it. There are also 500-mile straight railroads built around the same time between Makat and Nukus, and Yaroslavl and Archangel. All these railroads go through flat empty terrain that previously had only a few nomadic inhabitants—and no settlements to define a route. But in many ways such geometrical forms seem vastly simpler to imagine producing than for example the elaborate pattern of successive lightning strikes visible especially in the tropics from space.

▪ **Page 835 · Astronomical objects.** Stars and planets tend to be close to perfect spheres. Lagrange points and resonances often lead to simple geometrical patterns of orbiting bodies. (The orbits of most planets in our solar system are also close to perfect circles; see page 973.) Regular spirograph-like patterns can occur for example in planetary nebulas formed by solar mass exploding stars. Unexplained phenomena that could conceivably be at least in part artifacts include gamma ray bursts and ultra high-energy cosmic rays. The local positions of stars are generally assumed to be random. 88 constellations are usually named—quite a few presumably already identified by the Babylonians and Sumerians around 2000 BC.

▪ **Page 835 · Natural radio emissions.** Each of the few million lightning flashes that occur on the Earth each day produce bursts of radio energy. At kilohertz frequencies reflection from the ionosphere allows these signals to propagate up to thousands of miles around the Earth, leading to continual intermittent crackling and popping. Particularly at night such signals can also travel within the ionosphere, but different frequencies travel at different rates, leading to so-called tweeks involving ringing or pinging. Signals can sometimes travel through the magnetosphere along magnetic field lines from one hemisphere to the other, yielding so-called whistlers with frequencies that fall off in a highly regular way with time. (Occasionally the signals can also travel back and forth between hemispheres, giving more complex results.) Radio emission can also occur when charged particles from the Sun excite plasma waves in the magnetosphere. And particularly at dawn or when an aurora is present an elaborate chorus of different elements can be produced—and heard directly on a VLF radio receiver.

Sunspots and solar flares make the Sun the most intense radio source in the solar system. Artificial radio signals from the Earth come next. The interaction of the solar wind with the magnetosphere of Jupiter produces radio emissions that exhibit variations reminiscent of gusting.

Outside the solar system, gas clouds show radio emission at discrete gigahertz frequencies from rotational transitions in molecules and spin-flip transitions in hydrogen atoms. (The narrowest lines come from natural masers and have widths around 1 kHz.) The cosmic microwave background, and processes such as thermal emission from dust, radiation from electrons in ionized gases, and synchrotron radiation from relativistic electrons in magnetic fields yield radio emissions with characteristic continuous frequency spectra. A total of over a million radio sources inside and outside our galaxy have now been catalogued, most with frequency spectra apparently consistent with known natural phenomena. Variations of source properties on timescales of months or years are not uncommon; variations of signals on timescales of tens of minutes can be introduced by propagation through turbulence in the interstellar medium.

Most radio emission from outside the solar system shows little apparent regularity. The almost perfectly repetitive signals from pulsars are an exception. Pulsars appear to be rapidly rotating neutron stars—perhaps 10 miles across—whose magnetic fields trap charged particles that produce radio emissions. When they first form after a supernova pulsars have millisecond repetition rates, but over the course of a few million years they slow to repetition rates of seconds through a series of glitches, associated perhaps with cracking in their solid crusts or perhaps with motion of quantized vortices in their superfluid interiors. Individual pulses from pulsars show some variability, presumably largely reflecting details of plasma dynamics in their magnetospheres.

■ **Page 835 · Artificial radio signals.** In current technology radio signals are essentially always based on carriers of the form $Sin[\omega t]$ with frequencies $\omega/(2\pi)$. When radio was first developed around 1900 information was normally encoded using amplitude modulation (AM) $s[t]\,Sin[\omega t]$. In the 1940s it also became popular to use frequency modulation (FM) $Sin[(1+s[t])\,\omega t]$, and in the 1970s pulse code modulation (PCM) (pulse trains for $IntegerDigits[s[t], 2]$). All such methods yield signals that remain roughly in the range of frequencies $\{\omega-\delta, \omega+\delta\}$ where δ is the data rate in $s[t]$. But in the late 1990s—particularly for the new generation of cellular telephones—it began to be common to use spread spectrum CDMA methods, in which many signals with the same carrier frequency are combined. Each is roughly of the form $BitXor[u[t], s[t]]\,Sin[\omega t]$, where $u[t]$ is a pseudonoise (PN) sequence generated by a linear feedback shift register (LFSR) (see page 1084); the idea is that by using a different PN sequence for each signal the corresponding $s[t]$ can be recovered even if thousands are superimposed.

The radio spectrum from about 9 kHz to 300 GHz is divided by national and international legislation into about 460 bands designated for different purposes. And except when spread spectrum methods are used, most bands are then divided into between a few and a few thousand channels in which signals with identical structures but different frequencies are sent.

If one steps through frequencies with an AM radio scanner, one sometimes hears intelligible speech—from radio or TV broadcasts, or two-way radio communication. But in many frequency bands one hears instead either very regular or seemingly quite random signals. (A few bands allocated for example to distress signals or radio astronomy are normally quiet.) The regular signals come from such sources as navigation beacons, time standards, identification transponders and radars. Most have characteristic almost perfectly repetitive forms (radar pulses, for example, typically have the chirped form $Sin[(1+\alpha t)\,\omega t]$)—and some sound uncannily like pulsars. When there are seemingly random signals some arise say from transmission of analog video (though this typically has very rigid overall structure associated with successive lines and frames), but most are now associated with digital data. And when CDMA methods are used there can be spreading over a significant range of frequencies—with regularities being recognizable only if one knows or can cryptanalyze LFSR sequences.

In general to send many signals together one just needs to associate each with a function $f[i, t]$ orthogonal to all other functions $f[j, t]$ (see page 1072). Current electronics (with analog elements such as phase-locked loops) make it easy to handle functions $Sin[\omega t]$, but other functions can yield better data density and perhaps better signal propagation. And as faster digital electronics makes it easier to implement these it seems likely that it will become less and less common to have simple carriers with definite frequencies.

In addition, there is a continuing trend towards greater spatial localization of signals—whether by using phased arrays or by explicitly using technologies like fiber optics.

At present, the most intense overall artificial radio emission from the Earth is probably the 50 or 60 Hz hum from power lines. The most intense directed signals are probably from radars (such as those used for ballistic missile detection) that operate at a few hundred megahertz and put megawatts of power into narrow beams. (Some such systems are however being replaced by lower-power phased array systems.)

■ **Page 835 · SETI.** First claims of extraterrestrial radio signals were made by Nikola Tesla in 1899. More widely believed claims were made by Guglielmo Marconi in 1922, and for several years searches were done—notably by the U.S. military—for signals presumed to be coming from Mars. But it became increasingly accepted that in fact nothing beyond natural radio emissions such as whistlers (see note above) were actually being detected.

When galactic radio emission was first noticed by Karl Jansky in 1931 it seemed too random to be of intelligent origin. And when radio astronomy began to develop it essentially ignored extraterrestrial intelligence. But in 1959

Giuseppe Cocconi and Philip Morrison analyzed the possibility of interstellar radio communication, and in 1960 Frank Drake used a radio telescope to look for explicit signals from two nearby stars.

In 1965 a claim was made that there might be intensity variations of intelligent origin in radio emission from the quasar CTA-102—but this was quickly retracted. Then in 1967 when the first pulsar was discovered it was briefly thought that perhaps its precise 1.33730113-second repetition rate might be of intelligent origin.

Since the 1960s around a hundred different SETI (search for extraterrestrial intelligence) experiments have been done. Most use the same basic scheme: to look for signals that show a narrow band of frequencies—say only 1 Hz wide—perhaps changing in time. (The corresponding waveform is thus required to be an almost perfect sinusoid.) Some concentrate on specific nearby stars, while others look at the whole sky, or test the stream of data from all observations at a particular radio telescope, sometimes scanning for repetitive trains of pulses rather than single frequencies. The best current experiments could successfully detect radio emission at the level now produced on Earth only from about 10 light years away—or from about the nearest 10 stars. The detection distance increases like the square root of the signal strength, covering all 10^{11} stars in our galaxy when the signal uses the total power output of a star.

Most SETI has been done with specially built systems or with existing radio telescopes. But starting in the mid-1990s it became possible to use standard satellite receivers, and there are now plans to set up a large array of these specifically for SETI. In addition, it is now possible to use software instead of hardware to implement SETI signal-processing algorithms—both traditional ones and presumably much more general ones that can for example pick out much weaker signals.

Many SETI experiments look for signals in the so-called "water hole" between the 1420 MHz frequency associated with the 21 cm line of hydrogen and the 1720 MHz frequency associated with hydroxyl (OH). But although there are now practical constraints associated with the fact that on Earth only a few frequency regions have been left clear for radio astronomy I consider this to be a remarkable example of reliance on details of human intellectual development.

Already in the early 1960s it was suggested that lasers instead of radio could be used for interstellar communication, and there have been various attempts to detect interstellar optical pulses. Other suggested methods of communication have included optical solitons, neutrinos and as-yet-unknown faster-than-light quantum effects.

It is sometimes suggested that there must be fundamental limits to detection of radio signals based on such issues as collection areas, noise temperatures and signal degradation. But even existing technology has provided a steady stream of examples where limits like these have been overcome—most often by the use of more sophisticated signal processing.

▪ **Detection methods.** Ways to identify computational origins include looking for repeatability in apparently random signals and comparing with output from large collections of possible simple programs. At a practical level, the one-dimensional character of data from radio signals makes it difficult for us to apply our visual systems—which remain our most powerful general-purpose analysis tools.

▪ **Higher perception and analysis.** See page 632.

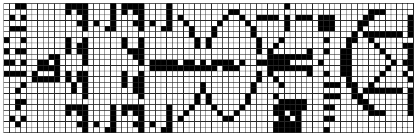
▪ **Page 837 · Messages to send.** The idea of trying to send messages to extraterrestrials has existed since at least the early 1800s. The proposed content and medium of the messages has however steadily changed, usually reflecting what seemed to be the most significant human achievements of the time—yet often seeming quaint within just a few decades.

Starting in the 1820s various scientists (notably Carl Friedrich Gauss) suggested signalling the Moon by using such schemes as cutting clearings in a forest to illustrate the Pythagorean theorem or reflecting sunlight from mirrors in different countries placed so as to mimic an observed constellation of stars. In the 1860s, with the rise of telegraphy, schemes for sending flashes of light to Mars were discussed, and the idea developed that mathematics should somehow be the basic language used. In the 1890s radio signals were considered, and were tried by Nikola Tesla in 1899. Discussion in the 1920s led to the idea of sending radio pulses that could be assembled into a bitmap image, and some messages intended for extraterrestrials were probably sent by radio enthusiasts.

There is a long history of attempts to formulate universal languages (see page 1181). The Lincos language of Hans Freudenthal from 1960 was specifically designed for extraterrestrial communication. It was based on predicate logic, and attempted to use this to build up first mathematics, then science, then a general presentation of human affairs.

When the Pioneer 10 spacecraft was launched in 1972 it carried a physical plaque designed by Carl Sagan and others. The plaque is surprisingly full of implicit assumptions based on details of human intellectual development. For example, it has line drawings of humans—whose interpretation inevitably seems very specific to our visual system and artistic culture. It also has a polar plot of the positions of 14 pulsars relative to the Sun, with the pulsars specified by giving their periods as base 2 integers—but with trailing zeros inserted to cover inadequate precision. Perhaps the most peculiar element, however, is a diagram indicating the 21 cm transition in hydrogen—by showing two abstract quantum mechanical spin configurations represented in a way that seems completely specific to the particular details of human mathematics and physics. In 1977 the Voyager spacecraft carried phonograph records that included bitmap images and samples of spoken languages and music.

In 1974 the bitmap image below was sent as a radio signal from the Arecibo radio telescope. At the left-hand end is a version of the pattern of digits from page 117—but distorted so it has no obvious nested structure. There follow atomic numbers for various elements, and bitvectors for components of DNA. Next are idealized pictures of a DNA molecule, a human, and the telescope. All these parts seem to depend almost completely on detailed common conventions—and I suspect that without all sorts of human context their meaning would be essentially impossible to recognize.



In all, remarkably few messages have been sent—perhaps in part because of concerns that they might reveal us to extraterrestrial predators (see page 1191). There has also been a strong tendency to make messages hard even for humans to understand—perhaps on the belief that they must then be more scientific and more universal.

The main text argues that it will be essentially impossible to give definitive evidence of intelligence. Schemes that might however get at least some distance include sending:

- waveforms made of simple underlying elements;
- long complicated sequences that repeat precisely;
- a diversity of kinds of sequences;
- something complicated that satisfies simple constraints.

Examples of the latter include pattern-avoiding sequences (see page 944), magic squares and other combinatorial designs, specifications of large finite groups, and maximal length linear feedback shift register sequences (see page 1084). Notably, the last of these are already being transmitted by GPS satellites and CDMA communications systems. (If cases could be found where the sequences as a whole were forced not to have any obvious regularities, then pattern-avoiding sequences might perhaps be good since they have constraints that are locally fairly easy to recognize.)

Extrapolation of trends in human technology suggest that it will become ever easier to detect weak signals that might be assumed distorted beyond recognition or swamped by noise.

▪ **Page 838 · P versus NP.** Given a constraint, it may be an NP-complete problem to find out what object satisfies it. So it may be difficult to generate the object from the constraint. But if one allows oneself to generate the object in any way at all, this may still be easy, even if P ≠ NP.

▪ **Science fiction.** Inhabitants of the Moon were described in stories by Lucian around 150 AD and Johannes Kepler in 1634—and in both cases were closely modelled on terrestrial organisms. Interest in fiction about extraterrestrials increased greatly at the end of the 1800s—perhaps because by then few parts of the Earth remained unexplored. And as science fiction developed, accounts of the future sometimes treated extraterrestrials as commonplace—and sometimes did not mention them at all. Most often extraterrestrials have been easy to recognize, being little more than simple combinations of terrestrial animals (and occasionally plants)—though fairly often with extra features like telepathy. Some stories have nevertheless explored extraterrestrial intelligence based for example on solids, gases or energy fields. An example is Fred Hoyle's 1957 *The Black Cloud* in which a large cloud of hydrogen gas achieves intelligence by exchanging electromagnetic signals between rocks whose surface molecular configurations store memories.

The most common fictional scenario for first contact with extraterrestrials is the arrival of spacecraft—often induced by us having passed a technology threshold such as radio, nuclear explosions or faster-than-light travel. Other scenarios sometimes considered include archeological discovery of extraterrestrial artifacts and receipt of radio signals.

In the movie *2001* a black cuboid with side ratios 1:4:9 detected on the Moon through its anomalous magnetic properties sends a radio pulse in response to sunlight. Later there are also a few frames of flashing octahedra, presumably intended to be extraterrestrial artifacts, or perhaps extraterrestrials themselves.

In *The Black Cloud* intelligence is suggested by responsiveness to radio stimuli. Communication is established—as often in science fiction—by the intelligence interpreting material that we supply, and then replying in the same format.

The movie *Contact* centers on a radio signal with several traditional SETI ideas: it is transmitted at $1420\,\pi$ MHz, and involves a sequence of primes to draw attention, an amplified TV signal from Earth and a description of a machine to build.

The various *Star Trek* television series depict many encounters with "new life and new civilizations". Sometimes intelligence is seen not associated with something that is considered a lifeform.

Particularly in short stories various scenarios have been explored where it is difficult ever to recognize intelligence. These include one-of-a-kind beings that have nothing to communicate with, as well as beings with inner intellectual activity but no effect on the outside world. When there are extraterrestrials substantially more advanced than humans few efforts have been made to describe their motives and purposes directly—and usually what is emphasized is just their effects on humans.

(See also page 1184.)

▪ **Page 839 · Practical arguments.** If extraterrestrials exist at all an obvious question—notably asked by Enrico Fermi in the 1940s—is why we have not encountered them. For there seems no fundamental reason that even spacecraft could not colonize our entire galaxy within just a few million years.

Explanations suggested for apparent absence include:

- Extraterrestrials are visiting, but we do not detect them;
- Extraterrestrials have visited, but not in recorded history;
- Extraterrestrials choose to exist in other dimensions;
- Interstellar travel is somehow infeasible;
- Colonization is somehow ecologically limited;
- Physical travel is not worth it; only signals are ever sent.

Explanations for apparent lack of radio signals include:

- Broadcasting is avoided for fear of conquest;
- There are active efforts to prevent us being contaminated;
- Extraterrestrials have no interest in communicating;
- Radio is the wrong medium;
- There are signals, but we do not understand them.

The so-called Drake equation gives a straightforward upper bound on the number of cases of extraterrestrial intelligence that could have arisen in our galaxy through the same basic chain of circumstances as humans. The result is a product of: rate of formation of suitable stars; fraction with planetary systems; number of Earth-like planets per system; fraction where life develops; fraction where intelligence develops; fraction where technology develops; time communicating civilizations survive. It now seems fairly certain that there are at least hundreds of millions of Earth-like planets in our galaxy. Biologists sometimes argue that intelligence is a rare development—though in the Darwinian approach it certainly has clear benefit. In addition, particularly in the Cold War period, it was often said that technological civilizations would quickly tend to destroy themselves, but now it seems more likely that intelligence—once developed—will tend to survive indefinitely, at least in machine form.

It is obviously difficult to guess the possible motivations of extraterrestrials, but one might expect that—just as with humans—different extraterrestrials would tend to do different things, so that at least some would choose to send out signals if not spacecraft. Out of about 6 billion humans, however, it is notable that only extremely few choose, say, to explore life in the depths of the oceans—though perhaps this is just because technology has not yet made it easy to do. In human history a key motivator for exploration has been trade. But trade requires that there be things of value to exchange; yet it is not clear that with sufficiently advanced technology there would be. For if the fundamental theory of physics is known, then everything about what is possible in our universe can in principle be worked out purely by a computation. Often irreducible work will be required, which one might imagine it would be worthwhile to trade. But as a practical matter, it seems likely that there will be vastly more room to do more extensive computations by using smaller components than by trading and collaborating with even millions of other civilizations. (It is notable that just a couple of decades ago, it was usually assumed that extraterrestrials would inevitably want to use large amounts of energy, and so would eventually for example tap all the output of a star. But seeing the increasing emphasis on information rather than mechanical work in human affairs this now seems much less clear.)

Extrapolating from our development, one might expect that most extraterrestrials would be something like immortal disembodied minds. And what such entities might do has to some extent been considered in the context of the notion of heaven in theology and art. And it is perhaps notable that while such activities as music and thought are often discussed, exploration essentially never is.

▪ **Physics as intelligence.** From the point of view of traditional thinking about intelligence in the universe it might seem like an extremely bizarre possibility that perhaps intelligence could exist at a very small scale, and in effect have spread throughout the universe, building as an artifact everything we see. But at least with a broad interpretation of intelligence this is at some level exactly what the Principle of Computational Equivalence suggests has actually happened. For it implies that even at the smallest scales the laws of physics will show the same computational sophistication that we normally associate with intelligence. So in some sense this supports the theological notion that there might be a kind of intelligence that permeates our universe. (See page 1195.)

Implications for Technology

▪ **Covering technology.** In writing this book I have tried to achieve some level of completeness in covering the obvious scientific implications of my ideas. But to cover technological implications at anything like the same level would require at least as long a book again. And in my experience many of the intellectually most interesting aspects of technology emerge only when one actually tries to build technology for real—and they are often in a sense best captured by the technology itself rather than by a book about it.

▪ **Page 840 · Applications of randomness.** Random drawing of lots has been used throughout recorded history as an unbiased way to distribute risks or rewards. Also common have been games of chance (see page 968). Randomness is in general a convenient way to allow local decisions to be made while maintaining overall averages. In biological organisms it is used in determining sex of offspring, as well as in achieving uniform sampling, say in foraging for food. (Especially in antiquity, all sorts of seemingly random phenomena have been used as a basis for fortune telling.)

The notion of taking random samples as a basis for making unbiased deductions has been common since the early 1900s, notably in polling and market research. And in the past few decades explicit randomization has become common as a way of avoiding bias in cases such as clinical trials of drugs.

In the late 1800s it was noted in recreational mathematics that one could find the value of π by looking at randomly dropped needles. In the early 1900s devices based on randomness were built to illustrate statistics and probability (see page 312), and were used for example to find the form of the Student t-distribution. With the advent of digital computers in the mid-1940s Monte Carlo methods (see page 968) were introduced, initially as a way to approximate processes like neutron diffusion. (Similar ideas had been used in 1901 by Kelvin to study the Boltzmann equation.) Such methods became increasingly popular, especially for simulating systems like telephone networks and particle detectors that have many heterogeneous elements—as well as in statistical physics. In the 1970s they also became widely used for high-dimensional numerical integration, notably for Feynman diagram evaluation in quantum electrodynamics. But eventually it was realized that quasi-Monte Carlo methods based on simple sequences could normally do better than ones based on pure randomness (see page 1085).

A convenient way to tell whether expressions are equal is to evaluate them with random numerical values for variables. (Care must be taken with branch cuts and bounding intervals for inexact numbers.) In the late 1970s it was noted that by evaluating *PowerMod[a, n - 1, n] == 1* for several random integers *a* one can with high probability quickly deduce *PrimeQ[n]*. (In the 1960s it had been noted that one can factor polynomials by filling in random integers for variables and factoring the resulting numbers.) And in the 1980s many such randomized algorithms were invented, but by the mid-1990s it was realized that most did not require any kind of true randomness, and could readily be derandomized and made more predictable. (See page 1085.)

There are all sorts of situations where in the absence of anything better it is good to use randomness. Thus, for example, many exploratory searches in this book were done randomly. And in testing large hardware and software systems random inputs are often used.

Randomness is a common way of avoiding pathological cases and deadlocks. (It requires no communication between components so is convenient in parallel systems.) Examples include message routing in networks, retransmission times after ethernet collisions, partitionings for sorting algorithms, and avoiding getting stuck in minimization procedures like simulated annealing. (See page 347.) As on page 333, it is common for randomness to add robustness—as for example in cellular automaton fluids, or in saccadic eye movements in biology.

In cryptography randomness is used to make messages look typical of all possibilities (see page 598). It is also used in roughly the same way in hashing (see page 622). Such randomness must be repeatable. But for cryptographic keys it should not be. And the same is true when one picks unique IDs, say to keep track of repeat web transactions with a low probability of collisions. Randomness is in effect also used in a similar way in the shotgun method for DNA sequencing, as well as in creating radar pulses that are difficult to forge. (In biological organisms random diversity in faces and voices may perhaps have developed for similar reasons.)

The unpredictability of randomness is often useful, say for animals or military vehicles avoiding predators (see page 1105). Such unpredictability can also be used in simulating human or natural processes, say for computer graphics, videogames, or mock handwriting. Random patterns are often used as a way to hide regularities—as in camouflage, security envelopes, and many forms of texturing and distressing. (See page 1077.)

In the past, randomness was usually viewed as a thing to be avoided. But with the spread of computers and consumer electronics that normally operate predictably, it has become increasingly popular as an option.

Microscopic randomness is implicitly used whenever there is dissipation or friction in a system, and generally it adds robustness to the behavior that occurs in systems.

■ **Page 841 · Self-assembly.** Given elements (such as pieces of molecules) that fit together only when certain specified constraints are satisfied it is fairly straightforward to force, say, cellular automaton patterns to be generated, as on page 221. (Notable examples of such self-assembly occur for instance in spherical viruses.)

■ **Page 841 · Nanotechnology.** Popular since the late 1980s, especially through the work of Eric Drexler, nanotechnology has mostly involved investigation of several approaches to making essentially mechanical devices out of small numbers of atoms. One approach extrapolates chip technology, and studies placing atoms individually on solid surfaces using for example scanning probe microscopy. Another extrapolates chemical synthesis—particularly of fullerenes—and considers large molecules made for example out of carbon atoms. And another involves for example setting up fragments of DNA to try to force particular patterns of self-assembly. Most likely it will eventually be possible to have a single universal system that can manufacture almost any rigid atomic-scale structure on the basis of some kind of program. (Ribosomes in biological cells already construct arbitrary proteins from DNA sequences, but ordinary protein shapes are usually difficult to predict.) Existing work has tended to concentrate on trying to make rather elaborate components suitable for building miniature versions of familiar machines. The discoveries in this book imply however that there are much simpler components that can also be used to set up systems that have behavior with essentially any degree of sophistication. Such systems can either have the kind of chemical and mechanical character most often considered in nanotechnology, or can be primarily electronic, for example along the lines of so-called quantum-dot cellular automata. Over the next several decades applications of nanotechnology will no doubt include much higher-capacity computers, active materials of various kinds, and cellular-scale biomedical devices.

■ **Page 842 · Searching for technology.** Many inventions are made by pure ingenuity (sometimes aided by mathematical calculation) or by mimicking processes that go on in nature. But there are also cases where systematic searches are done. Notable examples were the testing of thousands of materials as candidate electric light bulb filaments by Thomas Edison in 1879, and the testing of 606 substances for chemotherapy by Paul Ehrlich in 1910. For at least fifty years it has now been quite routine to test many hundreds or thousands of substances in looking, say, for catalysts or drugs with particular functions. (Other kinds of systematic searches done include ones for metal alloys, cooking recipes and plant hybrids.) Starting in the late 1980s the methods of combinatorial chemistry (see note below) began to make it possible to do biochemical tests on arrays of millions of related substances. And by the late 1990s, similar ideas were being used for example in materials science: in a typical case an array of different combinations of substances is made by successively spraying through an appropriate sequence of masks, with some physical or chemical test then applied to all the samples.

In the late 1950s maximal length shift register sequences (page 1084) and some error-correcting codes (page 1101) were found by systematic searches of possible polynomials. Most subsequent codes, however, have been found by explicit mathematical constructions. Optimal circuit blocks for operations such as addition and sorting (see page 1142) have occasionally been found by searches, but are more often found by explicit construction, progressive improvement or systematic logic minimization (see page 1097). In some compilers searches are occasionally done for optimal sequences of instructions to implement particular simple functions. And in recent years—notably in the building of *Mathematica*—optimal algorithms for operations such as function evaluation and numerical integration have sometimes been found through searches. In addition, my 1984 identification of rule 30 as a randomness generator was the result of a small-scale systematic search.

Particularly since the 1970s, many systematic methods have been tried for optimizing engineering designs by computer. Usually they are based on iterative improvement rather than systematic search. Some rely on linear programming or gradient descent. Others use methods such as simulated annealing, neural networks and genetic algorithms. But as discussed on page 342, except in very simple cases, the results are usually far from any absolute optimum. (Plant and animal breeding can be viewed as a simple form of randomized search done since the dawn of civilization.)

■ **Page 843 · Methodology in this book.** Much of what is presented in this book comes from systematic enumeration of all possible systems of particular types. However, sometimes I have done large searches for systems (see e.g. page 112). And especially in Chapter 11 I have occasionally explicitly constructed systems that show particular features.

■ **Chemistry.** Chemical compounds are a little like cellular automata and other kinds of programs. For even though the basic physical laws relevant to chemical compounds have been known since the early 1900s, it remains extremely difficult to predict the actual properties of a given compound. And I suspect that the ultimate reason for this—just as in the case of simple programs—is computational irreducibility.

For a single molecule, the minimum energy configuration can presumably always be found by a limited amount of computational work—though potentially increasing rapidly with the number of atoms. But if one allows progressively more molecules computational irreducibility can make it take progressively more computational work to see what will happen. And much as in determining whether constraints like those on page 213 can be satisfied for an infinite region, it can take an infinite amount of computational work to determine bulk properties of an infinite collection of molecules. Thus in practice it has typically been difficult to predict for example boiling and particularly melting points (see note below). So this means in the end that most of chemistry must be based on facts determined experimentally about specific compounds that happen to have been studied.

There are currently about 10 million compounds listed in standard chemical databases. Of these, most were first identified as extracts from biological or other natural systems. In trying to discover compounds that might be useful say as drugs the traditional approach was to search large libraries of compounds, then to study variations on those that seemed promising. But in the 1980s it began to be popular to try so-called rational design in which molecules were created that could at least to some extent specifically be computed to have relevant shapes and chemical functions. Then in the 1990s so-called combinatorial chemistry became popular, in which—somewhat in imitation of the immune system—large numbers of possible compounds were created by successively adding at random several different possible amino acids or other units. But although it will presumably change in the future it remained true in 2001 that half of all drugs in use are derived from just 32 families of compounds.

Doing a synthesis of a chemical is much like constructing a network by applying a specified sequence of transformations. And just like for multiway systems it is presumably in principle undecidable whether a given set of possible transformations can ever be combined to yield a particular chemical. Yet ever since the 1960s there have been computer systems like LHASA that try to find synthesis pathways automatically. But perhaps because they lack even the analog of modern automated theorem-proving methods, such systems have never in practice been extremely successful.

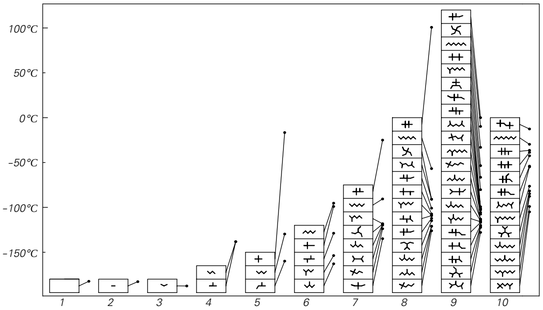
■ **Interesting chemicals.** The standard IUPAC system for chemical nomenclature assigns a name to essentially any possible compound. But even among hydrocarbons with fairly few atoms not all have ever been considered interesting enough to list in standard chemical databases. Thus for example the following compares the total number of conceivable alkanes (paraffins) to the number actually listed in the 2001 standard Beilstein database:

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*
total	*1*	*1*	*1*	*2*	*3*	*5*	*9*	*18*	*35*	*75*	*159*	*355*	*802*	*1858*	*4347*	*10359*
listed	*1*	*1*	*1*	*2*	*3*	*5*	*9*	*18*	*35*	*75*	*68*	*108*	*60*	*60*	*41*	*62*

Any tree with up to 4 connections at each node can in principle correspond to an alkane with chemical formula C_nH_{2n+2}. The total number of such trees—studied since 1875—increases roughly like $2.79^n n^{-5/2}$. If every node has say 4 connections, then eventually one gets dendrimers that cannot realistically be constructed in 3D. But long before this happens one runs into many alkanes that presumably exist, but apparently have never explicitly been studied. The small unbranched ones (methane, ethane, propane, butane, pentane, etc.) are all well known, but ones with more complicated branching are decreasingly known. In coal and petroleum a continuous range of alkanes occur. Branched octanes are used to reduce knocking in car engines. Biological systems contain many specific alkanes—often quite large—that happen to be produced through chemical pathways in biological cells. (The $n = 11$ and $n = 13$ unbranched alkanes are for example known to serve as ant pheromones.)

In general the main way large molecules have traditionally ended up being considered chemically interesting is if they occur in biological systems—or mimic ones that do. Since the 1980s, however, molecules such as the fullerenes that instead have specific regular geometrical shapes have also begun to be considered interesting.

■ **Alkane properties.** The picture on the facing page shows melting points measured for alkanes. (Note that even when alkanes are listed in chemical databases—as discussed above—their melting points may not be given.) Unbranched alkanes yield melting points that increase smoothly for n even and for n odd. Highly symmetrical branched alkanes tend to have high melting points, presumably because they pack well in space. No reliable general method for predicting melting points is however known (see note above), and in fact for large n alkanes tend to form jellies with no clear notion of melting.



Things appear somewhat simpler with boiling points, and as noticed by Harry Wiener in 1947 (and increasingly discussed since the 1970s) these tend to be well fit as being linearly proportional to the so-called topological index given by the sum of the smallest numbers of connections visited in getting between all pairs of carbon atoms in an alkane molecule.

■ **Page 843 · Components for technology.** The Principle of Computational Equivalence suggests that a vast range of systems in nature can all ultimately be used to make computers. But it is remarkable to what extent even the components of present-day computer systems involve elements of nature originally studied for quite different reasons. Examples include electricity, semiconductors (used for chips), ferrites (used for magnetic storage), liquid crystals (used for displays), piezoelectricity (used for microphones), total internal reflection (used for optical fibers), stimulated emission (used for lasers) and photoconductivity (used for xerographic printing).

■ **Future technology.** The purposes technology should serve inevitably change as human civilization develops. But at least in the immediate future many of these purposes will tend to relate to the current character of our bodies and minds. For certainly technology must interface with these. But presumably as time progresses it will tend to become more integrated, with systems that we have created eventually being able to fit quite interchangeably into our usual biological or mental setup. At first most such systems will probably tend either to be based on standard engineering, or to be quite direct emulations of human components that we see. But particularly by using the ideas and methods of this book I suspect that significant progressive enhancements will be possible. And probably there will be many features that are actually quite easy to take far beyond the originals. One example is memory and the recall of history. Human memory is in many ways quite impressive. Yet for ordinary physical objects we are used to the idea that they remember little of their history, for at a macroscopic level we tend to see only the coarsest traces. But at a microscopic scale something like the surface of a solid has in at least some form remarkably detailed information about its history. And as technological systems get smaller it should become possible to read and manipulate this. And much as in the discussion at the end of Chapter 10 the ability to interact at such a level will yield quite different experiences, which in turn will tend to suggest different purposes to pursue with technology.

Historical Perspectives

■ **Page 844 · Human uniqueness.** The idea that there is something unique and special about humans has deep roots in Judeo-Christian tradition—and despite some dilution from science remains a standard tenet of Western thought today. Eastern religions have however normally tended to take a different view, and to consider humans as just one of many elements that make up the universe as a whole. (See note below.)

■ **Page 845 · Animism.** Belief in animism remains strong in perhaps several hundred million indigenous people around the world. In its typical form, it involves not only explaining natural phenomena by analogy to human behavior but also assuming that they can be influenced as humans might be, say by offerings or worship. (See also page 1177.)

Particularly since Edward Tylor in 1871 animism has often been thought of as the earliest identifiable form of religion. Polytheism is then assumed to arise when the idea of localized spirits associated with individual natural objects is generalized to the idea of gods associated with types of objects or concepts (as for example in many Roman beliefs). Following their rejection in favor of monotheism by Judaism—and later Christianity and Islam—such ideas have however tended to be considered primitive and pagan. In Europe through the Middle Ages there nevertheless remained widespread belief in animistic kinds of explanations. And even today some Western superstitions center on animism, as do rituals in countries like Japan. Animism is also a key element of the New Age movement of the 1960s, as well as of such ideas as the Gaia Hypothesis.

Particularly since the work of Jean Piaget in the 1940s, young children are often said to go through a phase of animism, in which they interact with complex objects much as if they were alive and human.

■ **Page 845 · Universe as intelligent.** Whether or not something like thinking can be attributed to the universe has long been discussed in philosophy and theology. Theism and the standard Western religions generally attribute thinking to a person-like God who governs the universe but is separate from it. Deism emphasizes that God can govern the universe only according to natural laws—but whether or not this involves thinking is unclear. Pantheism generally identifies the universe and God. In its typical religious form in Eastern metaphysics—as well as in philosophical idealism—the contents of the universe are identified quite directly with the thoughts of God. In scientific pantheism the abstract order of the universe is identified with God (often termed "Nature's God" or "Spinoza's God"), but whether this means that thinking is involved in the operation of the universe is not clear. (See also pages 822 and 1191.)

▪ **Non-Western thinking.** Some of my conclusions in this book may seem to resonate with ideas of Eastern thinking. For example, what I say about the fundamental similarity of human thinking to other processes in nature may seem to fit with Buddhism. And what I say about the irreducibility of processes in nature to short formal rules may seem to fit with Taoism. Like essentially all forms of science, however, what I do in this book is done in a rational tradition—with limited relation to the more mystical traditions of Eastern thinking.

▪ **Aphorisms.** Particularly from ancient and more fragmentary texts aphorisms have survived that may sometimes seem at least vaguely related to this book. (An example from the pre-Socratics is "everything is full of gods".) But typically it is impossible to see with any definiteness what such aphorisms might really have been intended to mean.

▪ **Postmodernism.** Since the mid-1960s postmodernism has argued that science must have fundamental limitations, based on its general belief that any single abstract system must somehow be as limited—and as arbitrary in its conclusions—as the context in which it is set up. My work supports the notion that—despite implicit assumptions made especially in the physical sciences—context can in fact be crucial to the choice of subject matter and interpretation of results in science (see e.g. page 1105). But the Principle of Computational Equivalence suggests at some level a remarkable uniformity among systems, that allows all sorts of general scientific statements to be made without dependence on context. It so happens that some of these statements then imply intrinsic general limitations on science—but even the very fact that such statements can be made is in a sense an example of successful generality in science that goes against the conclusions of postmodernism. (See also page 1131.)

▪ **Microcosm.** The notion that a human mind might somehow be analogous to the whole universe was discussed by Plato and others in antiquity, and known in the Middle Ages. But it was normally assumed that this was something fairly unique to the human mind—and nothing with the generality of the Principle of Computational Equivalence was ever imagined.

▪ **Human future.** The Principle of Computational Equivalence and the results of this book at first suggest a rather bleak view of the end point of the development of technology. As I argued in Chapter 10 computers will presumably be able to emulate human thinking. And particularly using the methods of this book one will be able to use progressively smaller physical components as elements of computers. So before too long it will no doubt be possible to implement all the processes of thinking that go on in a single human—or even in billions of humans—in a fairly small piece of material. Each piece of human thinking will then correspond to some microscopic pattern of changes in the atoms of the material. In the past one might have assumed that these changes would somehow show fundamental evidence of representing sophisticated human thinking. But the Principle of Computational Equivalence implies that many ordinary physical processes are computationally just as sophisticated as human thinking. And this means that the pattern of microscopic changes produced by such processes can at some level be just as sophisticated as those corresponding to human thinking. So given, say, an ordinary piece of rock in which there is all sorts of complicated electron motion this may in a fundamental sense be doing no less than some system of the future constructed with nanotechnology to implement operations of human thinking. And while at first this might seem to suggest that the rich history of biology, civilization and technology needed to reach this point would somehow be wasted, what I believe instead is that this just highlights the extent to which such history is what is ultimately the defining feature of the human condition.

▪ **Philosophical implications.** The Principle of Computational Equivalence has implications for many issues long discussed in the field of philosophy. Most important are probably those in epistemology (theory of knowledge). In the past, it has usually been assumed that if we could only build up in our minds an adequate model of the world, then we would immediately know whatever we want about the world. But the Principle of Computational Equivalence now implies that even given a model it may be irreducibly difficult to work out its consequences. In effect, computational irreducibility introduces a new kind of limit to knowledge. And it implies that one needs a criterion more sophisticated than immediate predictability to assess a scientific theory—since when computational irreducibility is present this will inevitably be limited. In the past, it has sometimes been assumed that truths that can be deduced purely by operations like those in logic must somehow always be trivial. But computational irreducibility implies that in general they are not. Indeed it implies that even once the basic laws are known there are still an endless series of questions that are worth investigating in science. It is often assumed that one cannot learn much about the world just by studying purely formal systems—and that one has to rely on empirical input. But the Principle of Computational Equivalence implies that at some level there are inevitably common features across both abstract and natural systems. In ontology (theory of being) the Principle of Computational Equivalence implies that special components are vastly less necessary than might have been thought. For it shows that all sorts of sophisticated characteristics can emerge from the very same kinds of simple components. (My discussion of fundamental physics in Chapter 9 also suggests that no separate entities beyond simple rules are needed to capture space, time or matter.) Arguments in several areas of philosophy involve in effect considering fundamentally different intelligences. But the Principle of Computational Equivalence implies that in fact above a certain threshold there is an ultimate equivalence between possible intelligences. In addition, the Principle of Computational Equivalence implies that all sorts of systems in nature and elsewhere will inevitably exhibit features that in the past have been considered unique to intelligence—and this has consequences for the mind-body problem, the question of free will, and recognition of other minds. It has often been thought that traditional logic—and to some extent mathematics—are somehow fundamentally special and provide in a sense unique foundations. But the Principle of Computational Equivalence implies that in fact there are a huge range of other formal systems, equivalent in their ultimate richness, but different in their details, and in the questions to which they naturally lead. In philosophy of science the Principle of Computational Equivalence forces a new methodology based on formal experiments—that is ultimately the foundation for the whole new kind of science that I describe in this book.