Question 1.1

Bayes' Rule : $P(A|B)\cdot P(B) = P(B|A)\cdot P(A)$

$$P(t=1|\vec{x}) = P(\vec{x}|t=1)\cdot P(t=1)/P(\vec{x})$$

$$= P(\vec{x}|t=1)\cdot P(t=1) \Big/ \left(P(\vec{x}|t=1)\cdot P(t=1) + P(\vec{x}|t=0)\cdot P(t=0)\right)$$

$$P(\vec{x}|t=1) = \prod_{i=1}^{D} P(x_i|t=1) = \left(\frac{1}{\sqrt{2\pi}}\right)^D \cdot \prod_{i=1}^{D}\frac{1}{\sigma_i}\cdot \exp\left(\sum_{i=1}^{D} - \frac{(x_i-\mu_{i1})^2}{2\sigma_i^2}\right)$$

$$P(\vec{x}|t=0) = \prod_{i=1}^{D} P(x_i|t=0) = \left(\frac{1}{\sqrt{2\pi}}\right)^D \cdot \prod_{i=1}^{D}\frac{1}{\sigma_i}\cdot \exp\left(\sum_{i=1}^{D} - \frac{(x_i-\mu_{i0})^2}{2\sigma_i^2}\right)$$

$$P(t=1|\vec{x}) = \frac{1}{1 + \left(P(\vec{x}|t=0)P(t=0)\right)\Big/\left(P(\vec{x}|t=1)\cdot P(t=1)\right)}$$

In this question, we know the underlying true distribution or you are given the learned distribution

$$= \frac{1}{1 + (1-\alpha)/\alpha \ \exp\left(\sum_{i=1}^{D} - \frac{(x_i-\mu_{i0})^2}{2\sigma_i^2} + \frac{(x_i-\mu_{i1})^2}{2\sigma_i^2}\right)}$$

$$= \frac{1}{1 + \frac{1-\alpha}{\alpha}\exp\left(-\sum_{i=1}^{D}\frac{(x_i-\mu_{i0})^2-(x_i-\mu_{i1})^2}{2\sigma_i^2}\right)}$$

$$(x_i-\mu_{i0})^2 - (x_i-\mu_{i1})^2 = (x_i-\mu_{i0}+x_i-\mu_{i1})(x_i-\mu_{i0}-x_i+\mu_{i1})$$

$$= (2x_i-\mu_{i0}-\mu_{i1})(\mu_{i1}-\mu_{i0})$$

$$= 2x_i\mu_{i1} - \cancel{\mu_{i0}\mu_{i1}} - \mu_{i1}^2 - 2x_i\mu_{i0} + \mu_{i0}^2 + \cancel{\mu_{i1}\mu_{i0}}$$

$$= 2(\mu_{i1}-\mu_{i0})x_i + \mu_{i0}^2 - \mu_{i1}^2$$

$$-\sum_{i=1}^{D}\frac{(x_i-\mu_{i0})^2-(x_i-\mu_{i1})^2}{2\sigma_i^2} = -\sum_{i=1}^{D}\frac{(\mu_{i1}-\mu_{i0})}{\sigma_i^2}x_i - \sum_{i=1}^{D}\frac{\mu_{i0}^2-\mu_{i1}^2}{2\sigma_i^2}$$

$$\text{Thus } P(t=1|\vec{x}) = 1\Big/\left(1 + (1-\alpha)/\alpha \ \exp\left(-\sum_{i=1}^{D}\frac{(\mu_{i1}-\mu_{i0})}{\sigma_i^2}x_i - \sum_{i=1}^{D}\frac{\mu_{i0}^2-\mu_{i1}^2}{2\sigma_i^2}\right)\right)$$

$$= 1\Big/\left(1 + \exp\left(-\sum_{i=1}^{D}\frac{(\mu_{i1}-\mu_{i0})}{\sigma_i^2}x_i - \left(\sum_{i=1}^{D}\frac{\mu_{i0}^2-\mu_{i1}^2}{2\sigma_i^2} - \log\frac{1-\alpha}{\alpha}\right)\right)\right)$$

$$\boxed{\vec{W} = (w_1,\dots,w_D)^T \text{ where } w_i = \frac{\mu_{i1}-\mu_{i0}}{\sigma_i^2} \quad b = \sum_{i=1}^{D}\frac{\mu_{i0}^2-\mu_{i1}^2}{2\sigma_i^2} - \log\frac{1-\alpha}{\alpha}}$$

Question 1.2 the sampling process is independent

$$\mathcal{L}(\vec{w},b) = -\log p(t_1,t_2,\dots,t_n \mid x^{(1)},x^{(2)},\dots,x^{(n)}; \vec{w},b)$$

$$= -\log[p(t_1|\cdots)\cdot p(t_2|\cdots)\cdot \dots \cdot p(t_n|\cdots)]$$

$$= -\log p(t_1|\cdots) - \log p(t_2|\cdots) - \dots - \log p(t_n|\cdots)$$

$$= -\sum_{i=1}^{n} \log p(t_i|x^{(i)},\vec{w},b)$$

$$= -\sum_{i=1}^{n} \log[t_i\cdot p(t_i=1|x^{(i)},\vec{w},b) + (1-t_i)\cdot p(t_i=0|x^{(i)},\vec{w},b)]$$

since $t_i = 1$ or 0,

$$= -\sum_{i=1}^{n}\left(t_i \log p(t_i=1|x^{(i)},\vec{w},b) + (1-t_i)\log p(t_i=0|x^{(i)},\vec{w},b)\right)$$

$$= -\left(\sum_{i=1}^{n} t_i \log p(t_i=1|x^{(i)},\vec{w},b) + \sum_{i=1}^{n}(1-t_i)\log p(t_i=0|x^{(i)},\vec{w},b)\right)$$

$$\boxed{\mathcal{L}(\vec{w},b) = -\left[\sum_{i=1}^{n}\left(t_i \log\sigma(\vec{w}^T\vec{x}^{(i)}+b)\right) + \sum_{i=1}^{n}\left[(1-t_i)\log(1-\sigma(\vec{w}^T\vec{x}^{(i)}+b))\right]\right]}$$

Let $u_i = \vec{w}^T\vec{x}^{(i)}+b$

$$\frac{\partial\mathcal{L}(\vec{w},b)}{\partial\vec{w}} = -\left[\sum_{i=1}^{n}\frac{\partial(t_i\log\sigma(u^{(i)}))}{\partial\vec{w}} + \sum_{i=1}^{n}\frac{\partial[(1-t_i)\log(1-\sigma(u^{(i)}))]}{\partial\vec{w}}\right]$$

$$= -\left[\sum_{i=1}^{n} t_i\frac{\partial\log\sigma(u^{(i)})}{\partial\vec{w}} + \sum_{i=1}^{n}(1-t_i)\frac{\partial\log(1-\sigma(u^{(i)}))}{\partial\vec{w}}\right]$$

and $$\frac{\partial\log\sigma(u^{(i)})}{\partial\vec{w}} = \frac{1}{\sigma(u^{(i)})}\cdot\sigma'(u^{(i)})\cdot\frac{\partial u^{(i)}}{\partial\vec{w}}$$

$$\frac{\partial u^{(i)}}{\partial\vec{w}} = \frac{\partial(\vec{w}^T\vec{x}^{(i)}+b)}{\partial\vec{w}} = \vec{x}^{(i)} \qquad (1)$$

$$\sigma'(u^{(i)}) = \left(\frac{1}{1+\exp(-u^{(i)})}\right)' = -\frac{1}{(1+\exp(-u^{(i)}))^2}\cdot(-\exp(-u^{(i)}))$$

$$\sigma'(u^{(i)}) = \frac{\exp(-u^{(i)})}{(1+\exp(-u^{(i)}))^2} \qquad (2)$$

---

according to (1), (2)

$$\frac{\partial\log\sigma(u^{(i)})}{\partial\vec{w}} = \frac{1}{\sigma(u^{(i)})}\cdot\frac{\exp(-u^{(i)})}{(1+\exp(-u^{(i)}))^2}\cdot\vec{x}^{(i)}$$

$$\frac{\partial \log \sigma(u^{(i)})}{\partial \vec{w}} = \frac{\cancel{(1+\exp(-u^{(i)}))}\exp(-u^{(i)})}{(1+\exp(-u^{(i)}))^{\cancel{2}}} \cdot \vec{x}^{(i)}$$

$$= \frac{\exp(-u^{(i)})}{1+\exp(-u^{(i)})} \cdot \vec{x}^{(i)}$$

$$\frac{\partial \log(1-\sigma(u^{(i)}))}{\partial \vec{w}} = \frac{1}{1-\sigma(u^{(i)})} \cdot (-1) \cdot \sigma'(u^{(i)}) \cdot \frac{\partial u^{(i)}}{\partial \vec{w}}$$

$$= \frac{1}{\frac{1}{1+\exp(-u^{(i)})} - 1} \cdot \frac{\exp(-u^{(i)})}{(1+\exp(-u^{(i)}))^2} \cdot \vec{x}^{(i)}$$

$$= \frac{\cancel{1+\exp(-u^{(i)})}}{-\cancel{\exp(-u^{(i)})}} \cdot \frac{\cancel{\exp(-u^{(i)})}}{(1+\exp(-u^{(i)}))^{\cancel{2}}} \vec{x}^{(i)}$$

$$= \frac{1}{-(1+\exp(-u^{(i)}))} \cdot \vec{x}^{(i)}$$

$$\frac{\partial L(\vec{w}, b)}{\partial \vec{w}} = -\left[\sum_{i=1}^{n} t_i \frac{\partial \log \sigma(u^{(i)})}{\partial \vec{w}} + \sum_{i=1}^{n} (1-t_i) \frac{\partial \log(1-\sigma(u^{(i)}))}{\partial \vec{w}}\right]$$

$$= -\left[\sum_{i=1}^{n} \frac{t_i \exp(-u^{(i)})}{1+\exp(-u^{(i)})} \cdot \vec{x}^{(i)} + \sum_{i=1}^{n} \frac{(1-t_i)}{-(1+\exp(-u^{(i)}))} \cdot \vec{x}^{(i)}\right]$$

$$= -\left[\sum_{i=1}^{n} \left(\frac{t_i \exp(-u^{(i)}) + (t_i - 1)}{1+\exp(-u^{(i)})} \vec{x}^{(i)}\right)\right]$$

$$= -\sum_{i=1}^{n} \left(\frac{t_i(\exp(-u^{(i)})+1) - 1}{1+\exp(-u^{(i)})} \vec{x}^{(i)}\right)$$

$$= -\sum_{i=1}^{n} \left[\left(t_i - \frac{1}{1+\exp(-u^{(i)})}\right) \vec{x}^{(i)}\right]$$

$$\boxed{\frac{\partial L(\vec{w}, b)}{\partial \vec{w}} = -\sum_{i=1}^{n} \left[\left(t_i - \frac{1}{1+\exp(-w^T \vec{x}^{(i)} - b)}\right) \vec{x}^{(i)}\right]}$$

$$\frac{\partial \log \sigma(u^{(i)})}{\partial b} = \frac{1}{\sigma(u^{(i)})} \cdot \sigma'(u^{(i)}) \cdot \frac{\partial u^{(i)}}{\partial b}$$

$$= \frac{\exp(-u^{(i)})}{1+\exp(-u^{(i)})} \cdot 1 = \frac{\exp(-u^{(i)})}{1+\exp(-u^{(i)})}$$

$$\frac{\partial \log(1-\sigma(u^{(i)}))}{\partial b} = \frac{1}{1-\sigma(u^{(i)})} \cdot (-1) \cdot \sigma'(u^{(i)}) \cdot \frac{\partial u^{(i)}}{\partial b}$$

$$= -\frac{1}{(1+\exp(-u^{(i)}))}$$

Thus $$\frac{\partial \mathcal{L}(\vec{w}, b)}{\partial b} = -\left[\sum_{i=1}^{n} t_i \frac{\partial \log \sigma(u^{(i)})}{\partial b} + \sum_{i=1}^{n} (1-t_i) \frac{\partial \log(1-\sigma(u^{(i)}))}{\partial b}\right]$$

$$= -\left[\sum_{i=1}^{n} \frac{t_i \exp(-u^{(i)})}{1+\exp(-u^{(i)})} + \sum_{i=1}^{n} \frac{(1-t_i)}{-(1+\exp(-u^{(i)}))}\right]$$

$$= -\left[\sum_{i=1}^{n}\left(\frac{t_i \exp(-u^{(i)}) + (t_i - 1)}{1+\exp(-u^{(i)})}\right)\right]$$

$$= -\sum_{i=1}^{n}\left(\frac{t_i(\exp(-u^{(i)})+1) - 1}{1+\exp(-u^{(i)})}\right)$$

$$\frac{\partial \mathcal{L}(\vec{w}, b)}{\partial b} = -\sum_{i=1}^{n}\left(t_i - \frac{1}{1+\exp(-u^{(i)})}\right)$$

$$\boxed{\frac{\partial \mathcal{L}(\vec{w}, b)}{\partial b} = -\sum_{i=1}^{n}\left(t_i - \frac{1}{1+\exp(-\vec{w}^T\vec{x}^{(i)}-b)}\right)}$$

Q1.3: $$P(\vec{w}, b \mid \mathcal{D}) = P(\mathcal{D} \mid \vec{w}, b) \cdot P(\vec{w}, b) \,/\, P(\mathcal{D})$$

$$= P(\mathcal{D} \mid \vec{w}, b) \cdot P(\vec{w}) \cdot P(b) \,/\, P(\mathcal{D})$$

$$= P(\mathcal{D} \mid \vec{w}, b) \cdot P(\vec{w}) \,/\, P(\mathcal{D})$$

$$P(\mathcal{D} \mid \vec{w}, b) = \prod_{i=1}^{n}\left[t_i \sigma(\vec{w}^T\vec{x}^{(i)}+b) + (1-t_i)(1-\sigma(\vec{w}^T\vec{x}^{(i)}+b))\right]$$

$$P(\vec{w}) = \prod_{i=1}^{D} P(w_i) = \prod_{i=1}^{D} \frac{1}{\sqrt{2\pi\frac{1}{\lambda}}} \exp\left(-(w_i-0)^2 / \tfrac{2}{\lambda}\right) = \prod_{i=1}^{D} \frac{1}{\sqrt{2\pi/\lambda}} \exp(-\lambda w_i^2/2)$$

$$P(\mathcal{D}) = \prod_{i=1}^{n}\left[t_i \alpha + (1-t_i)(1-\alpha)\right]$$

$$= \left(\frac{1}{\sqrt{2\pi/\lambda}}\right)^D \exp\left(\sum_{i=1}^{D}(-\lambda w_i^2/2)\right)$$

$$\mathcal{L}_{post}(\vec{w}, b) = -\log P(\vec{w}, b \mid \mathcal{D})$$

$$= -\log P(\mathcal{D} \mid \vec{w}, b) - \log P(\vec{w}) + \log P(\mathcal{D})$$

$$= \mathcal{L}(\vec{w}, b) - \left(\log\left(\frac{1}{\sqrt{2\pi/\lambda}}\right)^D + \log \exp\left(\sum_{i=1}^{D}(-\lambda w_i^2/2)\right)\right) + \log P(\mathcal{D})$$

$$= \mathcal{L}(\vec{w}, b) - \sum_{i=1}^{D}(-\lambda w_i^2/2) + \log P(\mathcal{D}) - \log\left(\frac{1}{\sqrt{2\pi/\lambda}}\right)^D$$

$$= \mathcal{L}(\vec{w}, b) + \frac{\lambda}{2}\sum_{i=1}^{D} w_i^2 + \sum_{i=1}^{n} \log(t_i\alpha + (1-t_i)(1-\alpha)) - D\log\left(\frac{1}{\sqrt{2\pi/\lambda}}\right)$$

$$= \mathcal{L}(\vec{w}, b) + \frac{\lambda}{2}\sum_{i=1}^{D} w_i^2 + \sum_{i=1}^{n}\left[t_i \log\alpha + (1-t_i)\log(1-\alpha)\right] - D\log\left(\frac{1}{\sqrt{2\pi/\lambda}}\right)$$

$$\mathcal{L}_{post}(\vec{w}, b) = \mathcal{L}(\vec{w}, b) + \frac{\lambda}{2}\sum_{i=1}^{D} w_i^2 + \underbrace{\log\alpha \cdot \sum_{i=1}^{n} t_i + \log(1-\alpha) \cdot \sum_{i=1}^{n}(1-t_i) - D\log\left(\frac{1}{\sqrt{2\pi/\lambda}}\right)}_{C}$$

$$\frac{\partial L_{post}(\vec{w},b)}{\partial \vec{w}} = \frac{\partial L(\vec{w},b)}{\partial \vec{w}} + \frac{\lambda}{2}\frac{\partial \sum_{i=1}^{D} w_i^2}{\partial \vec{w}} + 0$$

$$= -\sum_{i=1}^{n}\left[\left(t_i - \frac{1}{1+\exp(-W^T\vec{x}^{(i)}-b)}\right)\vec{x}^{(i)}\right] + \frac{\lambda}{2}\cdot 2\vec{w}$$

$$\boxed{\frac{\partial L_{post}(\vec{w},b)}{\partial \vec{w}} = -\sum_{i=1}^{n}\left[\left(t_i - \frac{1}{1+\exp(-W^T\vec{x}^{(i)}-b)}\right)\vec{x}^{(i)}\right] + \lambda\vec{w}}$$

$$\boxed{\frac{\partial L_{post}(\vec{w},b)}{\partial b} = \frac{\partial L(\vec{w},b)}{\partial b} = -\sum_{i=1}^{n}\left(t_i - \frac{1}{1+\exp(-W^T\vec{x}^{(i)}-b)}\right)}$$