2.7 Error Probabilities and Integrals

We can obtain additional insight into the operation of a general classifier — Bayes or otherwise — if we consider the sources of its error. Consider first the two-category case, and suppose the dichotomizer has divided the space into two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in a possibly non-optimal way. There are two ways in which a classification error can occur; either an observation $\mathbf{x}$ falls in $\mathcal{R}_2$ and the true state of nature is ω_1, or $\mathbf{x}$ falls in $\mathcal{R}_1$ and the true state of nature is ω_2. Since these events are mutually exclusive and exhaustive, the probability of error is

$$\begin{aligned} P(error) &= P(\mathbf{x} \in \mathcal{R}_2, \omega_1) + P(\mathbf{x} \in \mathcal{R}_1, \omega_2) \\ &= P(\mathbf{x} \in \mathcal{R}_2|\omega_1)P(\omega_1) + P(\mathbf{x} \in \mathcal{R}_1|\omega_2)P(\omega_2) \\ &= \int\limits_{\mathcal{R}_2} p(\mathbf{x}|\omega_1)P(\omega_1)\; d\mathbf{x} + \int\limits_{\mathcal{R}_1} p(\mathbf{x}|\omega_2)P(\omega_2)\; d\mathbf{x}. \end{aligned} \tag{68}$$

This result is illustrated in the one-dimensional case in Fig. 2.17. The two integrals in Eq. 68 represent the pink and the gray areas in the tails of the functions $p(\mathbf{x}|\omega_i)P(\omega_i)$. Because the decision point x^* (and hence the regions $\mathcal{R}_1$ and $\mathcal{R}_2$) were chosen arbitrarily for that figure, the probability of error is not as small as it might be. In particular, the triangular area marked "reducible error" can be eliminated if the decision boundary is moved to x_B. This is the Bayes optimal decision boundary and gives the lowest probability of error. In general, if $p(\mathbf{x}|\omega_1)P(\omega_1) > p(\mathbf{x}|\omega_2)P(\omega_2)$, it is advantageous to classify $\mathbf{x}$ as in $\mathcal{R}_1$ so that the smaller quantity will contribute to the error integral; this is exactly what the Bayes decision rule achieves.

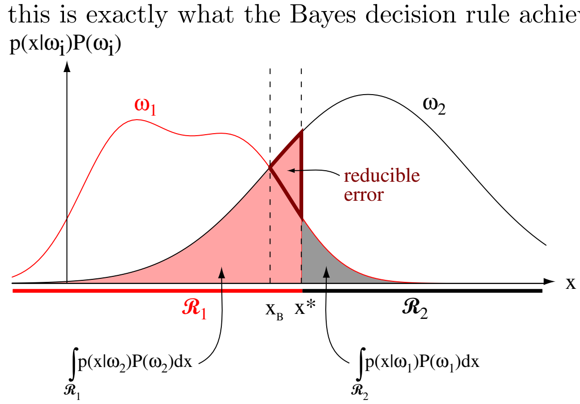

Figure 2.17: Components of the probability of error for equal priors and (non-optimal) decision point x^*. The pink area corresponds to the probability of errors for deciding ω_1 when the state of nature is in fact ω_2; the gray area represents the converse, as given in Eq. 68. If the decision boundary is instead at the point of equal posterior probabilities, x_B, then this reducible error is eliminated and the total shaded area is the minimum possible — this is the Bayes decision and gives the Bayes error rate.

In the multicategory case, there are more ways to be wrong than to be right, and it is simpler to compute the probability of being correct. Clearly

$$P(correct) = \sum_{i=1}^{c} P(\mathbf{x} \in \mathcal{R}_i, \omega_i)$$