1. (a) $f(x) = x^3 - 4x + 1$

$$f(0) = 1 > 0$$
$$f(1) = 1 - 4 + 1 = -2 < 0$$

so $f(c) = 0$ for some $0 < c < 1$

(b) $f(x) = 5\cos \pi x - 4$

$$f(0) = 5 - 4 = 1 > 0$$
$$f(1) = 5\cos \pi - 4 = -9 < 9$$

so $f(c) = 0$ for some $0 < c < 1$

(c) $f(x) = 8x^4 - 8x^2 + 1$

$$f(0) = 1 > 0$$
$$f(\frac{1}{2}) = 8\frac{1}{16} - 8\frac{1}{4} + 1 = -\frac{1}{2} < 0$$

so $f(c) = 0$ for some $0 < c < \frac{1}{2}$

2. (a) $f(x) = e^x$

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = e - 1$$
$$e^c = e - 1$$
$$c = \ln(e - 1)$$

(b) $f(x) = x^2$

$$f'(x) = 2x$$
$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 1$$
$$2c = 1$$
$$c = \frac{1}{2}$$

(c) $f(x) = 1/(x+1)$

$$f'(x) = -\frac{1}{(x+1)^2}$$
$$f(0) = 1$$
$$f(1) = \frac{1}{2}$$
$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = -\frac{1}{2}$$
$$-\frac{1}{(c+1)^2} = -\frac{1}{2}$$
$$c = \pm\sqrt{2} - 1$$

3. (a) $f(x) = x$, $g(x) = x$

$$\int_0^1 f(x)g(x)dx = \int_0^1 x^2 dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}$$
$$f(c)\int_0^1 g(x)dx = f(c)\int_0^1 xdx = \frac{1}{2}f(c)x^2|_0^1 = \frac{1}{2}f(c)$$
$$\frac{1}{2}f(c) = \frac{1}{3}$$
$$x = \frac{2}{3}$$

(b) $f(x) = x^2$, $g(x) = x$

$$\int_0^1 f(x)g(x)dx = \int_0^1 x^3 dx = \frac{1}{4}x^4|_0^1 = \frac{1}{4}$$
$$f(c)\int_0^1 g(x)dx = f(c)\int_0^1 xdx = \frac{1}{2}f(c)x^2|_0^1 = \frac{1}{2}c^2$$
$$\frac{1}{2}c^2 = \frac{1}{4}$$
$$c = \frac{\sqrt{2}}{2}$$

(c) $f(x) = x$, $g(x) = e^x$

$$\begin{aligned}\int_0^1 f(x)g(x)dx &= \int_0^1 xe^x dx \\ &= \int_0^1 x d(e^x) \\ &= xe^x|_0^1 - \int_0^1 e^x dx \\ &= e - e^x|_0^1 = 1\end{aligned}$$

$$\begin{aligned}f(c)\int_0^1 g(x)dx &= c\int_0^1 e^x dx \\ &= ce^x|_0^1 = c(e-1)\end{aligned}$$

$$\begin{aligned}c(e-1) &= 1 \\ c &= \frac{1}{e-1}\end{aligned}$$

4. (a) $f(x) = e^{x^2}$

$$\begin{aligned}f'(x) &= 2xe^{x^2} \\ f''(x) &= 2e^{x^2} + 2x \times 2xe^{x^2} = 2e^{x^2} + 4x^2e^{x^2} \\ f^{(3)}(x) &= 4xe^{x^2} + 8xe^{x^2} + 4x^2 \times 2xe^{x^2} = (12x + 8x^3)e^{x^2}\end{aligned}$$

$$\begin{aligned}P_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &= 1 + x^2\end{aligned}$$

The remainder term is:

$$\frac{f^{(3)}(c)}{3!}x^3 = \frac{(12c + 8c^3)e^{c^2}}{6}x^3 = (2c + \frac{4}{3}c^3)e^{c^2}x^3$$

(b) $f(x) = \cos 5x$

$$\begin{aligned}f'(x) &= -5\sin 5x \\ f''(x) &= -25\cos 5x \\ f^{(3)}(x) &= 125\sin 5x\end{aligned}$$

$$\begin{aligned} P_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &= 1 + \frac{25}{2}x^2 \end{aligned}$$

The remainder term is:

$$\frac{f^{(3)}(c)}{3!}x^3 = \frac{125\sin 5c}{6}x^3$$

(c) $f(x) = 1/(x+1)$

$$\begin{aligned} f'(x) &= -\frac{1}{(x+1)^2} \\ f''(x) &= \frac{2}{(x+1)^3} \\ f^{(3)}(x) &= -\frac{6}{(x+1)^4} \end{aligned}$$

$$\begin{aligned} P_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &= 1 - x + x^2 \end{aligned}$$

The remainder term is:

$$\frac{f^{(3)}(c)}{3!} = -\frac{1}{(c+1)^4}x^3$$

5. (a) $f(x) = e^{x^2}$

$$\begin{aligned} f'(x) &= 2xe^{x^2} \\ f''(x) &= 2e^{x^2} + 2x \times 2xe^{x^2} = (2+4x^2)e^{x^2} \\ f^{(3)}(x) &= 8xe^{x^2} + (2+4x^2) \times 2xe^{x^2} = (12x+8x^3)e^{x^2} \\ f^{(4)}(x) &= (12+24x^2)e^{x^2} + (12x+8x^3) \times 2xe^{x^2} = (12+48x^2+16x^4)e^{x^2} \\ f^{(5)}(x) &= (96x+64x^3)e^{x^2} + (24x+96x^3+32x^5)e^{x^2} = (120x+160x^3+32x^5)e^{x^2} \\ f^{(6)}(x) &= (120+480x^3+160x^4)e^{x^2} + (240x^2+320x^4+64x^6)e^{x^2} \end{aligned}$$

$$\begin{aligned} P_5(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\ &= 1 + \frac{2}{2}x^2 + \frac{12}{4!}x^4 \\ &= 1 + x^2 + \frac{1}{2}x^4 \end{aligned}$$

The remainder term is:

$$\frac{f^{(6)}(c)}{6!}x^6$$

(b) $f(x) = \cos 2x$

$$\begin{aligned}
f'(x) &= -2\sin 2x \\
f''(x) &= -4\cos 2x \\
f^{(3)}(x) &= 8\sin 2x \\
f^{(4)}(x) &= 16\cos 2x \\
f^{(5)}(x) &= -32\sin 2x \\
f^{(6)}(x) &= -64\cos 2x
\end{aligned}$$

$$\begin{aligned}
P_5(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\
&= 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 \\
&= 1 - 2x^2 + \frac{2}{3}x^4
\end{aligned}$$

The remainder term is:

$$\frac{f^{(6)}(c)}{6!}x^6$$

(c) $f(x) = \ln(1 + x)$

$$\begin{aligned}
f'(x) &= \frac{1}{1+x} \\
f''(x) &= -\frac{1}{(1+x)^2} \\
f^{(3)}(x) &= \frac{2}{(1+x)^3} \\
f^{(4)}(x) &= -\frac{6}{(1+x)^4} \\
f^{(5)}(x) &= \frac{24}{(1+x)^5} \\
f^{(6)}(x) &= -\frac{120}{(1+x)^6}
\end{aligned}$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$
$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$

The remainder term is:

$$\frac{f^{(6)}(c)}{6!}x^6$$

(d) $f(x) = \sin^2 x$

$$f'(x) = 2\sin x \cos x = \sin 2x$$
$$f''(x) = 2\cos 2x$$
$$f^{(3)}(x) = -4\sin 2x$$
$$f^{(4)}(x) = -8\cos 2x$$
$$f^{(5)}(x) = 16\sin 2x$$
$$f^{(6)}(x) = 32\cos 2x$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$
$$= x^2 - \frac{1}{3}x^4$$

The remainder term is:

$$\frac{f^{(6)}(c)}{6!}x^6$$

6. (a) $f(x) = x^{-2}$

$$f'(x) = -2x^{-3}$$
$$f''(x) = 6x^{-4}$$
$$f^{(3)}(x) = -24x^{-5}$$
$$f^{(4)}(x) = 120x^{-6}$$

$$P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$
$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4$$

The remainder term is:

$$\frac{f^{(5)}(c)}{5!}(x-1)^5 = -6c^{-7}(x-1)^5$$

(b)

$$f(0.9) = 1 + 0.2 + 0.03 + 0.004 + 0.0005 = 1.2345$$
$$f(1.1) = 1 - 0.2 + 0.03 - 0.004 + 0.0005 = 0.8265$$

(c) if $x = 0.9$, the error bound is:

$$-6c^{-7}(x-1)^5 = 0.00012545$$

if $x = 1.1$, the error bound is:

$$-6c^{-7}(x-1)^5 = 0.00006$$

(d) $f(0.9) = 1.2345679012345679012345679012346$
$1.2345679012345679012345679012346 - 1.2345 = 0.00006790123$
$f(1.1) = 0.82644628099173553719008264462810$
$0.82644628099173553719008264462810 - 0.8265 = 0.000053719$

7. $f(x) = \ln x$

(a)

$$f'(x) = x^{-1}$$
$$f''(x) = -x^{-2}$$
$$f^{(3)}(x) = 2x^{-3}$$
$$f^{(4)}(x) = -6x^{-4}$$
$$f^{(5)}(x) = 24x^{-5}$$

$$\begin{aligned} P_4(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \end{aligned}$$

The remainder term is:

$$\frac{f^{(5)}(c)}{5!}(x-1)^5 = \frac{24}{5!}c^{-5}(x-1)^5 = 0.2c^{-5}(x-1)^5$$

(b)

$$f(0.9) = -0.1 - 0.005 - 0.0003333 - 0.000025 = -0.10535833333$$
$$f(1.1) = 0.1 - 0.005 + 0.00033333 - 0.000025 = 0.0953083$$

(c) if $x = 0.9$, the error bound is:

$$0.2c^{-5}(x-1)^5 = -0.000002c^{-5} = 0.000003387$$

if $x = 0.9$, the error bound is:

$$0.2c^{-5}(x-1)^5 = -0.000002c^{-5} = 0.000002$$

(d) $f(0.9) = -0.105360515657826301227500980839 31$

$-0.10536051565782630122750098083931 + 0.10535833333 = -0.0000021823$

$f(1.1) = 0.095310179804324860043952123280 77$

$0.09531017980432486004395212328077 - 0.0953083 = 0.0000018798$

8. (a) $f(x) = \cos x$

$$\begin{aligned} f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \\ f^{(5)}(x) &= -\sin x \\ f^{(6)}(x) &= -\cos x \end{aligned}$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

The remainder term is:

$$\frac{f^{(6)}(c)}{6!}x^6$$

(b) $x \in [-\pi/4, \pi/4]$ the upper bound for the error is:

$$\frac{f^{(6)}(c)}{6!}x^6 = \frac{\cos(c)}{6!}(\frac{\pi}{4})^6 < \frac{\pi^6}{6! \times 4^6}$$

9. $f(x) = \sqrt{x+1}$

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x+1}} \\ f''(x) &= -\frac{1}{4}(x+1)^{-\frac{3}{2}} \end{aligned}$$

$$
\begin{aligned}
P_1(x) &= f(0) + f'(0)x \\
&= 1 + \frac{1}{2}x
\end{aligned}
$$

the remainder term is:

$$\frac{f''(c)}{2!}x^2$$

if $x = 0.02$

$$\frac{f''(c)}{2!}x^2 = -\frac{1}{4}(c+1)^{-\frac{3}{2}} \times \frac{1}{2} \times 0.0004 = 0.00005(c+1)^{-\frac{3}{2}}$$

$$
\begin{gathered}
\sqrt{1.02} = 1.009950493836208 \\
P_1(0.02) = 1 + \frac{1}{2} \times 0.02 = 1.01 \\
1.01 - 1.009950493836208 = 0.0000495
\end{gathered}
$$