solutions to Exercises 1.2

2019 年 12 月 27 日

1. (a) $\frac{3}{x}$

$$\frac{3}{x} = x$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$

(b) $x^2 - 2x + 2$

$$x^2 - 2x + 2 = x$$
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$
$$x = 1 \text{ or } x = 2$$

(c) $x^2 - 4x + 2$

$$x^2 - 4x + 2 = x$$
$$x^2 - 5x + 2 = 0$$
$$(-5)^2 - 4 \times 2 = 17$$
$$x = \frac{5 \pm \sqrt{17}}{2}$$

2. (a) $\frac{x+6}{3x-2}$

$$\frac{x+6}{3x-2} = x$$
$$x + 6 = 3x^2 - 2x$$
$$3x^2 - 3x - 6 = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \text{ or } x = -1$$

(b) $\frac{8+2x}{2+x^2}$

$$\frac{8+2x}{2+x^2} = x$$
$$8 + 2x = 2x + 2x^2$$
$$x^2 = 4$$
$$x = \pm 2$$

(c) x^5

$$x^5 = x$$
$$x^4 = 1$$
$$x = \pm 1$$

3. (a)

$$\frac{x^3 + x - 6}{6x - 10}$$

$$\frac{1+1-6}{6-10} = \frac{-4}{-4} = 1$$

$$\frac{2^3 + 2 - 6}{6 \times 2 - 10} = \frac{4}{2} = 2$$

$$\frac{3^3 + 3 - 6}{6 \times 3 - 10} = \frac{24}{8} = 3$$

(b)

$$\frac{6+6x^2-x^3}{11}$$

$$\frac{6+6-1}{11}=1$$

$$\frac{6+6\times 2^2-2^3}{11}=2$$

$$\frac{6+6\times 3^2-3^3}{11}=3$$

4. (a)

$$\frac{4x}{x^2+3}$$

$$\frac{-4}{1+3}=-1$$

$$\frac{0}{3}=0$$

$$\frac{4}{1+3}=1$$

(b)

$$\frac{x^2-5x}{x^2+x-6}$$

$$\frac{1+5}{1-1-6}=-1$$

$$\frac{0}{-6}=0$$

$$\frac{1-5}{1+1-6}=1$$

5. (a)

$$\frac{x}{\sqrt{3}}=\frac{\sqrt{3}}{\sqrt{3}}=1\neq\sqrt{3}$$

$\sqrt{3}$ is not a fixed point.

(b)

$$\frac{2x}{3} + \frac{1}{x} = \frac{2\sqrt{3}}{3} + \frac{1}{\sqrt{3}} = \sqrt{3}$$

$\sqrt{3}$ is a fixed point.

(c)

$$x^2 - x = 3 - \sqrt{3} \neq= \sqrt{3}$$

$\sqrt{3}$ is not a fixed point.

(d)

$$1 + \frac{2}{x+1} = 1 + \frac{2}{\sqrt{3}+1} = 1 + \frac{2(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \sqrt{3}$$

$\sqrt{3}$ is a fixed point.

6. (a)

$$\frac{5+7x}{x+7} = \frac{5+7\sqrt{5}}{\sqrt{5}+7} = \frac{(5+7\sqrt{5}}{(\sqrt{5}+7)\sqrt{5}}\sqrt{5} = \sqrt{5}$$

$\sqrt{5}$ is a fixed point.

(b)

$$\frac{10}{3x} + \frac{x}{3} = \frac{10}{3\sqrt{5}} + \frac{\sqrt{5}}{3} = \frac{10\sqrt{5}}{3 \times 5} + \frac{\sqrt{5}}{3} = \sqrt{5}$$

$\sqrt{5}$ is a fixed point.

(c)

$$x^2 - 5 = 0$$

$\sqrt{5}$ is not a fixed point.

(d)

$$1 + \frac{4}{x+1} = 1 + \frac{4}{\sqrt{5}+1} = \sqrt{5}$$

$\sqrt{5}$ is a fixed point.

7. (a)

$$g(x) = (2x-1)^{1/3}$$
$$g'(x) = \frac{2}{3}(2x-1)^{-2/3}$$
$$g'(1) = 0 < 1$$

Fixed-Point Iteration is locally convergent to the given fixed point $r = 1$.

(b)

$$\begin{aligned} g(x) &= \frac{x^3+1}{2} \\ g'(x) &= \frac{3}{2}x^2 \\ g'(1) &= \frac{3}{2} > 1 \end{aligned}$$

Fixed-Point Iteration is not locally convergent to the given fixed point $r = 1$.

(c)

$$\begin{aligned} g(x) &= \sin x + x \\ g'(x) &= \cos x + 1 \\ g'(0) &= 2 > 1 \end{aligned}$$

Fixed-Point Iteration is not locally convergent to the given fixed point $r = 0$.

8. (a)

$$\begin{aligned} g(x) &= \frac{2x-1}{x^2} = \frac{2}{x} - \frac{1}{x^2} \\ g'(x) &= -2x^{-2} + 2x^{-3} \\ g'(1) &= 0 < 1 \end{aligned}$$

Fixed-Point Iteration is locally convergent to the given fixed point $r = 1$.

(b)

$$\begin{aligned} g(x) &= \cos x + \pi + 1 \\ g'(x) &= -\sin x \\ g'(\pi) &= 0 \end{aligned}$$

Fixed-Point Iteration is locally convergent to the given fixed point $r = \pi$.

(c)

$$g(x) = e^{2x} - 1$$
$$g'(x) = 2e^{2x}$$
$$g'(x) = 2 > 1$$

Fixed-Point Iteration is not locally convergent to the given fixed point $r = 0$.

9. (a)

$$g(x) = \frac{1}{2}x^2 + \frac{1}{2}x$$
$$\frac{1}{2}x^2 + \frac{1}{2}x = x$$
$$\frac{1}{2}x^2 - \frac{1}{2}x = 0$$
$$x = 0 \text{ or } x = 1$$

$$g'(x) = x + \frac{1}{2}$$

If $x = 0$, $g'(0) = \frac{1}{2} < 1$, Fixed-Point Iteration of $g(x)$ is locally convergent to it.
If $x = 1$, $g'(\frac{1}{2}) = \frac{3}{2} > 1$, Fixed-Point Iteration of $g(x)$ is not locally convergent to it.

(b)

$$g(x) = x^2 - \frac{1}{4}x + \frac{3}{8}$$
$$x^2 - \frac{1}{4}x + \frac{3}{8} = x$$
$$x^2 - \frac{5}{4}x + \frac{3}{8} = 0$$
$$8x^2 - 10x + 3 = 0$$
$$x = \frac{10 \pm \sqrt{100 - 4 \times 8 \times 3}}{16}$$
$$x = \frac{3}{4} \text{ or } x = \frac{1}{2}$$

$$g'(x) = 2x - \frac{1}{4}$$

If $x = \frac{3}{4}$, $g'(\frac{3}{4}) = \frac{5}{4} > 1$, Fixed-Point Iteration of $g(x)$ is not locally convergent to it.
If $x = \frac{1}{2}$, $g'(\frac{1}{2}) = \frac{3}{4} < 1$, Fixed-Point Iteration of $g(x)$ is locally convergent to it.

10. (a)

$$\begin{gathered} x^2 - \frac{3}{2}x + \frac{3}{2} = x \\ x^2 - \frac{5}{2}x + \frac{3}{2} = 0 \\ 2x^2 - 5x + 3 = 0 \\ (x-1)(2x-3) = 0 \\ x = 1 \text{ or } x = \frac{3}{2} \end{gathered}$$

$$g'(x) = 2x - \frac{3}{2}x$$

If $x = 1$, $g'(1) = \frac{1}{2} < 1$, Fixed-Point Iteration of $g(x)$ is locally convergent to it.
If $x = \frac{3}{2}$, $g'(\frac{3}{2}) = 2\frac{3}{2} - \frac{3}{2}\frac{3}{2} = 3 - \frac{9}{4} = \frac{3}{4} < 1$, Fixed-Point Iteration of $g(x)$ is locally convergent to it.

(b)

$$\begin{gathered} x^2 + \frac{1}{2}x - \frac{1}{2} = x \\ x^2 - \frac{1}{2}x - \frac{1}{2} = 0 \\ 2x^2 - x - 1 = 0 \\ (x-1)(2x+1) = 0 \\ x = 1 \text{ or } x = -\frac{1}{2} \end{gathered}$$

$$g'(x) = 2x + \frac{1}{2}$$

If $x = 1$, $g'(1) = 2 + \frac{1}{2} > 1$, Fixed-Point Iteration of $g(x)$ is not locally convergent to it.
If $x = -\frac{1}{2}$, $|g'(-\frac{1}{2})| = \frac{1}{2} < 1$, Fixed-Point Iteration of $g(x)$ is locally convergent to it.

11. (a)

$$x^3 - x + e^x = 0$$

$$x^3 + e^x = x$$
$$\sqrt[3]{x - e^x} = x$$
$$\ln(x - x^3) = x$$

(b)

$$3x^{-2} + 9x^3 = x^2$$

$$3 + 9x^5 = x^4$$
$$\sqrt[5]{\frac{x^4 - 3}{9}} = x$$
$$3x^{-2} + 9x^3 - x^2 + x = x$$
$$x^2 - 3x^{-2} = 9x^3$$
$$\sqrt[3]{\frac{x^2 - 3x^{-2}}{9}} = x$$

12. (a)

$$g(x) = x^2 - 0.24$$
$$g'(x) = 2x$$
$$g'(0.2) = 0.4 < 0.5$$

Fixed-Point Iteration is locally convergent to $r = 0.2$, it is faster than the Bisection Method.

(b)

$$x^2 - 0.24 = x$$
$$x^2 - x - 0.24 = 0$$
$$(x - 1.2)(x + 0.2) = 0$$
$$x = 1.2 \text{ or } x = -0.2$$

If $x = 1.2$, $g'(1.2) = 1.2^2 - 0.24 = 1.2 > 1$, Fixed-Point Iteration is not locally convergent to $r = 1.2$,

13. (a)

$$\begin{aligned} 0.39 - x^2 &= x \\ x^2 + x - 0.39 &= 0 \\ (x - 0.3)(x + 1.3) &= 0 \\ x = 0.3 &\text{ or } x = -1.3 \end{aligned}$$

(b)

$$g'(x) = -2x$$

If $x = 0.3$, $|g'(0.3)| = |-0.6| < 1$, FPI converge to it.
If $x = -1.3$, $|g'(1.3)| > 1$, FPI does't converge to it.

(c) Because $|g'(0.3)| > 0.5$, FPI converge to this fixed point slower than the Bisection Method.

14. (A)

$$\begin{aligned} g(x) &= \frac{1}{2}x + \frac{1}{x} \\ g'(x) &= \frac{1}{2} - \frac{1}{x^2} \\ g'(\sqrt{2}) &= 0 < 1 \end{aligned}$$

(B)

$$\begin{aligned} g(x) &= \frac{2}{3}x + \frac{2}{3x} \\ g'(x) &= \frac{2}{3} - \frac{2}{3x^2} \\ g'(\sqrt{2}) &= \frac{1}{3} < 1 \end{aligned}$$

(C)

$$\begin{aligned} g(x) &= \frac{3}{4}x + \frac{1}{2x} \\ g'(x) &= \frac{3}{4} - \frac{1}{2x^2} \\ g'(\sqrt{2}) &= \frac{1}{2} < 1 \end{aligned}$$

Rank: $A > B > C$

15. (A)

$$g(x) = \frac{4}{5}x + \frac{1}{x}$$
$$g'(x) = \frac{4}{5} - \frac{1}{x^2}$$
$$g'(\sqrt{5}) = \frac{4}{5} - \frac{1}{5} = \frac{3}{5} = 0.6 < 1$$

(B)

$$g(x) = \frac{x}{2} + \frac{5}{2x}$$
$$g'(x) = \frac{1}{2} - \frac{5}{2x^2}$$
$$g'(\sqrt{5}) = \frac{1}{2} - \frac{5}{10} = 0 < 1$$

(C)

$$g(x) = \frac{x+5}{x+1}$$
$$g'(x) = \frac{x+1-(x+5)}{(x+1)^2} = \frac{-4}{(x+1)^2}$$
$$|g'(\sqrt{5})| = \frac{4}{(\sqrt{5}+1)^2} = 0.382 < 1$$

Rank $B > C > A$

16. (A)

$$g(x) = \frac{2}{\sqrt{x}}$$
$$|g'(x)| = |2 \times (-\frac{1}{2})x^{-3/2}| = |-x^{-3/2}|$$
$$|g'(\sqrt[3]{4})| = \frac{1}{\sqrt{4}} = \frac{1}{2} < 1$$

(B)

$$g(x) = \frac{3x}{4} + \frac{1}{x^2}$$
$$g'(x) = \frac{3}{4} - \frac{2}{x^3}$$
$$g'(x) = \frac{3}{4} - \frac{2}{4} = \frac{1}{4} < 1$$

(C)

$$g(x) = \frac{2x}{3} + \frac{4}{3x^2}$$
$$g'(x) = \frac{2}{3} - \frac{8}{3x^3}$$
$$g'(x) = \frac{2}{3} - \frac{8}{12} = 0 < 1$$

Rank: $C > B > A$

17.

$$g(x) = 2x^2 + x - 1$$
$$g(1/2) = \frac{1}{2} + \frac{1}{2} - 1 = 0$$
$$g(-1) = 2 - 1 - 1 = 0$$

$$2x^2 + x - 1 = 0$$
$$x^2 = \frac{1-x}{2}$$
$$f_1(x) = \sqrt{\frac{1-x}{2}} = \frac{1}{\sqrt{2}}(1-x)^{1/2}$$
$$f_2(x) = -\sqrt{\frac{1-x}{2}} = -\frac{1}{\sqrt{2}}(1-x)^{1/2}$$
$$f_1'(x) = \frac{1}{2\sqrt{2}} \times (1-x)^{-1/2} \times (-1) = -\frac{1}{2\sqrt{2}}(1-x)^{-1/2}$$
$$f_2'(x) = -\frac{1}{2\sqrt{2}} \times (1-x)^{-1/2} \times (-1) = \frac{1}{2\sqrt{2}}(1-x)^{-1/2}$$
$$|f_1'(1/2)| = \frac{1}{2\sqrt{2}}(\frac{1}{2})^{-1/2} = \frac{1}{2} < 1$$
$$|f_2'(-1)| = \frac{1}{2\sqrt{2}}(\frac{1}{2})^{-1/2} = \frac{1}{2} < 1$$

18. Suppose k is an integer, and $k > 1$,

$$x^2 = k$$
$$x = \frac{k}{x}$$
$$2x = x + \frac{k}{x}$$
$$\frac{1}{2}(x + \frac{k}{x}) = x$$
$$g(x) = \frac{1}{2}(x + \frac{k}{x})$$
$$g'(x) = \frac{1}{2}(1 - \frac{k}{x^2})$$

if $x^2 = k - 1$,

$$|g'(x)| = |\frac{1}{2}(1 - \frac{k}{k-1})| = |\frac{1}{2(k-1)}| < 1$$

if $x^2 = k + 1$,

$$|g'(x)| = |\frac{1}{2}(1 - \frac{k}{k+1})| = |\frac{1}{2(k+1)}| < 1$$

if $k - 1 < x^2 < k + 1$, $|g'(x)| < 1$

19.

$$x^3 = A$$
$$x = \frac{A}{x^2}$$
$$2x = x + \frac{A}{x^2}$$
$$x = \frac{1}{2}(x + \frac{A}{x^2})$$
$$g(x) = \frac{1}{2}(x + \frac{A}{x^2})$$
$$g'(x) = \frac{1}{2}(1 - \frac{2A}{x^3})$$

20.

21.

22.

23.