Linear Classifiers

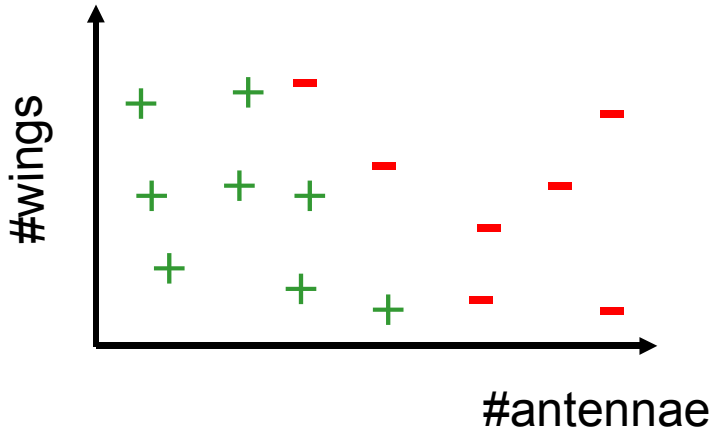
Outline

- Framework
- “Exact”
  - Minimize Mistakes (Perceptron Training)
  - Matrix inversion
- “Logistic Regression” Model
  - Max Likelihood Estimation (MLE) of P(y | **x**)
  - Gradient descent (MSE; MLE)
- “Linear Discriminant Analysis”
  - Max Likelihood Estimation (MLE) of P(y, **x**)
  - Direct Computation

Diagnosing Butterfly-itis

Classifier: Decision Boundaries

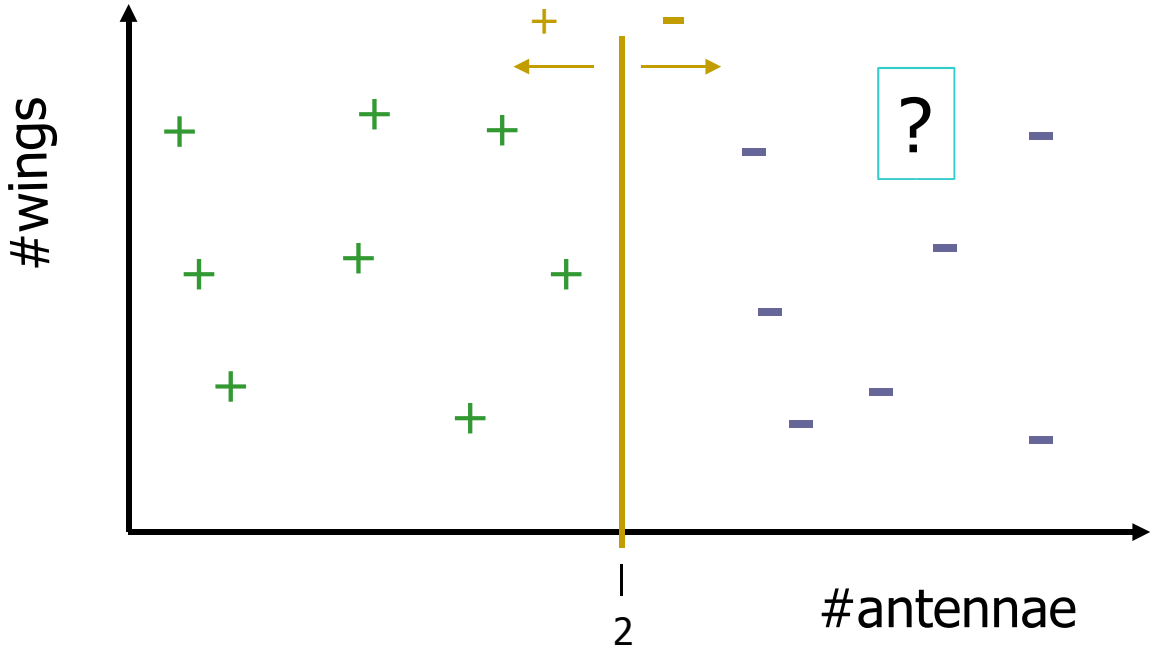
- **Classifier**: partitions input space X into "decision regions"



- L*inear threshold unit* has a linear decision boundary
- Defn: Set of points that can be separated by linear decision boundary is "linearly separable"

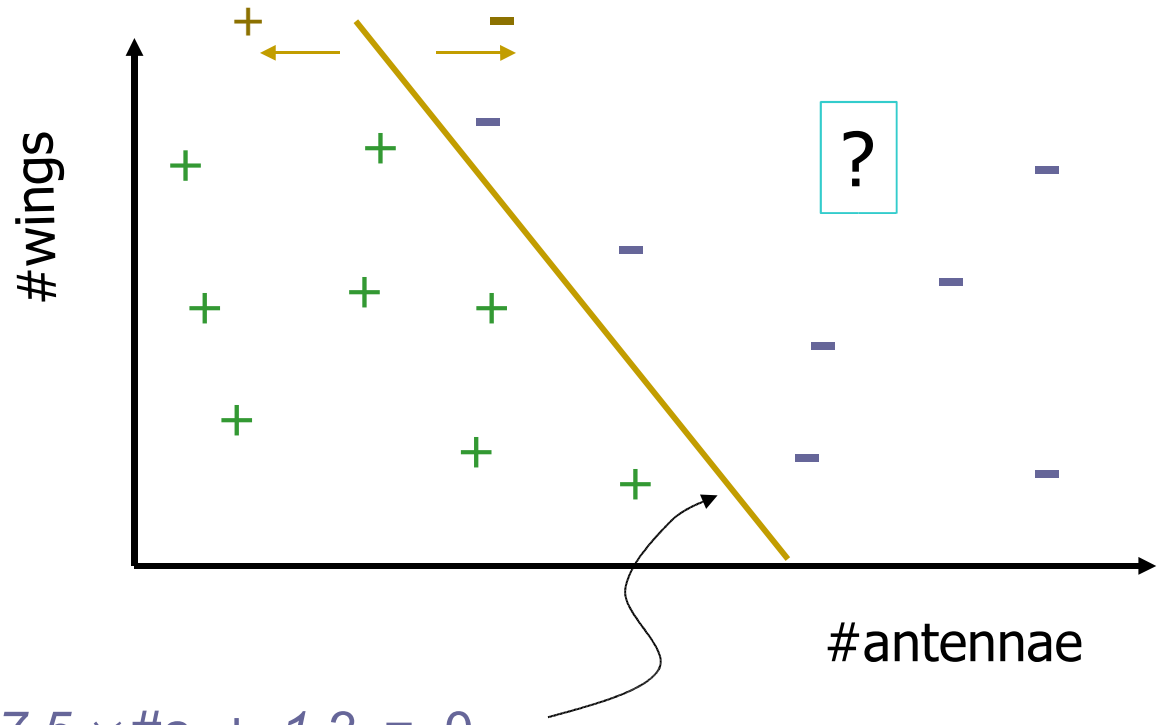
Linear Separators

- Draw “separating line”



- If #antennae ≤ 2, then butterfly-itis
- So ? is Not butterfly-itis.

Can be “angled”…



$2.3 \times \#w + 7.5 \times \#a + 1.2 = 0$

- If $2.3 \times \#Wings + 7.5 \times \#antennae + 1.2 > 0$ then butterfly-itis

Linear Separators, in General

- Given data (many features)

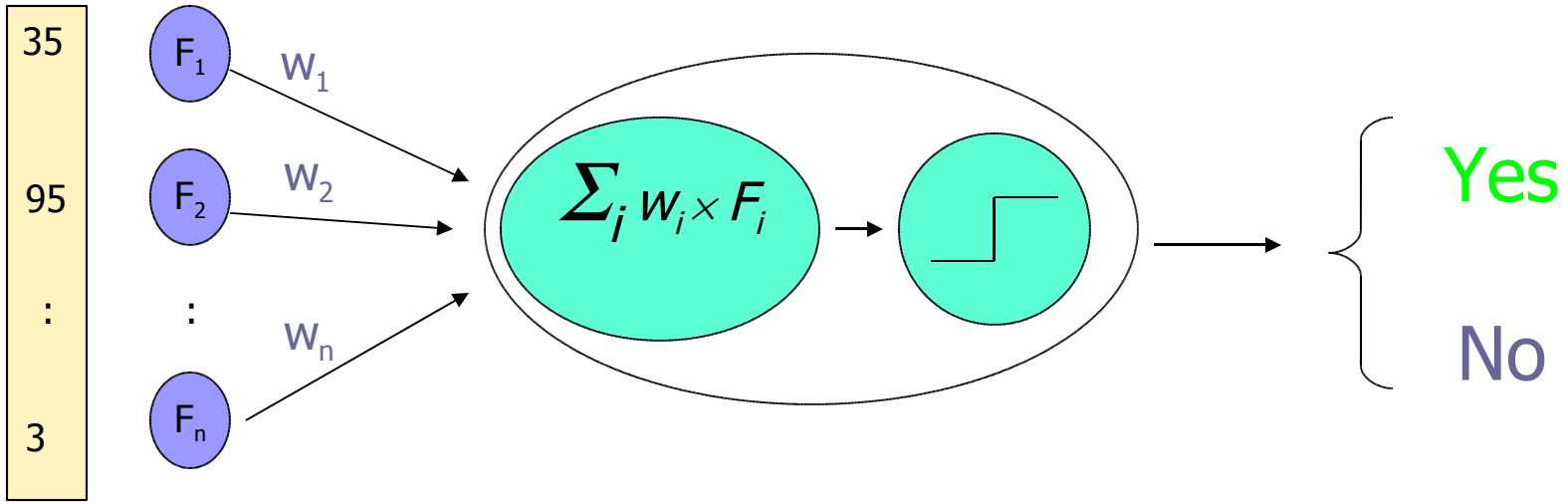
F_1	F_2	…	F_n	Class
35	95	…	3	No
22	80	…	-2	Yes
:	:		:	:
10	50	…	1.9	No

- find "weights" $\{w_1, w_2, \ldots, w_n, w_0\}$ such that

$$w_1 \times F_1 + \ldots + w_n \times F_n + w_0 > 0$$

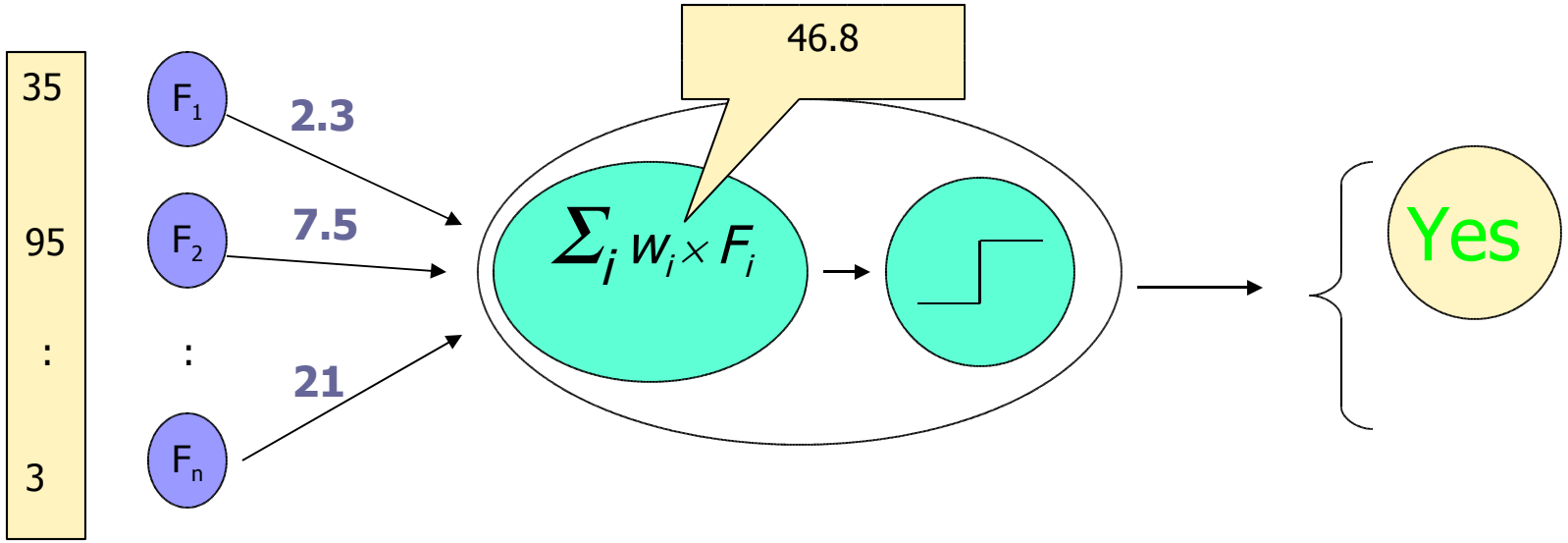
means $Class = Yes$

Linear Separator



Just view $F_0 = 0$, so w_0 …

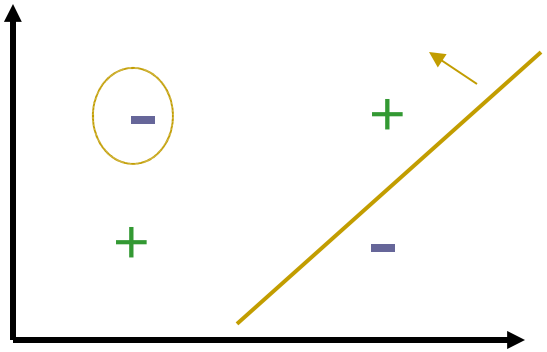
Linear Separator



- Performance
  - Given $\{w_i\}$, and values for instance, compute response
- Learning
  - Given labeled data, find "correct" $\{w_i\}$
- Linear Threshold Unit … "Perceptron"

Linear Separators – Facts

- GOOD NEWS:
  - If data is linearly separated,
  - Then FAST ALGORITHM finds correct $\{w_i\}$!
    - But…



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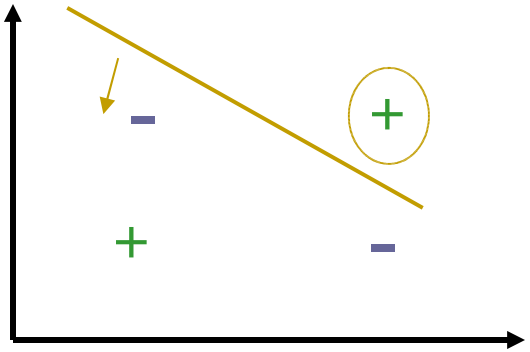


- Some “data sets” are NOT linearly separatable!

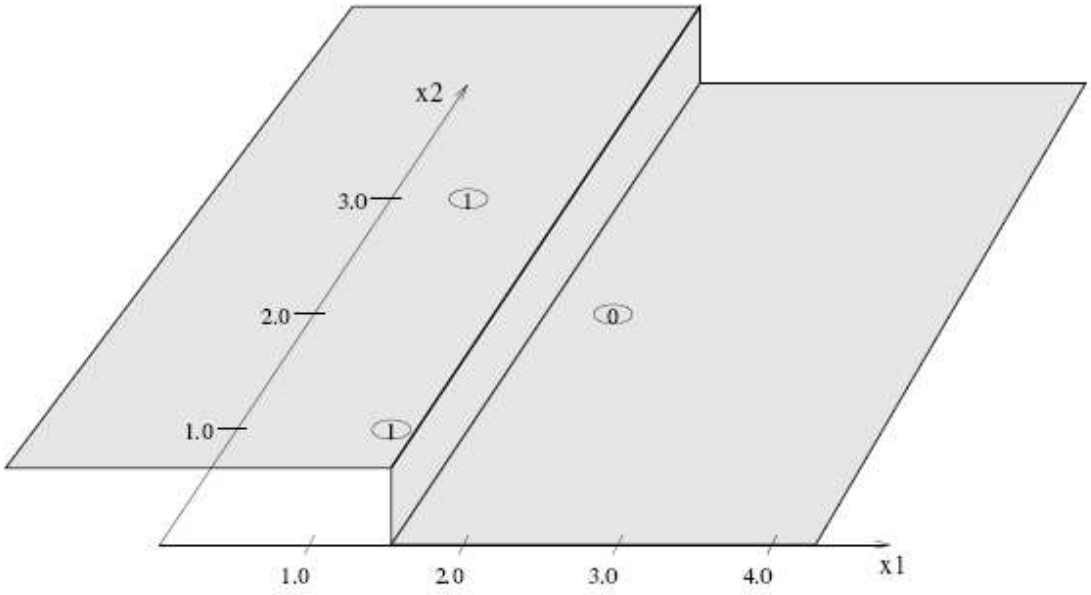
Stay tuned!

Geometric View

- Consider 3 training examples:

([1.0, 1.0]; 1)
([0.5; 3.0]; 1)
([2.0; 2.0]; 0)

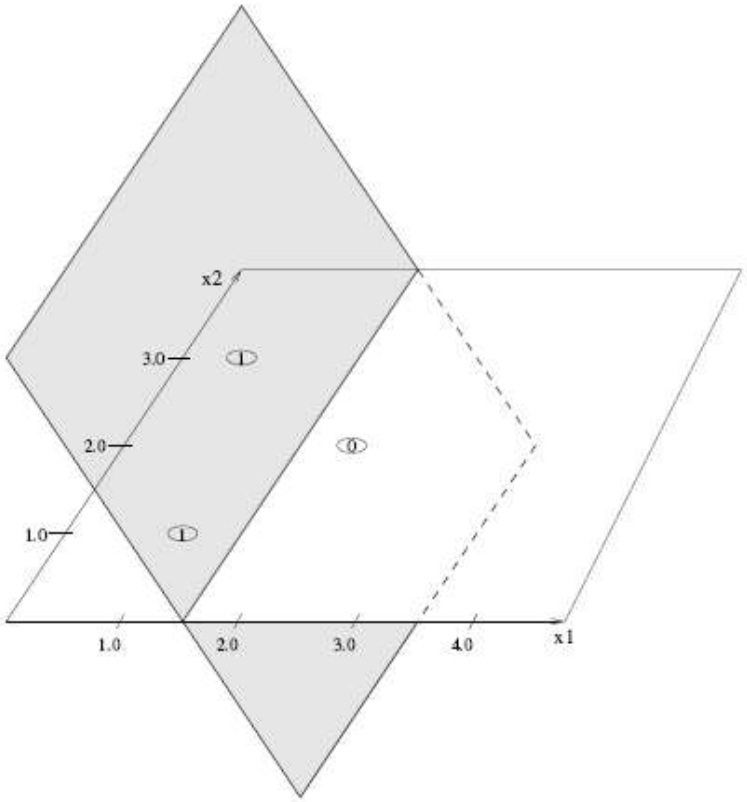
- Want classifier that looks like. . .



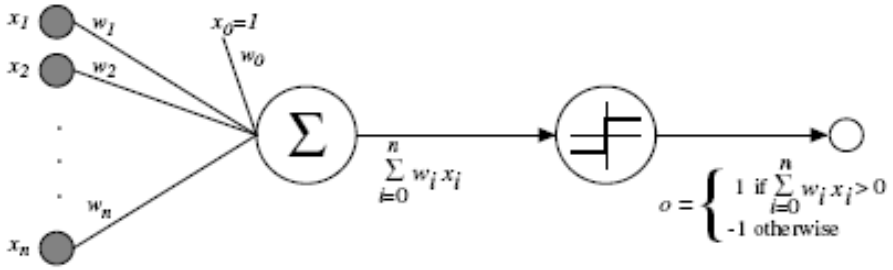
Linear Equation is Hyperplane

- Equation $\mathbf{w}\cdot\mathbf{x} = \sum_i w_i \cdot x_i$ is plane

$$y(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{w}\cdot\mathbf{x} > 0 \\ 0 & \text{otherwise} \end{cases}$$



Linear Threshold Unit: “Perceptron”



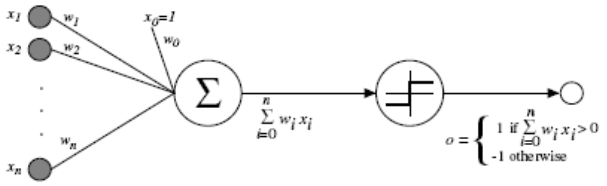
$$o_{\mathbf{w}}(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } w_0 + w_1 x_1 + \cdots + w_n x_n > 0 \\ -1 & \text{otherwise.} \end{cases}$$
$$= \text{sign}(\ (w_0, w_1, \ldots, w_n) \cdot (1, x_1, \ldots, x_n))$$

- Squashing function:
  sgn: $\Re \rightarrow$ {-1, +1 }

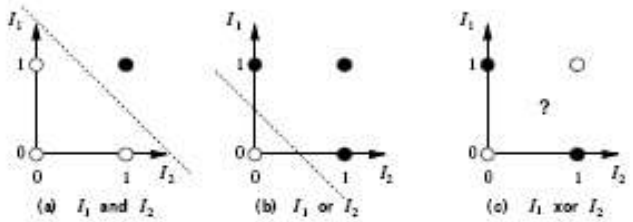
  $$\text{sgn}(r) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{otherwise} \end{cases}$$

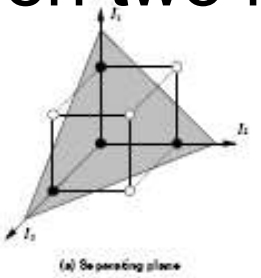
  (“heaviside”)
- Actually $\mathbf{w} \cdot \mathbf{x} > b$ but. . .
  Create extra input x_0 fixed at 1
  Corresponding w_0 corresponds to -b

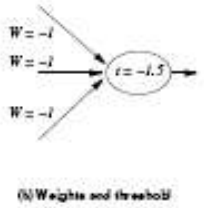
Learning Perceptrons



- Can represent Linearly-Separated surface
  . . . any hyper-plane between two half-spaces…







- Remarkable learning algorithm: [Rosenblatt 1960]

  If function *f* can be represented by perceptron,
  then ∃learning alg guaranteed to quickly converge to *f*!

⇒ enormous popularity, early / mid 60's

- But some simple fns cannot be represented
  (Boolean XOR) [Minsky/Papert 1969]
- Killed the field temporarily!

Perceptron Learning

- Hypothesis space is. . .
  - **Fixed Size:**
    ∃ $O(2^{n^2})$ distinct perceptrons over n boolean features
  - **Deterministic**
  - **Continuous** Parameters
- Learning algorithm:
  - Various: **Local** search, **Direct** computation, . . .
  - **Eager**
  - **Online** / **Batch**

Task

- Input: labeled data

x_1	x_2	x_3	$\cdots$	x_r	y
16.1	−22.7	0.3	$\cdots$	−4.0	+
−7.3	−5.1	9.1	$\cdots$	17.1	−
⋮				⋮	⋮
−16.2	−77.0	−1.2	$\cdots$	−4.0	+

Transformed to

x_0	x_1	x_2	x_3	$\cdots$	x_r	y
1	16.1	−22.7	0.3	$\cdots$	−4.0	+
1	−7.3	−5.1	9.1	$\cdots$	17.1	−
⋮	⋮				⋮	⋮
1	−16.2	−77.0	−1.2	$\cdots$	−4.0	+

- Output: $\mathbf{w} \in \Re^{r+1}$

  Goal: Want **w** s.t.

  $\forall i \;\; \text{sgn}(\mathbf{w} \cdot [1, \mathbf{x}^{(i)}]) = y^{(i)}$

  - . . . minimize mistakes wrt data . . .

Error Function

Given data $\{ [x^{(i)}, y^{(i)}] \}_{i=1..m}$, optimize...

- 1. Classification error

  Perceptron Training; Matrix Inversion

$$err_{Class}(w) = \frac{1}{m}\sum_{i=1}^{m} I[\, y^{(i)} \neq o_w(x^{(i)}) \,]$$

- 2. Mean-squared error

  Matrix Inversion; Gradient Descent

$$err_{MSE}(w) = \frac{1}{m}\sum_{i=1}^{m} \frac{1}{2}[\, y^{(i)} - o_w(x^{(i)}) \,]^2$$

- 3. (Log) Conditional Probability

  MSE Gradient Descent; LCL Gradient Descent

$$LCL(w) = \frac{1}{m}\sum_{i=1}^{m} \log P_w(\, y^{(i)} | x^{(i)})$$

- 4. (Log) Joint Probability

  Direct Computation

$$LL(w) = \frac{1}{m}\sum_{i=1}^{m} \log P_w(\, y^{(i)}, x^{(i)})$$

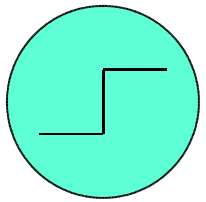
#1: Optimal Classification Error

- For each labeled instance [**x**, y]

  Err = y – $o_{\mathbf{w}}(\mathbf{x})$

  y = f(x) is target value

  $o_{\mathbf{w}}(\mathbf{x}) = \text{sgn}(\mathbf{w} \cdot \mathbf{x})$ is perceptron output



- **Idea**: Move weights in appropriate direction, to push Err → 0
- If Err > 0 (error on POSITIVE example)
  - need to increase sgn(**w** · **x**)

    ⇒ need to increase **w** · **x**
  - Input j contributes $w_j \cdot x_j$ to **w** · **x**
    - if $x_j > 0$, increasing w_j
    - if $x_j < 0$, decre

  ⇒ $w_i \leftarrow w_i + x_i$

- If Err < 0 (error on NEGATIVE example)

  ⇒ $w_j \leftarrow w_j - x_j$

#1a: Mistake Bound Perceptron Alg

Initialize **w** = 0
Do until bored
  Predict "+" iff **w** · **x** > 0
    else "–"
  Mistake on positive: **w** ←**w** + **x**
  Mistake on negative: **w** ←**w** – **x**

Weights	Instance	Action
[0 0 0]	#1	

Orig Data

$\langle\langle x_1$	$x_2\rangle$	$c(x)\rangle$
0	0	+
1	0	–
1	1	+

Data + "$x_0 = 1$"

	$\langle\langle x_0$	x_1	$x_2\rangle$	$c(x)\rangle$
i_1 :	1	0	0	+
i_2 :	1	1	0	–
i_3 :	1	1	1	+

Mistake Bound Theorem

Theorem: [Rosenblatt 1960]

If data is consistent w/some linear threshold **w**,

then number of mistakes is $\leq (1/\Delta)^2$,

where $\Delta = \min_x \frac{|\mathbf{w} \cdot x|}{|\mathbf{w}| \times |x|}$

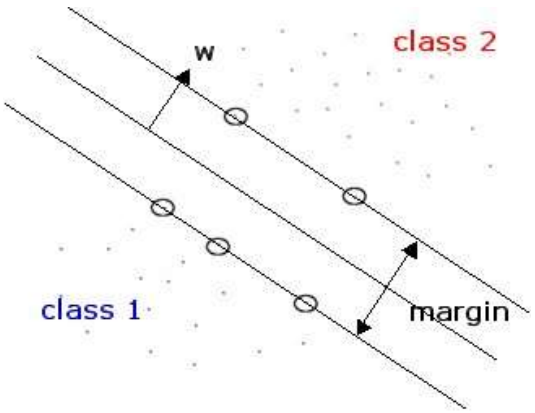
- Δ measures "wiggle room" available:

  If |x| = 1, then Δ is max, over all consistent planes, of minimum distance of example to that plane

- w is $\perp$ to separator, as $w \cdot x = 0$ at boundary

- So $|\mathbf{w} \cdot \mathbf{x}|$ is projection of **x** onto plane, PERPENDICULAR to boundary line

  … ie, is distance from **x** to that line (once normalized)



Proof of Convergence

For simplicity:

0. Use $x_0 \equiv 1$, so target plane goes thru 0
1. Assume target plane doesn't hit any examples
2. Replace negative point $\langle\langle x_0, x_1, \ldots, x_n\rangle\ 0\rangle$
   by positive point $\langle\langle -x_0, -x_1, \ldots, -x_n\rangle\ 1\rangle$
3. Normalize all examples to have length 1

x wrong wrt **w** iff $\mathbf{w} \cdot \mathbf{x} < 0$

- Let $\mathbf{w}^*$ be unit vector rep'ning target plane
  $\Delta = \min_{\mathbf{x}} \{ \mathbf{w}^* \cdot \mathbf{x} \}$
  Let **w** be hypothesis plane
- Consider: $\dfrac{(w \cdot w^*)}{|w|}$
- On each mistake, add **x** to **w**

$$\mathbf{w} = \sum_{\{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{w} < 0\}} \mathbf{x}$$

Proof (con't)

$$\frac{(w \cdot w^*)}{|w|}$$

$\Delta = \min_x \{ \mathbf{w}^* \cdot \mathbf{x} \}$

$\mathbf{w} = \sum_{\{x \mid \mathbf{x} \cdot \mathbf{w} < 0\}} \mathbf{x}$

If w is mistake…

Numerator increases by $x \cdot w^* \geq \Delta$

(denominator)^2 becomes

$(w+x)^2 = w^2 + x^2 + 2(w \cdot x) < w^2 + 1$

as $w \cdot x < 0$

As initially $w = \langle 0, \ldots, 0 \rangle$.

After m mistakes,

numerator is $\geq m \times \Delta$

(denominator)^2 is $\leq 0 + \underbrace{1 + \ldots + 1}_{m} = m$

so denominator $\leq \sqrt{m}$

- As $(w \cdot w^*)/|w|$ = cos(angle between w and w^*)
  it must be ≤ 1, so
  numerator $\leq$ denominator

$$\Rightarrow \quad \Delta * m \leq \sqrt{m} \quad \Rightarrow \quad m \leq \frac{1}{\Delta^2}$$

#1b: Perceptron Training Rule

- For each labeled instance [**x**, y]

  Err([**x**, y]) = $y - o_{\mathbf{w}}(\mathbf{x}) \in \{ -1, 0, +1 \}$

  - If Err([**x**, y]) = 0 Correct! … Do nothing!

    $\Delta w = 0 \equiv \text{Err}([\mathbf{x}, y]) \cdot \mathbf{x}$

  - If Err([**x**, y]) = +1 Mistake on positive! Increment by +x

    $\Delta w = +x \equiv \text{Err}([\mathbf{x}, y]) \cdot \mathbf{x}$

  - If Err([**x**, y]) = -1 Mistake on negative! Increment by -x

    $\Delta w = -x \equiv \text{Err}([\mathbf{x}, y]) \cdot \mathbf{x}$

In all cases... $\Delta w^{(i)} = \text{Err}([\mathbf{x}^{(i)}, y^{(i)}]) \cdot \mathbf{x}^{(i)} = [y^{(i)} - o_{\mathbf{w}}(\mathbf{x}^{(i)})] \cdot \mathbf{x}^{(i)}$

- Batch

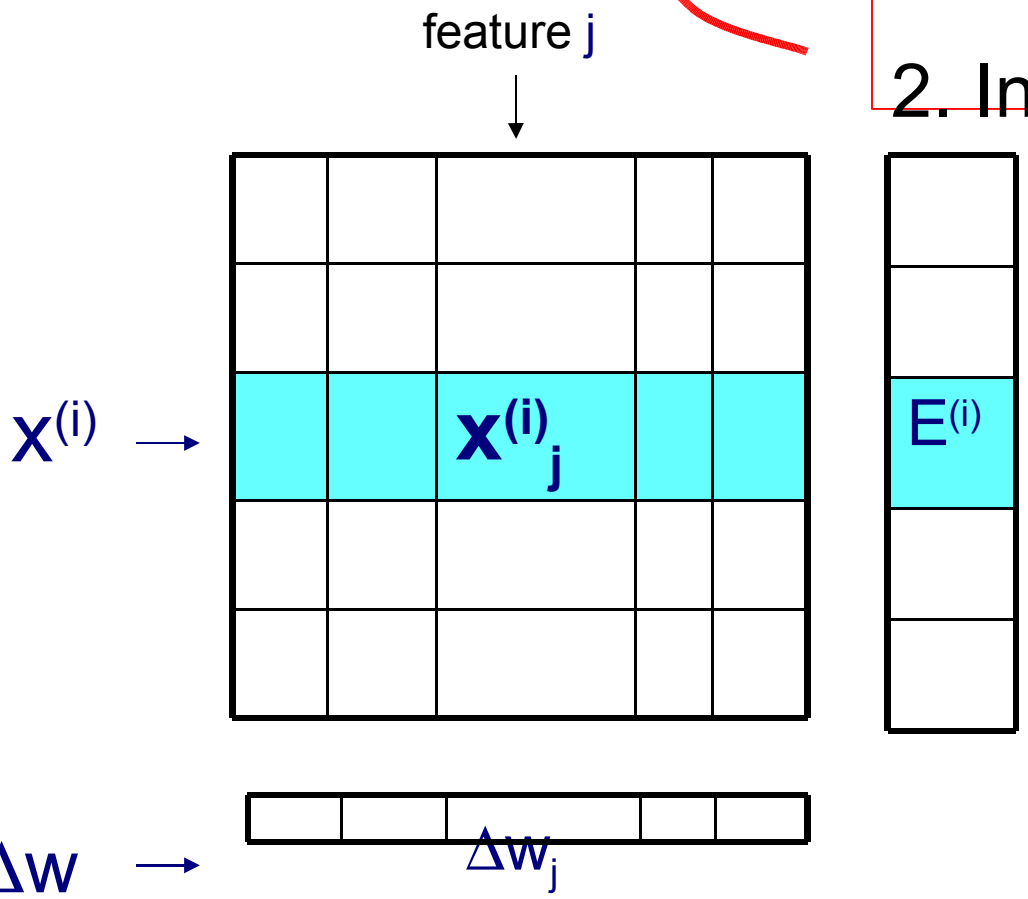
$$\Delta w_j = \sum_i \Delta w_j^{(i)}$$

$$= \sum_i x^{(i)}_j \left(y^{(i)} - o_{\mathbf{w}}(\mathbf{x}^{(i)}) \right)$$

$$w_j \mathrel{+}= \eta \, \Delta w_j$$

η is **learning rate** (small pos "constant" . . . ≈ 0.05?)

0. New **w**
1. For each row i, compute
   a. $\Delta \mathbf{w} = 0$
   b. $E^{(i)} = y^{(i)} - o_{\mathbf{w}}(\mathbf{x}^{(i)})$
   c. $\Delta \mathbf{w} \mathrel{+}= E^{(i)} \mathbf{x}^{(i)}$
      [… $\Delta w_j \mathrel{+}= E^{(i)} x^{(i)}_j$ …]
2. Increment $\mathbf{w} \mathrel{+}= \eta \, \Delta \mathbf{w}$



Correctness

- Rule is intuitive: **Climbs in correct direction. . .**
- Thrm: Converges to correct answer, if . . .
  - training data is linearly separable
  - sufficiently small η
- Proof: Weight space has **EXACTLY 1 minimum**!
  (no non-global minima)
  $\Rightarrow$ with enough examples, finds correct function!
- Explains early popularity
- If η too large, may overshoot
  If η too small, takes too long
- So often $\eta = \eta(k)$ … which decays with # of iterations, k

#1c: Matrix Version?

Task: Given $\{ \langle \mathbf{x}^i, y^i \rangle \}$

$y^i \in \{-1, +1\}$ is label

Find w_i s.t.

$$\begin{aligned} y^1 &= w_0 + w_1 x_1^1 + \cdots + w_n x_n^1 \\ y^2 &= w_0 + w_1 x_1^2 + \cdots + w_n x_n^2 \\ &\vdots \\ y^m &= w_0 + w_1 x_1^m + \cdots + w_n x_n^m \end{aligned}$$

- **Linear Equalities** $\mathbf{y} = \mathbf{X}\mathbf{w}$!

$$\mathbf{y} = [y^1, \ldots, y^m]^\top$$

$$\mathbf{X} = \begin{pmatrix} 1 & x_1^1 & \cdots & x_n^1 \\ 1 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ 1 & x_1^m & \cdots & x_n^m \end{pmatrix}$$

$$\mathbf{w} = [w_0, w_1, \ldots, w_n]^\top$$

- Solution: $\mathbf{w} = \mathbf{X}^{-1}\mathbf{y}$!

Issues

Task: Given $\{ \langle \mathbf{x}^i, y^i \rangle \}$ $y^i \in \{-1, +1\}$ is label
Find w_i s.t.

$$\begin{aligned} y^1 &= w_0 + w_1 x_1^1 + \cdots + w_n x_n^1 \\ y^2 &= w_0 + w_1 x_1^2 + \cdots + w_n x_n^2 \\ &\vdots \\ y^m &= w_0 + w_1 x_1^m + \cdots + w_n x_n^m \end{aligned}$$

1. Why restrict to only $y^i \in \{ -1, +1 \}$?
   - If from discrete set $y^i \in \{ 0, 1, \ldots, m \}$: General (non-binary) classification
   - If ARBITRARY $y^i \in \Re$: Regression

2. What if NO **w** works?

   ...X is singular; overconstrained ...

   Could try to minimize residual

   NP-Hard!

   $\sum_i \mathbb{I}[y^{(i)} \neq \mathbf{w} \cdot \mathbf{x}^{(i)}]$

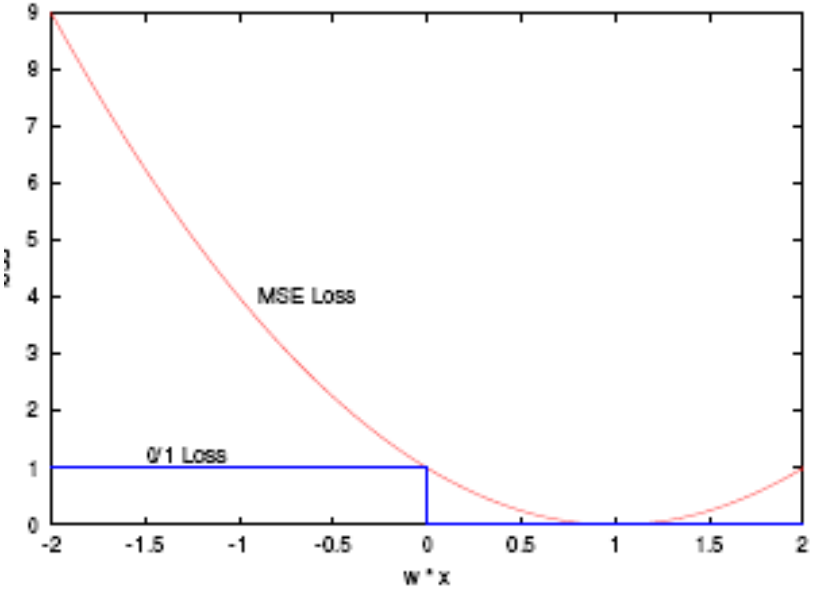
   $|| y - X \mathbf{w} ||_1 = \sum_i | y^{(i)} - \mathbf{w} \cdot \mathbf{x}^{(i)} |$

   $|| y - X \mathbf{w} ||_2 = \sum_i (y^{(i)} - \mathbf{w} \cdot \mathbf{x}^{(i)})^2$

   Easy!

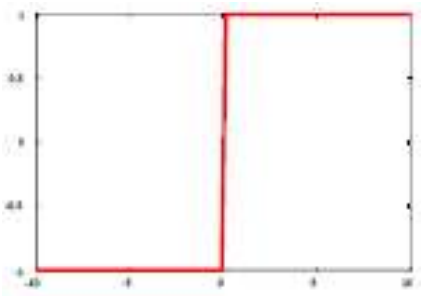
L_2 error vs 0/1-Loss

- "0/1 Loss function" not smooth, differentiable
- MSE error is smooth, differentiable… and is overbound...



Gradient Descent for Perceptron?

- Why not Gradient Descent for THRESHOLDed perceptron?
- Needs gradient (derivative), not



- Gradient Descent is General approach.
  Requires
  + continuously parameterized hypothesis
  + error must be differentiatable wrt parameters
  But. . .
  – can be slow (many iterations)
  – may only find LOCAL opt

#1. LMS version of Classifier

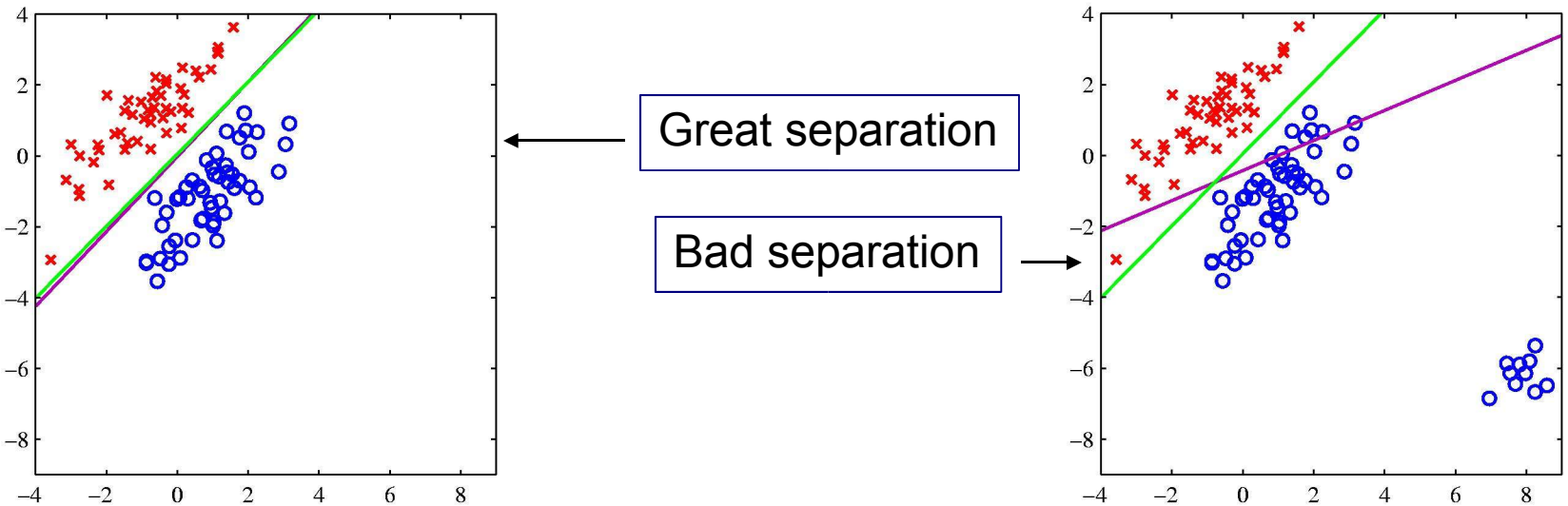
- View as Regression
  - Find "best" linear mapping **w** from **X** to Y
    - $w^* = \text{argmin } Err_{LMS}^{(\mathbf{X}, \mathbf{Y})}(w)$
    - $Err_{LMS}^{(\mathbf{X}, \mathbf{Y})}(w) = \sum_i (y^{(i)} - \mathbf{w} \cdot \mathbf{x}^{(i)})^2$
  - Threshold: if w.x > 0.5,
    return 1;
    else 0
- See Chapter 3

Use Linear Regression for Classification?

1. Use regression to find weights **w**
2. Classify new instance **x** as sgn(**w**·**x**)

- But … regression minimizes
  sum of squared errors on target function

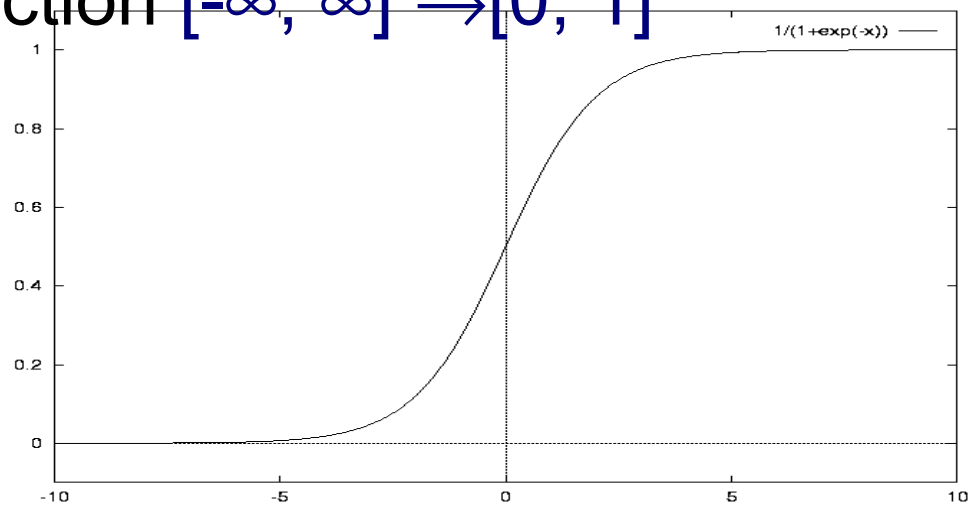
… which gives strong influence to outliers



#3: Logistic Regression

- Want to compute $P_w(y=1|\ \mathbf{x})$
  ... based on parameters **w**
- But …
  - **w**·**x** has range $[-\infty, \infty]$
  - probability must be in range $\in$ [0; 1]
- Need "squashing" function $[-\infty, \infty] \rightarrow [0, 1]$

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



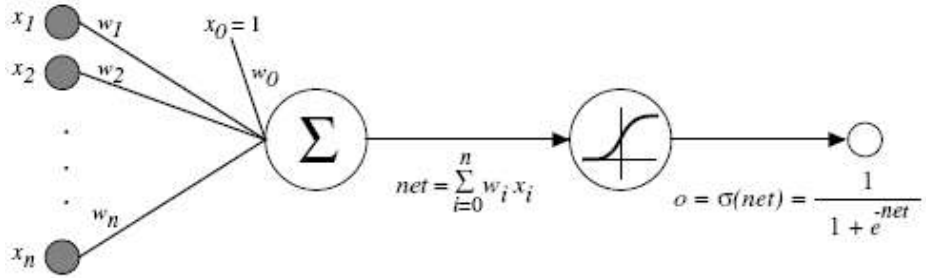
Alternative Derivation…

$$P(+y|x) = \frac{P(x|+y)P(+y)}{P(x|+y)P(+y) + P(x|-y)P(-y)}$$

$$= \frac{1}{1+\exp(-a)}$$

$$a = \ln \frac{P(x|+y)P(+y)}{P(x|-y)P(-y)}$$

Logistic Regression (con't)



- Assume 2 classes:

$$P_w(+y|x) = \sigma(w \cdot x) = \frac{1}{1+e^{-(x \cdot w)}}$$

$$P_w(-y|x) = 1 - \frac{1}{1+e^{-(x \cdot w)}} = \frac{e^{-(x \cdot w)}}{1+e^{-(x \cdot w)}}$$

- Log Odds:

$$\log \frac{P_w(+y|x)}{P_w(-y|x)} = x \cdot w$$

How to learn parameters **w** ?

- … depends on goal?
  - A: Minimize MSE?

$$\sum_i (y^{(i)} - o_{\mathbf{w}}(\mathbf{x}^{(i)}))^2$$

  - B: Maximize likelihood?

$$\sum_i \log P_{\mathbf{w}}(y^{(i)} \mid \mathbf{x}^{(i)})$$

MSError Gradient for Sigmoid Unit

- Error: $\sum_j (y^{(j)} - o_{\mathbf{w}}(\mathbf{x}^{(j)}))^2 = \sum_j E^{(j)}$

$$\sigma(z) = \frac{1}{1+e^{-z}}$$

For single training instance

- **Input**: $\mathbf{x}^{(j)} = [x^{(j)}_1, \ldots, x^{(j)}_k]$
- **Computed Output**: $o^{(j)} = \sigma(\sum_i x^{(j)}_i \cdot w_i) = \sigma(z^{(j)})$
  - where $z^{(j)} = \sum_i x^{(j)}_i \cdot w_i$ using current $\{ w_i \}$
- **Correct output**: $y^{(j)}$

Stochastic Error Gradient

$$\frac{\partial E}{\partial w_i} = \frac{\partial}{\partial w_i}\left[\frac{1}{2}(o-y)^2\right] = \frac{1}{2}\left[2(o-y)\frac{\partial}{\partial w_i}(o-y)\right]$$

$$= (o-y)\left(\frac{\partial o}{\partial w_i}\right) = (o-y)\frac{\partial \sigma(z)}{\partial z}\frac{\partial z}{\partial w_i}$$

Derivative of Sigmoid

$$\frac{d}{da}\sigma(a) = \frac{d}{da}\frac{1}{(1+e^{-a})}$$

$$= \frac{-1}{(1+e^{-a})^2}\frac{d}{da}(1+e^{-a}) = \frac{-1}{(1+e^{-a})^2}(-e^{-a})$$

$$= \frac{e^{-a}}{(1+e^{-a})^2} = \frac{1}{(1+e^{-a})}\frac{e^{-a}}{(1+e^{-a})} = \sigma(a)\left[1-\sigma(a)\right]$$

Updating LR Weights (MSE)

- $\frac{\partial E}{\partial w_i} = (o - y) \frac{\partial \sigma(z)}{\partial z} \frac{\partial z}{\partial w_i}$

- Using:

$$\frac{\partial \sigma(z)}{\partial z} = \sigma(z)\,(1 - \sigma(z)) = o(1 - o)$$

$$\frac{\partial z}{\partial w_i} = \frac{\partial(\sum_i w_i \cdot x_i)}{\partial w_i} = x_i$$

$$\Rightarrow \boxed{\frac{\partial E^{(j)}}{\partial w_i} = (o^{(j)} - y^{(j)})\, o^{(j)}\,(1 - o^{(j)})\, x_i^{(j)}}$$

Note: As already computed $o^{(j)} = \sigma(z^{(j)})$ to get answer, trival to compute $\sigma'(z^{(j)}) = \sigma(z^{(j)})\,(1 - \sigma(z^{(j)}))$!

- Update w_i += Δw_i where

$$\boxed{\Delta w_i = \eta \cdot \frac{\partial E^{(j)}}{\partial w_i}}$$

B: Or... Learn Conditional Probability

- As fitting probability distribution,
  better to return probability distribution (≈ **w**)
  that is most likely, given training data, S

$$\text{Goal: } \mathbf{w}^* = \underset{\mathbf{w}}{\operatorname{argmax}} P(\mathbf{w} \mid S)$$

$$= \underset{\mathbf{w}}{\operatorname{argmax}} \frac{P(S \mid \mathbf{w}) P(\mathbf{w})}{P(S)}$$

Bayes Rules

$$= \underset{\mathbf{w}}{\operatorname{argmax}} P(S \mid \mathbf{w}) P(\mathbf{w})$$

As P(S) does not depend on **w**

$$= \underset{\mathbf{w}}{\operatorname{argmax}} P(S \mid \mathbf{w})$$

As P(**w**) is uniform

$$= \underset{\mathbf{w}}{\operatorname{argmax}} \log P(S \mid \mathbf{w})$$

As log is monotonic

ML Estimation

- $P(S \mid \mathbf{w}) \equiv$ likelihood function

  $L(\mathbf{w}) = \log P(S \mid \mathbf{w})$

- $\mathbf{w}^* = \text{argmax}_{\mathbf{w}} L(\mathbf{w})$

  is “maximum likelihood estimator” (MLE)

Computing the Likelihood

- As training examples $[\mathbf{x}^{(i)}, y^{(i)}]$ are iid
  - drawn independently from same (unknown) prob $P_w(\mathbf{x}, y)$
- $\log P(S \mid \mathbf{w}) = \log \prod_i P_{\mathbf{w}}(\mathbf{x}^{(i)}, y^{(i)})$

  $= \sum_i \log P_{\mathbf{w}}(\mathbf{x}^{(i)}, y^{(i)})$

  $= \sum_i \log P_{\mathbf{w}}(y^{(i)} \mid \mathbf{x}^{(i)}) + \sum_i \log P_{\mathbf{w}}(\mathbf{x}^{(i)})$
- Here $P_{\mathbf{w}}(\mathbf{x}^{(i)}) = 1/n$ …
  not dependent on $\mathbf{w}$, over empirical sample S
- $\mathbf{w}^* = \text{argmax}_{\mathbf{w}} \sum_i \log P_{\mathbf{w}}(y^{(i)} \mid \mathbf{x}^{(i)})$

Fit Logistic Regression… by Gradient Ascent

- Want $\mathbf{w}^* = \text{argmax}_w J(\mathbf{w})$
  - $J(w) = \sum_i r(y^{(i)}, \mathbf{x}^{(i)}, \mathbf{w})$
  - For $y \in \{0, 1\}$

    $r(y, \mathbf{x}, \mathbf{w}) = \log P_{\mathbf{w}}(y \mid \mathbf{x}) =$

    $y \log(P_{\mathbf{w}}(y=1 \mid \mathbf{x}))+(1 - y) \log(1 - P_{\mathbf{w}}(y=1 \mid \mathbf{x}))$

- So climb along…

$$\frac{\partial J(w)}{\partial w_j} = \sum_i \frac{\partial r(y^{(i)}, x^{(i)}, w)}{\partial w_j}$$

Gradient Descent …

$$\frac{\partial\, r(y,x,w)}{\partial\, w_j} = \frac{\partial}{\partial\, w_j}[\, y\, \log(p_1) + (1-y)\log(1-p_1)$$

$$= \frac{y}{p_1}\frac{\partial\, p_1}{\partial\, w_j} + (-1)\times\frac{1-y}{1-p_1}\frac{\partial\, p_1}{\partial\, w_j} = \frac{y-p_1}{p_1(1-p_1)}\frac{\partial\, p_1}{\partial\, w_j}$$

$$\frac{\partial\, p_1}{\partial\, w_j} = \frac{\partial\, P_w(y=1|x)}{\partial\, w_j} = \frac{\partial}{\partial\, w_j}\left(\sigma(x\cdot w)\right)$$

$$= \sigma(x\cdot w)[1-\sigma(x\cdot w)]\frac{\partial}{\partial\, w_j}(x\cdot w) = p_1(1-p_1)\cdot x_j^{(i)}$$

$$\frac{\partial\, J(w)}{\partial\, w_j} = \sum_i \frac{\partial\, r(y^{(i)}, x^{(i)}, w)}{\partial\, w_j} = \sum_i \frac{y^{(i)}-p_1}{p_1(1-p_1)}\, p_1(1-p_1)\cdot x_j^{(i)}$$

$$= \sum_i (y^{(i)} - P_w(y=1|x))\cdot x_j^{(i)}$$

Gradient Ascent for Logistic Regression (MLE)

Given: training examples $\langle \mathbf{x}^{(i)}, y^{(i)} \rangle$, $i = 1..N$
Set initial weight vector $\mathbf{w} = \langle 0, 0, 0, 0, \ldots, 0 \rangle$
Repeat until convergence
  Let gradient vecto $\Delta w = \langle 0, 0, 0, 0, \ldots, 0 \rangle$
  For $i = 1$ to N do
    $p_1^{(i)} = 1/(1 + \exp[\mathbf{w} \cdot \mathbf{x}^{(i)}])$
    $\text{error}_i = y^{(i)} - p_1^{(i)}$
    For $j = 1$ to n **do**
      $\Delta w_j \mathrel{+}= \text{error}_i \cdot x_{ij}$
  $\mathbf{w} \mathrel{+}= \eta \, \Delta w$ % *step in direction of increasing gradient*

Comments on MLE Algorithm

- This is BATCH;

  ∃ obvious online alg
  (stochastic gradient ascent)
- Can use second-order (Newton-Raphson) alg for faster convergence
  - weighted least squares computation;
    aka
    "Iteratively-Reweighted Least Squares" (IRLS)

Use Logistic Regression for Classification

- Return YES iff

$$P(y=1|x) \quad ¿ \quad P(y=0|x)$$

$$\frac{P(y=1|x)}{P(y=0|x)} \quad ¿ \quad 1$$

$$\ln \frac{P(y=1|x)}{P(y=0|x)} \quad ¿ \quad 0$$

$$\ln \frac{1 \quad /(1+\exp(-w\cdot x))}{\exp(-w\cdot x)/(1+\exp(-w\cdot x))} \quad ¿ \quad 0$$

$$\ln \frac{1}{\exp(-w\cdot x)} = w\cdot x \quad > \quad 0$$

Logistic Regression learns a LTU!

Logistic Regression for K > 2 Classes

- To handle $K > 2$ classes
  - Let class K be "reference"
  - Represent each other class $k \neq K$ as logistic function of odds of class k versus class K:

$$\log \frac{P(y=1|\mathbf{x})}{P(y=K|\mathbf{x})} = \mathbf{w}_1 \cdot \mathbf{x}$$

$$\log \frac{P(y=2|\mathbf{x})}{P(y=K|\mathbf{x})} = \mathbf{w}_2 \cdot \mathbf{x}$$

$$\vdots$$

$$\log \frac{P(y=K-1|\mathbf{x})}{P(y=K|\mathbf{x})} = \mathbf{w}_{K-1} \cdot \mathbf{x}$$

- Apply gradient ascent to learn *all* $\mathbf{w}_k$ weight vectors, in parallel.
- Conditional probabilities:

$$P(y=k|\mathbf{x}) = \frac{\exp(\mathbf{w}_k \cdot \mathbf{x})}{1 + \sum_{\ell=1}^{K-1} \exp(\mathbf{w}_\ell \cdot \mathbf{x})}$$

and

$$P(y=K|\mathbf{x}) = \frac{1}{1 + \sum_{\ell=1}^{K-1} \exp(\mathbf{w}_\ell \cdot \mathbf{x})}$$

Note: k-1 different $\mathbf{w}_i$ weights, … each of dimension $|\mathbf{x}|$

Learning LR Weights

Task: Given data $\langle\langle \mathbf{x}^{(i)}, y^{(i)} \rangle\rangle$

find w in $p_{\mathbf{w}}(y|\mathbf{x}) = \begin{cases} \frac{1}{1+\exp(-w \cdot x)} & if \ y=1 \\ \frac{\exp(-w \cdot x)}{1+\exp(-w \cdot x)} & if \ y=0 \end{cases}$

s.t. $p_{\mathbf{w}}(y^{(i)}|\mathbf{x}^{(i)}) > \frac{1}{2}$ iff $y^{(i)} = 1$

Approach 1: MSE – "Neural nets"

Minimize $\sum_i (o^{(i)} - y^{(i)})^2$

Gradient: $\Delta \mathbf{w}^{(i)}_j = (o^{(i)} - y^{(i)})\, o^{(i)} (1 - o^{(i)})$

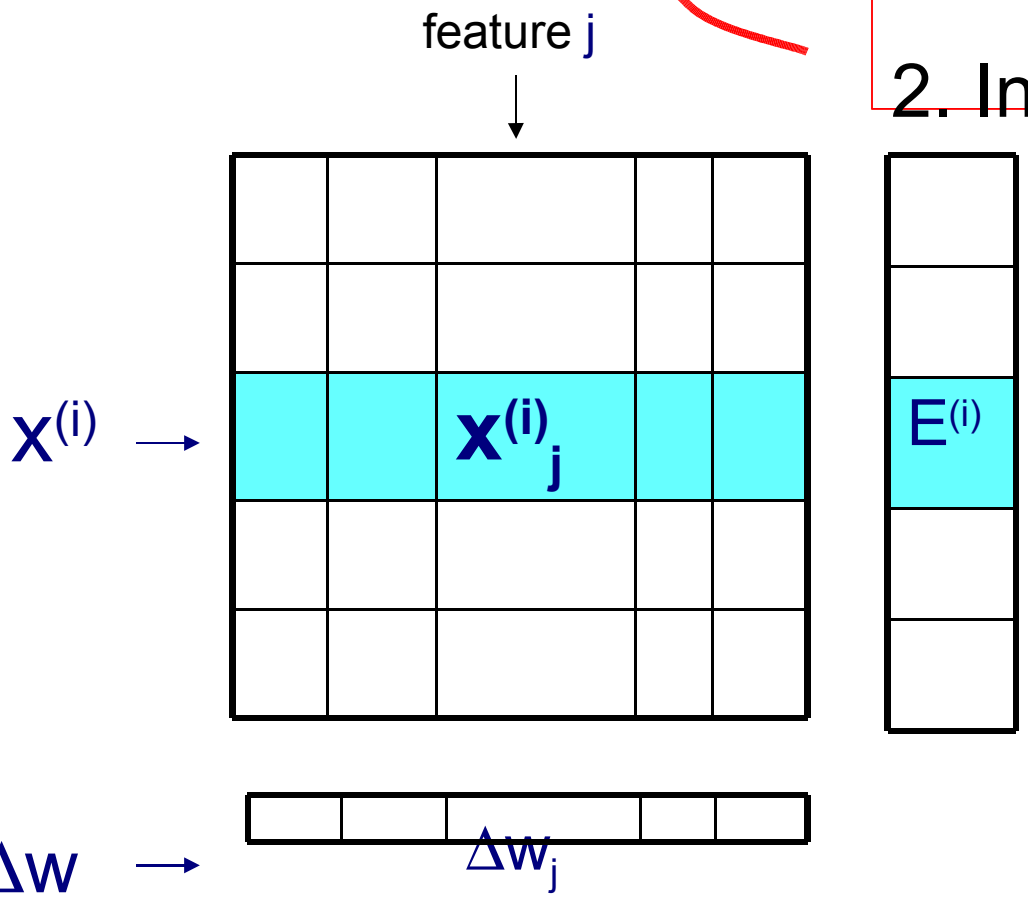
Approach 2: MLE – "Logistic Regression"

Maximize $\sum_i p_{\mathbf{w}}(y|\mathbf{x})$

Gradient: $\Delta \mathbf{w}^{(i)}_j = (y^{(i)} - p(1|x^{(i)}))\, x^{(i)}_j$

(MaxProb)

0. New **w**
1. For each row i, compute
   a. $\Delta \mathbf{w} = 0$
   b. $E^{(i)} = (y^{(i)} - p(1|x^{(i)}))$
   c. $\Delta \mathbf{w} \mathrel{+}= E^{(i)} \mathbf{x}^{(i)}$
      [… $\Delta w_j \mathrel{+}= E^{(i)} x^{(i)}_j$ …]
2. Increment $\mathbf{w} \mathrel{+}= \eta \Delta \mathbf{w}$



Logistic Regression Algs for LTUs

- Learns Conditional Probability Distribution $P(y \mid x)$

- **Local Search**:
  Begin with initial weight vector;
  iteratively modify to maximize objective function
  log likelihood of the data
  (ie, seek **w** s.t. probability distribution $P_{\mathbf{w}}(y \mid x)$ is most likely given data.)

- **Eager**: Classifier constructed from training examples, which can then be discarded.

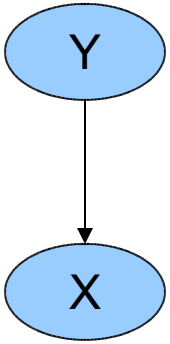
- **Online** or **batch**

#4: Linear Discriminant Analysis

- LDA learns joint distribution P(y, **x**)
  - As P(y, x) ≠ P(y | x);
    optimizing P(y, x) ≠ optimizing P(y | x)
- "generative model"
  - P(y,**x**) model of how data is generated
  - Eg, factor

    P(y, **x**) = P(y) P(**x** | y)
    - P(y) generates value for y; then
    - P(**x** | y) generates value for **x** given this y

- Belief net:



Linear Discriminant Analysis, con't

- $P(y, \mathbf{x}) = P(y) P(\mathbf{x} \mid y)$
- $P(y)$ is a simple discrete distribution
  - Eg: $P(y = 0) = 0.31$; $P(y = 1) = 0.69$

    (31% negative examples; 69% positive examples)
- Assume $P(\mathbf{x} \mid y)$ is multivariate normal, with mean μ_k and covariance Σ

$$P(\mathbf{x} \mid y = k) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x} - \mu_k)^\top \Sigma^{-1} (\mathbf{x} - \mu_k)\right]$$

Estimating LDA Model

- Linear discriminant analysis assumes form

$$P(\mathbf{x},y) = P(y)\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left[-\frac{1}{2}(\mathbf{x}-\mu_y)^\top\Sigma^{-1}(\mathbf{x}-\mu_y)\right]$$

- μ_y is mean for examples belonging to class y;

  covariance matrix Σ is shared by all classes !
- Can estimate LDA directly:

  m_k = #training examples in class y = k
  - Estimate of P(y = k): $p_k = m_k / m$

$$\hat{\mu}_k = \frac{1}{m}\sum_{\{i:\, y_i=k\}} x_i$$

$$\hat{\Sigma} = \frac{1}{m}\sum_i (x_i-\hat{\mu}_{y_i})(x_i-\hat{\mu}_{y_i})^T$$

(Subtract each x_i from corresponding $\hat{\mu}_{y_i}$ before taking outer product)

Example of Estimation

x_1	x_2	x_3	y
13.1	20.2	0.4	+
6.0	17.7	-4.2	+
8.2	18.2	-2.5	+
0.4	10.1	19.2	−
-4.2	12.8	5.1	−
-4.3	15.0	21.7	−
0.9	10.1	19.2	−

- m=7 examples;
  m_+ = 3 positive; m_- = 4 negative
  $\Rightarrow$ $p_+ = 3/7$ $p_- = 4/7$

- Compute $\hat{\mu}_i$ over each class

$$\hat{\mu}_+ = \frac{1}{3}\sum_{i:\ \langle y^{(i)}=+\rangle} \mathbf{x}^{(i)}$$

$$= \frac{1}{3}\begin{pmatrix} [13.1,\ 20.2,\ \ 0.4] + \\ [\ 6.0,\ 17.7,\ -4.2] + \\ [\ 8.2,\ 18.2,\ -2.5] \end{pmatrix}$$

$$= [9.1,\ 18.7,\ -2.1]$$

$$\hat{\mu}_- = \frac{1}{4}\sum_{i:\langle y^{(i)}=-\rangle} \mathbf{x}^{(i)} = [-1.8,\ 12.0,\ 16.3]$$

Note: do NOT pre-pend x_0=1!

Estimation…

x_1	x_2	x_3	y
13.1	20.2	0.4	+
6.0	17.7	-4.2	+
8.2	18.2	-2.5	+
0.4	10.1	19.2	−
-4.2	12.8	5.1	−
-4.3	15.0	21.7	−
0.9	10.1	19.2	−

- Compute common $\hat{\Sigma}$
  - "Normalize" each $\mathbf{z} := \mathbf{x} - \mu_{y(\mathbf{x})}$

$$\mathbf{z}^{(1)} := [13.1, 20.2, 0.4] - [9.1, 18.7, -2.1] = [4.0, 1.5, -1.7]$$

...

$$\mathbf{z}^{(4)} := [0.4, 10.1, 19.2] - [-1.8, 12.0, 16.3] = [2.2, -1.9, 2.9]$$

... $\mathbf{z}^{(7)}$:= ...

  - Compute covariance matrix, for each i:

For $\mathbf{x}^{(1)}$, via $\mathbf{z}^{(1)}$:

$$\mathbf{z}^{(1)} \times \mathbf{z}^{(1)\top} = \begin{bmatrix} 4.0 \\ 0.5 \\ -1.7 \end{bmatrix} \cdot [4.0, 0.5, -1.7]$$

$$= \begin{bmatrix} 4.0 \cdot 4.0 & 4.0 \cdot 0.5 & 4.0 \cdot -1.7 \\ 0.5 \cdot 4.0 & 0.5 \cdot 0.5 & 0.5 \cdot -1.7 \\ -1.7 \cdot 4.0 & -1.7 \cdot 0.5 & -1.7 \cdot -1.7 \end{bmatrix}$$

$$= \begin{bmatrix} 16.0 & 2.0 & -6.8 \\ 2.0 & 0.25 & -0.85 \\ -6.8 & -0.85 & -2.89 \end{bmatrix}$$

  - Set $\hat{\Sigma} = \frac{1}{m} \sum_i \mathbf{z}^{(i)} \mathbf{z}^{(i)\top}$

Classifying, Using LDA

- How to classify new instance, given estimates

$\{\hat{p}_i\}$ $\{\hat{\mu}_i\}$ $\hat{\Sigma}$

Eg, $\hat{p}_+ = 3/7 \quad \hat{p}_- = 4/7$

$\star\ \hat{\mu}_+ = [9.1,\ 18.7,\ -2.1]$

$\hat{\mu}_- = [-1.8,\ 12.0,\ 16.3]$

$$\star\ \hat{\Sigma} = \begin{bmatrix} 7.22 & -1.31 & 6.35 \\ -1.31 & 2.91 & 0.32 \\ 6.35 & 0.32 & 26.03 \end{bmatrix}$$

- Class for instance **x** = [5, 14, 6] ?

$$P(y=+,\ \mathrm{x}=[5,\ 14,\ 6]) = P(y=+)\,P([5,\ 14,\ 6]\,|\,y=+)$$
$$= \tfrac{3}{7} \times P(\mathrm{x}=[5,\ 14,\ 6]\,|\,\mathrm{x}\sim\mathcal{N}(\hat{\mu}_+,\hat{\Sigma}))$$
$$= \tfrac{3}{7} \times \tfrac{1}{(2\pi)^{3/2}|\hat{\Sigma}|^{1/2}} \exp\left[-\tfrac{1}{2}(\mathrm{x}-\hat{\mu}_+)^\top \hat{\Sigma}^{-1}(\mathrm{x}-\hat{\mu}_+)\right]$$
$$= 16.63\text{E-}11$$

$$P(y=-,\ \mathrm{x}=[5,\ 14,\ 6]) = P(y=-)\,P([5,\ 14,\ 6]\,|\,y=-)$$
$$= \tfrac{4}{7} \times \tfrac{1}{(2\pi)^{3/2}|\hat{\Sigma}|^{1/2}} \exp\left[-\tfrac{1}{2}(\mathrm{x}-\hat{\mu}_-)^\top \hat{\Sigma}^{-1}(\mathrm{x}-\hat{\mu}_-)\right]$$
$$= 43.33\text{E-}11$$

- $P(y=+\,|\,[5,\ 14,\ 6]) = \frac{P(y=+,\ [5,\ 14,\ 6])}{P(y=+,\ [5,\ 14,\ 6])+P(y=-,\ [5,\ 14,\ 6])} = 0.2774$

$P(y=-\,|\,[5,\ 14,\ 6]) = 0.7226$

LDA learns an LTU

- Consider 2-class case with a 0/1 loss function
- Classify $\hat{y} = 1$ if

$$\log \frac{P(y=1\,|\,\mathbf{x})}{P(y=0\,|\,\mathbf{x})} > 0 \quad \text{iff} \quad \log \frac{P(y=1,\mathbf{x})}{P(y=0,\mathbf{x})} > 0$$

$$\frac{P(\mathbf{x},y=1)}{P(\mathbf{x},y=0)} = \frac{P(y=1)\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left[-\frac{1}{2}(\mathbf{x}-\mu_1)^\top\Sigma^{-1}(\mathbf{x}-\mu_1)\right]}{P(y=0)\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left[-\frac{1}{2}(\mathbf{x}-\mu_0)^\top\Sigma^{-1}(\mathbf{x}-\mu_0)\right]}$$

$$= \frac{P(y=1)\exp\left[-\frac{1}{2}(\mathbf{x}-\mu_1)^\top\Sigma^{-1}(\mathbf{x}-\mu_1)\right]}{P(y=0)\exp\left[-\frac{1}{2}(\mathbf{x}-\mu_0)^\top\Sigma^{-1}(\mathbf{x}-\mu_0)\right]}$$

$$\ln \frac{P(\mathbf{x},y=1)}{P(\mathbf{x},y=0)} = \ln \frac{P(y=1)}{P(y=0)} - \frac{1}{2}\left[(\mathbf{x}-\mu_1)^\top\Sigma^{-1}(\mathbf{x}-\mu_1) - (\mathbf{x}-\mu_0)^\top\Sigma^{-1}(\mathbf{x}-\mu_0)\right]$$

LDA Learns an LTU (2)

- $(x-\mu_1)^T \Sigma^{-1} (x-\mu_1) - (x-\mu_0)^T \Sigma^{-1} (x-\mu_0)$
  $= x^T \Sigma^{-1} (\mu_0 - \mu_1) + (\mu_0 - \mu_1)^T \Sigma^{-1} x +$
  $\mu_1^T \Sigma^{-1} \mu_1 - \mu_0^T \Sigma^{-1} \mu_0$

- As Σ^{-1} is symmetric,
  $\ldots = 2\, x^T \Sigma^{-1} (\mu_0 - \mu_1) + \mu_1^T \Sigma^{-1} \mu_1 - \mu_0^T \Sigma^{-1} \mu_0$

$$\Rightarrow \ln \frac{P(\mathbf{x}, y=1)}{P(\mathbf{x}, y=0)} =$$

$$\ln \frac{P(y=1)}{P(y=0)} - \frac{1}{2}\left[(\mathbf{x}-\mu_1)^\top \Sigma^{-1} (\mathbf{x}-\mu_1) - (\mathbf{x}-\mu_0)^\top \Sigma^{-1} (\mathbf{x}-\mu_0)\right]$$

$$= \ln \tfrac{P(y=1)}{P(y=0)} + \mathbf{x}^\top \Sigma^{-1} (\mu_1 - \mu_0) + \tfrac{1}{2}\mu_0^\top \Sigma^{-1} \mu_0 - \tfrac{1}{2}\mu_1^\top \Sigma^{-1} \mu_1$$

$$= \mathbf{x}^\top \Sigma^{-1} (\mu_1 - \mu_0) + \ln \tfrac{P(y=1)}{P(y=0)} + \tfrac{1}{2}\mu_0^\top \Sigma^{-1} \mu_0 - \tfrac{1}{2}\mu_1^\top \Sigma^{-1} \mu_1$$

LDA Learns an LTU (3)

$$\ln \frac{P(\mathbf{x}, y=1)}{P(\mathbf{x}, y=0)} = \mathbf{x}^\top \Sigma^{-1}(\mu_1 - \mu_0) + \ln \frac{P(y=1)}{P(y=0)} + \frac{1}{2}\mu_0^\top \Sigma^{-1}\mu_0 - \frac{1}{2}\mu_1^\top \Sigma^{-1}\mu_1$$

- So let…

$$\mathbf{w} = \Sigma^{-1}(\mu_1 - \mu_0)$$

$$c = \ln \frac{P(y=1)}{P(y=0)} + \frac{1}{2}\mu_0^\top \Sigma^{-1}\mu_0 - \frac{1}{2}\mu_1^\top \Sigma^{-1}\mu_1$$

- Classify $\hat{y} = 1$ iff $\mathbf{w} \cdot \mathbf{x} + c > 0$

  LTU!!

Variants of LDA

- Covariance matrix Σ
- n features; k classes

Same for all classes?	Diagonal	#param's	Name
+	+	k	
+	-	n^2	LDA
-	+	k n	Naïve Gaussian Classifier
-	-	k n^2	General Gaussian Classifier

Generalizations of LDA

- **General Gaussian Classifier**

  Allow each class k to have its own Σ_k

  $\Rightarrow$ Classifier $\equiv$ quadratic threshold unit (not LTU)

- **Naïve Gaussian Classifier**

  Allow each class *k* to have its own Σ_k

  but require each Σ_k be diagonal.

  $\Rightarrow$ within each class,
  any pair of features x_i and x_j are independent

  - Classifier is still quadratic threshold unit
    but with a restricted form

Summary of Linear Discriminant Analysis

- **Learns Joint Probability Distr'n** P(y, **x**)
- **Direct Computation**.
  MLEstimate of P(y, **x**) computed directly from data without search.

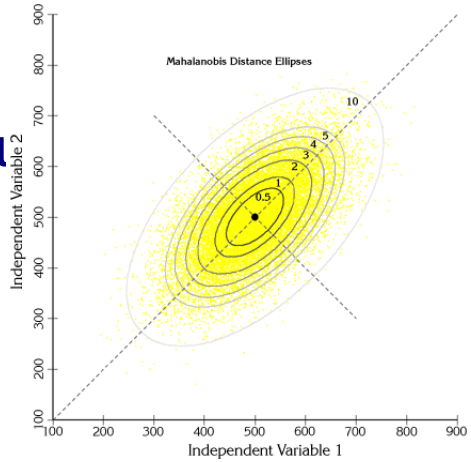
  But need to invert matrix, which is $O(n^3)$
- **Eager**:

  Classifier constructed from training examples, which can then be discarded.
- **Batch**: Only a batch algorithm.

  An online LDA alg requires online alg for incrementally updating Σ^{-1}

  [Easy if Σ^{-1} is diagonal. . .]

Two Geometric Views of LDA
View 1: Mahalanobis Distance



- Squared Mahalanobis distance between **x** and $\boldsymbol{\mu}$

$$D_M^2(\mathbf{x}, \boldsymbol{\mu}) = (\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})$$

  - $\Sigma^{-1} \approx$ linear distortion

  … converts standard Euclidean distance into Mahalanobis distance.

- LDA classifies **x** as 0 if

$$D_M^2(\mathbf{x}, \boldsymbol{\mu}_0) < D_M^2(\mathbf{x}, \boldsymbol{\mu}_1)$$

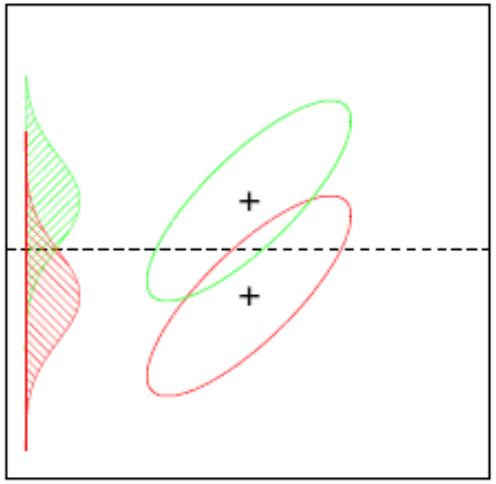
- $\log P(\mathbf{x} \mid y = k) \approx \log \pi_k - \frac{1}{2} D_M^2(\mathbf{x}, \boldsymbol{\mu}_k)$

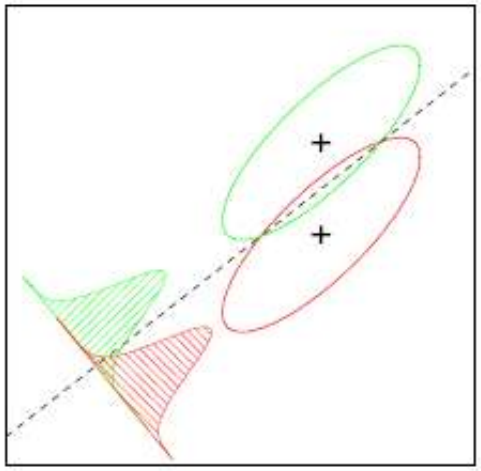
View 2: Most Informative Low Dimensional Projection

- LDA
  - Finds K–1 dim hyperplane
    (K = number of classes)
  - Project **x** and $\{ \mu_k \}$ to that hyperplane
  - Classify **x** as nearest μ_k within hyperplane

≈ Fisher's Linear Discriminant

- Goal: Hyperplane that maximally separates projection of **x**'s wrt $\sum^{-1}$





project onto: vertical axis LDA's w

Fisher Linear Discriminant

- Recall any vector **w** projects $\Re^n \rightarrow \Re$
- Goal: Want **w** that "separates" classes
  - Each $\mathbf{w} \cdot \mathbf{x}^+$ far from each $\mathbf{w} \cdot \mathbf{x}^-$

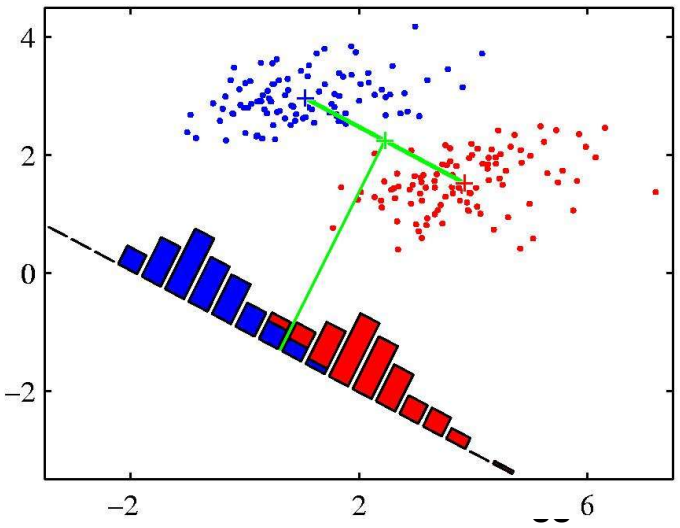
- Using $\mathbf{m}_+ = \frac{\sum_i y^{(i)} \cdot \mathbf{x}^{(i)}}{\sum_i y^{(i)}} \quad \mathbf{m}_- = \frac{\sum_i (1-y^{(i)}) \cdot \mathbf{x}^{(i)}}{\sum_i (1-y^{(i)})}$

Mean of x's projections:

$$m_+ = \frac{\sum_i y^{(i)} \mathbf{w}^\top \cdot \mathbf{x}^{(i)}}{\sum_i y^{(i)}} = \mathbf{w}^\top \cdot \mathbf{m}_+$$

$$m_- = \frac{\sum_i (1-y^{(i)}) \mathbf{w}^\top \cdot \mathbf{x}^{(i)}}{\sum_i (1-y^{(i)})} = \mathbf{w}^\top \cdot \mathbf{m}_-$$

- Perhaps project on $\mathbf{m}_+ - \mathbf{m}_-$?
- Still overlap… why?



Fisher Linear Discriminant

- Using $\mathbf{m}_+ = \frac{\sum_i y^{(i)} \cdot \mathbf{x}^{(i)}}{\sum_i y^{(i)}}$ $\mathbf{m}_- = \frac{\sum_i (1-y^{(i)}) \cdot \mathbf{x}^{(i)}}{\sum_i (1-y^{(i)})}$

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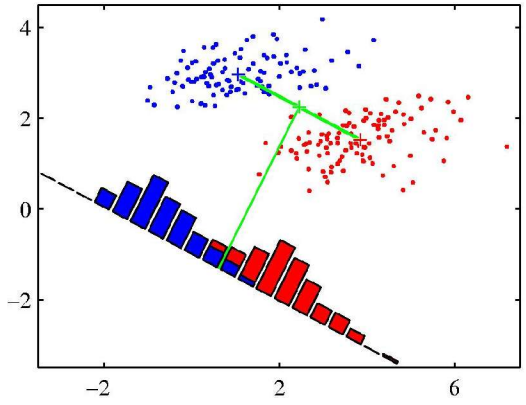

- Problem with $\mathbf{m}_+ - \mathbf{m}_-$:
- Doesn't consider "scatter" within class
- Goal: Want **w** that "separates" classes
  - Each $\mathbf{w} \cdot \mathbf{x}^+$ far from each $\mathbf{w} \cdot \mathbf{x}^-$
  - Positive $\mathbf{x}^+$'s: $\mathbf{w} \cdot \mathbf{x}^+$ close to each other
  - Negative $\mathbf{x}^-$'s: $\mathbf{w} \cdot \mathbf{x}^-$ close to each other
- "scatter" of +instance; –instance
  - $\mathbf{s}_+^2 = \sum_i y^{(i)} \left(\mathbf{w} \cdot \mathbf{x}^{(i)} - \mathbf{m}_+\right)^2$
  - $\mathbf{s}_-^2 = \sum_i (1 - y^{(i)}) \left(\mathbf{w} \cdot \mathbf{x}^{(i)} - \mathbf{m}_+\right)^2$

Fisher Linear Discriminant

- Recall any vector **w** projects $\Re^n \rightarrow \Re$
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- Using $\mathbf{m}_+ = \frac{\sum_i y^{(i)} \cdot \mathbf{x}^{(i)}}{\sum_i y^{(i)}} \qquad \mathbf{m}_- = \frac{\sum_i (1-y^{(i)}) \cdot \mathbf{x}^{(i)}}{\sum_i (1-y^{(i)})}$

  Mean of x's projections:

$$m_+ = \frac{\sum_i y^{(i)} \mathbf{w}^\top \cdot \mathbf{x}^{(i)}}{\sum_i y^{(i)}} = \mathbf{w}^\top \cdot \mathbf{m}_+$$

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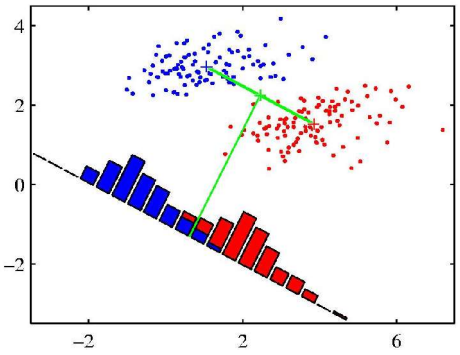

- "scatter" of +instance; –instance
  - $\mathbf{s}_+^2 = \sum_i y^{(i)} \left(\mathbf{w} \cdot \mathbf{x}^{(i)} - \mathbf{m}_+\right)^2$
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FLD, con't

- Separate means $\mathbf{m}_-$ and $\mathbf{m}_+$

  $\Rightarrow$ maximize $\mathbf{(m_- - m_+)^2}$
- Minimize each spead $\mathbf{s_+^2}$, $\mathbf{s_-^2}$

  $\Rightarrow$ maximize $(\mathbf{s_+^2 + s_-^2})$
- Objective function: maximize

$$J_s(w) = \frac{(m_+ - m_-)^2}{(s_+^2 + s_-^2)}$$

#1: $(m_- - m_+)^2 = (\mathbf{w}^T \mathbf{m}_+ - \mathbf{w}^T \mathbf{m}_-)^2$

$= \mathbf{w}^T (m_+ - m_-)(m_+ - m_-)^T \mathbf{w} = \mathbf{w}^T S_B \mathbf{w}$

"between-class scatter"

$$S_B = (m_+ - m_-)(m_+ - m_-)^T$$

FLD, III

$$J_S(w) = \frac{(m_+ - m_-)^2}{(s_+^2 + s_-^2)}$$

- $s_+^2 = \sum_i y^{(i)} \left(\mathbf{w} \cdot \mathbf{x}^{(i)} - m_+\right)^2$

  $= \sum_i \mathbf{w}^T y^{(i)} \left(\mathbf{x}^{(i)} - m_+\right)\left(\mathbf{x}^{(i)} - m_+\right)^T \mathbf{w}$

  $= \mathbf{w}^T S_+ \mathbf{w}$

$$S_+ = \sum_i y^{(i)} \left(\mathbf{x}^{(i)} - m_+\right)\left(\mathbf{x}^{(i)} - m_+\right)^T$$

… "within-class scatter matrix" for +

$$S_- = \sum_i (1 - y^{(i)}) \left(\mathbf{x}^{(i)} - m_-\right)\left(\mathbf{x}^{(i)} - m_-\right)^T$$

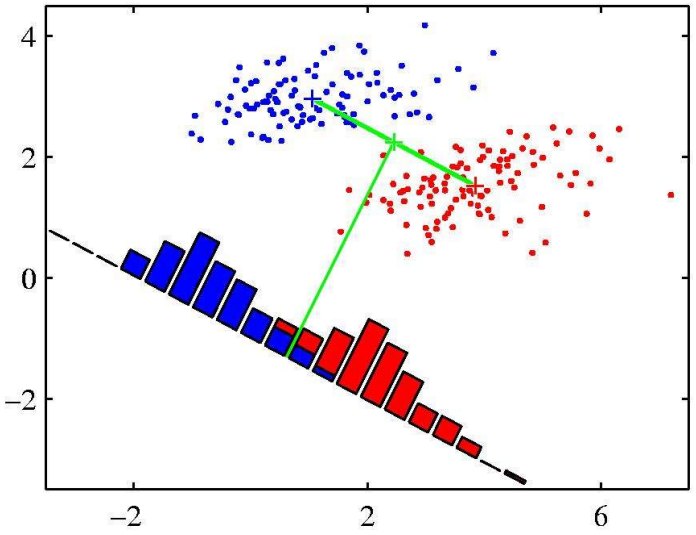
… "within-class scatter matrix" for –

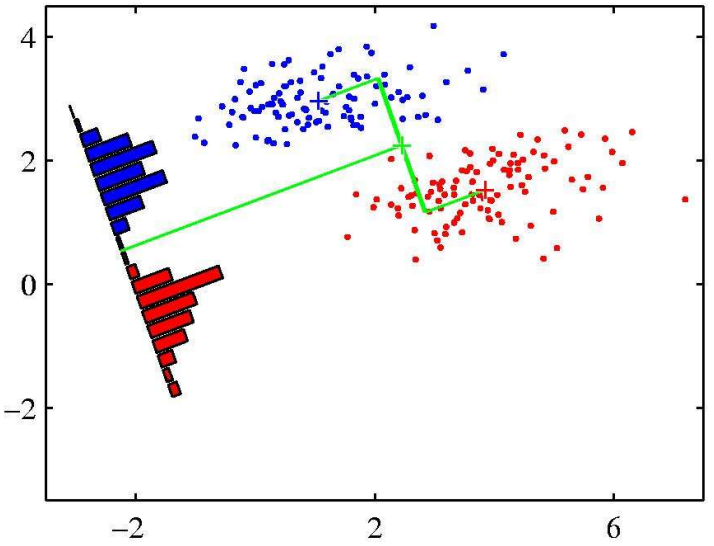
- $S_w = S_+ + S_-$ so $s_+^2 + s_-^2 = \mathbf{w}^T S_W \mathbf{w}$

FLD, IV

$$J_S(w) = \frac{(m_+ - m_-)^2}{(s_+^2 + s_-^2)} = \frac{w^T S_B w}{w^T S_w w} = \frac{(w^T (m_1 - m_2))^2}{w^T S_w w}$$

- Solving $\frac{\partial J_S(w)}{\partial w_j} = 0 \Rightarrow$ $\mathbf{w} = \alpha S_B^{-1}(m_+ - m_-)$





FLD, V

$$J_S(w) = \frac{(m_+ - m_-)^2}{(s_+^2 + s_-^2)} = \frac{w^T S_B w}{w^T S_w w} = \frac{(w^T (m_1 - m_2))^2}{w^T S_w w}$$

- Solving $\frac{\partial J_S(w)}{\partial w_j} = 0 \Rightarrow$ $\mathbf{w} = \alpha S_B^{-1}(m_+ - m_-)$

- When $P(x \mid y_i) \sim N(\mu_i; \Sigma)$

  $\exists$ LINEAR DISCRIMINANT: $\mathbf{w} = \Sigma^{-1}(\mu_+ - \mu_-)$

  $\Rightarrow$ FLD is optimal classifier,
  if classes normally distributed

- Can use even if not normal:

  After projecting *d*-dim to 1,
  just use any classification method

- Analogous derivation for K > 2 classes

Comparing LMS, Logistic Regression, LDA

- Which is best: *LMS*, *LR*, *LDA* ?
- Ongoing debate within machine learning community about relative merits of
  - direct classifiers [*LMS*]
  - conditional models $P(y \mid \mathbf{x})$ [*LR*]
  - generative models $P(y, \mathbf{x})$ [*LDA*]
- Stay tuned...

Issues in Debate

- **Statistical efficiency**
  If generative model P(y, x) is correct, then …
  usually gives better accuracy, particularly if training sample is small
- **Computational efficiency**
  Generative models typically easiest to compute
  (LDA computed directly, without iteration)
- **Robustness to changing loss functions**
  LMS must re-train the classifier when the loss function changes.
  … no retraining for generative and conditional models
- **Robustness to model assumptions**.
  Generative model usually performs poorly when the assumptions are violated.
  Eg, LDA works poorly if P(x | y) is non-Gaussian.
  Logistic Regression is more robust, … LMS is even more robust

- **Robustness to missing values and noise**.
  In many applications, some of the features x_{ij} may be missing or corrupted for some of the training examples.
  Generative models typically provide better ways of handling this than non-generative models.

Other Algorithms for learning LTUs

- **Naive Bayes** [Discuss later]

  For K = 2 classes, produces LTU

- **Winnow** [?Discuss later?]

  Can handle large numbers of “irrelevant" features

  - (features whose weights should be zero)