2019

最优化理论与方法

研究生学位课

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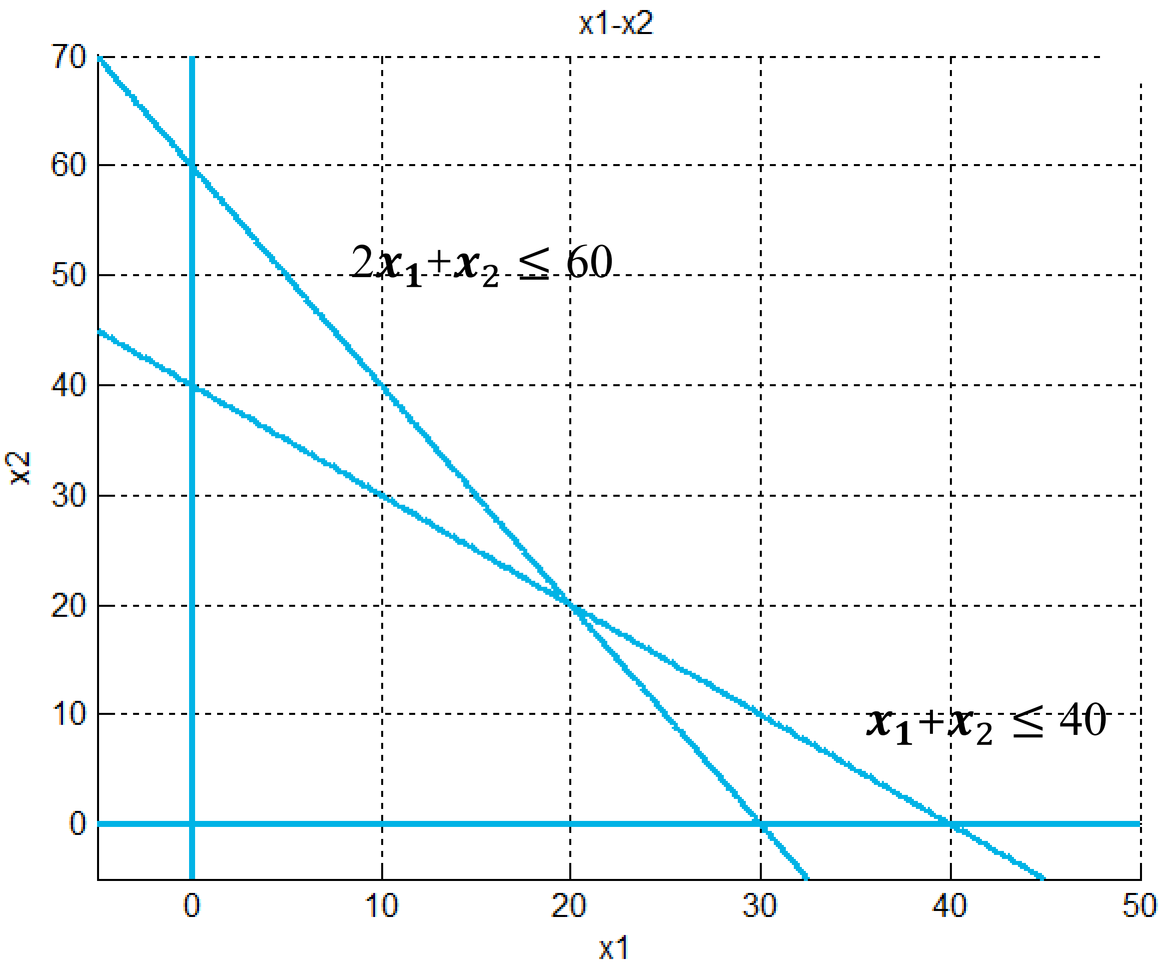
What is Linear Programming(LP)?

- **Optimize a linear objective function of decision variables subject to a set of linear constraints.**
- **Example**
- $$\begin{aligned} \text{Minimize} \quad & -3x_1 - 2x_2 \\ s.t. \quad & x_1 + x_2 \leq 40 \\ & 2x_1 + x_2 \leq 60 \\ & x_1 \,,\, x_2 \geq 0 \end{aligned}$$

- **Graphic Representation**

$$\begin{array}{llcccr} \text{Minimize} & -3x_1 & - & 2x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 40 \\ & 2x_1 & + & x_2 & \leq & 60 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$



$$P = \{(x_1, x_2) | x_1 + x_2 \leq 40, 2x_1 + x_2 \leq 60, x_1, x_2 \geq 0\}$$

- LP的线性表示

$$\max(\min)\ z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le (=, \ge) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le (=, \ge) b_2$$

$$\cdots \quad \cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le (=, \ge) b_m$$

$$x_1, x_2, \dots, x_n \ge 0$$

$$\max(\min)\ z = \sum_{j=1}^{n} c_j x_j$$

s.t.

$$\sum_{j=1}^{n} a_{ij}x_j \le (=, \ge) b_i \qquad \forall\, i \in \{1,2, \dots, m\}$$

$$x_1, x_2, \dots, x_n \ge 0$$

- **LP的矩阵表示**

$$\min z = \boldsymbol{c}^T \boldsymbol{x}$$
$$s.t. \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq 0$$

$$\max(\min) \ z = \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq (=, \geq) b_i \quad \forall\, i \in \{1,2,\dots,m\}$$
$$x_1, x_2, \dots, x_n \geq 0$$

- **LP的向量表示**

$$\min z = c^T x$$
$$s.t.\ Ax = b$$
$$x \geq 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$c^T = (c_1, c_2, \dots c_n)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (A_1, A_2, \dots A_n)$$

$$\max(\min)\ z = c^T x$$

s.t. $\sum_{j=1}^{n} A_j x \leq b$

$$x \geq 0$$

- **Linear Program**
  - **Recall that the standard form of LP:**

$$\text{Min } \boldsymbol{C^T X}$$

$$\boldsymbol{s.t.\ \ Ax = b}$$

$$\boldsymbol{x \geq 0}$$

Where $\boldsymbol{c} \in \boldsymbol{R^n}$, A is an $\boldsymbol{m} \times \boldsymbol{n}$ matrix with full row rank, $\boldsymbol{b} \in \boldsymbol{R^m}$

$$\text{Min} \sum_{j=1}^{n} \boldsymbol{c_j x_j}$$

$$\boldsymbol{s.t.} \qquad \sum_{j=1}^{n} \boldsymbol{a_{ij} x_j = b_i} \qquad \forall \boldsymbol{i} \in \{1,2,\ldots,m\}$$

$$\boldsymbol{x_j \geq 0} \qquad \forall \boldsymbol{j} \in \{1,2,\ldots,\boldsymbol{n}\}$$

- **Example 1**

$$
\begin{array}{llllll}
\text{Minimize} & -3x_1 & - & 2x_2 & & \\
s.t. & x_1 & + & x_2 & \leq 40 \\
 & 2x_1 & + & x_2 & \leq 60 \\
 & x_1 & , & x_2 & \geq 0
\end{array}
$$

  - Covert to standard form:

$$
\begin{array}{llllllllll}
\text{Minimize} & -3x_1 & - & 2x_2 & & & & & \\
s.t. & x_1 & + & x_2 & + & x_3 & & & = 40 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = 60 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq 0
\end{array}
$$

- **Example 1**

$$
\begin{array}{rrcrl}
\text{Minimize} & -3x_1 & - & 2x_2 & \\
s.t. & x_1 & + & x_2 & \leq 40 \\
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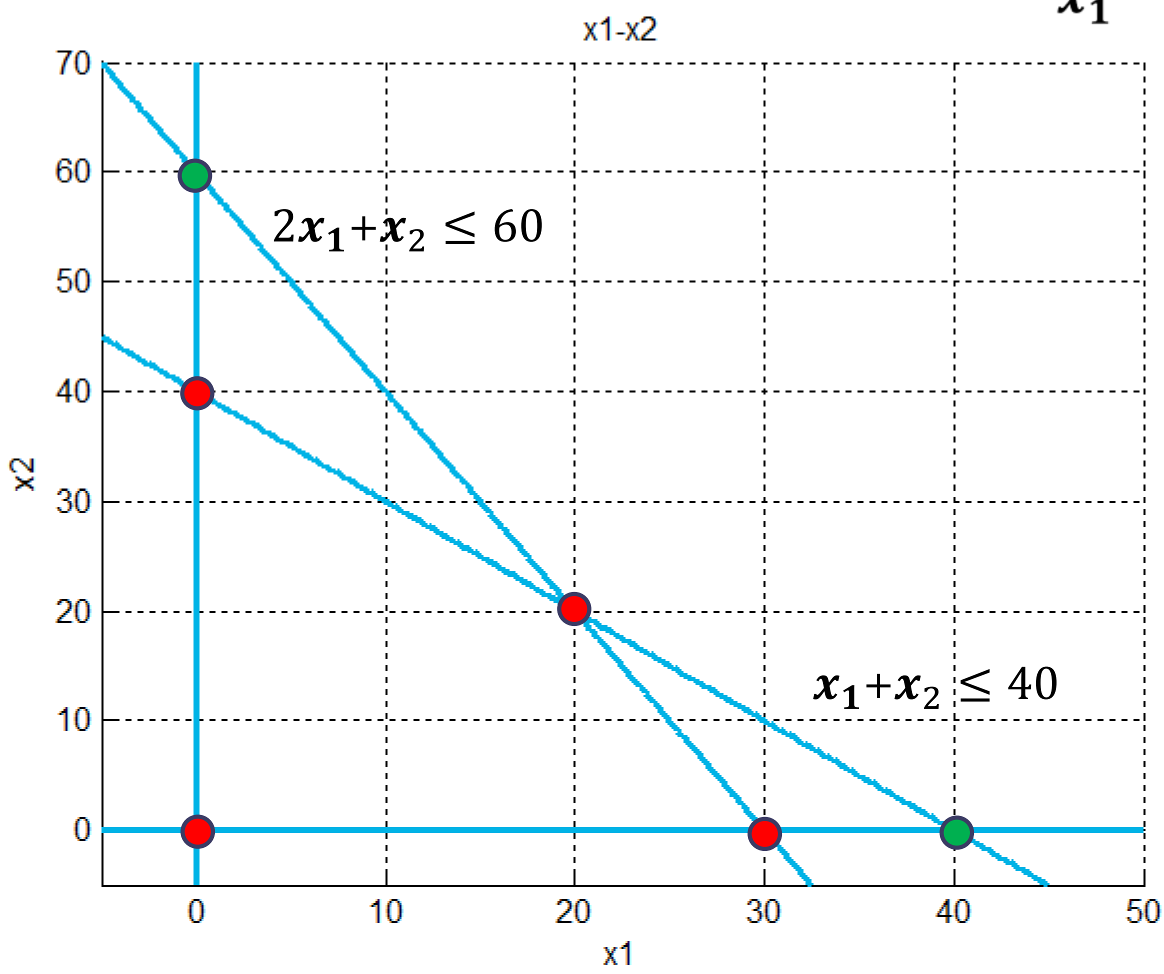
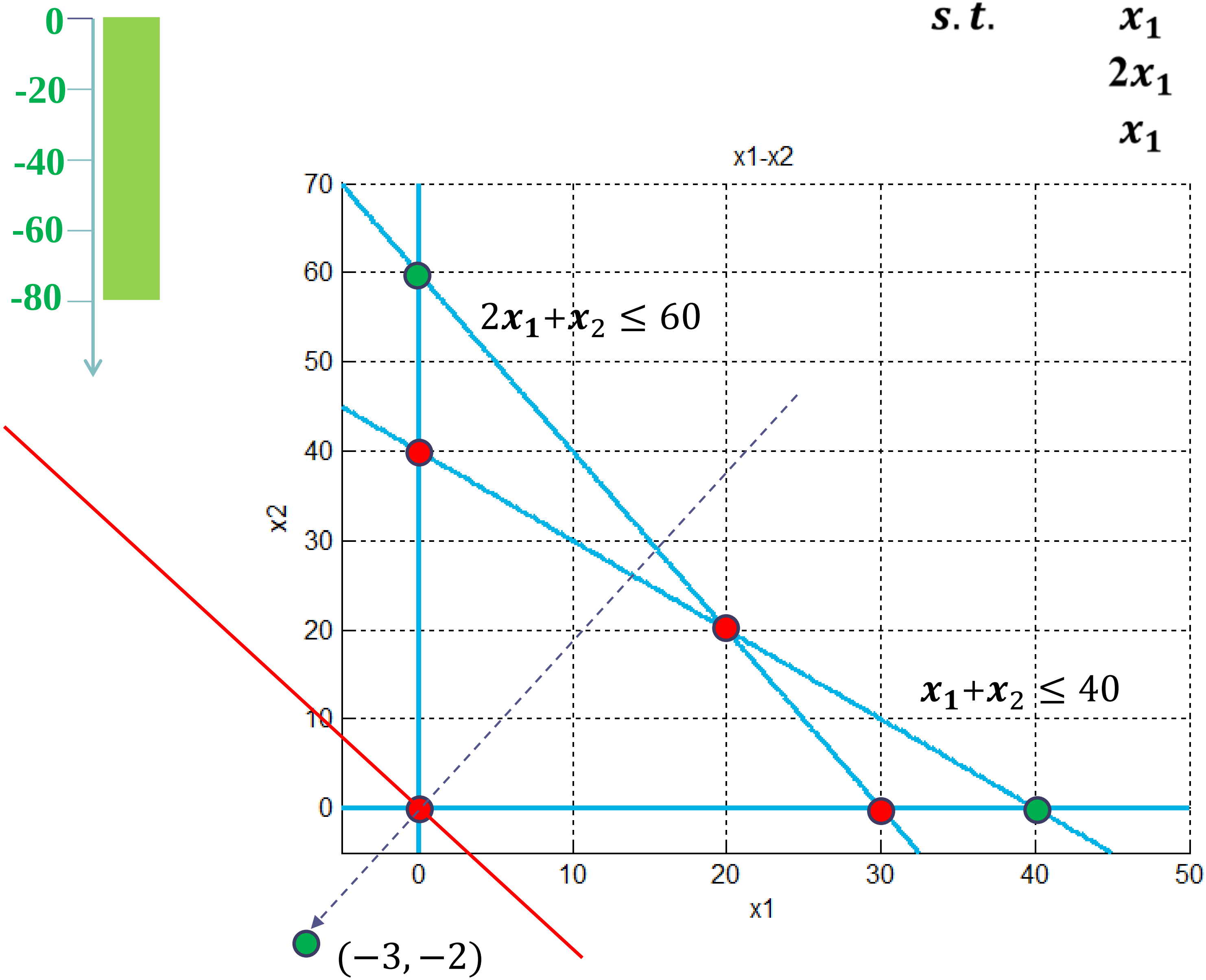
\end{array}
$$

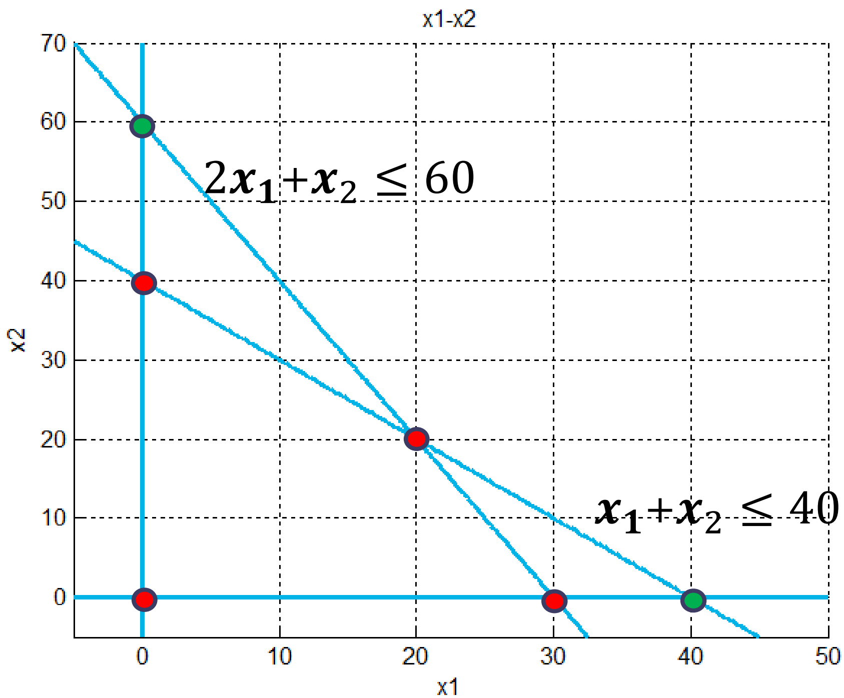


- **Example 1**

$$\begin{array}{llllll} \text{Minimize} & -3x_1 & - & 2x_2 & \\ s.t. & x_1 & + & x_2 & \le 40 \\ & 2x_1 & + & x_2 & \le 60 \\ & x_1 & , & x_2 & \ge 0 \end{array}$$



- **Example 1**



$$x^1=\begin{pmatrix}0\\0\\40\\60\end{pmatrix}\qquad x^2=\begin{pmatrix}0\\40\\0\\20\end{pmatrix}$$

$$x^3=\begin{pmatrix}20\\20\\0\\0\end{pmatrix}\qquad x^4=\begin{pmatrix}30\\0\\10\\0\end{pmatrix}$$

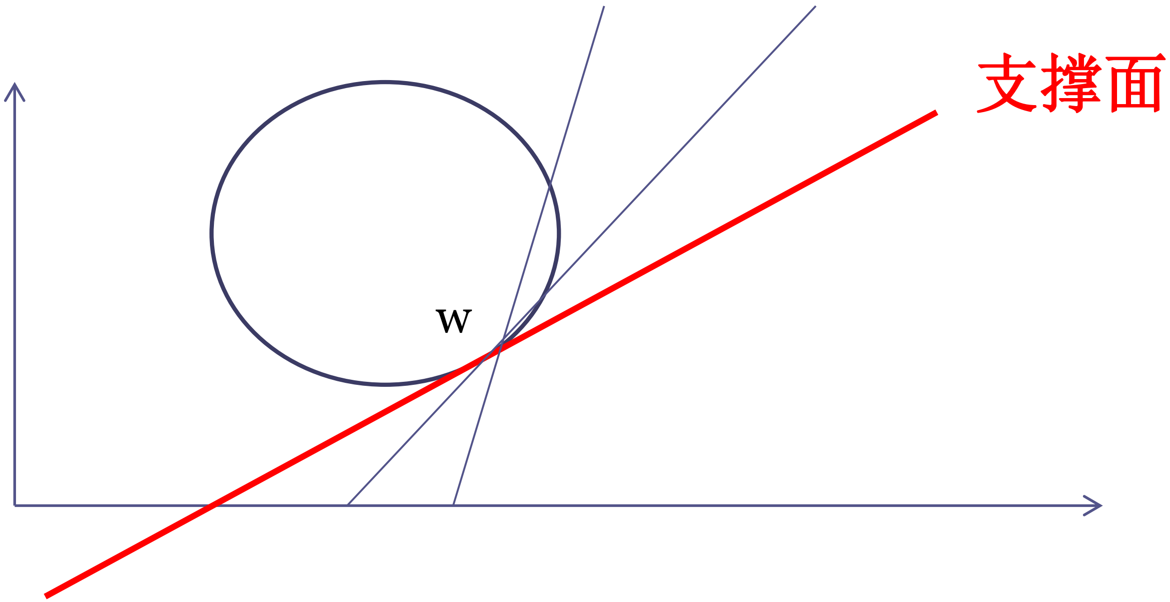
$$\begin{array}{llllllllll}
\text{Minimize} & -3x_1 & - & 2x_2 & & & & & & \\
s.t. & x_1 & + & x_2 & + & x_3 & & & & =40 \\
& 2x_1 & + & x_2 & & & + & x_4 & & =60 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & & \ge 0
\end{array}$$

超平面

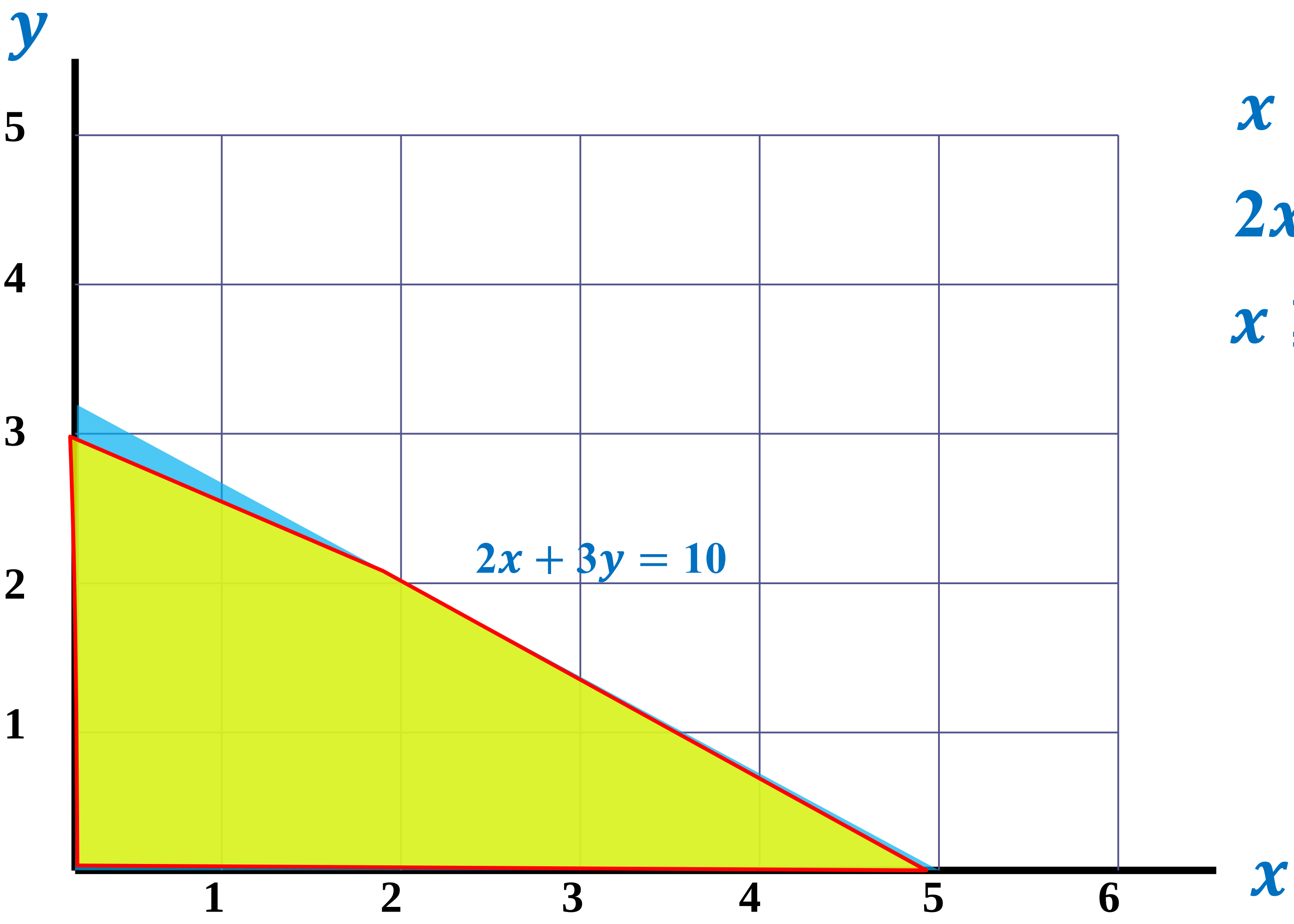
- 定义

$X = \{x | c^T x = z\} x \neq 0, z$ 为常数，那么：　称X为超平面

- 超平面将空间分成两部分：$c^T x \geq z, c^T x \leq z$



- **A single linear inequality determines a unique half-plane.**



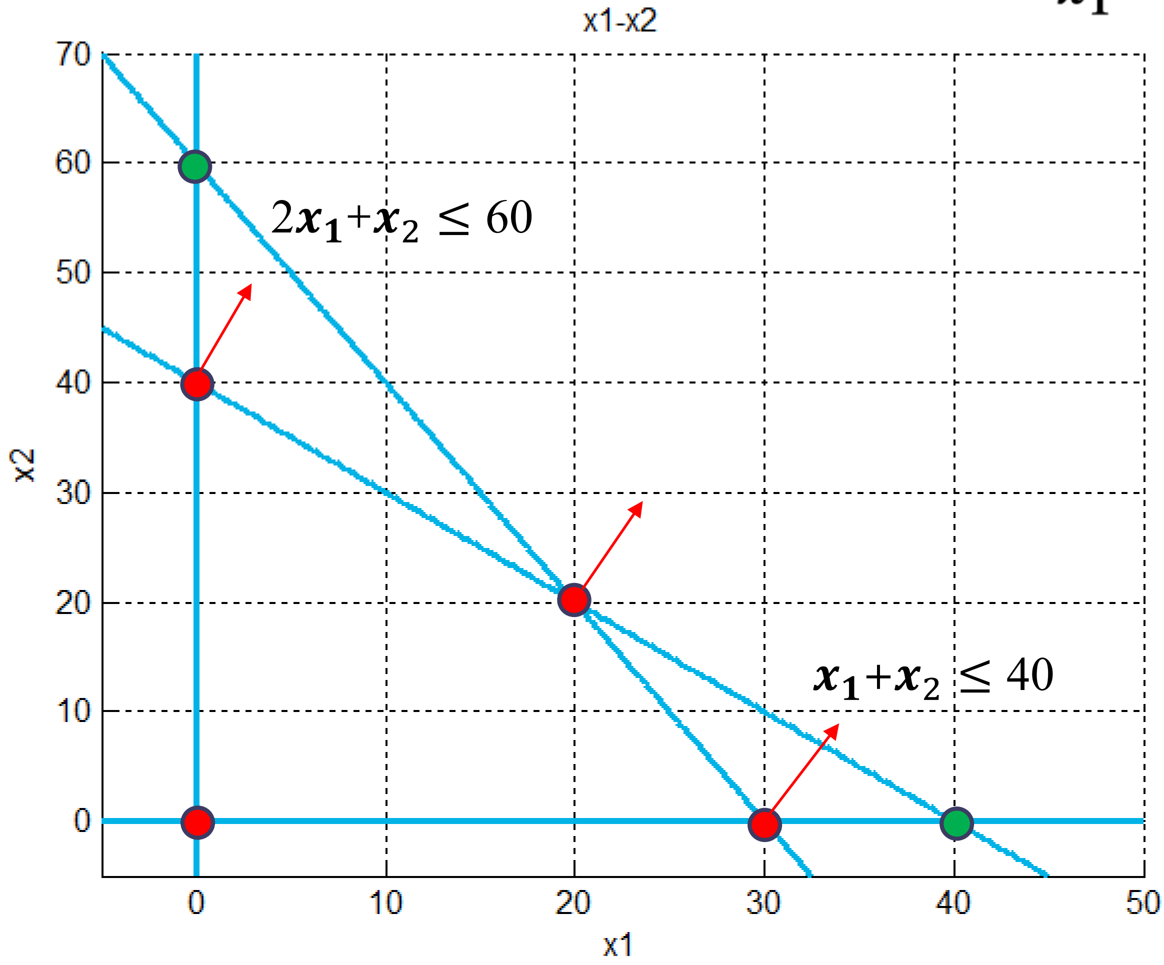
$$x + 2y \le 6$$

$$2x + 3y \le 10$$

$$x \ge 0, y \ge 0$$

- **Learning from Example**

$$\begin{array}{llllll} \text{Minimize} & -3x_1 & - & 2x_2 & \\ s.t. & x_1 & + & x_2 & \le 40 \\ & 2x_1 & + & x_2 & \le 60 \\ & x_1 & , & x_2 & \ge 0 \end{array}$$



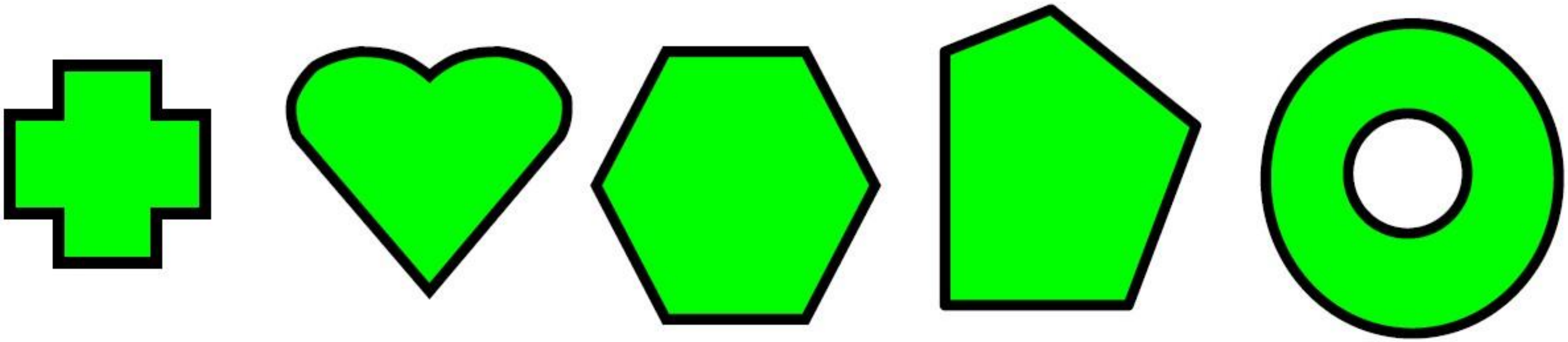
- **Linear Program**
  - **Recall that the standard form of LP:**

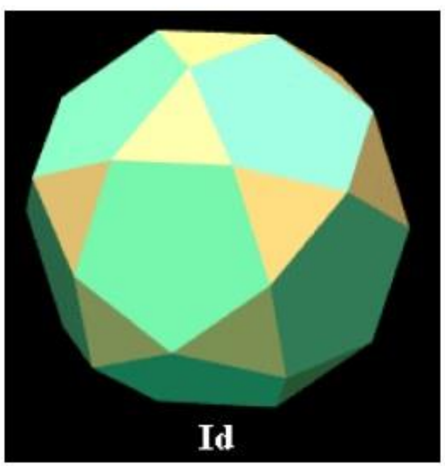
$$\min C^T X$$
$$s.t.\ Ax = b$$
$$x \geq 0$$

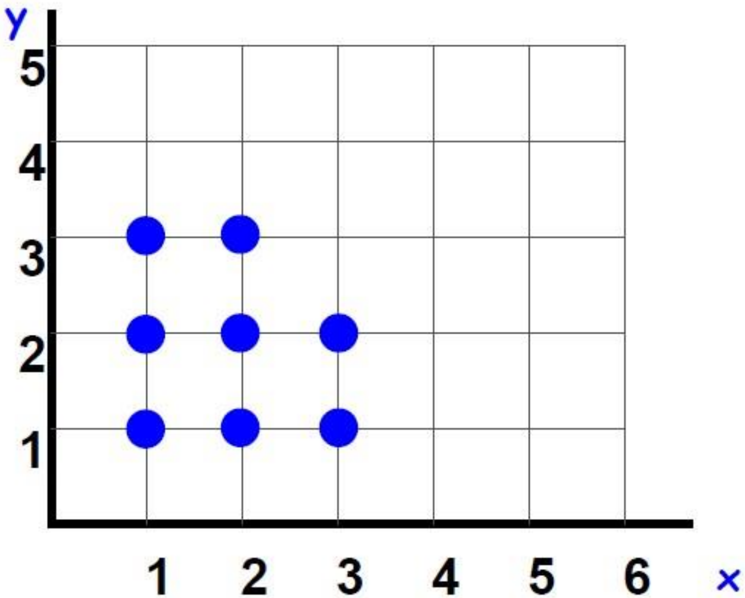
Where $\boldsymbol{c} \in \boldsymbol{R}^{\boldsymbol{n}}$, A is an $\boldsymbol{m} \times \boldsymbol{n}$ matrix with full row rank, $\boldsymbol{b} \in \boldsymbol{R}^{\boldsymbol{n}}$

  - **Polyhedron set & convex set :** $\{\boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{n}} | \boldsymbol{Bx} \geq \boldsymbol{d}\}$

- **Example**
  - **Which of the following are convex? Or not?**







- **Background knowledge**
  - **Definition: Let $x^1, x^2, \dots, x^p \in R^n, \lambda_1, \lambda_2, \dots, \lambda_p \in R$. And**

$$x = \sum_{i=1}^{p} \lambda_i x^i = \lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_p x^p$$

We say x is a linear combination of $\{x^1, \dots, x^p\}$.

If $\sum_{i=1}^{p} \lambda_i = 1$, we say x is an affine combination of $\{x^1, \dots, x^p\}$.

If $\lambda_i \geq 0$, we say x is a conic combination of $\{x^1, \dots, x^p\}$.

If $\sum_{i=1}^{p} \lambda_i = 1, \lambda_i \geq 0$ we say x is a convex combination of $\{x^1, \dots, x^p\}$.

- ## Affine set, convex set, and cone
  - **Definition: Let S be a subset of R^n.**

If the affine combination of any two points of S falls in S, then S is an affine set.

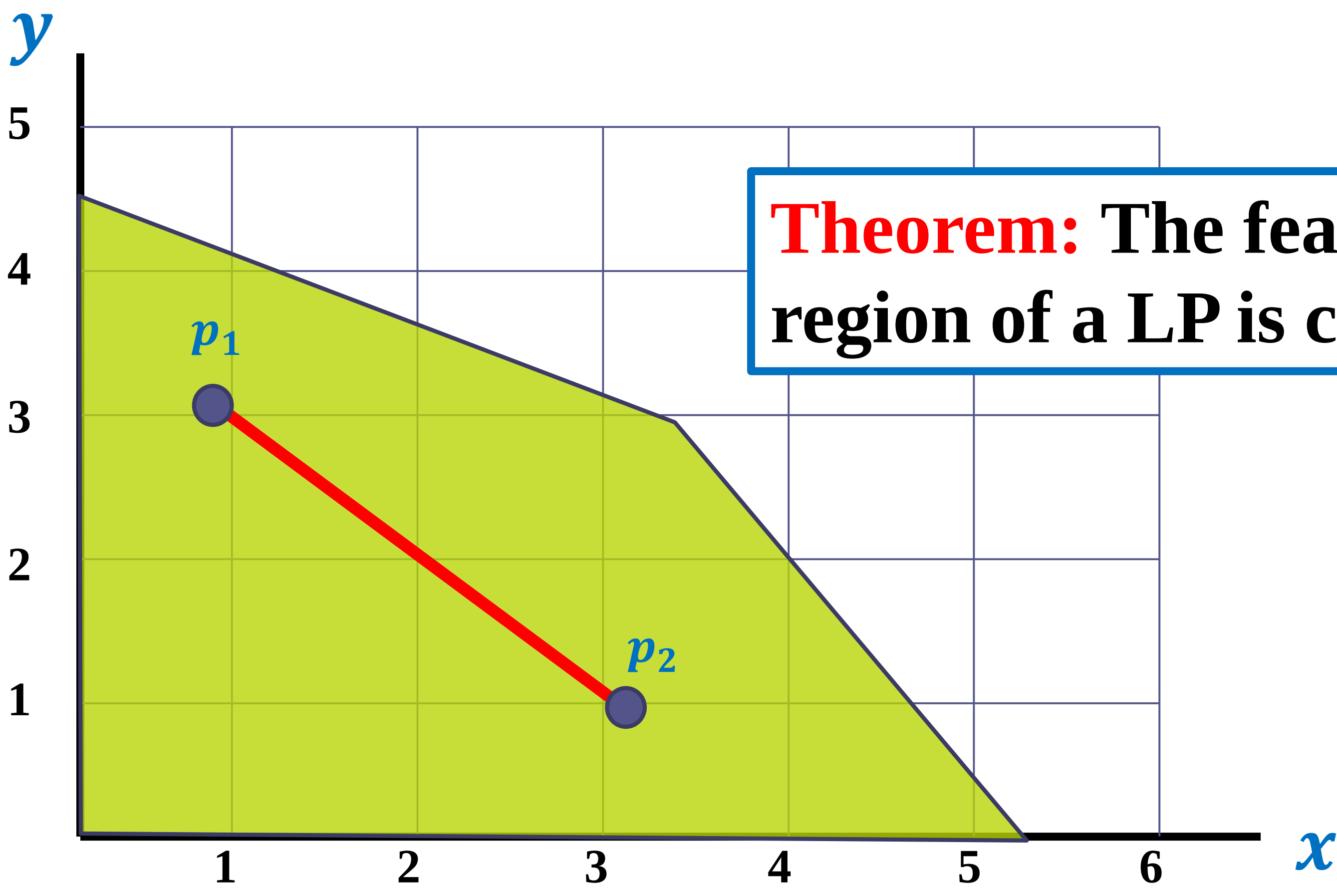
If the convex combination of any two points of S falls in S, then S is a convex set.

If $\lambda x \in S$ for all $x \in S$ and $\lambda \geq 0$, then S is a cone.

Convex set

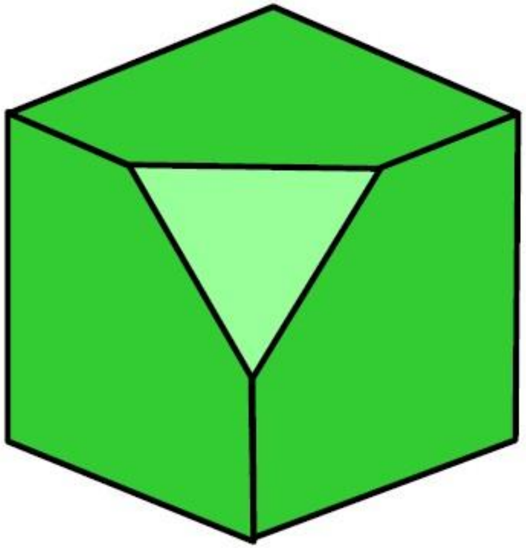
- Definition: A set S is convex if for every two points in the set, the line segment joining the points is also in the set; that is,

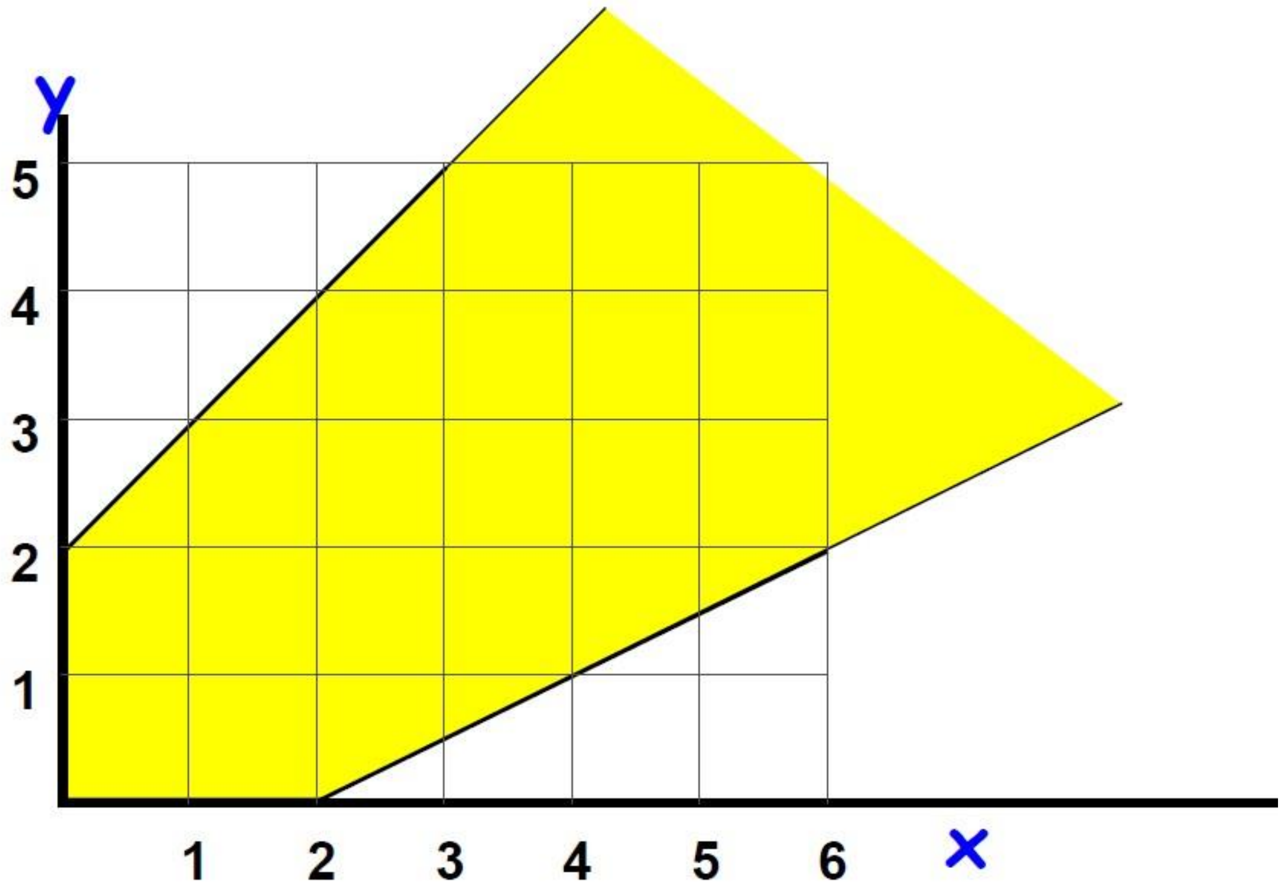
If $p_1, p_2 \in S$, then so is $(1-\lambda)p_1 + \lambda p_2$ for $\lambda \in [0,1]$.



Theorem: The feasible region of a LP is convex

- **The feasible region of a LP is convex**





$Ax = b$ and $x \geq 0$ means that the rhs vector b falls in the cone generated by the columns of constraint matrix A

$$A = (A_1|A_2|\dots|A_n)$$

$$A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix} \qquad Ax = (A_1|A_2|\dots|A_n)\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} x_j A_j \in R^m$$

- **Interior and boundary points**
  - **Given a set, what's the difference between an interior point and a boundary points?**
  - **Definition: Given a set $s \subset R^n$, a point $x \in s$ is an interior point of S, if**

$\exists \epsilon > 0$ such that the ball $B = \{y \in R^n | \|y - x\| \leq \epsilon\} \subset S$.

Otherwise, x is a boundary point of S.

We denote that

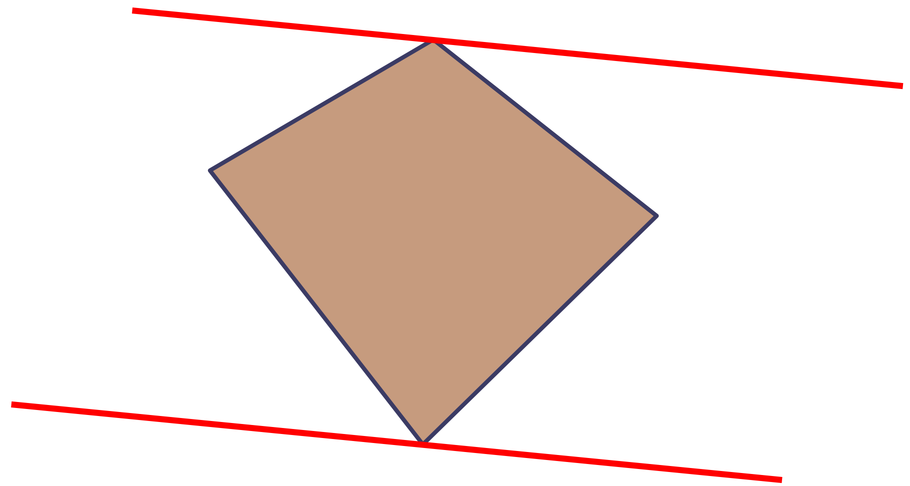
$int(s) = \{x \text{ is an interior point of } S\}$

$bdry(s) = \{x \text{ is an boundary point of } S\}$

Boundary points of convex sets

- What's special about boundary points of a convex set?
- Separation Theorem:

$s \subset R^n$ is convex, then $\forall x \in bdry(s), \exists\, n$ hyperplane H, such that $x \in H$ and either $s \subset H_L$ or $s \subset H_U$. Supporting hyperplane



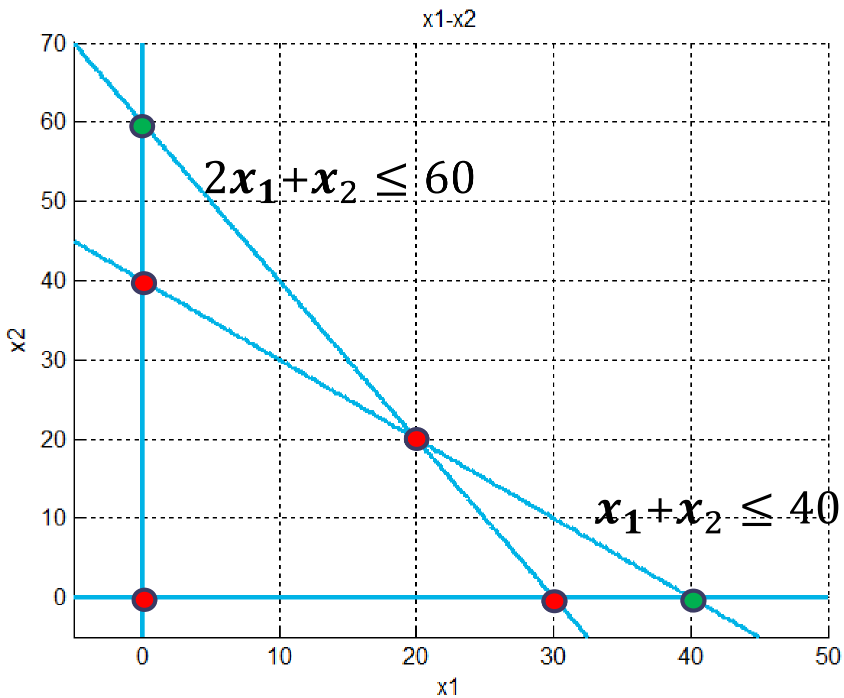
- **Are all boundary points the same?**

  - **Some sits on the shoulders of others, and some don't.**
  - **Definition: x is an extreme point of a convex set S. If x cannot be expressed as a convex combination of other points in S.**

  - **Question: Can you now see that if an LP has a finite optimal solution, then one vertex of P is optimal?**

  - **Let $P \in R^n$ be a given polyhedron. A vector $x \in P$ is an extreme point of P if there does not exist $y, z \in P$,and$\lambda \in (0, 1)$ such that $x = \lambda y + (1 - \lambda)z$**

- **Example 1**



$$x^1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \end{pmatrix} \qquad x^2 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \end{pmatrix}$$

$$x^3 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix} \qquad x^4 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \end{pmatrix}$$

$$\begin{array}{llllllllll}
\text{Minimize} & -3x_1 & - & 2x_2 & & & & & & \\
s.t. & x_1 & + & x_2 & + & x_3 & & & = 40 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = 60 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \ge 0
\end{array}$$

- **What's special?**

$$
\begin{array}{lrcrl}
\text{Minimize} & -3x_1 & - & 2x_2 & \\
s.t. & x_1 & + & x_2 & \le 40 \\
 & 2x_1 & + & x_2 & \le 60 \\
 & x_1 & , & x_2 & \ge 0
\end{array}
$$

  - Vertices

$$v^1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \end{pmatrix}, v^2 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \end{pmatrix}, v^3 = \begin{pmatrix} 20 \\ 30 \\ 0 \\ 0 \end{pmatrix}, v^4 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \end{pmatrix}$$

  - Edge

Interior

$$v^5 = \begin{pmatrix} 20 \\ 0 \\ 20 \\ 20 \end{pmatrix} \qquad v^6 = \begin{pmatrix} 15 \\ 15 \\ 10 \\ 15 \end{pmatrix}$$

$$n = 4, m = 2, n - m = 2$$

- **Finding extreme points**
  - **Theorem:**

A point $x \in P = \{x \in R^n | Ax = b, x \geq 0\}$ is an extreme point of P if and only if the columns of A corresponding to the positive components of x are linearly independent.

  - **Proof**

…

An extreme point of P is obtained by setting n-m variables to be zero and solving the remaining m variable in m equations.

- **Managing extreme points algebraically**
  - **Let A be an m by n matrix with $m < n$, we say A has full rank(full row rank) if A has m linearly independent columns.**
  - **In this, we can rearrange**

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$$ **where, x_B: basic variables x_N: non-basic variables**

$A = (B|N)$ **where, B: basics N: non-basics**

  - **Definition(basic solution and basic feasible solution)**
  - **If we set $x_N = 0$ and solve x_B for $Ax = Bx_B = b$, then x is a basic solution(bs)**
  - **Furthermore, if $x_B \geq 0$, then x is a basic feasible solution(bfs).**

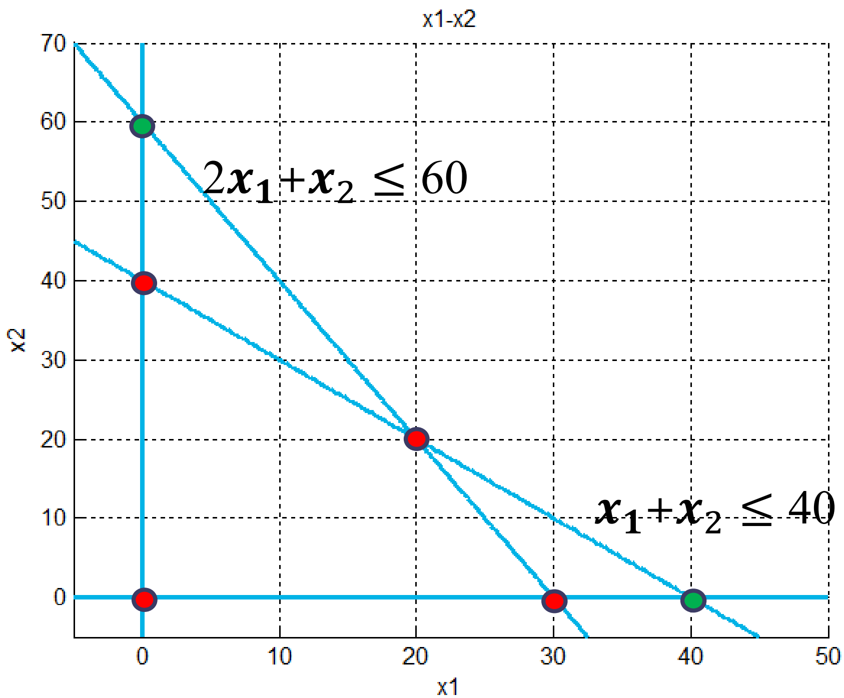
- **Example of basic and basic feasible solutions**

$$
\begin{array}{lrcrcrcrl}
\text{Minimize} & -3x_1 & - & 2x_2 & & & & & \\
s.t. & x_1 & + & x_2 & + & x_3 & & & = 40 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = 60 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq 0
\end{array}
$$

  - **Linear independence of the columns:**

$$\begin{pmatrix}1\\2\end{pmatrix}x_1+\begin{pmatrix}1\\1\end{pmatrix}x_2+\begin{pmatrix}1\\0\end{pmatrix}x_3+\begin{pmatrix}0\\1\end{pmatrix}x_4=\begin{pmatrix}40\\60\end{pmatrix}$$

- **Example of basic and basic feasible solutions**



$$x^1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \end{pmatrix} \quad x^3 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \end{pmatrix} \quad x^5 = \begin{pmatrix} 40 \\ 0 \\ 0 \\ -20 \end{pmatrix}$$

$$x^2 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix} \quad x^4 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \end{pmatrix} \quad x^6 = \begin{pmatrix} -20 \\ 0 \\ 60 \\ 0 \end{pmatrix}$$

$$\begin{array}{llllllll}
\text{Minimize} & -3x_1 & - & 2x_2 & & & & \\
s.t. & x_1 & + & x_2 & + & x_3 & & = 40 \\
& 2x_1 & + & x_2 & & & + \; x_4 & = 60 \\
& x_1 & , & x_2 & , & x_3 & , \; x_4 & \ge 0
\end{array}$$

- **Further results**
  - **Observation: when A does not have full rank, then either**

    (1) $Ax = b$ has no solution and hence $p = 0$, or

    (2) some constraints are redundant.

    For the second case, after remaining the redundant constraints, new A has full rank.
  - **Corollary: A point x in P is an extreme point of P if and only if x is a bfs corresponding to some basis B.**
  - **Corollary: The polyhedron P has only a finite number of extreme point. Proof:#of ways to choose m linearly independent columns from n columns**

    $$\leq C(n,m) = \frac{n!}{m!(n-m)!}$$

Extremal direction for unboundedness

- When P is unbounded, we need a direction leading to infinity.
- Definition:

A vector $\boldsymbol{d}(\neq 0) \in \boldsymbol{R}^{\boldsymbol{n}}$ is an extremal direction of P, if

$$\{\boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{n}} | \boldsymbol{x} = \boldsymbol{x}^{\mathbf{0}} + \boldsymbol{\lambda d}, \boldsymbol{\lambda} \geq 0\} \subset \boldsymbol{P}$$

For all $\boldsymbol{x}^{\mathbf{0}} \in \boldsymbol{P}$

- Observations:

(1) P is unbounded→ $\boldsymbol{P}$ has an extremal direction.

(2) d($\neq$ 0) is an extremal direction of $\boldsymbol{P} \to \boldsymbol{Ad} = 0$ and $\boldsymbol{d} \geq 0$

- **Basic solutions and Extreme Points**
  - **Let $\{x \in R^n | Ax = b, x \geq 0\}$, the feasible set of LP. Since A is full row rank, if the feasible set is not empty, then we must $m \leq n$, we assume that $m < n$.**
  - **Let $A = (B, N)$,where B is an $m \times m$ matrix with full rank, i.e., $det(B) \neq 0$. Then, B is called a basic.**
  - **Let $X = \begin{pmatrix} X_B \\ X_N \end{pmatrix}$. We have $BX_B + NX_N = b$. Setting $X_N = 0$, we have $X_B = B^{-1}b$. $X = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$ is called a basic solution. X_B is called basic variables, X_N is called nonbasic variables.**
  - **If the basic solution is also feasible, this is , $B^{-1}b \geq 0$, then X is called a basic feasible solution.**

- **Basic solutions and Extreme Points**
  - $\hat{x} \in S$ is an extreme point of S if and only if $\hat{x}$ is a basic feasible solution.
  - Two extreme points are adjacent if they differ in only one basic variable.
  - (Basic Theorem of LP) Consider the linear program: $min\{c^T x | Ax = b, x \geq 0\}$. If S has at least one extreme point and there exists an optimal solution, then there exists an optimal solution that is an extreme point.
  - Proof (representation of polyhedron)
  - The feasible set of standard form linear program has least one feasible point.

Basic solutions and Extreme Points

- **(Basic Theorem of LP) Consider the linear program:** $min\{c^T x | Ax = b, x \geq 0\}$ **. If S has at least one extreme point and there exists an optimal solution, then there exists an optimal solution that is an extreme point.**
- **Proof (representation of polyhedron)**
- **The feasible set of standard form linear program has least one feasible point.**
- **Therefore, we claim that the optimal value of a linear program is either $-\infty$, or is attained an extreme point(basic feasible solution) of the feasible set.**

- **Basic solutions and Extreme Points**
  - **Theorem: Let $V = \{v^i \in R^n | i \in I\}$ be a set of all extreme points of P, I is a finite index set, then $\forall x \in P$, we have**

$$x = \sum_{i \in I} \lambda_i v^i + d$$

where

$$\sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, \forall i \in I$$

and either d=0 or d is an external direction of P.

Basic solutions and Extreme Points

- **Theorem: For a standard form LP, if its feasible domain P is nonempty, then the optimal objective value of $z = c^T x$ over P is either unbounded below, or it is attained at an extreme point of P.**

Proof: There are two case1:

Case1: P has an extremal direction d such that

$c^T d < 0$. Hence P is unbounded and $z \to -\infty$,along d.

$$c^T x = c^T \left(\sum_{i \in I} \lambda_i v^i + d \right) = c^T \sum_{i \in I} \lambda_i v^i + c^T d$$

- **Basic solutions and Extreme Points**

Case2: P has no extremal direction d such that

$$c^T x = c^T \left(\sum_{i \in I} \lambda_i v^i + d \right) = c^T \sum_{i \in I} \lambda_i v^i + c^T d$$

$$\geq \sum_{i \in I} \lambda_i \left(c^T \ v^i \right)$$

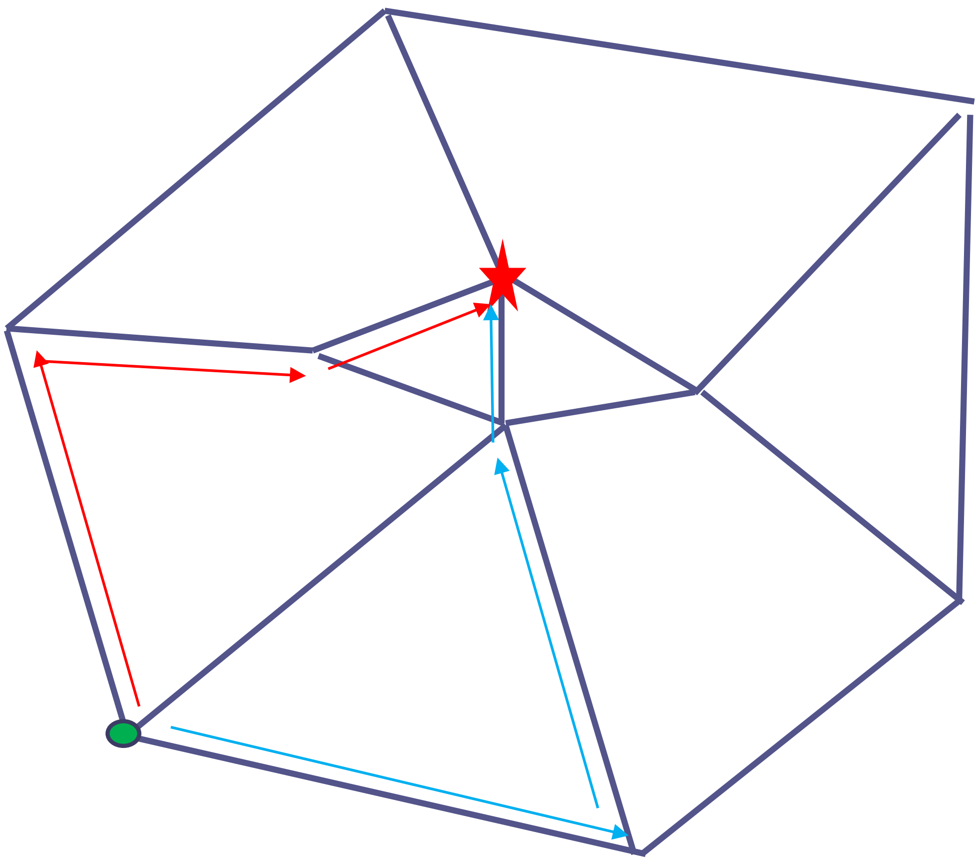
$$\geq min \{ (c^T \ v^i) \} \sum_{i \in I} \lambda_i$$

$$= min \{ (c^T \ v^i) \}$$

$$= c^T min(v^i)$$

- **Algorithm 1:Enumeration**
  - **Let $min\{c^T x|Ax = b, x \geq 0\}$ be a bounded LP**
  - **Enumerate all bases $B \in \{1, \ldots, n\}$, $C_n^m = o(n^m)$**
  - **Computer associated basic solution $x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$**
  - **Return the one which has largest objective function value among the feasible basic solutions.**
  - **Running time is $o(n^m \cdot m^3)$**

- **Algorithm 2:Simplex method**



- **Algorithm 2:Simplex method**

  - **Step1:(Starting)**

    Find an initial extreme point or declare P is null

  - **Step2: (Checking optimality)**

    If the current ep is optimal, STOP. Else Step3:

  - **Step3: (Pivoting)**

    Move to a better ep.

    Return to step 2.

- **Property 1: If a bfs x is nondegenerate, then x is uniquely determined by n hyperplanes.**

$$A = (B, N),\ x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$$

Let : M $= \begin{bmatrix} B & N \\ 0 & I \end{bmatrix}$, **Then M is nonsingular and**

$$\mathbf{Mx} = \begin{bmatrix} B & N \\ 0 & I \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = M^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} B^{-1} & -B^{-1}N \\ 0 & I \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix}$$

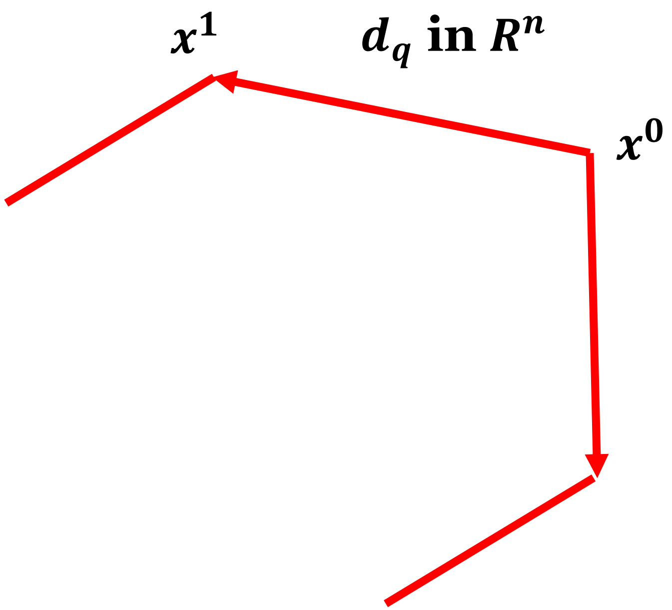
- Under nondegeneracy, every basic feasible solution(extreme point) has exactly n-m adjacent neighbors.

- For a bfs, each adjacent bfs can be reached by increasing one nonbasic variable from 0 to positive and decreasing one basic variable from positive to 0. -Pivoting
- See from the example.

- $x^1 = x^0 + \lambda d_q$ for $\lambda > 0$

d_q: edge direction ,λ: step length

- **Pivoting**
  - **One nonbasic variable enters (from 0 to positive) the basis and one basic variable leaves the basis (from positive to 0).**



  - $x^1 = x^0 + \lambda d_q$ **for** $\lambda > 0$

d_q**: edge direction ,**λ**: step length**

- **Where are these edge directions?**

$$\mathbf{M} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{N} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix}$$

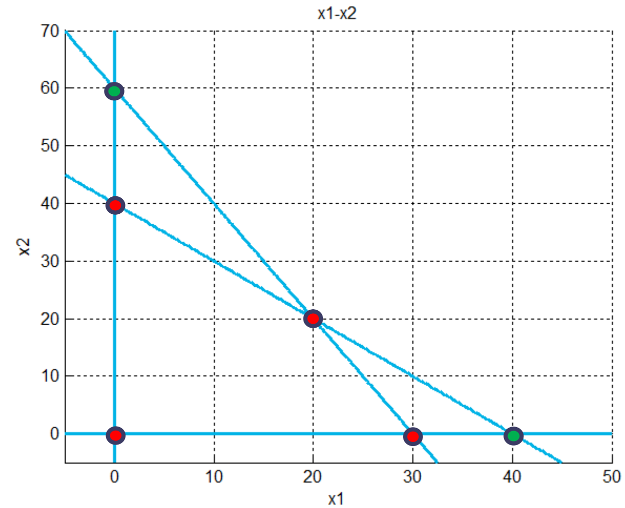
$$\mathbf{Mx} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{N} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{M}^{-1} = \begin{bmatrix} \boldsymbol{B}^{-1} & -\boldsymbol{B}^{-1}\boldsymbol{N} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \longrightarrow (-\boldsymbol{B}^{-1}A_{q1}, -\boldsymbol{B}^{-1}A_{q2}, \ldots, -\boldsymbol{B}^{-1}A_{q(n-m)})$$

$$\boldsymbol{n - m}$$

- **Example**

$$
\begin{array}{llllllllll}
\text{Minimize} & -3x_1 & - & 2x_2 & & & & & & \\
s.t. & x_1 & + & x_2 & + & x_3 & & & = 40 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = 60 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq 0
\end{array}
$$



$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

At vertex 1, $BV = \{x_3, x_4\}, NBV = \{x_1, x_2\}$

$$x^1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \end{pmatrix} \quad x^2 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$x^3 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix} \quad x^4 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **Which neighbor is a good one?**
- $\boldsymbol{x(\lambda) = x + \lambda d_q}$ **for** $\boldsymbol{\lambda > 0}$

$$z(x(\lambda)) = c^T x(\lambda) = c^T x + \lambda c^T d_q$$

$$= z(x) + \lambda\left(c_B^T, c_N^T\right)\begin{pmatrix} -B^{-1}A_q \\ e_q \end{pmatrix}$$

$$= z(x) + \lambda[c_q - c_B^T B^{-1} A_q]$$

$$= z(x) + \lambda r_q$$

$r_q = [c_q - c_B^T B^{-1} A_q]$: **reduced cost**

If $r_q < 0$**, then** d_q **is a good direction!**

- Optimality check by reduced cost

- **Analysis of step length (minimum ratio test)**
- **We have** $x(\alpha) = x + \alpha d_q$ **for** $\alpha > 0$

with $r_q = c^T d_q = c_q - c_B^T B^{-1} A_q < 0$

Case 1: if all $d_q \geq 0$, **then** $x(\alpha) \geq 0$.

$$c^T x(\alpha) = c^T x + \alpha c^T d_q$$

as $\alpha \to \infty, c^T x(\alpha) \to -\infty$

Case 2: d_q **has at least one component<0.**

To keep $x(\alpha) \geq 0$, **we have to choose**

$$\alpha = min\{\frac{x_i}{-d_q^i} | d_q^i < 0\}$$

- **Algorithm 2:Simplex method**
  - **Step1:(Starting)**

    Find a bfs x with A=[B|N]
  - **Step2: Check** $r_q = c^T d_q = c_q - c_B^T B^{-1} A_q$

    if all $r_q \geq 0$**, x is optimal.**

    else pick one $r_q < 0$**, Go to step 3**
  - **Step3: If all** $d_q \geq 0$ **, then LP is unbounded.**

    else find $\lambda = min\{\frac{x_i}{-d_q^i} | d_q^i < 0\}$

    Then $x = x + \lambda d_q$**, go to step 2.**

• Example 2

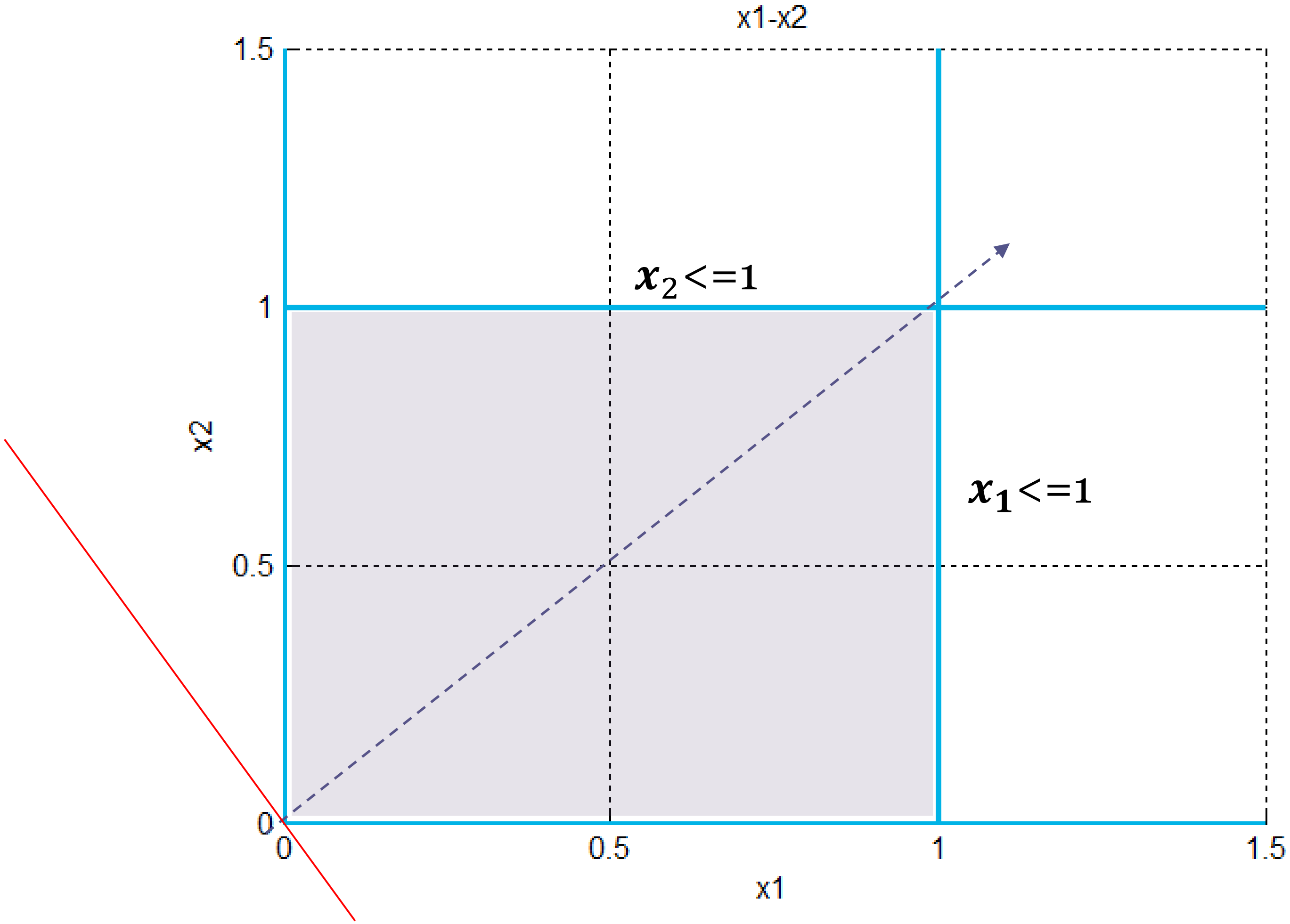
$$
\begin{array}{lrcrl}
\text{Minimize} & -x_1 & - & x_2 & \\
s.t. & x_1 & + & & \le 1 \\
 & & & x_2 & \le 1 \\
 & x_1 & , & x_2 & \ge 0
\end{array}
$$

- Covert to standard form:

$$
\begin{array}{lrcrcrcrl}
\text{Minimize} & -x_1 & - & x_2 & & & & & \\
s.t. & x_1 & & & + & x_3 & & & = 1 \\
 & & & x_2 & & & + & x_4 & = 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \ge 0
\end{array}
$$

- **Example 2**

$$\begin{array}{lrcrcrcrl}
\text{Minimize} & -x_1 & - & x_2 & & & & & \\
s.t. & x_1 & & & + & x_3 & & & = 1 \\
& & & x_2 & & & + & x_4 & = 1 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & \geq 0
\end{array}$$



- **How to start the simplex method?**
  - **How to get an initial basic feasible solution?**

    -eye inspection

    -randomly generate (test of luck)

    -systematic approach
  - **1. Two-phase method (Phase 1 problem)**
  - **2. big-M method**

- **Big-M method**
  - **Add a big penalty $M > 0$ to each artificial variable.**
  - **Combine phase I problem with the original problem to consider a big-M problem:**

$$
(PhI) \quad s.t. \quad \begin{aligned} \text{Min } & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} M u_i \\ & Ax + Iu = b (\geq 0) \\ & x, u \geq 0 \end{aligned}
$$

- **Homework**

1. **Code a lp algorithm using the simplex procedure and,**
2. **Solve the problem:**

$$\begin{array}{ll} \text{minimize} & 2x_1 + 4x_2 + x_3 + x_4 \\ \text{subject to} & x_1 + 3x_2 + x_4 \leqslant 4 \\ & 2x_1 + x_2 \leqslant 3 \\ & x_2 + 4x_3 + x_4 \leqslant 3 \\ & x_i \geqslant 0 \quad i = 1, 2, 3, 4. \end{array}$$

3. **For the lp exercise,**

   a) How much can the element of $b = (4,3,3)$ be changed without changing the optimal basis?

   b)How much can the elements of $c = (2,4,1,1)$be changed without changing the optimal basis.

   c) What happens to the optimal cost for small changes in b?

   d) what happens to the optimal cost for small changes in c?

Algorithm 2:Simplex method

- Simplex method was invented by George Dantzig (1914-2005)
- Suppose we have a basic feasible solution $\hat{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix}$,

  $\boldsymbol{A} = (\boldsymbol{B}, \boldsymbol{N})$, $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{c}_N \end{pmatrix}$
- $\boldsymbol{Ax} = \boldsymbol{b} \leftrightarrow \boldsymbol{Bx}_B + \boldsymbol{Nx}_N = \boldsymbol{b}$, and so: $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N$

$$\begin{aligned}
\boldsymbol{c}^T\boldsymbol{x} &= \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_N^T\boldsymbol{x}_N \\
&= \boldsymbol{c}_B^T(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N) + \boldsymbol{c}_N^T\boldsymbol{x}_N \\
&= \boldsymbol{c}^T\hat{\boldsymbol{x}} + (\boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N})\boldsymbol{x}_N
\end{aligned}$$