Assignment due date: Feb 10, 2017, 11:59pm

Hand-in to be submitted electronically in PDF format with code to the CDF server by the above due date

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I hereby affirm that all the solutions I provide, both in writing and in code, for this assignment are my own. I have properly cited and noted any reference material I used to arrive at my solution and have not share my work with anyone else. I am also aware that should my code be copied from somewhere else, whether found online, from a previous or current student and submitted as my own, it will be reported to the department.

Daiqing Li

Signature

(Note: -3 marks penalty for not completing properly the above section)

Part 1 total marks: 50
Part 2 total marks: 50
Total: 100

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(adapted from F. Estrada)

Part 1 - Written work: Be sure to provide clean, legible derivations and not omit any steps which your TA may need to fully understand your work. Use diagrams wherever appropriate.

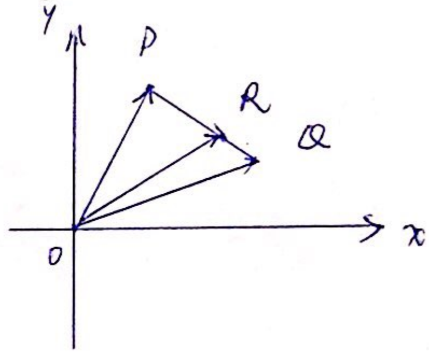
A) Lines, points, vectors, and dot-products

1) Let a line segment be defined by the points P and Q.

(4 marks) - Let R be the point on the line segment that is four times as far from P as it is from Q. Let $\vec{p} = \overline{OP}, \vec{q} = \overline{OQ}, \vec{r} = \overline{OR}$. Show that $\vec{r} = \frac{1}{5}\vec{p} + \frac{4}{5}\vec{q}$ and *draw* a sketch validating this.

$$\begin{cases} \vec{OR} = \vec{OP} + \vec{PR} \\ \vec{OR} = \vec{OQ} - \vec{RQ} \\ |\vec{PR}| = 4|\vec{RQ}| \end{cases} \Rightarrow \begin{cases} \vec{OR} = \vec{OP} + 4\vec{RQ} \\ 4\vec{OR} = 4\vec{OQ} - 4\vec{RQ} \end{cases}$$

$$\Rightarrow \vec{r} = \frac{1}{5}\vec{p} + \frac{4}{5}\vec{q}$$



(3 marks) - Show the parametric form of the line segment formed by P and Q, with parameter t, starting from P in the direction of Q (that is to say, positive values of the parameter will give points along the line towards Q). Be sure to define everything explicitly.

$\vec{f}(t) = \vec{P} + t(\vec{Q} - \vec{P})$ with $0 \le t \le 1$

(5 marks) Let the points have explicit representations $P(x_0, y_0)$, $Q(x_1, y_1)$ with respect to a coordinate system. Give a simple test to determine if an arbitrary point $R(x, y)$ is on the **line segment** or not.

let $f(x,y) = y - y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$, substitute $R(x,y)$ into $f(x,y)$, if $f = 0$, R is on the line ~~segment~~, else it is not. To make sure R is between P and Q, check $\min(x_0, x_1) \le x \le \max(x_0, x_1)$ and $\min(y_0, y_1) \le y \le \max(y_0, y_1)$

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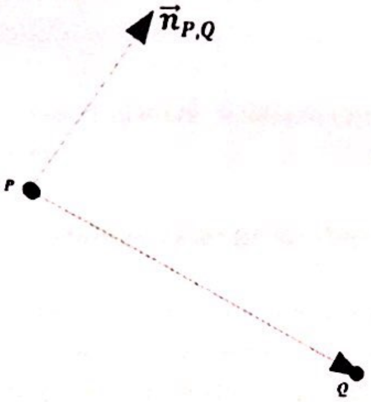
2)

(5 marks) - Derive a **unit** normal vector $\vec{n}$ to the line segment from above, expressed in terms of $\boldsymbol{P}$ and $\boldsymbol{Q}$. $\vec{n}$ should point to the left with respect to the direction of the line segment (the line segment is directed from P to Q).



$\vec{n}$ should satisfy $\vec{n} \cdot \vec{PQ} = 0$

let $dx = Q_x - P_x$, $dy = Q_y - P_y$

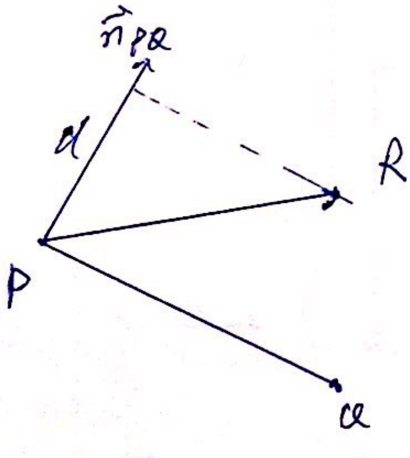
$(x', y') \cdot (dx, dy) = 0$

since x' points to positive, let $x' = -dy$

$y' = dx$. normalize it

$$\vec{n} = \left(\frac{P_y - Q_y}{|\vec{PQ}|}, \frac{Q_x - P_x}{|\vec{PQ}|} \right)$$

(5 marks) - Using $\vec{n}$ from 2), give an expression that computes the distance between an arbitrary point $R(x, y)$ and the line segment.



$$\vec{PR} = (x - P_x, y - P_y)$$

$$d = |\vec{PR} \cdot \vec{n}_{PQ}|$$

$$= \frac{|(x - P_x) \cdot (P_y - Q_y) + (y - P_y)(Q_x - P_x)|}{|\vec{PQ}|}$$

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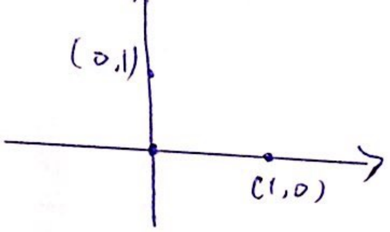
B. 1) it is not commute.

counter example, rotate point (1,0) and then translate it to origion



$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}0-0\\1-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

if translate first, $\begin{pmatrix}1-0\\0-1\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\neq\begin{pmatrix}0\\1\end{pmatrix}$$

2) it is commute

for two rotation $R_1=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$, $R_2=\begin{pmatrix}\cos\varphi&-\sin\varphi\\\sin\varphi&\cos\varphi\end{pmatrix}$

$$R_1\cdot R_2=\begin{pmatrix}\cos(\theta+\varphi)&-\sin(\theta+\varphi)\\\sin(\theta+\varphi)&\cos(\theta+\varphi)\end{pmatrix}=R_2\cdot R_1$$

3) no. counter example, consider reflection about $y=x$ and transition $\begin{pmatrix}x-0\\y-1\end{pmatrix}$. $P=(2,1)$; $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$

$\begin{pmatrix}1-0\\2-1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$. Now translation first, $\begin{pmatrix}2-0\\1-1\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}\neq\begin{pmatrix}1\\1\end{pmatrix}$$

4) no. consider shearing along x by 1 and uniform scale by 2.

$$\begin{pmatrix}2\\1\end{pmatrix}\xrightarrow{\text{shear}}\begin{pmatrix}3\\1\end{pmatrix}\xrightarrow{\text{scale}}\begin{pmatrix}6\\2\end{pmatrix},\quad\begin{pmatrix}2\\1\end{pmatrix}\xrightarrow{\text{scale}}\begin{pmatrix}4\\2\end{pmatrix}\xrightarrow{\text{shear}}\begin{pmatrix}5\\2\end{pmatrix}$$

5) no, consider rotation counterclockwise $90°$ and $\begin{pmatrix}1&0\\0&2\end{pmatrix}$ scale

$$\begin{pmatrix}1\\0\end{pmatrix}\xrightarrow{\text{rotate}}\begin{pmatrix}0\\1\end{pmatrix}\xrightarrow{\text{scale}}\begin{pmatrix}0\\2\end{pmatrix},\quad\begin{pmatrix}1\\0\end{pmatrix}\xrightarrow{\text{scale}}\begin{pmatrix}1\\0\end{pmatrix}\xrightarrow{\text{rotate}}\begin{pmatrix}0\\1\end{pmatrix}$$

B: *Transformations and transformation properties*

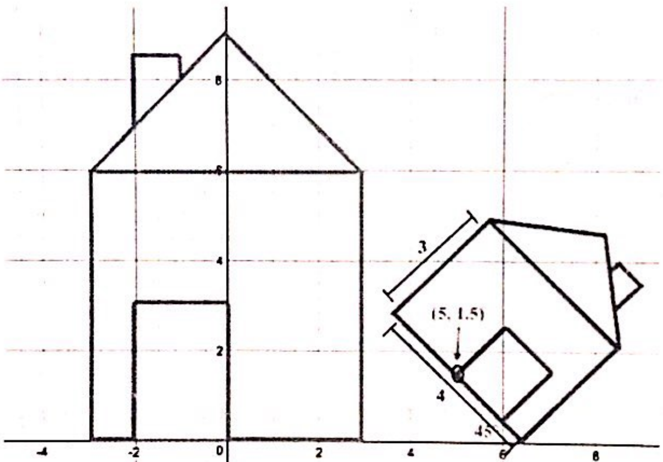
(3 marks each) - Prove whether or not the following pairs of transformations commute. The transformations are general, 2D affine transforms (*provide solutions on separate sheet*)

1) Rotations and Translations
2) Two rotations
3) Translation, Reflection
4) Shear wrt to x axis an uniform scaling
5) Rotation, non-uniform scaling

If not-commutative, a simple counter example will suffice. If commutative, algebraic proof is required.

C: Affine transformation properties

1) **(5 marks) - Give the sequence of 2D affine transformations that maps the object in the left figure to the object in the right figure. You can use R, T, S, Sh, Re, to express rotation, translation, scaling, shear, and reflection operations respectively.**



$$\xrightarrow{Re}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\xrightarrow{S}\begin{pmatrix}\frac{2}{3} & 0\\ 0 & \frac{3}{5}\end{pmatrix}\xrightarrow{R}\begin{pmatrix}\cos 45^\circ & \sin 45^\circ\\ -\sin 45^\circ & \cos 45^\circ\end{pmatrix}$$

$$\xrightarrow{T}\begin{pmatrix}x+5\\ y+1.5\end{pmatrix}$$

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2) (5 marks) A triangle can be expressed in parametric form as:

$$T(\alpha, \beta) = \overrightarrow{p_1} + \alpha\overrightarrow{d_1} + \beta\overrightarrow{d_2}$$

provided that $\alpha + \beta \leq 1$ and $\alpha, \beta \geq 0$ Prove that under an invertible affine transform, the triangle remains a triangle.

given a general affine transform:

$$A = \begin{bmatrix} a & b & f \\ c & e & g \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(A) = ae \neq 0$$

let $\vec{P_1} = \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}, \vec{d_1} = \begin{pmatrix} d_{1x} \\ d_{1y} \\ 0 \end{pmatrix}, \vec{d_2} = \begin{pmatrix} d_{2x} \\ d_{2y} \\ 0 \end{pmatrix}$

$$T(\alpha, \beta) = \begin{bmatrix} P_x + \alpha d_{1x} + \beta d_{2x} \\ P_y + \alpha d_{1y} + \beta d_{2y} \\ 1 \end{bmatrix}$$

$T'(\alpha, \beta) = A\, T(\alpha, \beta)$

$$= \begin{bmatrix} aP_x + bP_y + f + \alpha(ad_{1x} + bd_{1y}) + \beta(ad_{2x} \\ cP_x + eP_y + g + \alpha(cd_{1x} + ed_{1y}) + \beta(cd_{2x} + \\ 1 \end{bmatrix}$$

can be separated as

$\vec{P_1}' = \begin{pmatrix} aP_x + bP_y + f \\ cP_x + eP_y + g \end{pmatrix}$ $\quad T' = \vec{P_1}' + \alpha\vec{d_1}' + \beta$

$\vec{d_1}' = \begin{pmatrix} ad_{1x} + bd_{1y} \\ cd_{1x} + ed_{1y} \\ 0 \end{pmatrix}$ $\quad$ QED

$\vec{d_2}' = \begin{pmatrix} ad_{2x} + bd_{2y} \\ cd_{2y} + ed_{2y} \\ 0 \end{pmatrix}$

3) (3 marks) - Can affine transformations be represented with a formula where points can be represented in Cartesian coordinates instead of homogeneous coordinates? What is the main advantage of representing affine transformations in the same matrix form as general homographies?

Yes, as long as the affine transformation does not do translation. Otherwise, it is impossible. The main advantage is that we can combine many affine transformations into a single matrix by multiplying the respective matrices.