Finiteness of the Odd Perfect and Primitive Abundant Numbers with n Distinct Prime Factors

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Source: *American Journal of Mathematics*, Vol. 35, No. 4 (Oct., 1913), pp. 413-422
Published by: The Johns Hopkins University Press
Stable URL: http://www.jstor.org/stable/2370405
Accessed: 28-04-2017 02:33 UTC

*Finiteness of the Odd Perfect and Primitive Abundant Numbers with n Distinct Prime Factors.**

BY LEONARD EUGENE DICKSON.

1. Denote by $\sigma(a)$ the sum of all the divisors of a positive integer a. Then a is called abundant, perfect, or deficient, according as

$$\sigma(a) > 2a, \ = 2a, \ < 2a,$$

respectively. If $d_1, \ldots, d_k$ are the divisors of a, the divisors of ma include $md_1, \ldots, md_k$ and unity. Hence

$$\sigma(ma) > m\sigma(a) \qquad (m > 1),$$

so that any greater multiple of an abundant or perfect number is abundant. This implies that any smaller divisor of a deficient or perfect number is deficient.

A non-deficient number will be called *primitive* if it is not a multiple of a smaller non-deficient number. The set of all non-deficient numbers is identical with the set of all multiples of the primitive non-deficient numbers. Any perfect number is a primitive non-deficient number.

While there is an infinitude of non-deficient odd numbers having any given number n $(n > 2)$ of distinct prime factors, for example,

$$3^e 5 \cdot 7 \, p_4^{e_4} \ldots p_n^{e_n} \qquad (e \geqq 3, \ p\text{'s distinct primes} > 7),$$

we shall nevertheless prove the following

THEOREM. *There is only a finite number of primitive non-deficient odd*† *numbers having any given number of distinct prime factors.*

COROLLARY. *There is not an infinitude of odd perfect numbers with any given number of distinct prime factors.*

Sylvester‡ proved that there is no odd perfect number having five or fewer distinct prime factors.

Below are listed all primitive non-deficient odd numbers with four or fewer distinct prime factors. Since none of them are perfect, we obtain anew Sylvester's result, except for the case of five primes.

* Presented to the American Mathematical Society, December 31, 1912.

† The restriction to odd numbers is essential. For instance, if p is a prime, $2^m p$ is non-deficient if and only if $2^{m+1} > p+1$.

‡ *Collected Math. Papers*, Vol. IV (1912), pp. 588–629 (several articles).

2. LEMMA A. *Any set S of functions of the type*

$$F = x_1^{e_1} x_2^{e_2} \dots x_n^{e_n} \qquad (e\text{'s integers} \geqq 0) \tag{1}$$

contains a finite number of functions $F_1, \dots, F_k$ such that each function F of the set S can be expressed as a product $F_i f$, where f is of the form (1), *but is not necessarily in the set S.*

For $n=1$, we may take $k=1$, $F_1=x_1^l$, where l is the least e_1. To proceed by induction, let the lemma be true for $n-1$ variables. Select at random a function $x_1^{c_1} \dots x_n^{c_n}$ of the set S and call it F_1. Then any function (1) of S is of the desired form $F_1 f$ if $e_1 \geqq c_1, \dots, e_n \geqq c_n$, simultaneously. If there be further functions in S, those in which $e_i = v$, where i is a fixed integer $\leqq n$ and v a fixed integer $< c_i$, have the desired property. Indeed, after deleting the common factor x_i^v, we have a set S' of functions

$$F' = x_1^{e_1} \dots x_{i-1}^{e_{i-1}} x_{i+1}^{e_{i+1}} \dots x_n^{e_n},$$

which by hypothesis are expressible as products $F'_j f'$ $(j = 1, \dots, k_i)$. The number of cases arising by varying i and v is finite.

We may also derive* the lemma from a theorem due to Hilbert,† which may be stated in the following form:

Any set S of homogeneous polynomials in $x_1, \dots, x_n$ contains a finite number of polynomials $F_1, \dots, F_k$ such that any polynomial F of the set can be expressed in the form $f_1 F_1 + \dots + f_k F_k$, where $f_1, \dots, f_k$ are homogeneous polynomials in $x_1, \dots, x_n$, not necessarily in the set S.

Let each F be a monomial form (1). Since the f's are not uniquely determined, the resulting expression $\Sigma f_i F_i$ for F is not necessarily of the form $f_i F_i$. However, there is always at least one determination of the f's which gives this special expression for F. Indeed, we have $f_i \equiv 0$ unless the degree of F in x_j equals or is greater than the degree of F_i in x_j for each $j=1, \dots, n$. Since not every f_i vanishes identically, there is some value of i for which the monomial F is of the form $g_i F_i$, where g_i is monomial.

3. Lemma A may be stated and proved in an equivalent geometrical form. For example, let $n = 2$ and define the region $R(P)$ to be the quadrant of the plane determined by the two half-lines extending from the point P horizontally to the right and vertically upwards, respectively. Let S be a set of points (A, B) with integral coordinates $\geqq 0$. Let a be the least A, β the least B for the points (a, B), and set $P_1 = (a, \beta)$. Let b be the least B, α the least A for the points (A, b), and set $P_2 = (\alpha, b)$. All the points of S lie in $R(M)$,

* Professor E. H. Moore suggested to me that I undertake this derivation.

† *Mathematische Annalen*, Vol. XXXVI.

where $M=(a, b)$. In particular, if $\beta=b$, then $\alpha=a$ (and conversely), and $P_1=P_2=M$, so that the points of S lie in $R(P_1)$. In general, the points of S in neither $R(P_1)$ nor $R(P_2)$ lie *inside* the rectangle determined by P_1, P_2, M, and hence are finite in number.

By 2^n applications of Lemma A we obtain the theorem: Let the signs of the coordinates of a point $X'=(\xi'_1, \ldots, \xi'_n)$ be such that

$$\epsilon_1\xi'_1\geqq 0, \ldots, \epsilon_n\xi'_n\geqq 0 \qquad (\epsilon_1^2=1, \ldots, \epsilon_n^2=1),$$

and denote by $R_{\epsilon_1,\ldots,\epsilon_n}(X')$ the region of n dimensions which is composed of all the points $X=(\xi_1, \ldots, \xi_n)$ for which

$$\epsilon_i\xi_i\geqq\epsilon_i\xi'_i \qquad (i=1, \ldots, n).$$

Then any set S of points X with integral coordinates contains a finite number of points $X^{(i)}_{\epsilon_1,\ldots,\epsilon_n}$ such that every point of S lies in at least one region $R_{\epsilon_1,\ldots,\epsilon_n}(X^{(i)}_{\epsilon_1,\ldots,\epsilon_n})$.

The lemma may also be interpreted as a theorem on permutable operations O_i with positive or negative integral exponents. We replace x_i by $O_i^{\epsilon_i}$ and consider the aggregate of the resulting 2^n sets of operations.

4. From Lemma A we derive at once

LEMMA B. *If $p_1, \ldots, p_n$ are given integers, any set S of integers*

$$p_1^{e_1}p_2^{e_2}\ldots p_n^{e_n} \qquad (e\text{'s integers } \geqq 0) \tag{2}$$

contains a finite number of integers $F_1, \ldots, F_k$ such that every integer of the set S is a multiple of at least one F_i.*

5. Let $0<k<n$, $1<p'_i\leqq p_i$ $(i>k)$, and set

$$P=\prod_{i=1}^{n}\frac{\sigma(p_i^{e_i})}{p_i^{e_i}}=\prod_{i=1}^{n}\frac{p_i-1/p_i^{e_i}}{p_i-1},$$

$$P_0=\prod_{i=1}^{n}\frac{p_i}{p_i-1}, \quad P_k=\prod_{i=1}^{k}\frac{\sigma(p_i^{e_i})}{p_i^{e_i}}\cdot\prod_{i=k+1}^{n}\frac{p_i}{p_i-1},$$

$$P'_0=\prod_{i=1}^{n}\frac{p'_i}{p'_i-1}, \quad P'_k=\prod_{i=1}^{k}\frac{\sigma(p_i^{e_i})}{p_i^{e_i}}\cdot\prod_{i=k+1}^{n}\frac{p'_i}{p'_i-1}.$$

Let $p_1, \ldots, p_n$ be fixed distinct primes. By definition, an integer a, of the form (2), is deficient if $P<2$, non-deficient if $P\geqq 2$. Since

$$P'_m\geqq P_m>P \qquad (0\leqq m<n),$$

* For a case (like that of all non-deficient numbers built from $p_1, \ldots, p_n$) in which every multiple of any F_i is in the set S, we readily obtain the generators of the complementary set C of all integers (2) not in S. A number (2) is in C if and only if for each $i\leqq k$ there is some value of j such that $e_j<c_{ij}$ where

$$F_i=p_1^{c_{i1}}p_2^{c_{i2}}\ldots p_n^{c_{in}} \qquad (i=1, \ldots, k).$$

We may separate the numbers $1, \ldots, k$ into sets $I_1, \ldots, I_s$, where $s\leqq k$, and define corresponding distinct positive integers $j_1, \ldots, j_s$, each $\leqq n$, such that

$$e_{j_1}<c_{ij_1}\ (i \text{ in } I_1), \ldots, e_{j_s}<c_{ij_s}\ (i \text{ in } I_s),$$

while the remaining $n-s$ exponents e are subject to no restrictions.

a is deficient if $P_m \leqq 2$, also if $P'_m \leqq 2$. Since P_m is the limit of P for $e_i = \infty$ $(i = m+1, \ldots, n)$ and since $P < 2$ if a is deficient, we conclude that $P_m \leqq 2$ if a is deficient for all values of $e_{m+1}, \ldots, e_n$.

The case $P_m = 2$ occurs only when $a = 2^e$. For, if p is the greatest of the primes $p_{m+1}, \ldots, p_n$, no number in the denominator of P_m is divisible by p. Thus $P_m = 2$ implies that $p = 2$, $m = 0$, $n = 1$, $a = 2^e$.

LEMMA C. *If $P_m \leqq 2$ or $P'_m \leqq 2$,* (2) *is deficient. If an odd number* (2) *is deficient for all values of $e_{m+1}, \ldots, e_n$, then $P_m < 2$.*

6. We are now in a position to prove

LEMMA D. *All primitive non-deficient odd numbers having a given number n of distinct prime factors are formed from a finite number of sets of n primes.*

We consider numbers a of the form (2), where $p_1, \ldots, p_n$ are primes in ascending order of magnitude. If r is the positive real n-th root of 2, then $p_1 < r/(r-1)$. For,* if we take $p'_i = p_1$ $(i = 1, \ldots, n)$, we see that a is deficient if

$$P'_0 = \left(\frac{p_1}{p_1 - 1}\right)^n \leqq 2, \quad p_1 \geqq \frac{r}{r-1}.$$

To proceed by induction, assume that $p_1, \ldots, p_\nu$ $(\nu < n)$ is a particular one of a finite number of sets of ν distinct primes. Since a is to be a primitive non-deficient number, its divisor

$$\alpha = p_1^{e_1} p_2^{e_2} \ldots p_\nu^{e_\nu}$$

must be deficient. Noting that each divisor of a deficient α is deficient (or using the foot-note in § 4), we see that the deficient α's are the numbers in which certain exponents $e_{i_1}, \ldots, e_{i_k}$ are arbitrary, while each remaining exponent takes a limited number of values, and further numbers in which every exponent is limited. We consider one such type of deficient α's, thus treating one of a finite number of analogous cases. After permuting $p_1, \ldots, p_\nu$, we may assume that μ $(0 \leqq \mu \leqq \nu)$ is an integer such that $e_1, \ldots, e_\mu$ are limited, while e_i $(i = \mu+1, \ldots, \nu)$ takes all values. By Lemma C, the deficiency of these α's implies that

$$\Pi_\mu = \prod_{i=1}^{\mu} \frac{\sigma(p_i^{e_i})}{p_i^{e_i}} \cdot \prod_{i=\mu+1}^{\nu} \frac{p_i}{p_i - 1} < 2,$$

the second product being absent if $\mu = \nu$. Since there is a limited number of sets $e_1, \ldots, e_\mu$, each Π_μ is less than a constant $M < 2$. The use of P'_μ for

$$p'_i = p_i \ (i = \mu+1, \ldots, \nu), \qquad p'_i = p_{\nu+1} \ (i = \nu+1, \ldots, n),$$

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* A like simple proof by use of P_0' shows that also p_2 is limited (since $p_1 > 2$) and then that p_3 is limited, but fails for p_4 since

$$\frac{p_1}{p_1-1}\,\frac{p_2}{p_2-1}\,\frac{p_3}{p_3-1} > 2 \text{ if } p_1 = 3,\ p_2 = 5,\ p_3 \leqq 13.$$

shows that a is deficient if

$$\Pi_\mu \cdot \left(\frac{p_{\nu+1}}{p_{\nu+1}-1}\right)^{n-\nu} \leqq 2.$$

Hence a is deficient if

$$M\left(\frac{p_{\nu+1}}{p_{\nu+1}-1}\right)^{n-\nu} \leqq 2, \quad p_{\nu+1} \geqq \frac{g}{g-1}, \quad g = \sqrt[n-\nu]{2/M} > 1.$$

Hence in a non-deficient a, $p_{\nu+1}$ is less than the largest of the limits obtained in the various cases, finite in number.

7. Consider the set S of primitive non-deficient numbers having as distinct prime factors $p_1, \ldots, p_n$ a particular one of the finite (Lemma D) number of possible sets of n primes. Since any greater multiple of a non-deficient number is not primitive, the set S is finite by Lemma B. We have therefore proved the theorem of § 1.

8. THEOREM. *There is no odd abundant number with less than three distinct prime factors, and no odd perfect number with less than four distinct prime factors. The primitive odd abundant numbers with three distinct prime factors are:*

$$3^3 5 \cdot 7,\ 3^2 5^2 7,\ 3^2 5 \cdot 7^2,\ 3^3 5^2 11,\ 3^5 5^2 13,\ 3^4 5^3 13,\ 3^4 5^2 13^2,\ 3^3 5^3 13^2. \qquad (3)$$

We have $n > 2$ since $\frac{3}{2} \cdot \frac{5}{4} < 2$. Let $n = 3$. Since $\left(\frac{5}{4}\right)^3 < 2$, we have $p_1 = 3$. We have $p_2 = 5$ and then $p_3 < 17$ since

$$\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10} = \frac{77}{40} < 2, \quad \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{17}{16} = \frac{255}{128} < 2.$$

For $e_1 = 1$ or 2, we obtain a smaller limit for p_3 by use of $P_1 > 2$, viz., $p_3 < 6$, $p_3 < 11$, respectively. Hence $e_1 > 1$.

First, let $e_1 = 2$, so that $p_3 = 7$. Unless $e_2 = e_3 = 1$ (giving a deficient a), a is a multiple of (3_2) or (3_3).

Next, if $e_1 \geqq 3$ and $p_3 = 7$, a is a multiple of (3_1).

Finally, let $e_1 \geqq 3$, $p_3 = 11$ or 13. If $e_2 = 1$, a is deficient since

$$\frac{\sigma(5)}{5} \cdot \frac{3}{2} \cdot \frac{11}{10} = \frac{99}{50} < 2.$$

If $e_2 > 1$ and $p_3 = 11$, a is a multiple of (3_4). Let therefore $e_1 \geqq 3$, $e_2 > 1$, $p_3 = 13$. The conditions that $3^e 5^f 13$ be non-deficient for $f = 2$ and 3 are $3^e > 217$, $e > 4$; $3^e > 42$, $e > 3$, respectively; while $3^3 5^f 13$ is deficient since

$$\frac{\sigma(3^3)}{3^3} \cdot \frac{\sigma(13)}{13} \cdot \frac{5}{4} = \frac{700}{351} < 2.$$

Hence, for $e_3 = 1$, a is non-deficient if and only if it is a multiple of (3_5) or (3_6).

Next, $3^e 5^2 13^2$ and $3^3 5^f 13^2$ are non-deficient if and only if $3^e > 47$, $e > 3$; $5^f > 76$, $f > 2$, respectively. Hence for $e_3 = 2$, a is non-deficient if and only if it is a multiple of (3_7) or (3_8). For $e_3 \geqq 3$, a is a multiple of one of the latter numbers unless it be $3^3 5^2 13^e$, which is deficient.

9. Passing to the case $n = 4$, we have $p_1 = 3$, $p_2 = 5$ or 7 since

$$\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12} = \frac{1001}{576} < 2, \quad \frac{3}{2}\left(\frac{11}{10}\right)^3 = \frac{3993}{2000} < 2.$$

THEOREM. *The primitive non-deficient odd numbers with four distinct prime factors, of which the second is* 7, *are*

$3^2 7^2 11^2 13^3$, $3^2 7^2 11^3 13^2$, $3^2 7^3 11^2 13^2$, $3^3 7\cdot 11\cdot 13^2$, $3^3 7\cdot 11^2 13$, $3^3 7^2 11\cdot 13$, $3^4 7\cdot 11\cdot 13$; (4)
$3^3 7^2 11^2 17$, $3^3 7^3 11\cdot 17^2$, $3^4 7^2 11\cdot 17$, $3^5 7\cdot 11^2 17^3$, $3^5 7\cdot 11^3 17^2$, $3^6 7\cdot 11^2 17^2$; (5)
$3^3 7^2 11^3 19^2$, $3^3 7^3 11^3 19$, $3^3 7^3 11^2 19^2$, $3^4 7^2 11\cdot 19^2$, $3^4 7^2 11^2 19$, $3^4 7^3 11\cdot 19$, $3^5 7^2 11\cdot 19$; (6)
$3^4 7^3 11^2 23^2$, $3^4 7^4 11^3 23$, $3^5 7^2 11^2 23^2$, $3^5 7^2 11^4 23$, $3^5 7^3 11^2 23$, $3^6 7^2 11^2 23$; (7)
$3^4 7^2 13^3 17^3$, $3^4 7^3 13^2 17^2$, $3^5 7^2 13^2 17^2$, $3^5 7^3 13^2 17$, $3^6 7^2 13^3 17$, $3^6 7^3 13\cdot 17^2$, $3^7 7^2 13^2 17$; (8)
$3^6 7^4 13^3 19^3$, $3^7 7^3 13^4 19^4$, $3^7 7^3 13^5 19^3$, $3^7 7^4 13^3 19^2$, $3^8 7^3 13^3 19^3$, $3^8 7^4 13^2 19^3$, $3^{10} 7^3 13^4 19^2$. (9)

Henceforth we shall write a, b, c, d for $e_1, \ldots, e_4$ and p for p_4. Denote (2) by A. We have $p_3 < 17$. Consider first the case $p_3 = 11$. Then A is deficient if

$$\frac{3\cdot 3^a - 1}{2\cdot 3^a}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{p}{p-1} < 2, \quad 3^a(240 - 9p) \leqq 77p.$$

Hence, for a non-deficient A, $p < 27$ and $a \geqq 4$ if $p = 23$, $a \geqq 3$ if $p = 19$ or 17, $a \geqq 2$ if $p = 13$. Similarly, $b \geqq 2$ if $p \geqq 19$, $c \geqq 2$ if $p = 23$. First, let $p = 23$. Then A is a multiple of (7_6) unless $a \leqq 5$. For $a = 4$, we have $b > 2$; if $d > 1$, A is a multiple of (7_1); if $d = 1$, then $c > 2$, $b > 3$, and A is a multiple of (7_2). For $a = 5$, A is a multiple of (7_5) unless $b = 2$, in which case it is a multiple of (7_3) if $d > 1$; while, if $d = 1$, then $c \geqq 4$ and A is a multiple of (7_4).

Next, let $p = 19$. Then A is a multiple of (6_7) unless $a = 3$ or 4. First, let $a = 3$. Then $c > 1$. For $b = 2$, we have $c > 2$, $d > 1$, whence A is a multiple of (6_1). For $b \geqq 3$, A is a multiple of (6_3) if $d > 1$; while, if $d = 1$, we have $c \geqq 3$ and A is a multiple of (6_2). Next, let $a = 4$. Then A is a multiple of (6_4), (6_5) or (6_6) unless $b = 2$, $c = d = 1$, when A is deficient.

For $p = 17$, $a \geqq 3$. First, let $b = 1$. Then $c > 1$, $d > 1$, $a \geqq 5$, and A is a multiple of (5_6) unless $a = 5$. If $a = 5$, A is a multiple of (5_4) or (5_5) unless $c = d = 2$, in which case it is deficient. Next, if $b \geqq 2$, $a \geqq 4$, A is a multiple of (5_3). Finally, if $b \geqq 2$, $a = 3$, A is a multiple of (5_1) or (5_2) unless $c = d = 1$, or $c = 1$, $d > 1$, $b = 2$, in each of which two cases A is deficient.

The case $p = 13$ needs no special comment.

It remains to treat the coordinate case $p_3 = 13$. By P_1,

$$3^a(288 - 15p) > 91p,$$

whence $p = 19$, $a \geqq 6$, or $p = 17$, $a \geqq 4$. Similarly, $b \geqq 3$, $c \geqq 2$, $d \geqq 2$ if $p = 19$; $b \geqq 2$ if $p = 17$. For $p = 19$, A is a multiple of (9_1) unless $b = 3$ or $c = 2$ or $d = 2$. First, let $c = 2$. Then A is deficient if $a \leqq 7$ or $b = 3$ or $d = 2$; the remaining A's are multiples of (9_6). Next, let $d = 2$ $(c > 2)$. Then A is deficient if $a = 6$. Thus, if $b \geqq 4$, A is a multiple of (9_4). But if $b = 3$, A is deficient if $c = 3$ or $a \leqq 9$, while the remaining A's are multiples of (9_7). Finally, let $b = 3$ $(c > 2,\ d > 2)$. Now A is deficient if $a = 6$, and a multiple of (9_5) unless $a = 6$ or 7. For $a = 7$, we must have $c \geqq 4$. Then A is a multiple of (9_2) if $d \geqq 4$, of (9_3) if $d = 3$, unless also $c < 5$, when it is deficient.

Finally, let $p = 17$. Unless A is a multiple of (8_1), $c < 3$ or $d < 3$. First, let $c = 2$. Then A is a multiple of (8_2) or (8_3) unless $d = 1$ or $b = 2$, $a = 4$. In the latter case A is deficient. For $d = 1$, A is a multiple of (8_7) or (8_4) unless $b = 2$, $a < 7$ or $a = 4$, while A is then deficient. Second, let $c = 1$; then $a > 5$, $b > 2$, $d > 1$, and A is a multiple of (8_6). Third, let $d = 2$, $c \geqq 3$. Then A is a multiple (8_2) or (8_3) unless $b = 2$, $a = 4$, in which case A is deficient. Finally, let $d = 1$, $c \geqq 3$. If $a = 4$ or if $a = 5$, $b = 2$, A is deficient. If $a = 5$, $b \geqq 3$, A is a multiple of (8_4). If $a = 6$ or $a \geqq 7$, A is a multiple of (8_5) or (8_7), respectively.

10. In treating the more prolific case $p_2 = 5$, use is made of the

THEOREM. *If the prime p is not a divisor of the deficient number k, then kp^e is non-deficient if and only if*

$$\varkappa \equiv \frac{\sigma(k)}{2k - \sigma(k)} \geqq \frac{p^e}{\sigma(p^{e-1})}. \tag{10}$$

Since $2k = \sigma(k)\ (1 + 1/\varkappa)$, the condition is

$$\sigma(k)\,\sigma(p^e) \geqq 2kp^e, \quad \sigma(p^e) \geqq p^e(1 + 1/\varkappa).$$

But $\sigma(p^e) = p^e + \sigma(p^{e-1})$. Hence the inequality (10) follows.

COROLLARY. *kp (and hence also kp^e) is non-deficient if $p \leqq \varkappa$, while kp^e is deficient if $p \geqq \varkappa + 1$,*

To obtain the latter part of the corollary, we note that

$$\frac{p^e}{\sigma(p^{e-1})} = p - 1 + \frac{1}{\sigma(p^{e-1})}.$$

To give another proof, note that kp^e is deficient if

$$P_1 = \frac{\sigma(k)}{k} \cdot \frac{p}{p-1} \leqq 2, \quad p \geqq \frac{2k}{2k - \sigma(k)} = \varkappa + 1.$$

The corollary gives complete information concerning $k p^e$ unless there be a prime K between $\varkappa$ and $\varkappa + 1$. In the latter case, kK is deficient, while $kK^e (e > 1)$ is to be tested by the theorem itself.

11. For $n = 4$, there remains the case $p_1 = 3$, $p_2 = 5$. Since

$$\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{31}{30} \cdot \frac{37}{36} = \frac{1147}{576} < 2,$$

$p_3 \leqq 29$. First, let $p_3 = 7$. Then A is a multiple of (3_1), (3_2) or (3_3) unless $a = 1$ or $a = 2$, $b = c = 1$. In the latter case, A is abundant if $p \leqq \varkappa = 104$, deficient if $p > 104$. Let next $a = 1$. Then A is deficient if $p > 36$. For

$$k = 3 \cdot 5^2 7^2, \quad 3 \cdot 5 \cdot 7^2, \quad 3 \cdot 5^2 7, \quad 3 \cdot 5 \cdot 7,$$

the integer just $> \varkappa$ is 26, 14, 18, 11, respectively. For $k = 3 \cdot 5 \cdot 7$, we test $p = 11$ by the theorem and find that (10) holds for $e = 2$, since $\varkappa = 32/3$. Again, A is deficient if $b = 1$, $p \geqq 17$; $b = 2$, $c = 1$, $p \geqq 19$; $c = 1$, $p > 21$; $b = 2$, $c = 3$, $p = 29$; $b = 2$, $p = 31$; $b = 3$, $c = 2$, $d = 1$, $p = 31$. Hence all the primitives with $p_3 = 7$ are given in the first part of the following list. As in this case, we have rearranged the primitives, listing together those with the same set of primes. Two or more complete sets of primitives differing only in the value of the fourth prime are combined in the list. Although this rearranged list is somewhat longer than the list as initially determined, it has obvious advantages over the latter.

12. List of the primitive abundant numbers with exactly four distinct prime factors, the second prime being 5.

$3\cdot5\cdot7\cdot11^2$, $3\cdot5\cdot7^2 11$, $3\cdot5^2 7\cdot11$, $3^2 5\cdot7\cdot11$; $3\cdot5\cdot7^2 13$, $3\cdot5^2 7\cdot13$, $3^2 5\cdot7\cdot13$;
$3\cdot5^2 7\cdot17$, $3^2 5\cdot7\cdot17$; $3\cdot5^2 7^2 19$, $3\cdot5^3 7\cdot19$, $3^2 5\cdot7\cdot19$; $3\cdot5^2 7^2 23$, $3^2 5\cdot7\cdot23$;
$3\cdot5^2 7^4 29$, $3\cdot5^3 7^2 29$, $3^2 5\cdot7\cdot29$; $3\cdot5^3 7^2 31^2$, $3\cdot5^3 7^3 31$, $3^2 5\cdot7\cdot31$;
$3^2 5\cdot7p\,(37 \leqq p \leqq 103)$; $3^2 5\cdot11\cdot13$; $3^2 5\cdot11\cdot17$;
$3^2 5\cdot11^2 19^2$, $3^2 5\cdot11^3 19$, $3^2 5^2 11\cdot19$, $3^3 5\cdot11\cdot19$;
$3^2 5^2 11p$, $3^3 5\cdot11p\,(p = 23, 29, 31)$;
$3^2 5^2 11p$, $3^3 5\cdot11^2 p$, $3^4 5\cdot11p\,(p = 37, 41)$;
$3^2 5^2 11\cdot43^2$, $3^2 5^2 11^2 43$, $3^2 5^3 11\cdot43$, $3^3 5\cdot11^2 43^2$, $3^3 5\cdot11^3 43$, $3^4 5\cdot11\cdot43$;
$3^2 5^2 11^2 47$, $3^2 5^3 11\cdot47$, $3^4 5\cdot11^2 47$, $3^5 5\cdot11\cdot47$;
$3^2 5^2 11^2 53$, $3^2 5^3 11\cdot53$, $3^4 5\cdot11^2 53$, $3^6 5\cdot11\cdot53^2$, $3^7 5\cdot11\cdot53$;
$3^2 5^2 11^2 59$, $3^2 5^3 11\cdot59^2$, $3^2 5^3 11^2 59$, $3^2 5^4 11\cdot59$, $3^4 5\cdot11^2 59$;
$3^2 5^2 11^2 61$, $3^2 5^3 11^2 61$, $3^2 5^4 11\cdot61$, $3^4 5\cdot11^2 61$;
$3^2 5^2 11^4 67^2$, $3^2 5^3 11^2 67$, $3^4 5\cdot11^2 67^2$, $3^4 5\cdot11^3 67$, $3^5 5\cdot11^2 67$;
$3^2 5^3 11^2 71$, $3^4 5\cdot11^4 71^2$, $3^5 5\cdot11^2 71$; $3^2 5^3 11^2 p$, $3^5 5.11^2 p\,(p = 73, 79)$;
$3^2 5^3 11^2 83$, $3^5 5\cdot11^3 83$, $3^6 5\cdot11^2 83$; $3^2 5^3 11^2 89$, $3^6 5\cdot11^2 89^2$, $3^6 5\cdot11^3 89$, $3^7 5\cdot11^2 89$;

$3^25^311^297$, $3^75\cdot11^397^2$, $3^75\cdot11^497$, $3^85\cdot11^397$; $3^25^311^2p\,(p=101, 103)$;
$3^25^311^2107^2$, $3^25^411^2107$; $3^25^311^3p$, $3^25^411^2p\,(p=109, 113)$;
$3^25^411^3127$, $3^25^511^2127$; $3^25^411^3131$; $3^25^411^4137^2$, $3^25^511^3137$; $3^25^511^3139$;
3^25^213p, $3^35\cdot13p\,(p=17, 19)$; $3^25^213\cdot23$, $3^35\cdot13\cdot23^2$, $3^35\cdot13^223$, $3^45\cdot13\cdot23$;
$3^25^213^229$, $3^25^313\cdot29$, $3^35^213\cdot29$, $3^45\cdot13^229$, $3^55\cdot13\cdot29$;
$3^25^213^231$, $3^25^313\cdot31$, $3^35^213\cdot31$, $3^45\cdot13^231$, $3^55\cdot13\cdot31^2$, $3^65\cdot13\cdot31$;
$3^25^313^237$, $3^35^213\cdot37$, $3^55\cdot13^237^2$, $3^65\cdot13^237$; $3^25^313^241^2$, $3^25^313^341$, $3^35^213\cdot41$;
$3^25^413^243^2$, $3^25^413^343$, $3^25^513^243$, $3^35^213\cdot43$;
$3^35^213p\,(47\leqq p\leqq 89)$; $3^35^213^2p$, 3^35^313p, $3^45^213p\,(97\leqq p\leqq 181)$;
$3^35^213^3p$, 3^35^313p, $3^45^213p\,(191\leqq p\leqq 199)$; 3^35^313p, $3^45^213p\,(p=211, 223)$;
3^35^413p, $3^45^213p\,(227\leqq p\leqq 313)$; 3^35^513p, $3^45^213p\,(p=317, 331, 337)$;
$3^35^613\cdot347$, $3^45^213\cdot347$; $3^35^613\cdot349^2$, $3^35^713\cdot349$, $3^45^213\cdot349$;
$3^45^213p\,(353\leqq p\leqq 383)$;
$3^25^217\cdot19^2$, $3^25^217^219$, $3^25^317\cdot19$, $3^35^217\cdot19$, $3^45\cdot17^219$, $3^45\cdot17\cdot19^2$, $3^55\cdot17\cdot19$;
$3^25^317^223^2$, $3^25^417^223$, $3^35^217\cdot23$, $3^35^217p\,(p=29, 31)$;
$3^35^217^237$, $3^35^317\cdot37$, $3^45^217\cdot37$; $3^35^217^241^2$, $3^35^317\cdot41$, $3^45^217\cdot41$;
$3^35^317\cdot43$, $3^45^217\cdot43$; $3^35^317\cdot47^2$, $3^35^317^247$, $3^35^417\cdot47$, $3^45^217\cdot47$;
$3^35^317^253$, $3^45^217^253$, $3^45^317\cdot53$, $3^55^217\cdot53$;
$3^35^417^259^2$, $3^35^417^359$, $3^35^517^259$, $3^45^217^259$, $3^45^317\cdot59$, $3^55^217\cdot59$;
$3^45^217^361$, $3^45^217^261^2$, $3^45^317\cdot61$, $3^55^217^261$, $3^65^217\cdot61$;
3^45^317p, $3^55^217^2p\,(p=67, 71, 73)$;
$3^45^317^279$, $3^45^417.79$, $3^55^317\cdot79$, $3^65^217^279^2$, $3^65^217^379$, $3^75^217^279$;
$3^45^317^283$, $3^45^417\cdot83$, $3^55^317\cdot83$, $3^75^217^383^2$, $3^95^217^383$;
$3^45^317^289$, $3^65^317\cdot89$; $3^45^317^297$, $3^55^317\cdot97^2$, $3^55^417.97$, $3^65^317\cdot97$;
$3^45^317^2101$, $3^55^417\cdot101$, $3^65^317\cdot101$;
$3^45^317^3103$, $3^45^417^2103$, $3^55^317^2103$, $3^55^417\cdot103$, $3^65^317\cdot103$;
$3^45^417^2107$, $3^55^317^2107$, $3^55^417\cdot107$, $3^75^317\cdot107$;
$3^45^417^2109$, $3^55^317^2109$, $3^55^417\cdot109$, $3^75^317\cdot109^2$, $3^85^317.109$;
$3^45^417^2113$, $3^55^317^2113$, $3^55^517\cdot113^2$, $3^55^617\cdot113$, $3^65^417\cdot113$;
$3^55^317^2127$, $3^65^517\cdot127^2$, $3^75^517\cdot127$, $3^85^417\cdot127$; $3^55^317^2131$, $3^75^517\cdot131$;
$3^55^317^2p\,(p=137, 139)$; $3^45^417^2p$, $3^65^317^2p\,(p=149, 151, 157)$;
$3^45^417^2163$, $3^65^317^3163$, $3^75^317^2163$; $3^55^417^2167$, $3^65^317^3167^2$, $3^75^317^2167$;
$3^55^417^3173$, $3^55^517^2173$, $3^65^417^2173$, $3^75^317^3173$, $3^85^317^2173^2$, $3^95^317^2173$;
$3^55^517^2179$, $3^65^417^2179$, $3^85^317^3179^2$, $3^85^317^4179$, $3^95^317^3179$;
$3^55^517^3181$, $3^55^617^2181$, $3^65^417^2181$, $3^95^317^4181^2$, $3^{10}5^317^3181^2$;
$3^65^417^2p\,(p=191, 193, 197, 199)$; $3^65^417^3211$, $3^65^517^2211$, $3^75^417^2211$;
$3^65^517^3223$, $3^75^417^3223$, $3^75^517^2223$, $3^85^417^2223^2$, $3^95^417^2223$;
$3^65^617^3227^2$, $3^65^617^4227$, $3^65^717^3227$, $3^75^517^2227$, $3^75^417^3227^2$, $3^85^417^3227$;
$3^65^717^4229^2$, $3^75^517^2229$, $3^85^417^3229$;

$3^7 5^5 17^3 233$, $3^7 5^6 17^2 233^2$, $3^8 5^4 17^3 233^2$, $3^8 5^5 17^2 233$, $3^9 5^4 17^4 233$;
$3^7 5^5 17^3 239$, $3^8 5^6 17^2 239^2$, $3^8 5^7 17^2 239$, $3^9 5^6 17^2 239$, $3^{10} 5^5 17^2 239^2$;
$3^7 5^5 17^3 241^2$, $3^7 5^5 17^4 241$, $3^7 5^6 17^3 241$, $3^8 5^5 17^3 241$, $3^9 5^6 17^2 241^2$, $3^9 5^7 17^2 241$, $3^{10} 5^6 17^2 241$;
$3^8 5^6 17^3 251^2$, $3^8 5^7 17^4 251$, $3^9 5^6 17^3 251$, $3^{10} 5^5 17^4 251^2$, $3^{11} 5^5 17^3 251^2$;
$3^3 5^2 19 p\,(p = 23, 29)$; $3^3 5^2 19^2 31$, $3^3 5^3 19 \cdot 31$, $3^4 5^2 19 \cdot 31$;
$3^3 5^3 19 \cdot 37^2$, $3^3 5^3 19^2 37$, $3^3 5^4 19 \cdot 37$, $3^4 5^2 19 \cdot 37$;
$3^3 5^3 19^2 41^2$, $3^3 5^4 19^2 41$, $3^4 5^2 19^2 41$, $3^4 5^3 19 \cdot 41$, $3^5 5^2 19 \cdot 41$;
$3^3 5^4 19^2 43^2$, $3^3 5^5 19^3 43$, $3^4 5^2 19^2 43$, $3^4 5^3 19 \cdot 43$, $3^5 5^2 19 \cdot 43$;
$3^4 5^3 19 \cdot 47$, $3^5 5^2 19^2 47$, $3^7 5^2 19 \cdot 47^2$;
$3^4 5^3 19 \cdot 53^2$, $3^4 5^3 19^2 53$, $3^5 5^3 19 \cdot 53$, $3^7 5^2 19^2 53^2$, $3^9 5^2 19^2 53$;
$3^4 5^3 19^2 59$, $3^5 5^3 19 \cdot 59$; $3^4 5^3 19^2 61^2$, $3^4 5^3 19^3 61$, $3^5 5^3 19 \cdot 61$;
$3^4 5^4 19^3 67^2$, $3^4 5^5 19^2 67$, $3^5 5^3 19^2 67$, $3^5 5^4 19 \cdot 67^2$, $3^5 5^5 19 \cdot 67$, $3^6 5^4 19 \cdot 67$, $3^7 5^3 19 \cdot 67^2$;
$3^5 5^3 19^2 71$, $3^6 5^5 19 \cdot 71$, $3^7 5^4 19 \cdot 71$; $3^5 5^3 19^2 73$, $3^6 5^6 19 \cdot 73^2$, $3^7 5^5 19 \cdot 73$, $3^9 5^4 19 \cdot 73$;
$3^5 5^4 19^2 79$, $3^6 5^3 19^2 79^2$, $3^6 5^3 19^3 79$, $3^7 5^3 19^2 79$;
$3^5 5^5 19^2 83^2$, $3^6 5^4 19^2 83$, $3^9 5^3 19^3 83^2$; $3^6 5^4 19^3 89^2$, $3^6 5^5 19^2 89$, $3^7 5^4 19^2 89$;
$3^3 5^3 23^2 29^2$, $3^3 5^4 23 \cdot 29^2$, $3^3 5^4 23^2 29$, $3^4 5^2 23 \cdot 29^2$, $3^4 5^2 23^2 29$, $3^4 5^3 23 \cdot 29$;
$3^3 5^5 23^2 31^2$, $3^4 5^2 23^2 31^2$, $3^4 5^3 23 \cdot 31$, $3^5 5^2 23 \cdot 31$; $3^4 5^3 23^2 37$, $3^4 5^4 23 \cdot 37$, $3^5 5^3 23 \cdot 37$;
$3^4 5^4 23^2 41^2$, $3^5 5^3 23^2 41$, $3^5 5^4 23 \cdot 41$, $3^6 5^3 23 \cdot 41$;
$3^5 5^3 23^2 43^2$, $3^5 5^4 23^2 43$, $3^5 5^5 23 \cdot 43^2$, $3^6 5^3 23^2 43$, $3^6 5^4 23 \cdot 43$; $3^5 5^6 23^3 47^2$, $3^6 5^4 23^2 47$;
$3^5 5^3 29^2 31^2$, $3^5 5^4 29 \cdot 31^2$, $3^5 5^4 29^2 31$, $3^6 5^3 29 \cdot 31^2$, $3^6 5^3 29^2 31$, $3^6 5^4 29 \cdot 31$.

13. THEOREM. *The only primitive odd abundant numbers* < 15000 *are:*
$3^3 5 \cdot 7 = 945$, $3^2 5^2 7 = 1575$, $3^2 5 \cdot 7^2 = 2205$, $3^2 5 \cdot 7 \cdot 11 = 3465$,
$3^2 5 \cdot 7 \cdot 13 = 4095$, $3^2 5 \cdot 7 \cdot 17 = 5355$, $3 \cdot 5^2 7 \cdot 11 = 5775$, $3^2 5 \cdot 7 \cdot 19 = 5985$,
$3^2 5 \cdot 11 \cdot 13 = 6435$, $3 \cdot 5^2 7 \cdot 13 = 6825$, $3^2 5 \cdot 7 \cdot 23 = 7245$, $3^3 5^2 11 = 7425$,
$3 \cdot 5 \cdot 7^2 11 = 8085$, $3^2 5 \cdot 11 \cdot 17 = 8415$, $3 \cdot 5^2 7 \cdot 17 = 8925$, $3^2 5 \cdot 7 \cdot 29 = 9135$,
$3 \cdot 5 \cdot 7^2 13 = 9555$, $3^2 5 \cdot 7 \cdot 31 = 9765$, $3^2 5 \cdot 7 \cdot 37 = 11655$, $3 \cdot 5 \cdot 7 \cdot 11^2 = 12705$,
$3^2 5 \cdot 7 \cdot 41 = 12915$, $3^2 5 \cdot 7 \cdot 43 = 13545$, $3^2 5 \cdot 7 \cdot 47 = 14805$.

The number n of distinct prime factors is < 5, since

$$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 15015.$$

For $n < 4$, the primitives are given by (3). Since $3^4 5^2 13 = 26325$, the last four exceed 78000. Finally, let $n = 4$. The primitives (§ 9) with $p_2 = 7$ evidently exceed $3^2 7^2 11 \cdot 13 = 63063$. Next, let $p_2 = 5$. The primitives (§ 11) with $p_3 = 7$, which are < 15000, are readily listed. The primitives with $p_3 \geqq 11$ have the factor 3^2 since $3 \cdot 5^b p_3^c p^d$ is deficient. Since $3^2 5 \cdot 11 \cdot 13 = 6435$, whose product by 3 exceeds our limit, it remains to examine only $3^2 5\, p_3 p_4$. The latter is deficient if $p_3 > 11$, and if $p_3 = 11$, $p_4 > 17$.