MSE, Bias^2, and Variance of the ML-Estimator of the Standard Deviation

Setup:

- Assumptions: iid random sample $(X_1, \dots, X_n)$ with $X_i \sim N(\mu, \sigma^2)$ for all $i = 1, \dots, n$.
- Fix values for the mean μ, the variance σ^2, and the sample size n.
  For instance: $\mu = 3$, $\sigma = 1.5$, and $n = \{2, 4, 6, \dots, 30\}$.

Monte-Carlo Algorithm:

1. Simulate a realization from the iid random sample $(X_1, \dots, X_n)$.
2. Compute $s_n = \sqrt{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2}$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$

Repeat Steps 1-2 a large number of times, e.g., $R = 10,000$ times and save all estimates $s_{n,1}, \dots, s_{n,R}$. Then approximate the Mean Squared Error (MSE), the squared bias (Bias^2), and the variance (Var) by

$$\begin{aligned}
\mathrm{MSE}(s_n) &\approx \frac{1}{R}\sum_{r=1}^R (s_{n,r} - \sigma)^2 \\
\mathrm{Bias}^2(s_n) &\approx \left(\left(\frac{1}{R}\sum_{r=1}^R s_{n,r}\right) - \sigma\right)^2 \\
\mathrm{Var}(s_n) &\approx \frac{1}{R}\sum_{r=1}^R \left(s_{n,r} - \left(\frac{1}{R}\sum_{r=1}^R s_{n,r}\right)\right)^2
\end{aligned}$$

where $E(s_{n,r}) \approx R^{-1}\sum_{r=1}^R s_{n,r}$. (The *Law of Large Numbers* implies that the approximations become arbitrarily precise for $R \to \infty$.)

Presentation of the Simulation-Results:

- Plot your results (y-axis: MSE, Bias^2, and Var; x-axis: n)
- Add the corresponding results for the unbiased estimator $\tilde{s}_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2}$.