Powerfunction Simulation for the z-Test

Setup:

- Assumptions: Gaussian iid random sample $(X_1, \ldots, X_n)$, where $X_i \sim N(\mu, \sigma^2)$ for all $i = 1, \ldots, n$.
- Test H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$.
- Fix values for the significance level α, the mean μ, the variance σ^2, and the sample size n. For instance: $\alpha = 0.05$, $\mu \in \{-2, -1.5, \ldots, -0.5, 0, 0.5, \ldots, 1.5, , 2\}$, $\sigma = 1$, $\mu_0 = 0$, and $n = \{10, 15, 20\}$.

Monte-Carlo Algorithm:

1. Simulate a realization from the iid random sample $(X_1, \ldots, X_n)$.
2. Compute $Z_{obs} = \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}$, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$
3. Save test-decision:

Criterion	Decision	Save Value
$\lvert Z_{obs}\rvert > q_{1-\alpha/2}$	Rejection of H_0	$D = 1$
$\lvert Z_{obs}\rvert < q_{1-\alpha/2}$	No rejection of H_0	$D = 0$

Repeat Steps 1-3 a large number of times, e.g., $R = 10,000$ times and save all binary test-decisions $D_1, \ldots, D_R$. Then

$$\beta_{n,\alpha}(\mu) \approx \bar{D}_R := \frac{1}{R} \sum_{r=1}^{R} D_r,$$

where the approximation error $|\beta_{n,\alpha}(\mu) - \bar{D}_R|$ can be made arbitrarily small as $R \to \infty$.

Theoretical Justification:

- $D := 1_{(|Z_{obs}| > q_{1-\alpha/2})} \in \{0, 1\}$ is a Bernoulli random variable with probabilities $P(D = 1) = p$ and $P(D = 0) = 1 - p$.
- Note that p depends on α, n, σ^2, μ, and μ_0.
- By the Weak Law of Large Numbers[1] we have that

$$\bar{D}_R \to_P E(D) = p \quad \text{as} \quad R \to \infty,$$

  where $p = \beta_{n,\alpha}(\mu)$.
- That is, for instance under $\mathrm{H}_0 : \mu = \mu_0$, we have that $p = \alpha$.

[1] See, for instance, www.statlect.com/asymptotic-theory/law-of-large-numbers