# Exercises (with Solutions) · Chapter 3

1. **Problem**

Calculate the following sums and products (as far as they are defined):

$$\boldsymbol{A} = 3 \cdot \begin{bmatrix} 1 & -2 \\ 5 & 7 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -3 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 6 & 7 & -3 \\ -2 & -1 & 4 \end{bmatrix} \qquad \boldsymbol{D} = 6 + \begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$$

$$\boldsymbol{E} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 9 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -9 & 7 \end{bmatrix} \begin{bmatrix} -3 \\ -9 \end{bmatrix} \qquad \boldsymbol{H} = \begin{bmatrix} 6 & 3 & -3 \\ 4 & 1 & 2 \end{bmatrix}' \begin{bmatrix} 1 & -8 & -4 \\ 7 & 5 & 2 \end{bmatrix}$$

**Solution**

It holds that

$$\begin{aligned}
\boldsymbol{A} &= 3 \cdot \begin{bmatrix} 1 & -2 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 15 & 21 \end{bmatrix} \\
\boldsymbol{B} &= \begin{bmatrix} 3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \end{bmatrix} \\
\boldsymbol{C} &= \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 6 & 7 & -3 \\ -2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 20 & -5 \\ 12 & 14 & -6 \end{bmatrix} \\
\boldsymbol{F} &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 27 \\ 7 & 63 \end{bmatrix} \\
\boldsymbol{G} &= \begin{bmatrix} -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -9 & 7 \end{bmatrix} \begin{bmatrix} -3 \\ -9 \end{bmatrix} = \begin{bmatrix} -30 & -6 \end{bmatrix} \begin{bmatrix} -3 \\ -9 \end{bmatrix} = 144 \\
\boldsymbol{H} &= \begin{bmatrix} 6 & 3 & -3 \\ 4 & 1 & 2 \end{bmatrix}' \begin{bmatrix} 1 & -8 & -4 \\ 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -8 & -4 \\ 7 & 5 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 34 & -28 & -16 \\ 10 & -19 & -10 \\ 11 & 34 & 16 \end{bmatrix}
\end{aligned}$$

The matrices $\boldsymbol{D}$ and $\boldsymbol{E}$ are not defined.

2. **Problem**

   (a) Determine the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 5 & 0 & -1 & -6 \\ 4 & 0 & 2 & -2 \end{bmatrix}$$

(b) Determine the rank of the following matrix:

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

(c) Compute the inverse of the following matrix:

$$C = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

**Solution**

**Crash Course on Ranks:**

- The rank of a matrix can be defined as the maximal number of linearly independent columns of the matrix.
- If $\boldsymbol{A}$ is a $(n \times m)$ dimensional matrix, then $\operatorname{rank}(\boldsymbol{A}) \leq \min\{m, n\}$.
- The column rank and the row rank are always equal. So, it is sufficient to check either the columns or the rows.
- A column vector is said to be linearly dependent if it can be written as a linear combination of the other column vectors.
- A set of column vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be linearly independent if the vector equation
$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = 0$$
has only the trivial solution $c_1 = \cdots = c_p = 0$. Otherwise, the set of column vectors is linearly dependent.
- A $(n \times n)$ matrix is invertible if it has full rank $n$.

(a) The $(3 \times 4)$ Matrix $A$ has rank 2, i.e., there are two linear dependencies:
  - A first linear dependency is due to the zero vector in the second column of $A$. The vector equation
  $$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = 0$$
  is here fulfilled, for instance, by $c_1 = c_3 = c_4 = 0$ and any $c_2 \neq 0$.
  - A second linear dependency is due to the fact that the first column of $A$ multiplied by $-1$ plus the third column of $A$ equals the fourth column of $A$. I.e., the vector equation
  $$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = 0$$
  is here fulfilled by $c_1 = -1$, $c_2 = 0$, $c_3 = 1$, $c_4 = -1$.

(b) The $(2 \times 2)$ Matrix $B$ has rank 2 since both column vectors are linearly independent.

Side-Note: One can compute the rank of a matrix using R as following:

```
> library("Matrix")
> A <- rbind(c(1, 0, 3, 2),
+            c(5, 0,-1,-6),
+            c(4, 0, 2,-2))
> B <- rbind(c(1, -1),
+            c(-1, 0))
> rankMatrix(A)[1]
[1] 2
> rankMatrix(B)[1]
```

```
[1] 2
```

(c) Generally, the inverse of a $(2 \times 2)$ matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where $ad - bc$ is the determinant of the matrix, which has to be non-zero, otherwise the matrix is not invertible.

The inverse of $C$ is given by

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \cdot 6 - 7 \cdot 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

Remember it must be true that: $CC^{-1} = I$.
Side-Note: One can compute the inverse of a matrix using R as following:

```
> C <- rbind(c(4, 7),
+            c(2, 6))
> solve(C)
     [,1] [,2]
[1,]  0.6 -0.7
[2,] -0.2  0.4
```

3. **Problem**

Show that

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

for any two $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ that have full rank (i.e. $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{B}) = n$).

**Solution**

**Remember:** A $(n \times n)$ matrix $\boldsymbol{D}^{-1}$ is called the inverse of a $(n \times n)$ matrix $\boldsymbol{D}$, if

$$\boldsymbol{D}^{-1}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{D}^{-1} = \boldsymbol{I}_n.$$

We are asked whether the inverse of $\boldsymbol{AB}$, i.e. $(\boldsymbol{AB})^{-1}$, equals $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. So, we have to check the following two statements:

**1.** $(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{AB}) = \boldsymbol{I}_n$

**2.** $(\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{I}_n$

**To 1.**

$$(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{AB}) = \boldsymbol{B}^{-1}\underbrace{\boldsymbol{A}^{-1}\boldsymbol{A}}_{=\boldsymbol{I}_n}\boldsymbol{B} = \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}_n$$

**To 2.**

$$(\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{A}\underbrace{\boldsymbol{B}\boldsymbol{B}^{-1}}_{=\boldsymbol{I}_n}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$$

So, $(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})$ is indeed the inverse of $(\boldsymbol{AB})$, i.e., $(\boldsymbol{AB})^{-1} = (\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})$.

4. **Problem**

Let $A$ and $B$ be $(n \times n)$ matrices each with full rank and thus invertible. Assume that $(B + A)$ is also invertible. Calculate

a) $(AB)'(B^{-1}A^{-1})'$
b) $(A(A^{-1}+B^{-1})B)(B+A)^{-1}$

**Solution**

a) $(AB)'(B^{-1}A^{-1})' = B'A'(A^{-1})'(B^{-1})' = B'\underbrace{A'A'^{-1}}_{=I_n}B'^{-1}$

$= \underbrace{B'B'^{-1}}_{=I_n} = I_n$

b) $(A(A^{-1}+B^{-1})B)(B+A)^{-1} = ((\underbrace{AA^{-1}}_{=I_n}+AB^{-1})B)(B+A)^{-1} = (B+A\underbrace{B^{-1}B}_{=I_n})(B+A)^{-1} = (B+A)(B+A)^{-1} = I_n$

5. **Problem**

Consider the matrix

$$X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}',$$

where $X_1, X_2, \ldots, X_n \in \mathbb{R}^K$ are column vectors. Show that

$$\sum_{i=1}^n X_i X_i' = X'X\,.$$

**Solution**

First, let us check how $X$ looks like and what dimension this matrix has:

$$\begin{aligned} X &= \begin{bmatrix} \underset{(K\times1)}{X_1} & \underset{(K\times1)}{X_2} & \cdots & \underset{(K\times1)}{X_n} \end{bmatrix}' \\ &= \begin{bmatrix} \underset{(1\times K)}{X_1'} \\ \underset{(1\times K)}{X_2'} \\ \vdots \\ \underset{(1\times K)}{X_n'} \end{bmatrix} = \begin{bmatrix} X_{11} & \cdots & X_{1K} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nK} \end{bmatrix} \end{aligned}$$

OK, $X$ is a $(n \times K)$ matrix.

Now, we can show that $X'X = \sum_{i=1}^n X_i' X_i$:

$$\begin{aligned} \underset{(K\times K)}{X'X} &= \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}' \\ &= \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \\ \cdot \\ \cdot \\ \cdot \\ X_n' \end{bmatrix} \\ &= \underset{(K\times K)}{X_1X_1'} + \underset{(K\times K)}{X_2X_2'} + \cdots + \underset{(K\times K)}{X_nX_n'} \\ &= \sum_{i=1}^n X_i X_i' \end{aligned}$$

6. **Problem**

   (a) Show the following useful result:

   $$\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})=\sum_{i=1}^{n}(x_i-\overline{x})y_i$$

   (b) Derive the OLS estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ of a simple linear regression model by minimizing the sum of squared residuals, $S_n(b_0,b_1)$, for a given sample (i.e., for given data) $((Y_1,X_1),\dots,(Y_n,X_n))$, where

   $$\begin{aligned}S_n(b_0,b_1)&=\sum_{i=1}^{n}\hat{\varepsilon}_i^2\\&=\sum_{i=1}^{n}(Y_i-\hat{Y}_i)^2\\&=\sum_{i=1}^{n}(Y_i-b_0-b_1X_i)^2\\&=\sum_{i=1}^{n}(Y_i^2-2b_0Y_i-2b_1Y_iX_i+b_0^2+2b_0b_1X_i+b_1^2X_i^2).\end{aligned}$$

**Solution**

(a)

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) &= \sum_{i=1}^{n}(x_iy_i-x_i\overline{y}-\overline{x}y_i+\overline{xy})\\&= \sum_{i=1}^{n}x_iy_i-\overline{y}\sum_{i=1}^{n}x_i-\overline{x}\sum_{i=1}^{n}y_i+n\overline{x}\,\overline{y}\\&= \sum_{i=1}^{n}x_iy_i-\overline{y}n\overline{x}-\overline{x}\sum_{i=1}^{n}y_i+n\overline{x}\,\overline{y}\\&= \sum_{i=1}^{n}x_iy_i-\overline{x}\sum_{i=1}^{n}y_i\\&= \sum_{i=1}^{n}(x_iy_i-\overline{x}y_i)\\&= \sum_{i=1}^{n}(x_i-\overline{x})y_i\end{aligned}$$

Note that by similar arguments one can show that

$$\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})=\sum_{i=1}^{n}(y_i-\overline{y})x_i$$

and that

$$\sum_{i=1}^{n}(x_i-\overline{x})^2=\sum_{i=1}^{n}(x_i-\overline{x})(x_i-\overline{x})=\sum_{i=1}^{n}(x_i-\overline{x})x_i.$$

(b) Taking the partial differentiates of

$$S_n(b_0, b_1) = \sum_{i=1}^{n}(Y_i^2 - 2b_0Y_i - 2b_1Y_iX_i + b_0^2 + 2b_0b_1X_i + b_1^2X_i^2).$$

yields

$$\begin{aligned}\frac{\partial S_n(b_0, b_1)}{\partial b_0} &= \sum_{i=1}^{n}(-2Y_i + 2b_0 + 2b_1X_i)\\ \frac{\partial S_n(b_0, b_1)}{\partial b_1} &= \sum_{i=1}^{n}\left(-2Y_iX_i + 2b_0X_i + 2b_1X_i^2\right)\end{aligned}$$

Next, we want to find the minimizing arguments

$$(\hat\beta_0, \hat\beta_1)' = \min\arg_{(b_0,b_1)\in\mathbb{R}^2} S_n(b_0, b_1).$$

For this we set the two partial derivatives equal to zero which gives us two equations that fully determine the values $\hat\beta_0$ and $\hat\beta_1$:

$$\begin{aligned}n\hat\beta_0 - \sum_{i=1}^{n}Y_i + \hat\beta_1\sum_{i=1}^{n}X_i &= 0\\ \sum_{i=1}^{n}\left(-Y_iX_i + \hat\beta_0X_i + \hat\beta_1X_i^2\right) &= 0\end{aligned}$$

The two latter equations are known as the **least squares normal equations**. It is easy to see from the first normal equation that the OLS estimator of $\beta_0$ is

$$\hat\beta_0 = \bar Y - \hat\beta_1\bar X \tag{1}$$

Substituting $\hat\beta_0$ into the second normal equation gives

$$\begin{aligned}0 &= \sum_{i=1}^{n}\left(-Y_iX_i + (\bar Y - \hat\beta_1\bar X)X_i + \hat\beta_1X_i^2\right)\\ &= \sum_{i=1}^{n}\left(-X_i(Y_i - \bar Y) + \hat\beta_1X_i(X_i - \bar X)\right)\\ &= -\left(\sum_{i=1}^{n}X_i(Y_i - \bar Y)\right) + \hat\beta_1\left(\sum_{i=1}^{n}X_i(X_i - \bar X)\right)\end{aligned}$$

Solving for $\hat\beta_1$ gives

$$\begin{aligned}\hat\beta_1 &= \frac{\sum_{i=1}^{n}(Y_i - \bar Y)X_i}{\sum_{i=1}^{n}(X_i - \bar X)X_i}\\ &= \frac{\sum_{i=1}^{n}(Y_i - \bar Y)(X_i - \bar X)}{\sum_{i=1}^{n}(X_i - \bar X)(X_i - \bar X)}\\ &= \frac{\sum_{i=1}^{n}(X_i - \bar X)Y_i}{\sum_{i=1}^{n}(X_i - \bar X)^2}\end{aligned}$$

The last two lines follow from the useful result shown in (a).