Exercises Chapter 3

1. Calculate the following sums and products (as far as they are defined):

$$\boldsymbol{A} = 3 \cdot \begin{bmatrix} 1 & -2 \\ 5 & 7 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -3 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 6 & 7 & -3 \\ -2 & -1 & 4 \end{bmatrix} \qquad \boldsymbol{D} = 6 + \begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$$

$$\boldsymbol{E} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 9 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -9 & 7 \end{bmatrix} \begin{bmatrix} -3 \\ -9 \end{bmatrix} \qquad \boldsymbol{H} = \begin{bmatrix} 6 & 3 & -3 \\ 4 & 1 & 2 \end{bmatrix}' \begin{bmatrix} 1 & -8 & -4 \\ 7 & 5 & 2 \end{bmatrix}$$

2. (a) Determine the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 5 & 0 & -1 & -6 \\ 4 & 0 & 2 & -2 \end{bmatrix}$$

   (b) Determine the rank of the following matrix:

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

   (c) Compute the inverse of the following matrix:

$$C = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

3. Show that

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

   for any two $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ that have full rank (i.e. $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{B}) = n$).

4. Let A and B be $(n \times n)$ matrices each with full rank and thus invertible. Assume that $(B + A)$ is also invertible. Calculate

   a) $(AB)'(B^{-1}A^{-1})'$

   b) $(A(A^{-1} + B^{-1})B)(B + A)^{-1}$

5. Consider the matrix

$$X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}',$$

   where $X_1, X_2, \ldots, X_n \in \mathbb{R}^K$ are column vectors. Show that

$$\sum_{i=1}^{n} X_i X_i' = X'X \,.$$

6. (a) Show the following useful result:

$$\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = \sum_{i=1}^{n}(x_i - \overline{x})y_i$$

(b) Derive the OLS estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ of a simple linear regression model by minimizing the sum of squared residuals, $S_n(b_0, b_1)$, for a given sample (i.e., for given data) $((Y_1, X_1), \ldots, (Y_n, X_n))$, where

$$\begin{aligned}
S_n(b_0, b_1) &= \sum_{i=1}^{n}\hat{\varepsilon}_i^2 \\
&= \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 \\
&= \sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)^2 \\
&= \sum_{i=1}^{n}(Y_i^2 - 2b_0Y_i - 2b_1Y_iX_i + b_0^2 + 2b_0b_1X_i + b_1^2X_i^2).
\end{aligned}$$