Exercises Chapter 4

1. Let $(X_1, \dots, X_n)$ be a random sample with $X_i \sim N(\mu, \sigma^2)$ for all $i = 1, \dots, n$, where $\mu = 3$ and $\sigma = 1.5$. Consider different sample sizes of $n = \{2, 4, 6, \dots, 30\}$.

   (a) Conduct the following Monte Carlo simulation:

      1. Simulate a realization from the random sample $(X_1, \dots, X_n)$.
      2. Compute the corresponding realizations of the biased maximum likelihood estimator $s^2_{ml,n}$ and the unbiased estimator $s^2_{ml,n}$ for the variance $\sigma^2 = 1.5^2$:

$$s^2_{ml,n} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$
$$s^2_{ub,n} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

      where $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$.

   Repeat Steps 1-2 $B = 10,000$ times and save all estimates $s^2_{ml,n,(1)}, \dots, s^2_{ml,n,(B)}$ and $s^2_{ub,n,(1)}, \dots, s^2_{ub,n,(B)}$. Then approximate the Mean Squared Error (MSE), the squared bias (Bias^2), and the variance (Var) of $s_{ml,n}$ by

$$\text{MSE}(s^2_{ml,n}) \approx \frac{1}{B}\sum_{b=1}^{R}(s^2_{ml,n,(b)} - \sigma)^2$$
$$\text{Bias}^2(s^2_{ml,n}) \approx \left(\left(\frac{1}{B}\sum_{r=1}^{B} s^2_{ml,n,(b)}\right) - \sigma\right)^2$$
$$\text{Var}(s^2_{ml,n}) \approx \frac{1}{B}\sum_{b=1}^{B}\left(s^2_{ml,n,(b)} - \left(\frac{1}{B}\sum_{b=1}^{B} s^2_{ml,n,(b)}\right)\right)^2$$

   where $E(s^2_{ml,n}) \approx B^{-1}\sum_{b=1}^{B} s^2_{ml,n,(b)}$, and of $s_{ub,n}$ by

$$\text{MSE}(s^2_{ub,n}) \approx \frac{1}{B}\sum_{b=1}^{R}(s^2_{ub,n,(b)} - \sigma)^2$$
$$\text{Bias}^2(s^2_{ub,n}) \approx \left(\left(\frac{1}{B}\sum_{r=1}^{B} s^2_{ub,n,(b)}\right) - \sigma\right)^2$$
$$\text{Var}(s^2_{ub,n}) \approx \frac{1}{B}\sum_{b=1}^{B}\left(s^2_{ub,n,(b)} - \left(\frac{1}{B}\sum_{b=1}^{B} s^2_{ub,n,(b)}\right)\right)^2$$

   where $E(s_{ub,n}) \approx B^{-1}\sum_{b=1}^{B} s^2_{ub,n,(b)}$.

   Note: The *Law of Large Numbers* implies that the approximations become arbitrarily precise as $B \to \infty$.

   (b) Plot your results (y-axis: MSE, Bias^2, and Var; x-axis: n) for both estimators $s_{ml,n}$ and $s_{ub,n}$.