Exercises (with Solutions) · Chapter 2

1. **Problem**
   Let $P(A) = 0$ and show mathematically that in this case A is independent of every other event B.

   Solution
   To show that A is independent of B, we need to show that

   $$P(AB) = P(A)P(B)$$

   The right hand side of this equation is zero, since $P(A)P(B) = 0 \cdot P(B) = 0$. Thus, it remains to show that $P(AB) = 0$. For this, observe that

   $$AB \subset A$$

   and, therefore,

   $$P(AB) \leq P(A)$$

   which shows that $P(AB) = 0$ since

   $$0 \leq P(AB) \leq P(A) = 0.$$

2. **Problem**
   Suppose we toss a fair coin until we get exactly two Heads (H).
   Examples: $\omega = TTTTTTHH$, $\omega = HTTH$, $\omega = HH$, $\omega = THTTTTH$, etc.

   (a) Describe formally the corresponding sample space Ω.

   (b) Let $X(\omega) = m$ denote the random variable that gives the number of coin tosses m for each $\omega \in \Omega$. What is the probability mass function $P(X = m)$ for $m = 0, 1, 2, \dots$?

   Solution

   (a) Sample space of tossing a fair coin until one gets exactly two H:

   $\Omega = \{\omega = (\omega_1, \omega_2, \dots, \omega_m) \in \{H, T\}^m \mid (\omega_1, \dots, \omega_{m-1})$ contains exactly one H and $\omega_m = H\}$

   (b) The probability of stopping after m tosses is the probability of obtaining exactly one head H in the first $m-1$ tosses (i.e., $(m-1)/2^{m-1}$) times the probability of getting a H in the mth toss (i.e., $1/2$):

   $$P(X = m) = \frac{m-1}{2^{m-1}}\frac{1}{2} \quad \text{for all} \quad m = 2, 3, \dots,$$

   and where $P(X = 0) = 0$ and $P(X = 1) = 0$ since you need at least two coin tosses to fulfill the stopping criterion.

   > Explanation of the probability of obtaining exactly one head H in the first $m-1$ tosses: In total, there are 2^{m-1} combinations when tossing a coin $m-1$ times (denominator: 2^{m-1}), but there are only $m-1$ combinations that the $m-1$ tosses contain only one H (numerator: $m-1$).

3. **Problem**
   Consider a univariate, continuous random variable $X \in \mathbb{R}$ with density function f_X, where

   - f_X has a compact support $[0,1] \subset \mathbb{R}$, i.e.,
     $f_X(x) > 0$ for all $x \in [0,1]$ and $f_X(x) = 0$ for all $x \notin [0,1]$, and
   - f_X is bounded, i.e., $\max_{x\in[0,1]} f_X(x) \leq c$, for some constant $0 < c < \infty$.

   (a) Show that the kth moment $E(X^k)$ is finite for each $k = 1, 2, \ldots$.

   (b) What is the value of $\lim_{k\to\infty} E(X^k)$?

   Solution

   (a) Since $X^k \in [0,1]$ for all $k = 1, 2, \ldots$, $E(X^k) \geq 0$ for all $k = 1, 2, \ldots$. The kth moment $E(X^k)$ can be bounded from above by

   $$\begin{aligned}\mathbb{E}(X^k) = \int_0^1 x^k f_X(x)dx &\leq c\int_0^1 x^k dx\\ &= c\left[\frac{1}{k+1}x^{k+1}\right]_0^1\\ &= \frac{c}{k+1}\left(1^{k+1} - 0^{k+1}\right) = \frac{c}{k+1}\end{aligned}$$

   That is, $0 \leq \mathbb{E}(X^k) \leq \frac{c}{k+1} < \infty$ for all $k = 1, 2, \ldots$.

   (b) $\mathbb{E}(X^k)$ is bounded from below by zero and

   $$0 \leq \lim_{k\to\infty} \mathbb{E}(X^k) \leq \lim_{k\to\infty} \frac{c}{k+1} \to 0.$$

   Thus $\mathbb{E}(X^k)$ converges to zero as $k \to \infty$.

   Side-Note:
   What we applied here is called the Sandwich Theorem or Squeeze Theorem:
   If (a_n), (b_n), and (c_n) are three real-valued sequences satisfying $a_n \leq b_n \leq c_n$ for all n, and if furthermore $a_n \to \ell$ and $c_n \to \ell$, then $b_n \to \ell$ as $n \to \infty$.