MS-E1992 Harmonic Analysis

Exercise 1

Lien Tran

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Problem 1. Show that the dyadic subcubes in $\mathbb{R}^n$ have the following properties

(a) If $Q, Q' \in \mathcal{D}$, either one is contained in the other or the interiors of the cubes are disjoint.

Proof. Assume $Q, Q' \in \mathcal{D}_k$, for some $k \in \mathbb{N}^*$ (where $\mathbb{N}^*$ contains 0) and $Q \neq Q'$ (otherwise the claim is clear), then by definition of $\mathcal{D}_k$, a collection of dyadic cubes with side length 2^{-k}, the interiors of cubes in $\mathcal{D}_k$ are pairwise disjoint, or Q, Q' are almost disjoint.
If $Q \neq Q'$ belong to different collections of dyadic cubes, say $Q \in \mathcal{D}_k$ and $Q' \in \mathcal{D}_j$, for some $j, k \in \mathbb{N}^*, k < j$, then Q' is bisected some cube in $\mathcal{D}_k$, say $\tilde{Q}$. If $Q = \tilde{Q}$, obviously, $Q' \subsetneq Q$. If $Q \neq \tilde{Q}$, by definition of $\mathcal{D}_k$, Q and $\tilde{Q}$ are almost disjoint, thus $Q' \subsetneq \tilde{Q} \not\subset Q$. □

(b) If $Q' \in \mathcal{D}_k$ and $j < k$, there is exactly one parent cube in D_k, which contains Q'.

Proof. For a sake of contradiction, assume there were two parent cubes in D_k, say Q and $\tilde{Q}$, which contains Q', thus $Q' \subset Q \cap \tilde{Q}$ which is a contradiction to the definition of D_k that the interiors cubes in D_k are pairwise disjoint. □

(c) Every cube $Q' \in \mathcal{D}_k$ is a union of exactly 2^n children cubes $Q'' \in \mathcal{D}_{k+1}$ with $l(Q'') = 2^{-k} l(Q')$ and $|Q''| = 2^n |Q'| = 2^{-nk} |Q'|$.

Proof. By definition, every cube in $\mathcal{D}_k$ has side length of $l2^{-k}$. Bisect every cube $Q' \in \mathcal{D}_k$, we obtain 2^n subcubes with side length of $l2^{-k-1}$, thus

$$|Q''| = (2^{-k-1} l(Q))^n = 2^{-n} 2^{-nk} (l(Q))^n = 2^{-n} (2^{-k} l(Q))^n = 2^{-n} |Q'|$$
$$|Q'| = (l(Q'))^n = (2^{-k} l(Q))^n = 2^{-nk} |Q|.$$

□

(d) For every cube Q in $\mathbb{R}^n$ there is a dyadic cube $Q' \in \mathcal{D}$ such that $Q' \subset Q \subset 5Q'$.

Proof. Let Q be a cube in $\mathbb{R}^n$ of side length $l > 0$. If we bisect the side length of Q k-times to get dyadic subcubes Q' of length $2^{-k}l$, obviously $Q' \subset Q$. Further more, if we enlarge the length of Q' 5-times such that $Q \subset 5Q'$, it requires the length of Q to be 5-times smaller than that of Q', i.e.,

$$l < \frac{5l}{2^k} \implies 2^k < 5$$

Thus, the claim always holds whenever $k \in \{0, 1, 2\}$. □

Problem 2. For any subcollection $\mathcal{Q} \subset \mathcal{D}$ of dyadic cubes whose union is a bounded set, there is a subcollection of pairwise disjoint maximal cubes with the same union.

Proof. Let $\mathcal{Q} \subset \mathcal{D}$ be any subcollection of dyadic cubes whose union is a bounded set, thus $\mathcal{Q}$ contains finite countably many dyadic cubes. Define $\mathcal{Q}^*$ to be a collection of those cube $Q \in \mathcal{Q}$ which are maximal with respect to set inclusion. Thus dyadic cubes in $\mathcal{Q}*$ are not contained in any other cube in $\mathcal{Q}$. From nesting property, every cube in $\mathcal{Q}$ is contained in exactly one maximal cube in $\mathcal{Q}^*$ and any two such maximal cubes in $\mathcal{Q}^*$ are almost disjoint. We can easily check this property by taking two arbitrary cube in $\mathcal{Q}^*$, say Q and Q'. If they were not disjoint, their intersection must be either Q or Q'. Suppose for instance $Q \cap Q' = Q'$, which means $Q \subset Q'$, and since $Q \neq Q'$, thus $Q \subsetneq Q'$ is a contradiction to the maximality of Q. □

Problem 3. Show the every nonempty open set can be represented as a union of countably many pairwise disjoint dyadic cubes.

Proof. Let $U \in \mathbb{R}^n$ open, for any $x \in U$, by definition there is an open ball centered at x that is contained in U. Since we can always fit a cube in side a ball, we can conclude that there is also a closed dyadic cube containing x that is contained in U. Let $\mathcal{Q}$ be a collection of all the dyadic cube Q that is contained in U, thus

$$U \supset \cup_{Q \in \mathcal{Q}}$$

Note that there are only countably many dyadic cubes, thus $\mathcal{Q}$ is at most countable. Furthermore, dyadic cubes in $\mathcal{Q}$ can overlap with their child cubes. Define $\mathcal{Q}^*$ to be a collection of those cube $Q \in \mathcal{Q}$ which are maximal with respect to set inclusion. Thus dyadic cubes in $\mathcal{Q}*$ are not contained in any other cube in $\mathcal{Q}$. From nesting property, every cube in $\mathcal{Q}$ is contained in exactly one maximal cube in $\mathcal{Q}^*$ and any two such maximal cubes in $\mathcal{Q}^*$ are almost disjoint. Hence we can conclude

$$U = \cup_{Q \in \mathcal{Q}^*}$$

$\mathcal{Q}^*$ is at most countable and contains countably many pairwise disjoint dyadic cubes. □

Problem 4. Assume that $U \in \mathbb{R}^n$ is an open set with $|U| < \infty$. Show that there exist dyadic cubes $Q_i, i = 1, 2, \dots$, such that $U \subset \bigcup_{i=1}^{\infty} Q_i$,

$$|U \cap Q_i| \le \frac{1}{2}|Q_i| \le |Q_i \cap (\mathbb{R}^n \backslash U)| \quad \text{and} \quad |U| \le \sum_{i=1}^{\infty} |Q_i| \le 2^{n+1}|U|.$$

Proof. Assume that $U \in \mathbb{R}^n$ is an open set with $|U| < \infty$, for any $x \in U$, pick up a dyadic cube Q_x with $x \in Q$ such that

$$2^{-n-1}|Q| \le |U \cap Q| \le \frac{1}{2}|Q|$$

Let $\mathcal{Q}$ be a collection of such dyadic cubes,

$$\mathcal{Q} = \{Q_x : x \in U \quad \text{and} \quad 2^{-n-1}|Q| \le |U \cap Q| \le \frac{1}{2}|Q|\}$$

Thus $U \subset \bigcup_{x \in U}^{\infty} Q_x$. Since $|U| < 0$, then $\mathcal{Q}$ is a bounded set containing at most countably many such dyadic cubes, by problem 2, there is a subcollection $\mathcal{Q}^*$ of pairwise disjoint maximal cubes with the same union. Since any cube Q is Lebesgue measurable, thus for any cube in $\mathcal{Q}^*$, we get

$$\begin{aligned}
|Q| &= |Q \cap U| + |Q \backslash U| = |Q \cap U| + |Q \cap (\mathbb{R}^n \backslash U)| \\
&\le \frac{1}{2}|Q| + |Q \cap (\mathbb{R}^n \backslash U)| \\
\implies & \frac{1}{2}|Q| \le |Q \cap (\mathbb{R}^n \backslash U)|
\end{aligned}$$

and the lower bound gives

$$|Q| \le 2^{n+1} |U \cap Q|$$

Thus,

$$|U| \le \sum_{i=1}^{\infty} |Q_i| \le \sum_{i=1}^{\infty} 2^{n+1} |U \cap Q_i|$$

$$= 2^{n+1} \left| \bigcup_{i=1}^{\infty} (U \cap Q_i) \right| = 2^{n+1} \left| U \cap \bigcup_{i=1}^{\infty} Q_i \right| = 2^{n+1}|U|.$$

□

Problem 5. Assume that $f \in L^1_{loc}(\mathbb{R}^n)$. Define

$$E_k f(x) = \sum_{Q \in \mathcal{D}_k} \Big(\frac{1}{|Q|} \int_Q f(y)\, dy \Big) \mathcal{X}_Q(x)$$

where the sum is taken over dyadic cubes in $\mathcal{D}_k, k \in \mathbb{Z}$.

(a) Show that

$$\int_{\mathbb{R}^n} E_k f(x)dx = \int_{\mathbb{R}^n} f(x)dx$$

for every $k \in \mathbb{Z}$.

Proof. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} E_k f(x)dx &= \sum_{Q\in\mathcal{D}_k} \int_{\mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q f(y)\,dy\Big)\mathcal{X}_Q(x)\,dx \\
&= \sum_{Q\in\mathcal{D}_k} \frac{1}{|Q|}\int_Q \Big(\int_Q f(y)\,dx\Big)dy \quad \text{(by Fubini's theorem)} \\
&= \sum_{Q\in\mathcal{D}_k} \frac{1}{|Q|}\int_Q f(y)|Q|dy \\
&= \sum_{Q\in\mathcal{D}_k} \int_Q f(y)dy = \int_{\bigcup_{Q\in\mathcal{D}_k}} f(y)dy \\
&= \int_{\mathbb{R}^n} f(x)dx.
\end{aligned}$$

□

(b) Show that if $f \in L^p(\mathbb{R}^n), 1 \le p < \infty$, then $E_k f \in L^p(\mathbb{R}^n)$

Proof.

$$\begin{aligned}
\|E_k f\|^p &= \int_{\mathbb{R}^n} |E_k f(x)|^p\,dx \\
&= \int_{\mathbb{R}^n} \left|\sum_{Q\in\mathcal{D}_k} \Big(\frac{1}{|Q|}\int_Q f(y)\,dy\Big)\mathcal{X}_Q(x)\right|^p dx \\
&\le \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{D}_k} \frac{1}{|Q|^p}\int_Q |f(y)|^p\, \mathcal{X}_Q(x)\,dy\,dx \quad \text{(elementary's inequality)} \\
&= \sum_{Q\in\mathcal{D}_k} \frac{1}{|Q|^p}\int_Q\int_{\mathbb{R}^n} |f(y)|^p\, \mathcal{X}_Q(x)\,dx\,dy \quad \text{(by Fubini's theorem)} \\
&= \sum_{Q\in\mathcal{D}_k} \frac{|Q|}{|Q|^p}\int_Q |f(y)|^p\, dy = \sum_{Q\in\mathcal{D}_k} |Q|^{1-p}\int_{\bigcup_{Q\in\mathcal{D}_k}} |f(y)|^p\, dy \\
&= \int_{\mathbb{R}^n} |f(y)|^p\, dy \left|\bigcup_{Q\in\mathcal{D}_k} Q\right|^{1-p} < \infty \quad \text{if } \mathcal{D}_k \text{ is bounded.}
\end{aligned}$$

□

(c) Show that if $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, then $E_k f \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$

Proof. From part a, we have $E_k f = f$ almost every where in $\mathbb{R}^n$ for every $k \in \mathbb{Z}$. For case $p = 1$,

$$\lim_{k\to\infty} \|E_k f - f\|_{L^1(\mathbb{R}^n)} = 0$$

For case $1 < p < \infty$, by part b, $E_k f \in L_p(\mathbb{R}^n)$ and by Minkowski's inequality,

$$\|E_k f - f\|_p \leq \|E_k f\|_p + \|f\|_p$$

By Lebesgue dominated convergence theorem and Lebesgue differentiation theorem, we get

$$\begin{aligned}
\lim_{k\to\infty} \|E_k f - f\|^p &= \lim_{k\to\infty} \int_{\mathbb{R}^n} |E_k f(x) - f(x)|^p \, dx \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^n} \left| \sum_{Q\in\mathcal{D}_k} \Big(\frac{1}{|Q|} \int_Q f(y)\, dy \Big) \mathcal{X}_Q(x) - f(x) \right|^p dx \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^n} \left| \sum_{Q\in\mathcal{D}_k} \Big(\frac{1}{|Q|} \int_Q f(y) - f(x)\, dy \Big) \right|^p \mathcal{X}_Q(x)\, dx \\
&= \int_{\mathbb{R}^n} \left| \lim_{k\to\infty} \sum_{Q\in\mathcal{D}_k} \Big(\frac{1}{|Q|} \int_Q f(y) - f(x)\, dy \Big) \right|^p \mathcal{X}_Q(x)\, dx \to 0 \quad \text{as } k \to \infty
\end{aligned}$$

□