MS-E1992 Harmonic Analysis

Exercise 2

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1. Assume that $f \in L^1_{loc}(\mathbb{R}^n)$ and let Q be a cube in $\mathbb{R}^n$. Show that

$$\begin{aligned} \frac{1}{|Q|}\int_Q |f - f_Q|\, dx &= \frac{2}{|Q|}\int_{\{x\in Q: f(x) > f_Q\}} (f - f_Q)\, dx \\ &= \frac{2}{|Q|}\int_{\{x\in Q: f(x) < f_Q\}} (f_Q - f)\, dx. \end{aligned}$$

Proof.

$$\begin{aligned} 0 = f_Q - f_Q &= \frac{1}{Q}\int_Q (f - f_Q)\, dx \\ &= \frac{1}{|Q|}\Big(\int_{\{x\in Q: f(x) > f_Q\}} (f - f_Q)\, dx + \int_{\{x\in Q: f(x) < f_Q\}} (f - f_Q)\, dx\Big) \\ &+ \frac{1}{|Q|}\int_{\{x\in Q: f(x) = f_Q\}} (f - f_Q)\, dx \\ &= \frac{1}{|Q|}\Big(\int_{\{x\in Q: f(x) > f_Q\}} (f - f_Q)\, dx + \int_{\{x\in Q: f(x) < f_Q\}} (f - f_Q)\, dx\Big) \end{aligned}$$

which implies

$$\int_{\{x\in Q: f(x) > f_Q\}} (f - f_Q)\, dx = -\int_{\{x\in Q: f(x) < f_Q\}} (f - f_Q)\, dx$$

and

$$|\{x \in Q : f(x) > f_Q\}| = |\{x \in Q : f(x) < f_Q\}|$$

Thus we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f-f_Q|\,dx &= \frac{1}{|Q|}\int_{\{x\in Q: f(x)>f_Q\}} (f-f_Q)\,dx + \frac{1}{|Q|}\int_{\{x\in Q: f(x)<f_Q\}} (f_Q-f)\,dx \\
&= \frac{2}{|Q|}\int_{\{x\in Q: f(x)>f_Q\}} (f-f_Q)\,dx \\
&= \frac{2}{|Q|}\int_{\{x\in Q: f(x)<f_Q\}} (f_Q-f)\,dx.
\end{aligned}$$

□

2. Assume that $f\in L^1_{loc}(\mathbb{R}^n)$ and let Q be a cube in $\mathbb{R}^n$. Show that

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f-f_Q|\,dx &\le \frac{1}{|Q|^2}\int_Q\int_Q |f(x)-f(y)|\,dy\,dx \\
&= \frac{2}{|Q|}\int_Q |f-f_Q|\,dx.
\end{aligned}$$

Proof.

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f-f_Q|\,dx &= \frac{1}{|Q|}\int_Q \left| f(x) - \frac{1}{|Q|}\int_Q f(y)\,dy \right| dx \\
&= \frac{1}{|Q|}\int_Q \left| \frac{1}{|Q|}\int_Q f(x) - f(y)\,dy \right| dx \\
&\le \frac{1}{|Q|^2}\int_Q\int_Q |f(x)-f(y)|\,dy\,dx. \\
&\le \frac{1}{|Q|^2}\int_Q\int_Q |f(x)-f_Q+f_Q-f(y)|\,dy\,dx \\
&\le \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx + \frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy \\
&= \frac{2}{|Q|}\int_Q |f-f_Q|\,dx.
\end{aligned}$$

□

3. Assume that $f\in L^1_{loc}(\mathbb{R}^n)$ and let Q be a cube in $\mathbb{R}^n$. Show that

$$\frac{1}{|Q|}\int_Q |f-f_Q|\,dx \le \operatorname*{ess\,sup}_Q f - \operatorname*{ess\,inf}_Q f.$$

Proof. For any $x, y \in Q$ we have

$$|f(x) - f(y)| \le \operatorname*{ess\,sup}_{x\in Q} f(x) - \operatorname*{ess\,inf}_{y\in Q} f(y)$$

By problem 2, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f - f_Q|\, dx &\le \frac{1}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|\, dy\, dx \\ &\le \frac{1}{|Q|^2}\int_Q\int_Q \Big(\operatorname*{ess\,sup}_{x\in Q} f(x) - \operatorname*{ess\,inf}_{y\in Q} f(y)\Big)\, dy\, dx \\ &= \operatorname*{ess\,sup}_{x\in Q} f(x) - \operatorname*{ess\,inf}_{y\in Q} f(y).\end{aligned}$$

□

4. Assume that $f \in \mathrm{BMO}$ and $0 < \gamma < 1$. Show that $|f|^\gamma \in \mathrm{BMO}$ and

$$\||f|^\gamma\|_* \le 2\, \|f\|_*^\gamma .$$

Proof. Claim: $|f|^\gamma \in \mathrm{BMO}$. Observe that for $0 < \gamma < 1$

$$\begin{aligned}|a|^\gamma = \Big((|a| - |b|) + |b|\Big)^\gamma &\le (|a| - |b|)^\gamma + |b|^\gamma \\ \implies ||a|^\gamma - |b|^\gamma| &\le ||a| - |b||^\gamma \le |a - b|^\gamma\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{Q}\int_Q \Big||f|^\gamma - |f|_Q^\gamma\Big|\, dx &\le \frac{1}{Q}\int_Q |f - f_Q|^\gamma\, dx \\ &\le \frac{1}{|Q|}\int_Q |f(x) - f_Q|\, dx.\end{aligned}$$

Taking supremum over all cubes in $\mathbb{R}^n$ yiels

$$\||f|^\gamma\|_* = \sup_{Q\subset\mathbb{R}^n, Q\ni x} \frac{1}{Q}\int_Q \Big||f|^\gamma - |f|_Q^\gamma\Big|\, dx \le \sup_{Q\subset\mathbb{R}^n, Q\ni x} \frac{1}{Q}\int_Q |f - f_Q|\, dx < \infty$$

□

Proof. Claim: $\||f|^{\gamma}\|_* \leq 2\|f\|_*^{\gamma}$.
Reason:

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \Big||f(x)|^{\gamma} - |f|_Q^{\gamma}\Big|\,dx &= \frac{1}{|Q|}\int_Q \Big||f(x)|^{\gamma} - \frac{1}{Q}\int_Q |f(y)|^{\gamma}\,dy\Big|\,dx \\
&\leq \frac{1}{|Q|^2}\int_Q\int_Q ||f(x)|^{\gamma} - |f(y)|^{\gamma}|\,dy\,dx \\
&\leq \frac{1}{|Q|^2}\int_Q\int_Q ||f(x)| - |f(y)||^{\gamma}\,dy\,dx \\
&= \frac{1}{|Q|^2}\int_Q\int_Q \Big|\big(|f(x)| - |f|_Q\big) - \big(|f(y)| - |f|_Q\big)\Big|^{\gamma}\,dy\,dx \\
&\leq \frac{1}{|Q|^2}\int_Q\int_Q \Big(|f(x) - f_Q|^{\gamma} + |f(y) - f_Q|^{\gamma}\Big)\,dy\,dx \\
&= \frac{1}{|Q|}\int_Q |f(x) - f_Q|^{\gamma}\,dx + \frac{1}{Q}\int_Q |f(y) - f_Q|^{\gamma}\,dy \\
&\leq \frac{2}{|Q|}\int_Q |f(x) - f_Q|^{\gamma}\,dx \\
&\leq \frac{2}{|Q|}(|Q|^{1-1/p})(\int_Q |f(x) - f_Q|^{p\gamma}\,dx)^{\frac{1}{p}} \quad \text{(Holder's inequality)} \\
&\leq \frac{2}{|Q|^{\gamma}}(\int_Q |f(x) - f_Q|\,dx)^{\gamma} = 2\,\big(\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx\,\big)^{\gamma}.
\end{aligned}$$

The last inequality holds if we pick $p = \frac{1}{\gamma}$. Since $0 < \gamma < 1 \implies p > 1$. Taking supremum over all cubes in $\mathbb{R}^n$ completes the proof.

□

5. Assume that $f \in \text{BMO}$ and let $f_k(x) = \min(\max(f(x), -k), k), k = 1, 2, \dots$.
   (a) Show that $f_k \in \text{BMO}$ with $\|f_k\|_* \leq \frac{9}{4}\|f\|_*$ for every $k = 1, 2, \dots$.

   Proof. By theorem 3.7 (page 33 - Harmonic Analysis)

$$\begin{aligned}
\|f_k\|_* &= \frac{3}{2}\|\max(f(x), -k)\|_* \\
&\leq \frac{3}{2}\frac{3}{2}\|f\|_* = \frac{9}{4}\|f\|_*
\end{aligned}$$

□

   (b) Show that $f_k \to f$ pointwise as $k \to \infty$. We can write

$$f_k(x) = \min(\max(f(x), -k), k) = \begin{cases} k, & f(x) > k, \\ f(x), & -k \leq f(x) \leq k, \\ -k, & f(x) < -k, \end{cases}$$

Observer that f_k is bounded in $\mathbb{R}^n$ for any $k = 1, 2\ldots$ and the sequence f_k is increasing, thus

$$\lim_{k\to\infty} f_k(x) = f(x)$$

(c) Show that $f_k \to f$ in $L^1_{loc}(\mathbb{R}^n)$ as $k \to \infty$.

Proof. From 5.b $f_k \to f$ pointwise as $k \to \infty$ which implies $f_k \to f$ almost anywhere in R^n as $k \to \infty$. The sequence (f_k) is an increasing sequence, then by Lebesgue monotone convergence theorem,

$$\begin{aligned}\|f_k - f\|_{L^1(K)} = \int_K |f_k - f| \; dx &= \int_{\mathbb{R}^n} |f_k - f| \, \mathcal{X}_K \, dx \\ &\to 0 \text{ as } k \to \infty\end{aligned}$$

Thus $\lim\limits_{k\to\infty} \|f_k - f\|_{L^1(K)}\,, \quad$ for any K compact subset of $\mathbb{R}^n$

$\square$