MS-E1992 Harmonic Analysis

Exercise 2

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December 14, 2018

1. Show that the dyadic maximal operator is sublinear.

   Proof. For any $f, g \in L^p(\mathbb{R}^n)$,

$$\begin{aligned} |M_d(f+g)(x)| &= \sup \fint_Q |f(y)+g(y)|\,dy \\ &\le \sup \fint_Q \big(|f(y)|+|g(y)|\big)dy \\ &= \sup \fint_Q |f(y)|\,dy + \sup \fint_Q |g(y)|\,dy \\ &= |M_d f(x)| + |M_d g(x)|\,. \end{aligned}$$

   where the supremum is taken over all dyadic cubes Q containing x, and

$$\begin{aligned} |M_d(af)(x)| &= \sup \fint_Q |af(y)|\,dy \\ &= |a| \sup \fint_Q |f(y)|\,dy = |a|\,|M_d(f)(x)|\,, \quad a \in \mathbb{R}. \end{aligned}$$

   □

2. (a) Construct the Caldern-Zygmund decomposition for $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \mathcal{X}_{[0,1]^n}(x)$

   Proof. Observe that $f(x) \le 1$ for almost everywhere in $[0,1]^n$ and obviously, $f \in L^1(\mathbb{R}^n)$. Define

$$f = f\mathcal{X}_{\{|f|\le 1\}} + f\mathcal{X}_{\{|f|>1\}} = g(x) + b(x)$$

   Take a cube $Q = [0,1]^n$, thus

$$|Q| = 1 \ge \|f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|\mathcal{X}_{[0,1]^n}\right| dx = |[0,1]^n| = 1$$

We take the cube Q and bisect it into 2^n children dyadic cubes Q_i. Then by construction, $Q_i \subset Q$, $|Q_i| = 2^{-n} |Q|$

$$1 \leq \frac{1}{Q_i} \int_{Q_i} f(y)\, dy \leq 2^n$$

The interiors of cubes Q_i are pairwise disjoint in sense of set induction. We have, for any $i = 1, 2, \ldots,$

$$|Q_j| < \int_{Q_i} |f(y)|\, dy$$

By summing, we obtain

$$|\cup_{i=1}^{\infty} Q_i| \leq \|f\|_1$$

Define

$$g(x) = \begin{cases} f(x), & \text{if} \quad x \in [0,1]^n \\ \frac{1}{Q_i} \int_{Q_i} f(y)\, dy, & \text{if} \quad x \in \mathbb{R}^n \backslash [0,1]^n \end{cases}$$

by construction, $g(x) < 2^n$ almost anywhere. Thus

$$\int_{[0,1]^n} |g|^p \, dx \leq \|f\|_{L^1(\mathbb{R}^n)}$$

and

$$\int_{[0,1]^n} \leq 2^{np} |\cup_{i=1}^{\infty} Q_i| \leq 2^{np} \|f\|_{L^1(\mathbb{R}^n)}$$

Obviously, $b = f - g$ is defined almost everywhere and vanishes on $[0,1]^n$ and satisfies

$$\int_{Q_i} b(y)\, dy = 0.$$

□

(b) Show that the dyadic maximal function is not of strong type $(1,1)$.

Proof. For $f \in L^1(\mathbb{R}^n)$, we have

$$\begin{aligned} \|M_d f\|_1 &= \int_{\mathbb{R}^n} |M_d f(x)|\, dx \\ &= \int_0^{\infty} |\{x \in \mathbb{R}^n : |M_d f(x)| > t\}|\, dt, \quad t > 0, \quad \text{(Cavalieri's principle)} \\ &\geq \int_1^{\infty} |\{x \in \mathbb{R}^n : M_d f(x) > t\}|\, dt \\ &\geq \int_1^{\infty} \frac{2^{-n}}{t} \int_{\{x \in \mathbb{R}^n : |M_d f(x)| > t\}} |f(y)|\, dy\, dt \end{aligned}$$

□

3. Show that
$$\|M_d f\|_p \leq \frac{p}{p-1}\|f\|_p, \qquad 1<p<\infty.$$

Proof. For $f \in L^p(\mathbb{R}^n)$, we have

$$\begin{aligned}
\|M_d f\|_p^p &= \int_{\mathbb{R}^n} |M_d f(x)|^p\,dx \\
&= p\int_0^\infty t^{p-1}\,|\{x\in\mathbb{R}^n : |M_d f(x)|>t\}|\,dt, \quad t>0, \quad \text{(Cavalieri's principle)} \\
&\leq p\int_0^\infty \frac{t^{p-1}}{t}\int_{\{x\in\mathbb{R}^n:|M_d f(x)|>t\}} |f(y)|\,dy\,dt \\
&\leq p\int_{\mathbb{R}^n} |f(y)| \int_0^{|f(x)|} t^{p-2}dt\,dy, \quad \text{(Fubini's theorem)} \\
&\leq \frac{p}{p-1}\int_{\mathbb{R}^n} |f(x)|^{p-1}\,|f(y)|\;dy \\
&= \frac{p}{p-1}\int_{\mathbb{R}^n} |f(x)|^p\;dy
\end{aligned}$$

□

4. Show that for every cube $Q \subset \mathbb{R}^n$, we have
$$\int_Q M_d f(x)\,dx \leq 2\big(\,|Q| + \int_{\mathbb{R}^n} \log^+|f(x)|\;dx\,\big),$$

Where $\log^+ = \mathcal{X}_{|f|>1}\log|f|$ is the positive part of $\log|f|$.

Proof.

$$\begin{aligned}
\int_Q M_d f(x)\,dx &\leq |Q| + \int_{\{x\in\mathbb{R}^n: M_d f\leq 1\}} M_d f(x)\,dx \\
&\leq |Q| + |\{x\in\mathbb{R}^n : M_d f\leq 1\}| + 2\int_1^\infty |\{x\in\mathbb{R}^n : M_d f\leq 2t\}|\;dt \\
&\leq 2\,|Q| + 2\int_1^\infty \int_{\{x\in\mathbb{R}^n: M_d f\leq t\}} \frac{1}{t}\,|f(x)|\;dx\,dt, \quad t>0 \\
&\leq 2\,|Q| + 2\int_{\mathbb{R}^n} |f(y)| \int_1^{\max\{|f(x)|,1\}} \frac{1}{t}\,dt\,dx, \quad \text{(Fubini's theorem)} \\
&= 2\Big(\,|Q| + \int_{\mathbb{R}^n} |f(x)|\log|f(y)\}|\,\mathcal{X}_{x\in\mathbb{R}^n:|f(x)|>1}\,dx\Big) \\
&= 2\big(\,|Q| + \int_{\mathbb{R}^n} \log^+|f(x)|\;dx\,\big).
\end{aligned}$$

□

5. Assume that $f \in L^1(\mathbb{R}^n)$ is compactly supported supported function. Show that $M_d f \in L^1_{\text{loc}}(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)| \; dx < \infty$$

Proof. " $\Longleftarrow$ " from problem 4, we have

$$\int_K M_d f(x)\, dx \leq 2\big(|K| + \int_{\mathbb{R}^n} \log^+ |f(x)| \; dx \big) < \infty$$

for any compact $K \subset \mathbb{R}^n$

" $\Longrightarrow$ " assume $M_d f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\begin{aligned}
\infty &> \int_K M_d f(x)\, dx \\
&\geq \int_1^\infty |\{x \in \mathbb{R}^n : M_d f(x) > t\}|\, dt \\
&\geq \int_1^\infty \frac{2^{-n}}{t} \int_{\{x \in \mathbb{R}^n : |M_d f(x)| > t\}} |f(y)| \; dy\, dt \\
&\geq \int_{\{x \in \mathbb{R}^n : |M_d f(x)| > t\}} |f(y)| \int_1^{\max\{|f(x)|, 1\}} \frac{2^{-n}}{t}\, dt\, dy \\
&= 2^{-n} \int_{\mathbb{R}^n} |f(x)| \log |f(y)\}| \, \mathcal{X}_{x \in \mathbb{R}^n : |f(x)| > 1}\, dx \\
&= 2^{-n} \int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)| \; dx.
\end{aligned}$$

□