MS-E1992 Harmonic Analysis

Exercise 4

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1. Let $0 < q < p < \infty$. Show that there exists a constant c, depending only on n, p and q, such that we have
$$\|f\|_p \leq c \|f\|_q^{\frac{q}{p}} \|f\|_*^{1-\frac{q}{p}}$$
for every $f \in L^q(\mathbb{R}^n) \cap$ BMO.

Proof. Assume $\|f\|_* \neq 0$. Applying the L^q version of the Calderón-Zygmund decomposition for function $|f|^q$ at level $\|f\|_*^q$ in $\mathbb{R}^n$. Thus, we obtain pairwise disjoint dyadic subcubes $Q_i, i = 1, 2, \ldots,$ of $\mathbb{R}^n$ such that

$$\|f\|_*^q < \frac{1}{|Q_i|} \int_{Q_i} |f(y)|^q \, dy < 2^n \|f\|_*^q, \quad i = 1, 2, \cdot,$$

We get
$$\left| \bigcup_{i=1}^{\infty} Q_i \right| < \frac{1}{\|f\|_*^q} \int_{\mathbb{R}^n} |f|^q \, dx = \frac{\|f\|_q^q}{\|f\|_*^q}$$

By Hölder's inequality

$$|f|_{Q_i} = \frac{1}{Q_i} \int_{Q_i} |f(y)| \, dy \leq \big(\frac{1}{Q_i} \int_{Q_i} |f(y)|^q \, dy\big)^{\frac{1}{q}} \leq 2^{n/q} \|f\|_*$$

Thus, for $t > 2^{n/q} \|f\|_*$, we have

$$
\begin{aligned}
|\{x \in \mathbb{R}^n : |f| > t\}| &= \left| \bigcup_{i=1}^{\infty} \{x \in Q_i : |f(x)| > t\} \right| \\
&\leq \sum_{i=1}^{\infty} |\{x \in Q_i : |f(x) - f_{Q_i}| > t - |f_{Q_i}|\}| \\
&\leq \sum_{i=1}^{\infty} \left|\{x \in Q_i : |f(x) - f_{Q_i}| > t - 2^{n/q} \|f\|_*\}\right| \\
&\leq \sum_{i=1}^{\infty} c_1 \exp\Big(- \frac{c_2}{\|f\|_*} \big(t - 2^{n/q} \|f\|_*\big)\Big) |Q_i| \\
&\leq c_1 \exp\Big(- \frac{c_2}{\|f\|_*} \big(t - 2^{n/q} \|f\|_*\big)\Big) \frac{\|f\|_q^q}{\|f\|_*^q}
\end{aligned}
$$

By Cavalieri's principle:

$$
\begin{aligned}
&\|f\|_p^p = \int_{\mathbb{R}^n} |f|^p \, dx = p \int_0^{\infty} t^{p-1} |\{x \in \mathbb{R}^n : |f(x)| > t\}| \, dt \\
&= p \int_0^{2^{n/q}\|f\|_*} t^{p-1} |\{x \in \mathbb{R}^n : |f(x)| > t\}| \, dt + p \int_{2^{n/q}\|f\|_*}^{\infty} t^{p-1} |\{x \in \mathbb{R}^n : |f(x)| > t\}| \, dt \\
&\leq p \int_0^{2^{n/q}\|f\|_*} t^{p-1} \frac{\|f\|_q^q}{t^q} \, dt + p \int_{2^{n/q}\|f\|_*}^{\infty} t^{p-1} c_1 \exp\Big(- \frac{c_2}{\|f\|_*} \big(t - 2^{n/q} \|f\|_*\big)\Big) \frac{\|f\|_q^q}{\|f\|_*^q} \, dt \\
&= \frac{p}{p-q} 2^{(n/q)(p-q)} \|f\|_*^{p-q} \|f\|_q^q + \frac{pc_1}{c_2} 2^{(n/q)(p-1)} \|f\|_*^{p-q} \|f\|_q^q \\
&= C(n, q, p) \|f\|_*^{p-q} \|f\|_q^q .
\end{aligned}
$$

□

2. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and define the BMO norm over balls as

$$
\|f\|_{*,b} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx,
$$

where the supremum is taken over balls B in $\mathbb{R}^n$. Show that there exist constants c_1 and c_2, depending only on dimension n, such that

$$
c_1 \|f\|_{*,b} \leq \|f\|_* \leq c_2 \|f\|_{*,b}
$$

Proof. Observe that every cube $B(y,r) \subset \mathbb{R}^n$ with $r > 0$ can be covered by a cube Q center at y and length $l = 2r$. This implies

$$\begin{aligned}
\frac{1}{|B|}\int_B |f(x) - f_B| \, dx &= \frac{1}{|B|}\int_B |f(x) - f_Q + f_Q - f_B| \, dx \\
&\leq \frac{1}{|B|}\int_B |f(x) - f_Q| \, dx + |f_Q - f_B| \\
&\leq \frac{1}{|B|}\int_B |f(x) - f_Q| \, dx + \left|\frac{1}{|B|}\int_B f(x) - f_Q \, dx\right| \\
&\leq \frac{2}{|B(y,r)|}\int_{B(y,r)} |f(x) - f_Q| \, dx \\
&\leq \frac{2\,|Q(y,2r)|}{|B(y,r)|\,|Q(y,2r)|}\int_{Q(y,2r)} |f(x) - f_Q| \, dx \\
&\leq \frac{2^{n+1}}{|B(0,1)|}\sup_{Q\subset\mathbb{R}^n}\int_Q |f(x) - f_Q| \, dx = \frac{2^{n+1}}{|B(0,1)|}\|f\|_*
\end{aligned}$$

Take supremum over balls B in $\mathbb{R}^n$ for the left hand side, we get

$$|B(0,1)|\, 2^{-n-1}\, \|f\|_{*,b} \leq \|f\|_* \quad \text{or} \quad c_1 \|f\|_{*,b} \leq \|f\|_* \,.$$

Similarly, every cube $Q \subset \mathbb{R}^n$ with $l > 0$ is contained in the ball B center at $z \in Q$ and radius $\sqrt{n}l$. This implies

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(x) - f_Q| \, dx &\leq \frac{1}{|Q|}\int_Q |f(x) - f_B + f_B - f_Q| \, dx \\
&\leq \frac{1}{|Q|}\int_Q |f(x) - f_B| \, dx + |f_Q - f_B| \\
&\leq \frac{1}{|Q|}\int_Q |f(x) - f_B| \, dx + \left|\frac{1}{|Q|}\int_Q f(x) - f_B \, dx\right| \\
&\leq \frac{2}{|Q(y,l)|}\int_{Q(y,l)} |f(x) - f_B| \, dx \\
&\leq \frac{2\,|B(z,\sqrt{n}l)|}{|Q(y,l)|\,|B(z,\sqrt{n}l)|}\int_{B(z,\sqrt{n}l)} |f(x) - f_B| \, dx \\
&\leq \frac{2n^{n/2}}{|B(0,1)|}\int_B |f(x) - f_B| \, dx \\
&\leq c_2 \sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B |f(x) - f_B| \, dx = c_2 \|f\|_{*,b}
\end{aligned}$$

Take supremum of all cube Q in $\mathbb{R}^n$ in the left hand side, we obtain $\|f\|_* \le c_2 \|f\|_{*,b}$.

$\square$

3. Let $a > 1$ and assume that $f \in \mathrm{BMO}$. Show that there exists constant c, depending only on dimension n, such that

$$\left|f_{B(x,ar)} - f_{B(x,r)}\right| \le c\log(a+1)\|f\|_* .$$

Proof. We have $B(x,r) \subset B(x,ar)$

$$\begin{aligned}
\left|f_{B(x,ar)} - f_{B(x,r)}\right| &= \frac{1}{|B(x,r)|}\int_{B(x,r)} \left|f(x) - f_{B(x,ar)}\right| \, dx \\
&\le \frac{|B(x,ar)|}{|B(x,r)|\,|B(x,ar)|}\int_{B(x,ar)} \left|f(x) - f_{B(x,ar)}\right| \, dx \\
&= a^n \frac{1}{|B(x,ar)|}\int_{B(x,ar)} \left|f(x) - f_{B(x,ar)}\right| \, dx \\
&\le c(n) a^n \|f\|_*
\end{aligned}$$

If $1 < a \le 2$, we have

$$\left|f_{B(x,ar)} - f_{B(x,r)}\right| \le c(n)\log(a+1)\|f\|_*$$

If $a > 2$, pick k to be the integer such that $B(x, 2^k r) \subset B(x, ar) \subset B(x, 2^{k+1} r)$ and

$$\begin{aligned}
\left|f_{B(x,ar)} - f_{B(x,r)}\right| &\le \left|f_{B(x,ar)} - f_{B(x,2^k r)}\right| + \sum_{i=1}^{k} \left|f_{B(x,2^i r)} - f_{B(x,2^{i-1} r)}\right| \\
&\le c(n)(1+k)\|f\|_* \le c(n)\log(a+1)\|f\|_*
\end{aligned}$$

$\square$

4. Assume that $f \in \mathrm{BMO}$. Let $b > 1$ and let $B(x_1, r)$ and $B(x_2, r)$ be balls in $\mathbb{R}^n$, whose centers are at distance br.

   (a) Show that there exists constant c, depending only on dimension n, such that

   $$\left|f_{B(x_1,r)} - f_{B(x_2,r)}\right| \le c\log(b+1)\|f\|_*$$

   Proof. By problem 3 we have

   $$\begin{aligned}
   \left|f_{B(x_1,r)} - f_{B(x_2,r)}\right| &\le \left|f_{B(x_1,r)} - f_{B(x_1,2br)}\right| + \left|f_{B(x_1,2br)} - f_{B(x_2,br)}\right| + \left|f_{B(x_2,br)} - f_{B(x_2,r)}\right| \\
   &\le c(n)\log(b+1)\|f\|_*
   \end{aligned}$$

   $\square$

(b) Show that there exists constant c, depending only on dimension n, such that

$$\left|f_{B(x_1,(b+1)r)} - f_{B(x_2,r)}\right| \leq c\log(b+1)\,\|f\|_*$$

5. Let $\gamma > 0$ and consider $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} |\log|x||^{\gamma}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

(a) Show that $|\{x \in [-1,1] : f(x) > t\}| = 2e^{-t^{1/\gamma}}$ for every $t > 0$,

Proof. For any $x \in [-1,1]$ and any $t > 0$,

$$f(x) > t \iff |\log|x||^{\gamma} > t \iff |\log|x|| > t^{1/\gamma} \iff |x| < e^{-t^{1/\gamma}}.$$

This implies $|\{x \in [-1,1] : f(x) > t\}| = 2e^{-t^{1/\gamma}}$ for every $t > 0$ $\square$

(b) Does f belong to BMO whenever $\gamma > 1$?

Proof. Assume for a sake of contradiction f belonged to BMO whenever $\gamma > 1$. Fix $t > \gamma\left|f_{[-1,1]}\right|$

$$\{x \in \mathbb{R} : |f(x)| > t\} \subset \{x \in \mathbb{R} : \left|f(x) - f_{[-1,1]}\right| > \frac{t}{\gamma}\}$$

by John-Nirenberg lemma

$$\left|\{x \in \mathbb{R} : \left|f(x) - f_{[-1,1]}\right| > \frac{t}{\gamma}\}\right| \leq 2c_1 e^{-\frac{c_2 t}{\gamma\|f\|_*}}$$

If f was in BMO, the measure $|\{x \in [-1,1] : |f(x)| > t\}|$ would decay exponentially in t. However

$$|\{x \in \mathbb{R} : |f(x)| > t\}| = 2e^{-t^{1/\gamma}} \geq 2e^{-\frac{t}{\gamma\|f\|_*}}$$

which decays slower.

$\square$

(c) Does f belong to BMO whenever $0 < \gamma < 1$?

Proof. Since $|\log|x|| \in$ BMO, there exists a constant $M < \infty$ for all interval $I \subset \mathbb{R}$ such that

$$\frac{1}{|I|^2}\int_I\int_I ||\log|x|| - |\log|y||| \; dy\,dx < M$$

For $0 < \gamma < 1$ we have $|\log|x||^\gamma < |\log|x||$

$$\implies ||\log|x||^\gamma - |\log|y||^\gamma| < ||\log|x|| - |\log|y|||$$

Thus

$$\begin{aligned}
\frac{1}{|I|}\int_I |f(x) - f_I| \; dx &\leq \frac{1}{|I|^2}\int_I\int_I |f(x) - f(y)| \; dy\, dx \\
&= \frac{1}{|I|^2}\int_I\int_I ||\log|x||^\gamma - |\log|y||^\gamma| \; dy\, dx \\
&\leq \frac{1}{|I|^2}\int_I\int_I ||\log|x|| - |\log|y||| \; dy\, dx < M
\end{aligned}$$

Thus f belongs to BMO whenever $0 < \gamma < 1$

$\square$