MS-E1992 Harmonic Analysis

Exercise 6

Lien Tran - 510299

January 18, 2019.

1. (a) Assume that w is a weight for which there are constants m and M with $0 \le m \le M < \infty$ such that $m \le w(x) \le M$ for almost every $x \in \mathbb{R}^n$. Show that $w \in A_p$ for every p with $1 \le p < \infty$.

Proof. Assume that w is a weight, that is, $w \in L^1_{loc}(\mathbb{R}^n)$ and $w \ge 0$ almost everywhere in $\mathbb{R}^n$, for which there are constants m and M with $0 \le m \le M < \infty$ such that $m \le w(x) \le M$ for almost every $x \in \mathbb{R}^n$.

- Case $p = 1$, there is a contant $c \ge \frac{M}{m}$ such that

$$w(x) \le c\, m \qquad \text{for almost every } x \in \mathbb{R}^n$$

This implies

$$\frac{1}{|Q|}\int_Q w(x)dx \le c\, m \le c \operatorname*{ess\,inf}_{x\in Q} w(x).$$

Thus $w \in A_1$ and

$$\frac{1}{w(x)} \le c\frac{|Q|}{w(Q)}$$

- Case $1 < p < \infty$, since $m \le w(x) \le M$ for almost every $x \in \mathbb{R}^n$, we get

$$\frac{1}{|Q|}\int_Q w(x)dx \le M$$

$$\text{and} \quad \Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\Big)^{p-1} \le \frac{|Q|}{w(Q)}$$

Implies

$$\fint_Q w(x)dx\Big(\fint_Q w(x)^{1-p'}dx\Big)^{p-1} \le M\frac{|Q|}{w(Q)}.$$

$\square$

(b) Assume that g is a Lebesgue measurable function with the properties $g(x) \ge 0$ for almost every $x \in \mathbb{R}^n$, $g \in L^\infty(\mathbb{R}^n)$ and $\frac{1}{g} \in L^\infty(\mathbb{R}^n)$. Show that if $w \in A_p$, then $gw \in A_p$.

Proof. Since $g \in L^\infty(\mathbb{R}^n)$, we get

$$\fint_Q g(x)w(x)dx \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} g(x) \fint_Q w(x)dx,$$

and since $\frac{1}{g} \in L^\infty(\mathbb{R}^n)$

$$\begin{aligned}
\Big(\fint_Q \big(g(x)w(x)\big)^{1-p'}dx\Big)^{p-1} &= \Big(\fint_Q \big(\frac{1}{g(x)}\big)^{p'-1}w(x)^{1-p'}dx\Big)^{p-1} \\
&\le \big(\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{1}{g(x)}\big)^{(p'-1)(p-1)}\Big(\fint_Q w(x)^{1-p'}dx\Big)^{p-1} \\
&\le \Big(\frac{1}{\operatorname{ess\,inf}_{x\in\mathbb{R}^n} g(x)}\Big)^{-1}\Big(\fint_Q w(x)^{1-p'}dx\Big)^{p-1} \\
&\le \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} g(x)\Big(\fint_Q w(x)^{1-p'}dx\Big)^{p-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\fint_Q g(x)w(x)dx\Big(\fint_Q \big(g(x)w(x)\big)^{1-p'}dx\Big)^{p-1} \\
&\qquad\le \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} g(x) \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} g(x) \fint_Q w(x)dx\Big(\fint_Q w(x)^{1-p'}dx\Big)^{p-1}.
\end{aligned}$$

□

2. Let $1 \le p < \infty$ and assume that $w \in A_p$. Show that $w^\delta \in A_p$ for every δ with $0 < \delta < 1$.

Proof.
- Case $p = 1$, the assumption that $w \in A_1$ implies that

$$\fint_Q w(x)^\delta\, dx \le c\,(\operatorname*{ess\,inf}_{x\in Q} w(x))^\delta = c\, \operatorname*{ess\,inf}_{x\in Q} w(x)^\delta.$$

  This shows $w^\delta \in A_1$ and
$$\frac{1}{w(x)^\delta} \le c\frac{|Q|}{w(Q)}$$

- Case $1 < p < \infty$ Since $0 < \delta < 1$, we have $\frac{1}{\delta} > 1$, by Holder inequality, we get

$$\begin{aligned}
\fint_Q w(x)^\delta dx &\le \frac{1}{|Q|}\Big(\int_Q \big(w(x)^\delta\big)^{\frac{1}{\delta}}dx\Big)^\delta\Big(\int_Q 1^{\frac{1}{1-\delta}}dx\Big)^{1-\delta} \\
&= |Q|^\delta\Big(\int_Q w(x)dx\Big)^\delta = \Big(\fint_Q w(x)dx\Big)^\delta < \infty,
\end{aligned}$$

and

$$
\begin{aligned}
\fint_Q \left(w(x)^\delta\right)^{1-p'} dx &= \fint_Q \left(w(x)^{1-p'}\right)^\delta dx \\
&\le \Big(\fint_Q w(x)^{1-p'} dx\Big)^\delta < \infty,
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\fint_Q w(x)^\delta dx \Big(\fint_Q \left(w(x)^\delta\right)^{1-p'} dx\Big)^{p-1} \\
&\qquad \le \Big(\fint_Q w(x) dx\Big)^\delta \Big(\fint_Q w(x)^{1-p'} dx\Big)^{\delta(p-1)}.
\end{aligned}
$$

$\square$

3. Let $1 \le p < \infty$ and assume that $w \in A_p$.

   (a) Show that for any $0 < \alpha < 1$ there exists $0 < \beta < 1$ such that for every Lebesgue measurable set $A \subset Q$ with $\omega(A) \le \beta w(Q)$ we have $|A| \le \alpha |Q|$.

   Proof. Let $A \subset Q$ be a Lebesgue measurable set with $w(A) \le \beta w(Q)$, we have

$$
\begin{aligned}
\frac{w(A)}{|Q|} &= \frac{1}{|Q|} \int_Q \chi_A(x) w(x)\, dx \\
&\le \frac{1}{|Q|} \Big(\int_Q \chi_A^{\frac{1+\delta}{\delta}}\, dx\Big)^{\frac{\delta}{1+\delta}} \Big(\int_Q w(x)^{1+\delta}\, dx\Big)^{\frac{1}{1+\delta}}, \quad (\delta > 0)(\text{Hölder's inequality}) \\
&\le \frac{1}{|Q|} |A|^{\frac{\delta}{1+\delta}} |Q|^{\frac{1}{1+\delta}} \Big(\fint_Q w(x)^{1+\delta}\, dx\Big)^{\frac{1}{1+\delta}} \\
&\le c \Big(\frac{|A|}{|Q|}\Big)^{\frac{\delta}{1+\delta}} \fint_Q w(x)\, dx \quad (\text{Reverse Hölder's inequality}) \\
&= c\, \frac{1}{|Q|} \Big(\frac{|A|}{|Q|}\Big)^{\frac{\delta}{1+\delta}} w(Q)
\end{aligned}
$$

   Plug $Q \backslash A$ into the above result, we get

$$
c\left(1 - \frac{|A|}{|Q|}\right)^{\frac{\delta}{1+\delta}} \ge \frac{w(Q) - w(A)}{w(Q)} \ge \frac{w(Q) - \beta w(Q)}{w(Q)} = (1-\beta).
$$

   Raise both sides to the power $\frac{1+\delta}{\delta}$

$$1 - \frac{|A|}{|Q|} \geq \Big(\frac{1-\beta}{c}\Big)^{\frac{1+\delta}{\delta}}$$

which implies

$$\frac{|A|}{|Q|} \leq 1 - \Big(\frac{1-\beta}{c}\Big)^{\frac{1+\delta}{\delta}}$$

Choose δ so small such that $\alpha = 1 - \Big(\frac{1-\beta}{c}\Big)^{\frac{1+\delta}{\delta}} < 1$.

□

(b) Show that for any $0 < \alpha < 1$ there exists $0 < \beta < 1$ such that for every Lebesgue measurable set $A \subset Q$ with $|A| \leq \alpha|Q|$ we have $\omega(A) \leq \beta w(Q)$.

Proof. Let $1 \leq p < \infty$ and assume that $w \in A_p$, if $A \subset Q$ is Lebesgue measurable set, we have

$$w(Q)\Big(\frac{|A|}{|Q|}\Big)^p \leq c\, w(A)$$

Plug $Q \backslash A$ into the above result, we get

$$c\frac{w(Q)-w(A)}{w(Q)} \geq \Big(1 - \frac{|A|}{|Q|}\Big)^p \geq \Big(1 - \frac{\alpha\,|Q|}{|Q|}\Big)^p \geq (1-\alpha)^p$$

which implies,

$$\frac{w(A)}{w(Q)} \leq 1 - \frac{(1-\alpha)^p}{c} =: \beta.$$

□

4. Show that cubes may be replaced with balls in the definition of the A_p weights with $1 \leq p < \infty$ and that these definitions give the same classes of weights.

Proof. The motivation of Muckenhoupt weight arises from the purpose of looking for a weight such that Hardy-Littlewood maximal operator is of weighted strong type (p,p) with $1 \leq p < \infty$,

$$\int_{\mathbb{R}^n} \big(Mf(x)\big)^p w(x)\,dx \leq \int_{\mathbb{R}^n} |f(x)|^p\, w(x) dx$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$ and the corresponding weak type (p,p) estimate, with $1 < p < \infty$

$$\begin{aligned} w(\{x \in \mathbb{R}^n : Mf(x) > t\}) &= \int_{x \in \mathbb{R}^n : Mf(x) > t} w(x) dx \\ &\leq \frac{c}{t^p} \int_{\mathbb{R}^n} |f(x)|^p\, w(x) dx, \quad t > 0, \end{aligned}$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$ hold true. The Hardy-Little maximal operator

$$Mf(x) = \sup \frac{1}{|B|} \int_B |f(y)| \, dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. Following the analysis in lecture note (p.54), we end up with

$$\Big(\frac{1}{|B|} \int_B |f(x)| \, dx\Big)^p w(B) \leq c \int_B |f(x)|^p \, w(x) dx,$$

which is equivalent to

$$\begin{aligned} \fint_B |f(x)| \, dx &= \frac{1}{|B|} \int_B |f(x)| \, dx \\ &\leq c \Big(\frac{1}{w(B)} \int_B |f(x)|^p \, w(x) \, dx\Big)^{\frac{1}{p}} \end{aligned}$$

if $A \subset B$ is a measure set, choose $f = \mathcal{X}_A$, we get

$$w(B) \Big(\frac{|A|}{|B|}\Big)^p \leq c w(A)$$

- The case $p = 1$, we have

  $$\fint_B w(x) dx \leq \operatorname*{ess\,inf}_{x \in B} w(x)$$

  Since we can cover a ball B by a Q , we get

  $$\fint_B w(x) dx \leq c \fint_Q w(x) dx \leq c \operatorname*{ess\,inf}_{x \in Q} w(x) \leq c \operatorname*{ess\,inf}_{x \in B} w(x),$$

  for any ball B in $\mathbb{R}^n$.
- The case $1 < p < \infty$, an analogous analysis to lecture note (p.58), we get

  $$\fint_B w(x) dx \Big(\fint_B w(x)^{1-p'} \, dx\Big)^{p-1} \leq c$$

  for every ball $B \in \mathbb{R}^n$. Similarly, cover a ball B by a Q , we get

  $$\fint_B w(x) dx \Big(\fint_B w(x)^{1-p'} \, dx\Big)^{p-1} \leq \fint_Q w(x) dx \Big(\fint_Q w(x)^{1-p'} \, dx\Big)^{p-1} \leq c$$

$\square$

5. Let $w : \mathbb{R}^n \to [0, \infty], w(x) = |x|^\alpha$.

   (a) Show that $w \in A_p$ with $1 < p < \infty$ whenever $-n < \alpha < n(p-1)$.

   Proof. Let $Q(y,l) \subset \mathbb{R}^n$ be a cube containing x, we observe that $Q(y,l) \subset B(y, \sqrt{n}l)$, thus

$$\begin{aligned}
\fint_Q w(x)\,dx &= \frac{1}{|Q(y,l)|}\int_{Q(y,l)} |x|^\alpha \; dx \le \frac{1}{|Q(y,l)|}\int_{B(y,\sqrt{n}l)} |x|^\alpha \; dx \\
&\le \frac{1}{|Q(y,l)|}\int_{B(0,|y|+\sqrt{n}l)} |x|^\alpha \; dx = \frac{1}{|Q(y,\sqrt{n}l)|}\int_0^{|y|+l}\int_{\partial B(0,r)} |r|^\alpha \, dSdr \\
&= \frac{1}{|Q(y,l)|}\omega_{n-1}\int_0^{|y|+\sqrt{n}l} r^\alpha \cdot r^{n-1} dr \\
&= \frac{\omega_{n-1}}{\alpha+n}\left(|y| + \sqrt{n}l\right)^{\alpha} < \infty \text{ if } -n < \alpha,
\end{aligned}$$

   where ω_{n-1} denotes $n-1$-dimensional surface measure of the unit sphere. Similarly,

$$\begin{aligned}
\fint_Q w(x)^{1-p'}\,dx &= \frac{1}{|Q(y,l)|}\int_{Q(y,l)} |x|^{\alpha(1-p')} \; dx \le \frac{1}{|Q(y,l)|}\int_{B(y,\sqrt{n}l)} |x|^{\alpha(1-p')} \; dx \\
&\le \frac{1}{|Q(y,l)|}\int_{B(0,|y|+l)} |x|^{\alpha(1-p')} \; dx \\
&= \frac{1}{|Q(y,l)|}\int_0^{|y|+\sqrt{n}l}\int_{\partial B(0,r)} |r|^{\alpha(1-p')} \, dSdr \\
&= \frac{1}{|Q(y,l)|}\omega_{n-1}\int_0^{|y|+\sqrt{n}l} r^{\alpha(1-p')} \cdot r^{n-1} dr \\
&= \frac{\omega_{n-1}}{\alpha(1-p')+n}\left(|y| + \sqrt{n}l\right)^{\alpha(1-p')} < \infty
\end{aligned}$$

   if $\alpha(1-p') + n > 0 \iff \alpha < \frac{-n}{1-p'} = \frac{n}{p'-1} = n(p-1)$, where p and p' are Hölder conjugate.
   Thus by multiplying the previous two inequality, we conclude $w \in A_p$ with $1 < p < \infty$ whenever $-n < \alpha < n(p-1)$.

   $\square$

   (b) Show that $w \in A_1$ whenever $-n < \alpha \le 0$.
   Let $Q(y,l) \subset \mathbb{R}^n$ be a cube containing x, we observe that $Q(y,l) \subset B(y, \sqrt{n}l) :=$

$B(y,r)$, thus

$$
\begin{aligned}
\fint_Q w(x)\,dx &= \frac{1}{|Q(y,l)|}\int_{Q(y,l)} |x|^\alpha \; dx \le \frac{1}{|Q(y,l)|}\int_{B(y,r)} |x|^\alpha \; dx \\
&= \frac{|y|^n}{|Q(y,l)|}\int_{B(\frac{y}{|y|},\frac{r}{|y|})} ||y|\,z|^\alpha \; dz \quad (COV: z = \frac{x}{|y|} \implies |y|^n\,dz = dx) \\
&= \frac{|y|^{n+\alpha}}{|Q(y,l)|}\int_{B(\frac{y}{|y|},\frac{r}{|y|})} |z|^\alpha \; dz \\
&= \frac{|y|^n}{|Q(y,l)|} w(y) \int_{B(\frac{y}{|y|},\frac{r}{|y|})} |z|^\alpha \; dz
\end{aligned}
$$