MS-E1992 Harmonic Analysis

Exercise 5

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1. Assume that μ is a doubling measure on $\mathbb{R}^n$. Show that $\mu(\mathbb{R}^n) = \infty$.

Proof. Observe that for any integer k such that $1 \leq k < 2$, we have $B(x, kr) \subset B(x, 2r)$ and by property of μ

$$\begin{aligned}\mu(B(x,kr)) &\leq \mu(B(x,2r)) \leq c_\mu \mu(B(x,r)) = 2^{\log_2 c_\mu} \mu(B(x,r)) \\ &\leq (2k)^{\log_2 c_\mu} \mu(B(x,r)) = c_\mu (k)^{\log_2 c_\mu} \mu(B(x,r)).\end{aligned}$$

In case $k \geq 2$ we can find such $n \in \mathbb{N}$ such that $2^n \leq k \leq 2^{n+1}$, we have

$$\begin{aligned}\mu(B(x,kr)) &\leq \mu(B(x,2^{n+1}r)) \leq c_\mu^{n+1} \mu(B(x,r)) \\ &= c_\mu\, 2^{n \log_2 c_\mu} \mu(B(x,r)) = c_\mu (k)^{\log_2 c_\mu} \mu(B(x,r)).\end{aligned}$$

Thus if we cover entire space $\mathbb{R}^n$ by concentric balls $B(x,k)$ center at $x \in \mathbb{R}^n, k = 1, 2, \ldots$ and by the property of μ

$$\begin{aligned}\mu(\mathbb{R}^n) = \mu\Big(\bigcup_{k=1}^{\infty} B(x,k)\Big) &\leq \sum_{k=1}^{\infty} \mu\big(B(x,k) \\ &= c_\mu \sum_{k=1}^{\infty} \mu\big(B(x,k)\big) = c_\mu\, \mu\big(B(x,1)\big) \sum_{k=1}^{\infty} k^{\log_2 c_\mu} = \infty\end{aligned}$$

$\square$

2. (a) Let $\alpha > -n$. Show that the measure

$$\mu(A) = \int_A |x|^\alpha \, dx,$$

where A is a Lebesgue measurable subset of $\mathbb{R}^n$, is doubling.

Proof. Let $y \in \mathbb{R}^n$ and $r > 0$, we have

$$
\begin{aligned}
\mu(B(y,2r)) &= \int_{B(y,2r)} |x|^\alpha \, dx = \int_{B(0,2r)} |x|^\alpha \, dx, \\
&= \int_{\mathbb{R}^n} |x|^\alpha \, \mathcal{X}_{B(0,2r)}(x) \, dx = 2^n \int_{\mathbb{R}^n} |2y|^\alpha \, \mathcal{X}_{B(0,2r)}(2y) \, dy \\
&\text{(change of variable } x = 2y \implies dx = 2^n dy) \\
&= 2^{n+\alpha} \int_{\mathbb{R}^n} |x|^\alpha \, \mathcal{X}_{B(0,r)}(x) \, dx = 2^{n+\alpha} \int_{B(0,r)} |x|^\alpha \, dx \\
&= 2^{n+\alpha} \int_{B(y,r)} |x|^\alpha \, dx = 2^{n+\alpha} \mu(B(y,r)).
\end{aligned}
$$

Thus μ is a doubling measure if $\alpha > -n$. □

(b) Construct a nondoubling Borel measure μ with the property

$$
\mu(B(x,r)) > 0 \quad \text{for every } x \in \mathbb{R}^n \text{ and } 0 < r < \infty.
$$

Proof. Part (a) implies that the condition for which μ fails to be a doubling measure is $\alpha \leq -n$. Thus define $\mu : \mathbb{R}^n \to [0, \infty]$,

$$
\mu(A) = \int_A |x|^\alpha \, dx
$$

where $\alpha = -n$ and A is a Lebesgue measurable subset of $\mathbb{R}^n$. □

Thus by construction, μ is Borel outer measure with the property

$$
\begin{aligned}
\mu(B(x,r)) &= \int_{B(x,r)} |y|^{-\alpha} \, dy = \int_{B(0,r)} |y|^{-\alpha} \, dy \\
&= \omega_{n-1} \int_0^r \rho^\alpha \rho^{n-1} \, d\rho = \frac{\omega_{n-1}}{\alpha + n} r^{n+\alpha} = \infty,
\end{aligned}
$$

where ω_{n-1} is $(n-1)$-dimensional volume of the unit sphere. Thus by this measure, any ball with different radii will have a same measure.

3. Assume that μ is a doubling measure on $\mathbb{R}^n$. Let $a > 1$ and let $B(x_1, r_1)$ and $B(x_2, r_2)$ be balls in $\mathbb{R}^n$ such that the distance of the centers is at most ar_1 and $\frac{r_1}{a} \leq r_2 \leq ar_1$. Show that there exits a constant $c \geq 1$, depending only on a and doubling constant c_μ, such that

$$
\frac{1}{c}\mu(B(x_1, r_1)) \leq \mu(B(x_2, r_2)) \leq c\mu(B(x_1, r_1)).
$$

Proof. Let $a > 1$ and $\frac{r_1}{a} \leq r_2 \leq ar_1$ then $B(x_1, r_1) \subset B(x_1, 2ar_2)$ and $B(x_2, r_2) \subset B(x_1, 2ar_1)$. By property of μ and result from problem 1, we have

$$\mu(B(x_1, r_1)) \leq \mu(x_2, 2ar_2) \leq c_\mu \mu(B(x_2, ar_2)) \leq c_\mu a^{\log_2 c_\mu} \mu(B(x_2, r_2))$$

and

$$\mu(B(x_2, r_2)) \leq \mu(B(x_1, 2ar_1)) \leq c_\mu \mu(B(x_1, ar_1)) \leq c_\mu a^{\log_2 c_\mu} \mu(B(x_1, r_1))$$

Hence,

$$\frac{1}{c}\mu(B(x_1, r_1)) \leq \mu(B(x_2, r_2)) \leq c\mu(B(x_1, r_1)).$$

□

4. Assume that μ is a doubling measure on $\mathbb{R}^n$. Show that there exist constants c and Q, depending only on doubling constant c_μ, such that

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq c\Big(\frac{R}{r}\Big)^Q$$

for every $x \in \mathbb{R}^n$ and $0 < r < R < \infty$.

Proof. Choose $k \in \mathbb{N}$ such that $2^{k-1}r < R \leq 2^k r$ then the balls $B(x, r) \subset B(x, R) \subset B(x, 2^k r)$ and the doubling property of μ gives

$$\begin{aligned} \mu(B(x, R)) &\leq c_\mu^k \, \mu(B(x, r)) = c_\mu c_\mu^{k-1} \, \mu(B(x, r)) \\ &= c_\mu \, 2^{(k-1)\log_2 c_\mu} \mu(B(x, r)) \leq c_\mu (\frac{R}{r})^{\log_2 c_\mu} \mu(B(x, r)) \\ \implies \frac{\mu(B(x, R))}{\mu(B(x, r))} &\leq c\Big(\frac{R}{r}\Big)^Q. \end{aligned}$$

□

5. Assume that μ is a doubling measure on $\mathbb{R}^n$. Show that there exist constants $c > 0$ and $Q > 0$, depending only on doubling constant c_μ, such that

$$c\Big(\frac{R}{r}\Big)^Q \leq \frac{\mu(B(x, R))}{\mu(B(x, r))}$$

for every $x \in \mathbb{R}^n$ and $0 < r < R < \infty$.

Proof. Choose $k \in \mathbb{N}$ such that $2^{k-1}r < R \leq 2^k r$ then the balls $B(x,r) \subset B(x, 2^{k-1}r) \subset B(x,R) \subset B(x,2^k r)$ and the doubling property of μ gives

$$\begin{aligned}\mu(B(x,R)) \geq c_\mu^{k-1}\mu(B(x,r)) &= \frac{c_\mu^k}{c_\mu}\mu(B(x,r)) \\ &= \frac{2^{k\log_2 c_\mu}}{c_\mu}\mu(B(x,r)) \geq \frac{1}{c_\mu}\left(\frac{R}{r}\right)^{\log_2 c_\mu}\mu(B(x,r)) \\ \implies c\left(\frac{R}{r}\right)^Q &\leq \frac{\mu(B(x,R))}{\mu(B(x,r))}.\end{aligned}$$

□