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(a).Since $y = F^{-1}(z+w)$:

$$\begin{aligned}
\implies F(y) &= z + w \\
\implies z &= F(y) + w \\
\implies \pi(z|F(y)) &= \pi_w(z + F(y)) \\
\implies \pi(z|F(y)) &= \pi_w(F(x) + F(y))
\end{aligned}$$

By the change of variables formula:

$$\pi_Y(y) = \pi_Z(F(y))\|detDF(y)\| = \pi_Z(z)\|detDF(y)\|\pi(y|z)\pi(z) \quad = \pi(z|y)\pi(y)$$

$$\pi(y|z)\pi(z) = \pi(F(x) - F(y))\pi(z)\|detDF(y)\|$$

$$\Rightarrow \pi(y|x) = \|detDF(y)\|\pi_w(F(x) - F(y))$$

(b).

$$\begin{aligned}
x &= (x_1, x_2) \in \mathcal{D} \subset \mathbb{R} \\
F(x) &= \frac{1}{1-|x|^2}(x_1, x_2) = (\frac{x_1}{1-|x|^2}, \frac{x_2}{1-|x|^2}) \\
DF(y) &= \begin{bmatrix} \frac{\partial F1}{\partial x_1}(y) & \frac{\partial F1}{\partial x_2}(y) \\ \\ \frac{\partial F2}{\partial x_1}(y) & \frac{\partial F2}{\partial x_2}(y) \end{bmatrix} = \begin{bmatrix} \frac{x_1^2 - x_2^2 - 1}{(1-(x_1^2+x_2^2))^2} & \frac{2x_1x_2}{(1-(x_1^2+x_2^2))^2} \\ \\ \frac{2x_2x_2}{(1-(x_1^2+x_2^2))^2} & \frac{x_2^2 - x_1^2 - 1}{(1-(x_1^2+x_2^2))^2} \end{bmatrix} \\
\implies det(DF) &= \frac{(x_1^2 - x_2^2 - 1)(x_2^2 - x_1^2 - 1)}{(1-(x_1^2+x_2^2))^4} - \frac{4x_1^2x_2^2}{(1-(x_1^2+x_2^2))^4} \\
&= \frac{-x^4 - x_2^4 + 1 - 2x_1^2x_2^2}{(1-(x_1^2+x_2^2))^4} \\
&= \frac{1-(x_1^2+x_2^2)^2}{(1-||x||^2)^4} \\
|det(DF)| &= |\frac{1-||x||^4}{(1-||x||^2)^4}| = |\frac{(1-||x||^2)(1-||x||^2}{(1-||x||^2)^3}| \\
&= |\frac{1-||x||^2}{(1-||x||^2)^3}|
\end{aligned}$$

(c).

$$\text{Let } y = F^{-1}(z) \implies F(y) = z$$

$$\implies \frac{y}{1-|y|^2} = z \implies \frac{|y|}{1-|y|^2} = |z|$$

$$\implies |z||y|^2 + |y| - |z| = 0 \implies |y| = \frac{\sqrt{1+4|z|^2}-1}{2|z|}$$

Since F^{-1} reverse the polar angle, we have $y = |y|(\cos\phi, \sin\phi)$,

where $z = |z|(\cos\phi, \sin\phi) = (z_1, z_2) \implies \cos\phi = \frac{z_1}{|z|}, \sin\phi = \frac{z_2}{|z|}$

$$\implies y = |y|(\frac{z_1}{|z|}, \frac{z_2}{|z|}) = \frac{|y|}{|z|}(z_1, z_2) = (\frac{\sqrt{1+4|z|^2}-1}{2|z|^2})z$$

$$\implies F^{-1}(z) = y = (\frac{\sqrt{1+4|z|^2}-1}{2|z|^2})z$$

(d).The distribution is:

$$\pi_{post}(x) = \pi(x|v) \propto \exp\left(-\frac{\|v-qf(x)\|^2}{2\sigma^2}\right)$$

The steps of the Metropolis-Hastings algorithm:

1. Calculate the transformed variable $z = F(x)$
2. Draw a random vector distributed as $W \sim N(0, \gamma^2 I)$ and calculate $F(y)$ as $F(y) = z + w$
3. Calculate $y = F^{-1}(z + w)$
4. Calculate the values of $\pi_{post}(x), \pi_{post}(y), \pi(y|x)$ and $\pi(x|y)$ and determine the acceptance probability.
5. Draw a sample from the uniform distribution to see if the move gets accepted. If yes, set $x = y$ and repeat. Otherwise, keep x and repeat.

Results:

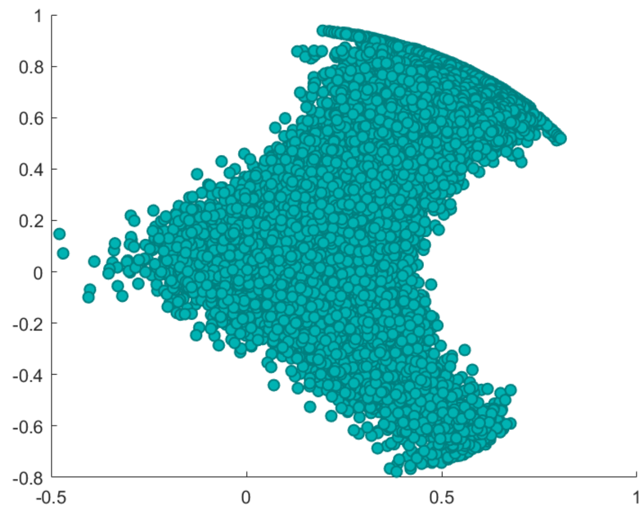

Figure 1 : Visualization of the samples $\gamma = 0.5$

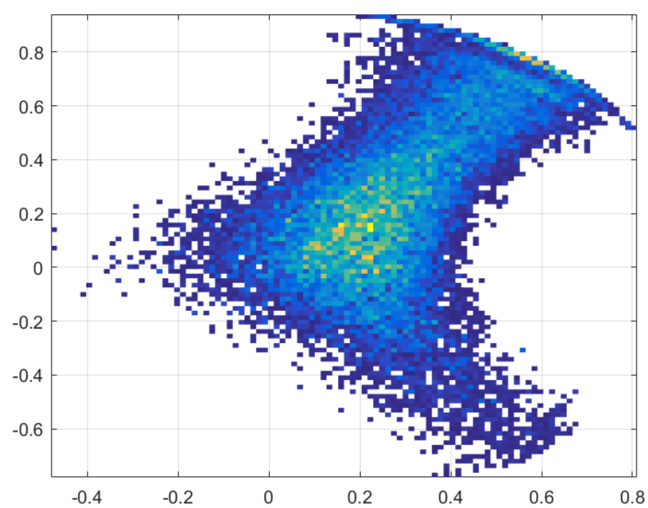

Figure 2 : Histogram plot of the samples $\gamma = 0.5$

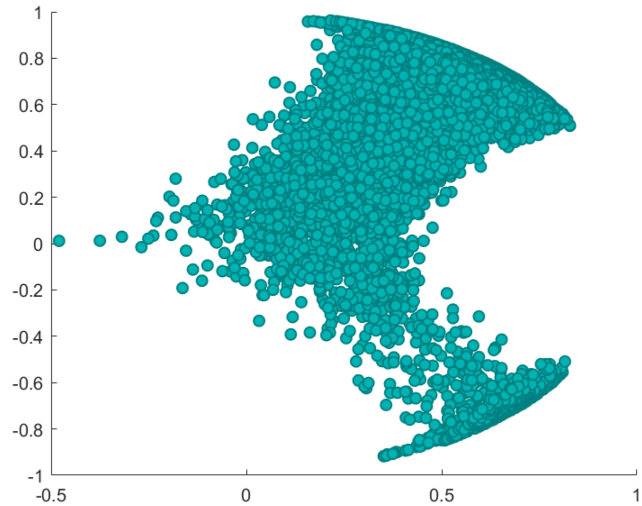

Figure 3 : Visualization of the samples $\gamma = 1$

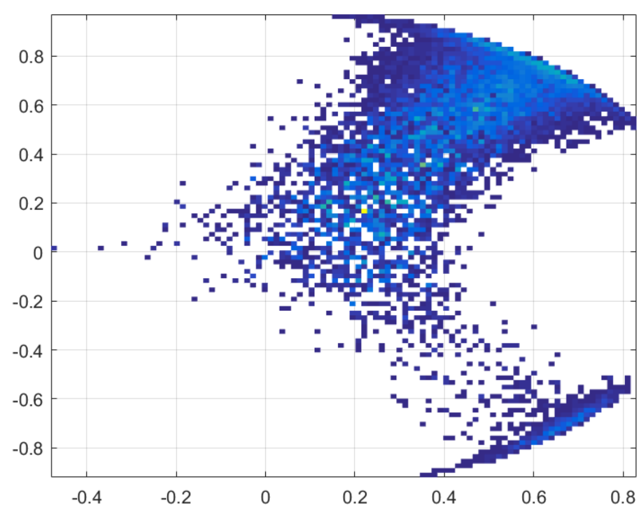

Figure 4 : Histogram plot of the samples $\gamma = 1$

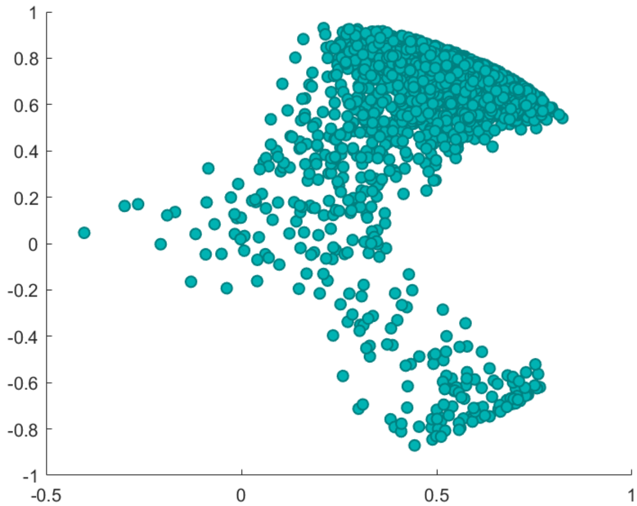

Figure 5 : Visualization of the samples $\gamma = 2$

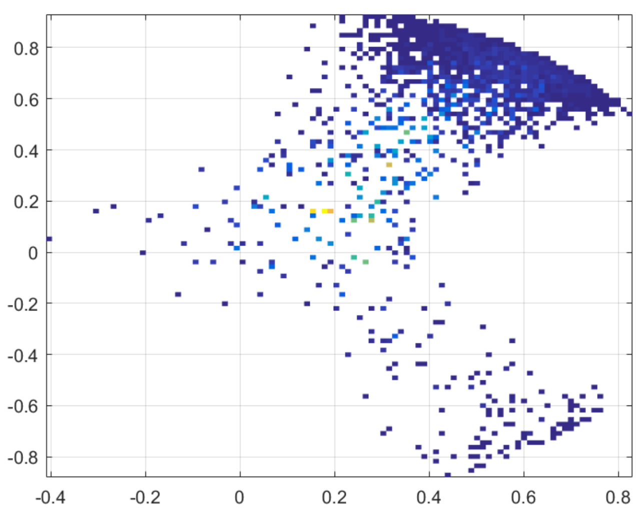

Figure 6 : Histogram plot of the samples $\gamma = 2$

```
close all
clc

gamma = 0.5;
x0 =  [0.25; 0.25];
f = @(p,x) 1./norm(x-p);
F = @(x) x./(1-norm(x)^2);
```

```
DFx = @(x) (1+norm(x)^2)/(1-norm(x)^2)^3;
q = @(x,y) abs(DFx(y)) * (1.0/(2*pi*gamma*gamma))...
*exp((-1.0/(2*gamma*gamma)) * norm( F(x) - F(y) )^2);


% Sensor locations
[ss1x,ss1y] = pol2cart(0       , 1);
[ss2x,ss2y] = pol2cart(2*pi/3  , 1);
[ss3x,ss3y] = pol2cart(4*pi/3  , 1);

p1   = [ss1x; ss1y];
p2   = [ss2x; ss2y];
p3   = [ss3x; ss3y];

qfx1_0 = f(p1, x0);
qfx2_0 = f(p2, x0);
qfx3_0 = f(p3, x0);

sigma  = 0.15*max(abs([qfx1_0, qfx2_0, qfx3_0]));

v1   = qfx1_0 + normrnd(0, sigma);
v2   = qfx2_0 + normrnd(0, sigma);
v3   = qfx3_0 + normrnd(0, sigma);

V  = [v1; v2; v3];


max_iter = 50000;

% Set initial x
x  = x0;

xs = zeros(max_iter,2);
alphas  = zeros(max_iter,1);
post_xs = zeros(max_iter,1);
post_ys = zeros(max_iter,1);
kernel  = zeros(max_iter,1);


for i = 1:max_iter
z = F(x);
z_w = z + normrnd(0, gamma^2,[2,1]);
[theta, rho_z] = cart2pol(z_w(1), z_w(2));
rho_y    = (sqrt(4*(rho_z*rho_z) + 1)-1)/(2*rho_z);
[yx, yy] = pol2cart(theta, rho_y);
```

```
y        = [yx; yy];

qfx1    = f(p1,x);
qfx2    = f(p2,x);
qfx3    = f(p3,x);

pi_post_x = exp((norm(vv - [qfx1; qfx2; qfx3])^2)/(-2*sigma*sigma));
post_xs(i) = pi_post_x;

qfy1    = f(p1,y);
qfy2    = f(p2,y);
qfy3    = f(p3,y);

post_y = exp((norm(vv - [qfy1;qfy2;qfy3])^2)/(-2*sigma*sigma));
post_ys(i) = post_y;

% Compute transition kernel for x and y
q_xy = q(x,y);
q_yx = q(y,x);

kernel(i) = q_xy;

% Compute alpha
alpha = ( post_y * q_yx )/( pi_post_x * q_xy );
alphas(i) = alpha;

% Sample t ~ Uni(0,1)

t = unifrnd(0,1);

if alpha > t
x = y;
else
x = x;
end

xs(i,:) = x;
end
figure
scatter(xs(:,1),xs(:,2),'MarkerEdgeColor',[0 .5 .5],..
'MarkerFaceColor',[0 .7 .7],'LineWidth',1)

figure
histogram2(xs(:,1),xs(:,2),[100 100],'DisplayStyle','tile')
```

$$N'_{\phi\phi} + csc\theta N_{\phi\phi} - \cos\theta N_{\phi\phi} + Q_\theta + R\rho g t \sin\theta$$

$$\nu$$