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Nonlinear partial differential equation

Exercise 2.

1. Let $1 < p < \infty$. Assume that $u \in W^{1,p}(\Omega)$ is a weak solution to the p-Laplace equation $-\operatorname{div}(|Du|^{p-2} Du) = 0$ in Ω, that is

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

Show that

$$\int_\Omega |Du|^{p-2} Du \cdot Dv \, dx = 0 \quad \text{for every } v \in W_0^{1,p}(\Omega)$$

Does the same result hold true under assumption $u \in W^{1,p}_{loc}(\Omega)$

<u>Proof</u>:

For every $v \in W_0^{1,p}(\Omega)$, by definition, there exists smooth functions $v_j \in C_0^\infty(\Omega)$ such that $v_j \to v$ in $W^{1,p}(\Omega)$ as $j \to \infty$. By assumption, we have

$$\int_\Omega |Du|^{p-2} Du \cdot Dv_j \, dx = 0 \quad , \; j = 1, \dots, n.$$

By Hölder inequality we have

$$\left| \int_\Omega \left(|Du|^{p-2} Du \cdot Dv - |Du|^{p-2} Du \cdot Dv_j \right) dx \right|$$

$$\leq \int_\Omega |Du|^{p-2} |Du| \, |Dv - Dv_j| \, dx = \int_\Omega |Du|^{p-1} |Dv - Dv_j| \, dx$$

$$\leq \left(\int_\Omega |Du|^p \, dx \right)^{1-\frac{1}{p}} \left(\int_\Omega |Dv - Dv_j|^p \right)^{1/p} dx$$

$$= \|Du\|^{p-1}_{L^p(\Omega)} \left(\int_\Omega |Dv - Dv_j|^p \right)^{1/p} dx \longrightarrow 0 \quad \text{as } j \to \infty$$

$\square$.

The same result holds true under assumption $u \in W^{1,p}_{loc}(\Omega)$ if and only if every point has a neighbourhood where u is a weak solution. to the p-Laplace equation in Ω.

$\square$.

2. Assume that u is a weak solution to p-Laplace equation in $\mathbb{R}^n$ and $u \in W^{1,p}(\mathbb{R}^n)$

Show that $u = 0$

Proof:

From Sobolev spaces, we have the fact that the standard Sobolev space and the Sobolev space with zero boundary value coincide in the whole space, i.e.,

$$u \in W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$$

By result from problem 1, we get

$$0 = \int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot Du \, dx = \int_{\mathbb{R}^n} |Du|^{p-2} |Du|^2 dx$$

$$= \int_{\mathbb{R}^n} |Du|^p dx = 0$$

$\Rightarrow$ $Du = 0$ almost every where in $\mathbb{R}^n$, which implies u is a constant almost every where in $\mathbb{R}^n$. Since u vanishes on the boundary of $\mathbb{R}^n$, u must ~~~~ have zero value almost everywhere in $\mathbb{R}^n$.

$\square$.

3. Assume that $g \in W^{1,p}(\Omega)$, $u \in W^{1,p}(\Omega)$ with $v - g \in W_0^{1,p}(\Omega)$ and consider the variational integral

$$I(u) = \frac{1}{p}\int_\Omega |Du|^p dx$$

Show that

$$I(u) = \inf\{ I(v) : v \in W^{1,p}(\Omega),\ v - g \in W_0^{1,p}(\Omega)\}$$

If and only If

$$\int |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \qquad \text{for every } \varphi \in C_0^\infty(\Omega)$$

Proof

" $\Rightarrow$ " Assume $I(u) = \inf\{I(v) : v \in W^{1,p}(\Omega), v - g \in W_0^{1,p}(\Omega)\}$. Let $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$, by Lagrange method of variation, we have

$$0 = \frac{\partial I(u+\varepsilon\varphi)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{1}{p}\int_\Omega \frac{\partial |D(u+\varepsilon\varphi)|^p}{\partial\varepsilon} dx \Big|_{\varepsilon=0}$$

$$= \frac{1}{p}\int_\Omega \frac{\partial}{\partial\varepsilon}\left(\sum_{j=1}^n \left(D_j(u+\varepsilon\varphi)\right)^2\right)^{p/2} dx \Big|_{\varepsilon=0}$$

$$= \frac{1}{p}\int_\Omega \sum_{j=1}^n 2\,\frac{p}{2}\, D_j(u+\varepsilon\varphi)\left(\sum_{j=1}^n (D_j(u+\varepsilon\varphi))^2\right)^{\frac{p-2}{2}} D_j\varphi \, dx \Big|_{\varepsilon=0}$$

$$= \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx .$$

" $\Leftarrow$ " Assume $u \in W^{1,p}(\Omega)$ with $u - v \in W_0^{1,p}(\Omega)$, where $v \in W^{1,p}(\Omega)$, $v - g \in W_0^{1,p}(\Omega)$.

we have

$$0 = \int_\Omega |Du|^{p-2} Du \cdot D(u - v)\, dx = \int_\Omega \left(|Du|^p - |Du|^{p-2} Du \cdot Dv\right) dx$$

$$\Rightarrow \left|\int_\Omega |Du|^p dx\right| = \left|\int_\Omega |Du|^{p-2} Du \cdot Dv \, dx\right| \le \left|\int_\Omega |Du|^{p-1} |Dv| \, dx\right|$$

$$\le \int_\Omega \frac{(p-1)}{p} |Du|^p dx + \int_\Omega \frac{|Dv|^p}{p} dx \quad \text{(Hölder's inequality)}$$

$$\Rightarrow \frac{1}{p}\int_\Omega |Du|^p dx \le \int \frac{|Dv|^p}{p} dx \quad \text{for every } v \in W^{1,p}(\Omega), v - g \in W_0^{1,p}(\Omega)$$

$\square$.

4. Assume that Ω is a bounded open subset of $\mathbb{R}^n$ and $g \in W^{1,p}(\Omega)$. Show that there exists $u \in W^{1,p}(\Omega)$ with $u - g \in W_0^{1,p}(\Omega)$, which satisfies

$$I(u) = \inf \{ I(v) : v \in W^{1,p}(\Omega), \ v - g \in W_0^{1,p}(\Omega) \}.$$

Proof

1) Consider the variational integral from problem 3.

$$I(u) = \frac{1}{p} \int_\Omega |Du|^p \, dx \ , \quad 1 < p < \infty$$

$\Rightarrow$ $I(u) \geq 0$, thus it is bounded from below in $W^{1,p}(\Omega)$

We denote

$$0 \leq m = \inf \{ I(v) : v \in W^{1,p}(\Omega), v - g \in W_0^{1,p}(\Omega) \}. \leq \frac{1}{p}\int_\Omega |Dg|^p dx < \infty$$

By the definition of infimum, it implies that there exists a minimizing sequence $u_k \in W^{1,2}(\Omega)$, $u_k - g \in W_0^{1,p}(\Omega)$, $k = 1, 2, \dots$, such that

$$\lim_{k\to\infty} I(u_k) = \inf_{\substack{v \in W^{1,p}(\Omega) \\ v - g \in W_0^{1,p}(\Omega)}} I(v) = m$$

The existance of the limit $\lim_{k\to\infty} I(u_k)$ implies the sequence $(I(u_k))$ is bounded, that is

$$(I(u_k)) \leq M. \ , \ k = 1, 2, \dots \ , \text{for some constant } M < \infty.$$

2) By Poincaré inequality, we get

$$\int_\Omega |u_k - g|^p dx + \int_\Omega |D(u_k - g)|^p dx \leq c\,\mathrm{diam}(\Omega)^p \int_\Omega |D(u_k - g)|^p dx + \int_\Omega |D(u_k - g)|^p dx$$

$$\leq (1 + c(p)\,\mathrm{diam}(\Omega)^p) \int_\Omega |Du_k - g|^p dx$$

$$\leq (1 + c(p)\,\mathrm{diam}(\Omega)^p)\, 2^{p-1} \left(\int_\Omega |Du_k|^p dx + \int_\Omega |Dg|^p dx \right) \frac{p}{p}$$

$$\leq p\, 2^{p-1} (1 + c(p)\,\mathrm{diam}(\Omega)^p) \left(M + \frac{1}{p} \int |Dg|^p dx \right) < \infty$$

For every $k = 1, 2, \dots$. This shows that $(u_k - g)$ is a bounded sequence in $W_0^{1,p}(\Omega)$

3). By reflexivity of $W_0^{1,p}(\Omega)$, $1 < p < \infty$, there exists a subsequence $(u_{k_j} - g)$ and a function $u \in W^{1,2}(\Omega)$ with $u - g \in W_0^{1,2}(\Omega)$, such that $u_{k_j} \to u$ weakly in $L^p(\Omega)$ and $D_i u_{k_j} \to D_i u$, $i = 1, \dots, n$ weakly in $L^p(\Omega)$ as $j \to \infty$. By lower semicontinuity of L^p-norm with respect to weak convergence, we have:

$$\frac{1}{p}\int_\Omega |Du|^p\,dx \le \liminf_{j\to\infty} \frac{1}{p}\int_\Omega |Du_{k_j}|^p\,dx = \lim_{k\to\infty}\frac{1}{p}\int_\Omega |Du_k|^p\,dx$$

Since $u \in W^{1,p}(\Omega)$ with $u-g \in W_0^{1,p}(\Omega)$, we have

$$m \le \frac{1}{p}\int |Du|^p\,dx \le \lim_{k\to\infty} \frac{1}{p}\int |Du_k|^p\,dx = m$$

which implies $\frac{1}{p}\int |Du|^p\,dx = m$.

$\square$.

5. Show that the solution of the pevious problem is unique

a) By using the PDE.

Proof:

For a sake of contradiction, assume u_1, u_2 are minimizers of $I(v)$, where $u_1, u_2 \in W^{1,p}(\Omega)$ with $u_1 - g \in W_0^{1,p}(\Omega)$ and $u_2 - g \in W_0^{1,p}(\Omega)$.

Assume $u_1 \neq u_2$, that is $|\{x \in \Omega : u_1(x) \neq u_2(x)\}| > 0$.

By Poincare's inequality we have

$$0 < \frac{1}{p}\int_\Omega |u_1 - u_2|^p dx \leq c \, \mathrm{diam}(\Omega)^p \frac{1}{p} \int |Du_1 - Du_2|^p dx$$

which implies $|\{x \in \Omega : Du_1(x) \neq Du_2(x)\}| > 0$

Let $v = u_1 - u_2 = (u_1 - g) - (u_2 - g) \in {}_0^{1,p}(\Omega)$, thus we can use v as a test function (from problem 1)

$$0 = \int_\Omega |Du_1|^{p-2} Du_1 \cdot Dv \, dx - \int_\Omega |Du_2|^{p-2} Du_2 \cdot Dv \, dx$$

$$= \int_\Omega \left(|Du_1|^{p-2} Du_1 - |Du_2|^{p-2} Du_2\right) \cdot (Du_1 - Du_2) \, dx > 0$$

Thus $0 > 0$ gives the contradiction $\Rightarrow u_1 = u_2$. $\square$

b) By using the variational integral: Let $v = \dfrac{u_1 + u_2}{2}$.

By strict convexity of $\xi \mapsto |\xi|^p$, we get $\left|\frac{1}{2}(\xi_1 + \xi_2)\right|^p < \frac{1}{2}\left(|\xi_1|^p + |\xi_2|^p\right)$ whenever $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$. Thus

$$\frac{1}{p}\int_\Omega |Dv|^p dx = \frac{1}{p}\int_\Omega \left|\frac{1}{2}(Du_1 + Du_2)\right|^p dx$$

$$< \frac{1}{p}\int_\Omega \frac{1}{2}\left(|Du_1|^p + |Du_2|^p\right) dx = \frac{1}{2}\frac{1}{p}\int |Du_1|^p dx + \frac{1}{2}\frac{1}{p}\int |Du_2|^p dx$$

$$= \frac{1}{2} m + \frac{1}{2} m = m$$

which shows that $v = \dfrac{u_1 + u_2}{2}$ is minimizer of $I(w)$, hence it is a contradict to the fact that u_1 and u_2 are minimizers

$\square$.