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Nonlinear partial differential equations

Exercise 4.

1. Assume that u is a weak solution to the p-Laplace equation in Ω. Show that $\max\{u,k\}$, $k\in\mathbb{R}$, is a weak subsolution to the p-Laplace equation in Ω.

Proof

Observe that $\max(u,k) = \max(u-k,0)+k$, $k\in\mathbb{R}$, and $\max\{u,k\}\in W^{1,p}_{loc}(\Omega)$

Let $\varphi\in C_0^\infty(\Omega)$ with $\varphi\geq 0$. Denote $\tilde{u} = \max\{u,k\}$ and

$v_\ell = \min\{\ell\tilde{u},1\}$, $\ell = 1,2,\dots$. Then (v_ℓ) is an increasing sequence,

$0\leq v_\ell\leq 1$, $\ell=1,2\dots$, $\lim_{\ell\to\infty} v_\ell(x) = \chi_{\{x\in\Omega:\, u(x)-k>0\}}(x)$, $x\in\Omega$;

Choose $v_\ell\varphi\in W_0^{1,p}(\Omega)$ to be a test function, note that $v_\ell\varphi\geq 0$ and

$$D_j v_\ell = \begin{cases} \ell D_j u & \text{almost everywhere in } \{x\in\Omega : 0<\ell(u-k)<1 \\ 0 & \text{almost everywhere in } \{x\in\Omega : \ell(u-k)\geq 1\}\cup\{u-k\leq 0\}\end{cases}$$

Since u is a weak solution to the p-Laplace equation in Ω if and only if it is both super- and subsolution, thus

$$0\geq\int_\Omega |Du|^{p-2}Du\cdot D(v_\ell\varphi)\,dx = \int |Du|^{p-2}Du\cdot(\varphi Dv_\ell + v_\ell D\varphi)\,dx$$

$$= \ell\int_{\{x\in\Omega:\, 0<u(x)-k<\frac{1}{\ell}\}} |Du|^p\varphi\,dx + \int_\Omega v_\ell |Du|^{p-2}Du\cdot D\varphi\,dx$$

$$\Leftrightarrow \int_\Omega v_\ell |Du|^{p-2}Du\cdot D\varphi\,dx \leq -\ell\int_{\{x\in\Omega:\, 0<u(x)-k<\frac{1}{\ell}\}} |Du|^p\varphi\,dx \leq 0$$

since

$$\left| v_\ell |Du|^{p-2}Du\cdot D\varphi\right| \leq |Du|^p|v_\ell|\,|D\varphi| \leq \|D\varphi\|_{L^\infty(\Omega)} |Du|^p \in L^1(\Omega)$$

By Lebesgue dominated convergence theorem

$$\int_\Omega |D\max\{u,k\}|^p D\max\{u,k\}\cdot D\varphi\,dx = \int_\Omega \lim_{k\to\infty} v_\ell |Du|^{p-2}Du\cdot D\varphi\,dx$$

$$= \lim_{k\to\infty}\int_\Omega v_\ell |Du|^{p-2}Du\cdot D\varphi\,dx \leq 0$$

for every $\varphi\in C_0^\infty(\Omega)$ with $\varphi\geq 0$.

Thus we can conclude $\max\{u,k\}$ is a weak subsolution to p-Laplace equation in Ω.

$\square$

2. Assume that $u \geq \varepsilon > 0$ is a weak supersolutions to the p- Laplace equation in Ω
Show that $\frac{1}{u}$ is a weak subsolution to the p- Laplace equation in Ω.

<u>Proof</u>

Let $v := \frac{1}{u}$ then $Dv = - u^{-2} Dv$

Note that $|Dv|^{p-2} Dv = - u^{-2(p-1)} |Du|^{p-2} Du$.

$\varphi \in W^{1,p}(\Omega)$

Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$ and let $\varphi = \psi u^{-2(p-1)}$ be a test function,

Since u is a weak supersolution to the p- Laplace equation, we have

$$0 \leq \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = \int_\Omega |Du|^{p-2} Du \cdot D(\psi u^{-2(p-1)}) \, dx$$

$$= \int_\Omega |Du|^p (-2p+2) u^{-2p+1} \psi \, dx + \int_\Omega |Du|^{p-2} Du \cdot D\psi \, u^{-2(p-1)} dx$$

$$\Rightarrow - \int u^{-2(p-1)} |Du|^{p-2} Du \cdot D\psi \leq -2(p-1) \int_\Omega |Du|^p u^{-2p+1} \psi \, dx \leq 0$$

(since $1 \leq p < n$) $\square$.

3. Assume that u and v are weak solution to the p-Laplace equation in Ω and $u, v \in W^{1,p}(\Omega)$

a) Show that if $\min\{v-u, 0\} \in W_0^{1,p}(\Omega)$, then $u \leq v$ almost everywhere in Ω.

Proof

Assume that u and v are solution to the p-Laplace equation in Ω.

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

and

$$\int_\Omega |Dv|^{p-2} Dv \cdot D\varphi \, dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$$\Rightarrow \int_\Omega \left(|Dv|^{p-2} Dv - |Du|^{p-2} Du \right) \cdot D\varphi \, dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega) \quad (1)$$

Since $\min\{v-u, 0\} \in W_0^{1,p}(\Omega)$, we can test (1) with $\min\{v-u, 0\}$

Observe that

$$D\left(\min\{v-u,0\}\right) = \begin{cases} Dv - Du & \text{in } \{x \in \Omega : v-u < 0\} \\ 0 & \text{in } \{x \in \Omega : v-u \geq 0\} \end{cases}$$

Thus

$$0 = \int_\Omega \left(|Dv|^{p-2} Dv - |Du|^{p-2} Du \right) \cdot D \min\{v-u, 0\} \, dx$$

$$= \int_{\{x \in \Omega : v-u < 0\}} \left(|Dv|^{p-2} Dv - |Du|^{p-2} Du \right) \cdot (Dv - Du) \, dx$$

- For $1 < p < 2$

$$0 = \int_{\{x \in \Omega : v-u < 0\}} \left(|Dv|^{p-2} Dv - |Du|^{p-2} Du \right) \cdot (Dv - Du) \, dx$$

$$\geq \int_{\{x \in \Omega : v-u < 0\}} c(p) \left(|Dv| + |Du| \right)^{p-2} (Dv - Du)^2 \, dx$$

- For $p \geq 2$

$$0 = \int_{\{x \in \Omega : v-u < 0\}} \left(|Dv|^{p-2} Dv - |Du|^{p-2} Du \right) \cdot D(v-u) \, dx \geq c(p) \int_{\{x \in \Omega : v-u < 0\}} |Dv - Du|^p \, dx$$

From here we can conclude that $\left| \{x \in \Omega : v-u < 0\} \right| = 0$

Thus $u \leq v$ almost everywhere in Ω

$\square$.

3b. Show that if $u, v \in C(\bar{\Omega})$ and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω

Proof

We have $u \leq v$ on $\partial\Omega$ $\Rightarrow$ $\min\{v-u, 0\} = 0$ on $\partial\Omega$

By part a and the continuity of u and v in $\bar{\Omega}$ $\Rightarrow$ $u \leq v$ in Ω.

$\square$.

4 Assume that u is a nonnegative solution to the p-Laplace equation in Ω.
Let $\Omega \in C_0^\infty(\Omega)$ and $\alpha > 0$. Show that

$$\int_\Omega |Du|^p u^{-1-\alpha} |\varphi|^p dx \leq \left(\frac{p}{\alpha}\right)^p \int_\Omega u^{p-1-\alpha} |D\varphi|^p dx$$

Proof

Assume that u is a nonnegative weak solution to the p-Laplace equation in Ω. Let $\varepsilon > 0$ and denote $u_\varepsilon = u + \varepsilon$, thus $u_\varepsilon \geq \varepsilon > 0$. Let $\varphi \in C_0^\infty(\Omega)$ and define $\varphi \geq 0$, $\psi_\varepsilon := |\varphi|^p u_\varepsilon^{-\alpha}$, $\alpha > 0$, $\psi_\varepsilon \in W^{1,p}(\Omega)$ can be a test function.

Note that: $D\psi_\varepsilon = D(u_\varepsilon^{-\alpha}|\varphi|^p) = -\alpha u_\varepsilon^{-\alpha-1} Du |\varphi|^p + p u_\varepsilon^{-\alpha} \varphi |\varphi|^{p-2} D\varphi$

Thus

$$0 = \int_\Omega |Du|^{p-2} Du \cdot D\psi_\varepsilon dx = -\int_\Omega |Du|^p |\varphi|^p \alpha u_\varepsilon^{-1-\alpha} dx + p \int_\Omega u_\varepsilon^{-\alpha} \varphi |\varphi|^{p-2} D\varphi \cdot Du |Du|^{p-2} dx$$

$$\Rightarrow \left| \int_\Omega |Du|^{p-2} Du \cdot D\psi dx \right| = \left| \frac{p}{\alpha} \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, u_\varepsilon^{-\alpha} \varphi |\varphi|^{p-2} dx \right|$$

$$\leq \frac{p}{\alpha} \int_\Omega |Du|^{p-1} |\varphi|^{p-1} u_\varepsilon^{-\alpha} |D\varphi| dx$$

$$\text{(Young's ineq.)} \leq \frac{p-1}{p} \int_\Omega |Du|^p u_\varepsilon^{-1-\alpha} |\varphi|^p dx + \frac{1}{p}\left(\frac{p}{\alpha}\right)^p \int u_\varepsilon^{p-1-\alpha} |D\varphi|^p dx$$

$$\left(\text{Here we write } u_\varepsilon^{-\alpha} = u_\varepsilon^{(-1-\alpha)(p-1)/p} \, u_\varepsilon^{(p-1-\alpha)\frac{1}{p}} \right)$$

Absorb the similar term in RHS to LHS we get

$$\int_\Omega |Du|^p u_\varepsilon^{-1-\alpha} |\varphi|^p dx \leq \left(\frac{p}{\alpha}\right)^p \int_\Omega u_\varepsilon^{p-1-\alpha} |D\varphi|^p dx$$

By dominated convergence theorem,

$$\int_\Omega |Du|^p u^{-1-\alpha} |\varphi|^p dx = \int_\Omega \lim_{\varepsilon\to 0} |Du|^p u_\varepsilon^{-1-\alpha} |\varphi|^p dx$$

$$= \lim_{\varepsilon \to 0} \int_\Omega |Du|^p u_\varepsilon^{-1-\alpha} |\varphi|^p dx$$

$$\leq \lim_{\varepsilon\to 0} \left(\frac{p}{\alpha}\right)^p \int_\Omega u_\varepsilon^{p-1-\alpha} |D\varphi|^p dx$$

$$\cancel{\leq} = \left(\frac{p}{\alpha}\right)^p \int_\Omega \lim_{\varepsilon\to 0} u_\varepsilon^{p-1-\alpha} |D\varphi|^p dx$$

$$\leq \left(\frac{p}{\alpha}\right)^p \int_\Omega u^{p-1-\alpha} |D\varphi|^p dx \quad \square.$$

5. Assume that u is a nonnegative weak solution to the p-Laplace equation in Ω with $p>n$.

a) Explain why there exists $C_1 = C_1(n,p)$ such that

$$|u(z) - u(y)| \leq C_1 |z-y|^{1-n/p} \|Du\|_{L^p(B(x,r))} \quad , z,y \in B(x,r) \Subset \Omega$$

Observe that this shows $u \in C^{0,\alpha}_{loc}(\Omega)$

<u>Reason</u>

For any $B(x,r) \Subset \Omega$, there exist $\varepsilon > 0$ such that $B(x, r+\varepsilon) \subset \Omega$. Define $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(y) = 1 \quad \forall y \in B(x,r)$ and

$\varphi(y) = 0 \quad \forall y \in \Omega \setminus B(x, r+\varepsilon)$

Thus $u\varphi \in W_0^{1,p}(\Omega)$ and $u\Big|_{B(x,r)} = u\varphi\Big|_{B(x,r)}$

Choose $r = z-y$, by Morrey's inequality, there exists $C_1 = C_1(n,p)$ such that

$$|u(z) - u(y)| \leq C_1 |z-y|^{1-n/p} \|Du\|_{L^p(B(x,r))} \quad , z,y \in B(x,r) \Subset \Omega$$

$\square$.

b. Show that there exists $C_2 = C_2(n,p)$ such that

$$\|D \log u\|_{L^p(B(x,r))} \leq C_2 r^{\frac{n}{p}-1} \quad , \text{whenever } B(x,2r) \subset \Omega.$$

<u>Proof</u>

Let $\varphi \in C_0^\infty(B(x,2r))$ such that $0 \leq \varphi \leq 1$ and $\varphi(y) = 1 \ \forall y \in B(x,r)$ and $|D\varphi| \leq \frac{2}{r}$. Choose $u^{+1-p}|\varphi|^p \in W^{1,p}(\Omega)$ to be a test function

$$0 = \int_\Omega |Du|^{p-2} Du \cdot D(u^{1-p}|\varphi|^p)\, dx$$

By problem problem 5 in the set 3 ~~and by Hölder's inequality~~ we have

$$\Big(\int_{B(x,r)} |D\log u|^p |\varphi|^p dx\Big)^{1/p} \leq \Big(\int_{B(x,2r)} |D\log u|^p |\varphi|^p dx\Big)^{1/p} = \Big(\int_\Omega |D\log u|^p |\varphi|^p dx\Big)^{1/p}$$

$$\leq \Big[\Big(\frac{p}{p-1}\Big)^p \int_\Omega |D\varphi|^p dx\Big]^{1/p} = \frac{p}{p-1}\Big(\int_{B(x,2r)} |D\varphi|^p dx\Big)^{1/p}$$

$$\leq \frac{p}{p-1}\Big(\int_{B(x,2r)} \Big(\frac{2}{r}\Big)^p dx\Big)^{1/p}$$

$$= \frac{p}{p-1} \frac{2}{r} |B(x,2r)|^{1/p} = c(n,p)\, r^{\frac{n}{p}-1}$$

$\square$.

c. Show that

$$\left|\log \frac{u(y)}{u(z)}\right| \leq C_1 C_2 .$$

Proof

Let $\varepsilon > 0$ and $u_\varepsilon = u + \varepsilon$, for $y, z \in B(x,r)$, we have

$$\left|\log \frac{u_\varepsilon(y)}{u_\varepsilon(z)}\right| = \left|\log\left(u_\varepsilon(y) - u_\varepsilon(z)\right)\right|$$

(by part a) $\leq C_1 \| D \log u_\varepsilon \|_{L^p(B(x,r))}$

(by part b) $\leq C_1 \tilde{C}_2 \, r^{n/p - 1} \leq C_1 C_2$

By Lebesgue dominated convergence theorem, let $\varepsilon \to 0$. $\square$.

d. Conclude that

$$\sup_{B(x,r)} u \leq C \inf_{B(x,r)} u \quad , C = C(n,p) = e^{C_1 C_2} \quad , z, y \in B(x,r),$$

whenever $B(x, 2r) \subset \Omega$.

Proof

From 5c, we have

$$|u(y) - u(z)| \leq e^{|u(y) - u(z)|}$$

$$\leq e^{C_1 |z-y|^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}} = e^{C_1 C_2}$$