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Nonlinear partial differential equation

Exercise 3.

1. Let $p > 2$. Show that $u : \mathbb{R}^n \to \mathbb{R}$

$$u(x) = \frac{p-1}{p} |x|^{\frac{p}{p-1}}$$

is a weak solution to $-\operatorname{div}(|Du|^{p-2} Du) = -n$ in $\mathbb{R}^n$

Proof:

a) We have $D_j u(x) = x_j |x|^{(2-p)/(p-1)}$, $Du(x) = \sum_{j=1}^n D_j u(x) = x |x|^{(2-p)/(p-1)}$

$$|Du(x)| = |x|^{1/(p-1)}$$

By Gauss-Green theorem, let $\varphi \in C_0^\infty(\Omega)$, we get

$$\int_\Omega -n \varphi \, dx = -\int_\Omega \operatorname{div}(|Du|^{p-2} Du) \varphi \, dx$$

$$= -\int_\Omega \operatorname{div}(|Du|^{p-2} Du) \varphi \, dx + \int_{\partial\Omega} \frac{\partial}{\partial \nu} (|Du|^{p-2} Du) \varphi \, dS(x)$$

(ν is unit normal pointing outward from $\partial\Omega$., $\varphi = 0$ on $\partial\Omega$)

$$= +\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = +\int_\Omega x \cdot D\varphi \, dx$$

$$= +\sum_{j=1}^n \int_\Omega x_j \frac{\partial \varphi}{\partial x_j} dx = +\sum_{j=1}^n \left(-\int_\Omega \frac{\partial x_j}{\partial x_j} \varphi \, dx + \int_{\partial\Omega} x_j \varphi \nu_j \, dS(x) \right)$$

$$= -\int \sum_{j=1}^n \varphi \, dx = -\int n \varphi \, dx.$$

(Integration by parts and $\varphi = 0$ on $\partial\Omega$)

□

Thus u is a weak solution to $-\operatorname{div}(|Du|^{p-2} Du) = -n$.

b). If $p = 1$, checking the difference quotion, let $\Omega' \Subset \Omega = \mathbb{R}^n$, $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$

$$\frac{u(x + he_j) - u(x)}{h} = \frac{p-1}{hp} \left(|x + he_j|^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}} \right)$$

$$\leq \frac{p-1}{hp} |h|^{\frac{p}{p-1}} \longrightarrow \infty \quad \text{as } h \to 0 \text{ and } p \to 1$$

Thus $u \notin C^{1,1}_{loc}(\mathbb{R}^n)$.

□

2. Show that $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the p-Laplace equation in Ω if and only if every point has a neighbourhood where u is a weak solution.

Proof

"$\Rightarrow$" Assume $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the p-Laplace equation.

Let $\Omega_j = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{j}\} \cap B(0,j)$.

Thus $\Omega_1 \Subset \dots \Subset \Omega$, $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Let $\{\eta_j\}_{j=1}^{\infty}$ is a partition of unity subordinate to the covering $\{\Omega_j\}$. Choose $\varepsilon_j > 0$ such that $\text{supp}(\phi_{\varepsilon_j} * (\eta_j u)) \subset \Omega_{j+2} \setminus \overline{\Omega}_{j-1}$,

$$0 = \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = \sum_{j=1}^{\infty} \int_{\Omega_j} |Du|^{p-2} Du \cdot D\varphi_j \, dx$$

$$\Rightarrow \int_{\Omega_j} |Du|^{p-2} Du \cdot D\varphi_j \, dx = 0, \text{ where } \varphi_j := \phi_{\varepsilon_j} * (\eta_j u) \text{ and } \varphi := \sum_{j=1}^{\infty} \varphi_j.$$

"$\Leftarrow$" Assume for any $x_0 \in \Omega$, there exists a neighbourhood U_{x_0} such that u is a weak solution to the p-Laplace equation in U_{x_0}, which implies $u \in W^{1,p}_{loc}(U_{x_0})$ For any $x_0 \in \Omega$, thus $u \in W^{1,p}_{loc}(\Omega = \bigcup_{x_0 \in \Omega} U_{x_0})$.

Let $\varphi \in C_0^\infty(\Omega)$, which means $\text{supp}\,\varphi$ is a compact subset of Ω. By compactness there exists a finite subcover such that $\text{supp}\,\varphi \subset \bigcup_{j=1}^{k} U_j$.

Next we construct a smooth partition of unity $\{\varphi_j\}_{j=1}^{k}$ to the covering $\{U_j\}_{j=1}^{k}$

Thus $\varphi\varphi_i \in C_0^\infty(U_i)$, $\sum_{j=1}^{k} \varphi_j(x) = 1$ for every $x \in \Omega$.

Use $\varphi\varphi_i \in C_0^\infty(U_i)$ as a test function

$$0 = \int_{U_j} |Du|^{p-2} Du \cdot D(\varphi\varphi_i) \, dx = \sum_{j=1}^{k} \int_{U_j} |Du|^{p-2} Du \cdot D(\varphi\varphi_i) \, dx$$

$$= \sum_{j=1}^{k} \int_{U_j} \varphi_j |Du|^{p-2} Du \cdot D\varphi \, dx + \sum_{j=1}^{k} \int_{U_i} |Du|^{p-2} Du \cdot D\varphi_i \, \varphi \, dx$$

$$= \int_{\bigcup_{j=1}^{k} U_j = \text{supp}\,\varphi} |Du|^{p-2} Du \cdot D\varphi \, dx + \int_{\bigcup_{j=1}^{k} U_j} |Du|^{p-2} Du \cdot \underbrace{D\left(\sum_{j=1}^{k} \varphi_j\right)}_{=0} \varphi \, dx$$

$$= \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx \quad \text{for } \varphi \in C_0^\infty(\Omega)$$

$\square$.

3. Assume that $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the p-Laplace equation in Ω

a) Show that

$$\int_\Omega |Du|^p |\varphi|^p dx \le p^p \int_\Omega |u|^p |D\varphi|^p dx$$

for every $\varphi \in C_0^\infty(\Omega)$

Proof

Assume that $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the p-Laplace equation in Ω. Let's take $u|\varphi|^p \in C_0^\infty(\Omega)$ as a test function, we get

$$0 = \int_\Omega |Du|^{p-2} Du . D(u|\varphi|^p)\, dx = \int_\Omega |Du|^{p-2} Du . Du\, |\varphi|^p dx + \int_\Omega |Du|^{p-2} Du . (u D|\varphi|^p) dx$$

$$\Rightarrow \left|\int_\Omega |Du|^p |\varphi|^p dx\right| \le \left|\int_\Omega |Du|^{p-2} Du . (u D|\varphi|^p) dx\right|$$

$$\Rightarrow \int_\Omega |Du|^p |\varphi|^p dx \le \int_\Omega |Du|^{p-2} |Du|\, |u|\, |(p|\varphi|^{p-1} D\varphi)|\, dx$$

$$\le \int_\Omega p |Du|^{p-1} |\varphi|^{p-1} |u| |D\varphi|\, dx$$

(Young's inequality) $$\le \int_\Omega \frac{(p-1)}{p} |Du|^p |\varphi| dx + \int_\Omega \frac{p^p |u|^p |D\varphi|^p}{p} dx$$

since $p \ge 1$,

$$\Rightarrow \int_\Omega |Du|^p |\varphi|^p dx \le p^p \int_\Omega |u|^p |D\varphi|^p dx \quad \text{for every } \varphi \in C_0^\infty(\Omega)$$

$\square$

b). Show that there is a constant $c = c(p)$ such that

$$\int_{B(x,r)} |Du|^p dy \le c r^{-p} \int_{B(x,2r)} |u|^p dy \quad \text{, whenever } B(x,2r) \subset \Omega.$$

Proof

Let φ be a cutoff function, $\varphi \in C_0^\infty(B(x,2r))$, $0 \le \varphi \le 1$ such that $\varphi \equiv 1$ in $B(x,r)$ and $|D\varphi| \le \frac{2}{r}$. We have

$$\int_{B(x,r)} |Du|^p dx = \int_{B(x,r)} |Du|^p \varphi^p dx \le \int_\Omega |Du|^p \varphi^p dx \le p^p \int_\Omega |u|^p |D\varphi|^p dx$$

(from part a)

$$\le p^p \int_{B(x,2r)} |u|^p \left|\frac{2}{r}\right|^p dx = c r^{-p} \int_{B(x,2r)} |u|^p dy$$

$\square$

4. Assume that $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the p-Laplace equation in Ω. Show that there exists $q < p$ and $c = c(n,p,q)$ such that

$$\left(\fint_{B(x,r)} |Du|^p \, dy \right)^{1/p} \leq c \left(\fint_{B(x,2r)} |Du|^q \, dy \right)^{1/q}.$$

whenever $B(x,2r) \subset \Omega$.

Proof

Assume $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the p-Laplace equation in Ω, thus $u - u_{B(x,2r)}$ is also a weak solution. Then we have

$$\left(\fint_{B(x,r)} |Du|^p dx \right)^{1/p} = \left(\fint_{B(x,r)} |D(u-u_{B(x,2r)})|^p dy \right)^{1/p} = \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |D(u-u_{B(x,r)})|^p dy \right)^{1/p}$$

$$(\text{From problem 3}) \leq \left(\frac{|B(x,2r)|}{|B(x,r)|} \frac{cr^{-p}}{|B(x,2r)|} \int_{B(x,2r)} |u-u_{B(x,2r)}|^p dy \right)^{1/p}$$

$$= (2^n c r^{-p})^{1/p} \left(\fint_{B(x,2r)} |u-u_{B(x,2r)}|^p dy \right)^{1/p}$$

- If $p \geq \frac{n}{n-1}$, choose $q < p$ such that $\frac{nq}{n-q} = p$

By Poincaré's inequality, we get

$$\left(\fint_{B(x,r)} |Du|^p dx \right)^{1/p} \leq (2^n c r^{-p})^{1/p} r \tilde{c}(n,q) \left(\fint_{B(x,2r)} |Du|^q dy \right)^{1/q}$$

$$= c(n,p,q) \left(\fint_{B(x,2r)} |Du|^q dy \right)^{1/q}.$$

- If $1 < p < \frac{n}{n-1}$

First by Jensen's inequality we get

$$\left(\fint_{B(x,2r)} |u-u_{B(x,2r)}|^p dy \right)^{1/p} \leq \left(\fint_{B(x,2r)} |u-u_{B(x,2r)}|^{n/(n-1)} dy \right)^{\frac{n-1}{n}}$$

Thus, by choosing $q = 1$ and Sobolev's–Poincaré's inequality, we have

$$\left(\fint_{B(x,r)} |Du|^p dx \right)^{1/p} = c(n,p,q) \left(\fint_{B(x,2r)} |Du|^q dy \right)^{1/q}$$

$\square$,

5. Assume that $u \in W^{1,p}_{loc}(\Omega)$ is a nonnegative weak solution to the p-Laplace equation in Ω.

a) Show that

$$\int_\Omega |D\log u|^p |\varphi|^p \, dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |D\varphi|^p dx \quad , \text{for every } \varphi \in C_0^\infty(\Omega)$$

<u>Proof</u>

Assume $u \in W^{1,p}_{Loc}(\Omega)$, $u \geq 0$, let $\varepsilon > 0$, thus $u+\varepsilon > 0$ and $(u+\varepsilon)^{1-p} \in W^{1,p}_{loc}(\Omega)$ and $(u+\varepsilon)^{1-p}|\varphi|^p \in C_0^\infty(\Omega)$. Hence we have $Du = D(u+\varepsilon)$,

$$0 = \int_\Omega |Du|^{p-2} Du \cdot D\left((u+\varepsilon)^{1-p}|\varphi|^p\right) dx$$

$$= \int_\Omega (1-p)(u+\varepsilon)^{-p}|\varphi|^p |D(u+\varepsilon)|^p dx + \int_\Omega |Du|^{p-2} Du \cdot \left((u+\varepsilon)^{1-p} p\, \varphi |\varphi|^{p-2} D\varphi\right) dx$$

$$\Rightarrow (p-1)\int_\Omega \frac{|D(u+\varepsilon)|^p}{(u+\varepsilon)^{+p}} |\varphi|^p dx \leq \int_\Omega |Du|^{p-2}|Du|\, p(u+\varepsilon)^{1-p}|\varphi|^{p-1}|D\varphi| dx.$$

$$\Rightarrow \int_\Omega |D\log(u+\varepsilon)|^p |\varphi|^p dx \leq \int_\Omega \frac{p}{p-1} \frac{|D(u+\varepsilon)|^{p-1}}{(u+\varepsilon)^{p-1}} |\varphi|^{p-1} |D\varphi| dx$$

(by Young's inequality) $$\leq \int_\Omega |D\log(u+\varepsilon)|^p |\varphi|^p \frac{(p-1)}{p} dx + \int_\Omega \left(\frac{p}{p-1}\right)^p \frac{|D\varphi|^p}{p} dx$$

since $p \geq 1$, we have

$$\int_\Omega |D\log(u+\varepsilon)|^p |\varphi|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |D\varphi|^p dx$$

$$\Rightarrow \liminf_{\varepsilon\to 0} \int_\Omega |D\log(u+\varepsilon)|^p |\varphi|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |D\varphi|^p dx$$

By Fatou's Lemma, we can conclude

$$\int_\Omega |D\log u|^p |\varphi|^p dx = \int \liminf_{\varepsilon\to 0} |D\log(u+\varepsilon)|^p |\varphi|^p dx \leq \liminf_{\varepsilon\to 0} \int_\Omega |D\log(u+\varepsilon)|^p |\varphi|^p dx$$

$$\leq \left(\frac{p}{p-1}\right)^p \int_\Omega |D\varphi|^p dx$$

$\square$

b: Let $v = \log u$. Show that there exists $c = c(n,p) < \infty$ such that

$$\fint_{B(x,r)} |v - v_{B(x,r)}| dy \leq c \quad , \text{whenever } B(x,2r) \subset \Omega.$$

<u>Proof</u>

Let $\varphi \in C_0^\infty(B(x,2r))$ be a cutoff function such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B(x,r)$ and $|D\varphi| \leq \frac{2}{r}$. Thus we have

$$\fint_{B(x,r)} |v - v_{B(x,r)}|\, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} |v - v_{B(x,r)}|\, dy$$

(Hölder's inequality) $$\leq \frac{1}{|B(x,r)|} \left(\int_{B(x,r)} 1\, dy \right)^{1-\frac{1}{p^*}} \left(\int_{B(x,r)} |v - v_{B(x,r)}|^{p^*} dy \right)^{1/p^*}$$

$$= \frac{|B(x,r)|^{1-1/p^*}}{|B(x,r)|} |B(x,r)|^{1/p^*} \left(\fint_{B(x,r)} |v - v_{B(x,r)}|^{p^*} dy \right)^{1/p^*}$$

(Sobolev-Poincaré's ineq) $$\leq cr \left(\fint_{B(x,r)} |Dv|^p dx \right)^{1/p} = \frac{cr}{|B(x,r)|^{1/p}} \left(\int_{B(x,r)} |D\log u|^p |\varphi|^p dx \right)^{1/p}$$

(From part a) $$\leq \cancel{\left(\frac{p}{p-1}\right)} \frac{cr}{|B(x,r)|^{1/p}} \left(\int_{\Omega} |D\log u|^p |\varphi|^p dx \right)^{1/p}$$

$$\leq \left(\frac{p}{p-1}\right) \frac{cr}{|B(x,r)|^{1/p}} \left(\int_{\Omega} |D\varphi|^p dx \right)^{1/p}$$

$$= \frac{p}{p-1} \frac{cr}{|B(x,r)|^{1/p}} \left(\int_{B(x,2r)} |D\varphi|^p dx \right)^{1/p}$$

$$\leq \frac{p}{p-1} \frac{cr}{|B(x,r)|^{1/p}} \left(\int_{B(x,2r)} \left(\frac{2}{r}\right)^p dx \right)^{1/p}$$

$$= \frac{p}{p-1} \frac{cr\,|B(x,2r)|^{1/p}}{|B(x,r)|^{1/p}} \frac{2}{r} = c(n,p)$$

$\square$.