1. Let $1 < p < \infty, u \in C^2(\Omega)$ with $Du(x) \neq 0$ for every $x \in \Omega$. Show that

$$\operatorname{div}\Big( |Du|^{p-2} Du\Big) = |Du|^{p-4} \left( |Du|^2 \Delta u - (p-2) \sum_{i,j=1}^{n} D_i u \, D_j u \, D_{ij} u\right)$$
$$= |Du|^{p-4} \left( |Du|^2 \Delta u - \frac{(p-2)}{2} Du \cdot D(|Du|^2)\right)$$

Is the assumption $Du(x) \neq 0$ needed when $p \geq 2$.

**Proof:**

$$\operatorname{div}\Big( |Du|^{p-2} Du\Big) = \sum_{i=1}^{n} D_i\Big( |Du|^{p-2} D_i u\Big)$$
$$= \sum_{i=1}^{n} \left( D_i |Du|^{p-2} D_i u + |Du|^{p-2} D_i D_i u\right)$$
$$= \sum_{i=1}^{n} \left( D_i \left|\sum_{j=1}^{n} (D_j u)^2\right|^{(p-2)/2} D_i u + |Du|^{p-2} D_i D_i u\right)$$
$$= \sum_{i=1}^{n} \left( (p-2) \sum_{j=1}^{n} D_j u \, D_i D_j u \, |Du|^{p-4} D_i u + |Du|^{p-2} D_i D_i u\right)$$
$$= |Du|^{p-4} \left( |Du|^2 \Delta u - (p-2) \sum_{i,j=1}^{n} D_i u \, D_j u \, D_{ij} u\right)$$
$$= |Du|^{p-4} \left( |Du|^2 \Delta u - \frac{(p-2)}{2} Du \cdot D(|Du|^2)\right)$$

The assumption $Du(x) \neq 0$ is needed when $p \geq 2$, otherwise we get $0 = 0$ $\square$

2. Let $1 < p < \infty$. Assume that $u \in W^{1,p}_{loc}(\Omega)$ is weak solution to p-Laplace equation $-\operatorname{div}(|Du|^{p-2} Du) = 0$ in $\Omega$, that is

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \quad \text{for every } \varphi \in C_0^\infty(\Omega)$$

Show that a classical solution of the p-Laplace equation is a weak solution.

**Proof:**

Assume $u \in C^2(\Omega)$, by Gauss-Green theorem we have

$$\begin{aligned}
&\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx \\
&= \int_\Omega \operatorname{div}(|Du|^{p-2} Du)\varphi \, dx + \int_{\partial\Omega} \frac{\partial}{\partial \nu}(|Du|^{p-2} Du)\varphi \, dx \\
&\text{(where } \nu \text{ is unit normal pointing outward from } \partial\Omega) \\
&= \int_\Omega \operatorname{div}(|Du|^{p-2} Du)\varphi \, dx \quad \text{for every } \varphi \in C_0^\infty(\Omega).
\end{aligned}$$

since $-\operatorname{div}(|Du|^{p-2} Du) = 0$ in $\Omega$, which implies

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

□

3. Assume that $u$ is a weak solution to the p-Laplace equation. Show that $au + b, a, b \in \mathbb{R}$ is a weak solution. Is the p-Laplace equation linear?

**Proof:** Straightforward computing, we get

$$\begin{aligned}
&\int_\Omega |D(au+b)|^{p-2} D(au+b) \cdot D\varphi \, dx \\
&= \int_\Omega |aDu|^{p-2} aDu \cdot D\varphi \, dx \\
&= \int_\Omega a\,|a|^{p-1} |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \quad \text{for every } \varphi \in C_0^\infty(\Omega).
\end{aligned}$$

which implies $au + b, a, b \in \mathbb{R}$ is a weak solution to p-Laplace equation. To see if p-Laplace equation is linear, we compute $au + bv, a, b \in \mathbb{R}, u, v$ are classical solution to p-Laplace equation:

$$\begin{aligned}
&-\operatorname{div}\Big( |D(au+bv)|^{p-2} D(au+bv)\Big) \\
&= -\operatorname{div}\Big( |aDu + bDv|^{p-2} (aDu + bDv)\Big) \\
&\neq -a\,|a|^{p-1} \operatorname{div}\Big( |Du|^{p-2} Du\Big) - b\,|b|^{p-1} \operatorname{div}\Big( |Dv|^{p-2} Dv\Big)
\end{aligned}$$

which implies p-Laplace equation is not linear . □

4. Show that $\Phi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$

$$\Phi(x) = \begin{cases} |x|^{\frac{p-n}{p-1}}, & p \neq n \\ \log|x|, & p = n, \end{cases}$$

is a classical solution to the p-Laplace equation in $\mathbb{R}^n \backslash \{0\}$. Is it a weak solution to the p-Laplace equation in $\mathbb{R}^n$ ?

**Proof:** Note that

$$D\Phi(x) = \begin{cases} \frac{p-n}{p-1} \, x \, |x|^{\frac{2-p-n}{p-1}} , & p \neq n \\ \frac{1}{x}, & p = n, \end{cases}$$

Let $\varepsilon > 0$,

$$\begin{aligned} &\int_{\mathbb{R}} |D\Phi|^{p-2} \, D\Phi \cdot D\varphi \, dx \\ &= \int_{\mathbb{R}^n \backslash B(0,\varepsilon)} |D\Phi|^{p-2} \, D\Phi \cdot D\varphi \, dx + \int_{B(0,\varepsilon)} |D\Phi|^{p-2} \, D\Phi \cdot D\varphi \, dx \\ &= I_\varepsilon + II_\varepsilon \end{aligned}$$

Since $\Phi$ is classical solution to the p-Laplace equation in $\mathbb{R}^n \backslash \{0\}$, it is also a weak solution to the p-Laplace equation in $\mathbb{R}^n \backslash B(0,\varepsilon) \subset \mathbb{R}^n \backslash \{0\}$. Hence $I_\varepsilon = 0$ and

$$\begin{aligned} |II_\varepsilon| &\leq \int_{B(0,\varepsilon)} |D\Phi|^{p-1} \, |D\varphi| \, dx \\ &\leq \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |D\Phi|^{p-1} \, dx, \end{aligned}$$

Thus

$$\begin{aligned} |II_\varepsilon| &\leq \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \frac{p-n}{p-1} \int_{B(0,\varepsilon)} \left( |x|^{\frac{1-n}{p-1}} \right)^{p-1} dx, && \text{if } n \neq p \\ &\leq \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \frac{p-n}{p-1} \int_{B(0,\varepsilon)} |x|^{1-n} \, dx, && \text{if } n \neq p \\ &\leq \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \frac{p-n}{p-1} \int_{B(0,\varepsilon)} \varepsilon^{1-n} dx, && \text{if } n \neq p \\ &\leq \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \frac{p-n}{p-1} \varepsilon \, |B(0,1)| \to 0 \text{ as } \varepsilon \to 0, && \text{if } n \neq p \end{aligned}$$

and

$$
\begin{aligned}
|II_\varepsilon| &\leq \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} \left(|x|^{-1}\right)^{p-1} dx && \text{if } n = p \\
&= \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |x|^{1-p}\, dx && \text{if } n = p \\
&= \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} \varepsilon^{1-n}\, dx && \text{if } n = p \\
&= \|D\varphi\|_{L^\infty(\mathbb{R}^n)}\, \varepsilon\, |B(0,1)| \to 0 \text{ as } \varepsilon \to 0, && \text{if } n = p.
\end{aligned}
$$

Hence $\Phi$ is weak solution to p-Laplace equation. □

5. Show that there exists $c = c(n,p)$ such that -div$\Big(|D\Phi|^{p-2} D\Phi\Big) = c\delta$ in $\mathbb{R}^n$ that is,

$$\int_\Omega |D\Phi|^{p-2} D\Phi \cdot D\varphi\, dx = c\varphi(0) \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega).$$

**Proof:**
Note that:

$$DD\Phi(x) = \begin{cases} \frac{n(p-n)}{p-1} |x|^{\frac{2-p-n}{p-1}} + \frac{(p-n)(2-p-n)}{(p-1)^2} |x|^{\frac{2-p-n}{p-1}}, & p \neq n \\ -\frac{1}{x^2}, & p = n, \end{cases}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, there exists $R > 0$ such that $supp\, \varphi \subset B(0,R)$ If $0 \notin supp\, \varphi$

If $0 \in supp\, \varphi$, let $\varepsilon > 0$ , we have

$$
\begin{aligned}
&\int_{B(0,R)} |D\Phi|^{p-2} D\Phi \cdot D\varphi\, dx \\
&= \int_{B(0,R)\setminus B(0,\varepsilon)} |D\Phi|^{p-2} D\Phi \cdot D\varphi\, dx + \int_{B(0,\varepsilon)} |D\Phi|^{p-2} D\Phi \cdot D\varphi\, dx \\
&= I_\varepsilon + II_\varepsilon
\end{aligned}
$$

where,

$$
\begin{aligned}
|II_\varepsilon| &\le \int_{B(0,\varepsilon)} |D\Phi|^{p-1} |D\varphi| \, dx \\
&\le \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |D\Phi|^{p-1} \, dx,
\end{aligned}
$$

Thus

$$
\begin{aligned}
|II_\varepsilon| &\le \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \frac{p-n}{p-1} \int_{B(0,\varepsilon)} \left( |x|^{\frac{1-n}{p-1}} \right)^{p-1} dx, && \text{if } n \ne p \\
&\le \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \frac{p-n}{p-1} \int_{B(0,\varepsilon)} |x|^{1-n} \, dx, && \text{if } n \ne p \\
&\le \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \frac{p-n}{p-1} \int_{B(0,\varepsilon)} \varepsilon^{1-n} dx, && \text{if } n \ne p \\
&\le \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \frac{p-n}{p-1} \varepsilon \, |B(0,1)| \to 0 \text{ as } \varepsilon \to 0, && \text{if } n \ne p
\end{aligned}
$$

and

$$
\begin{aligned}
|II_\varepsilon| &\le \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} \left( |x|^{-1} \right)^{p-1} dx && \text{if } n = p \\
&= \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |x|^{1-p} \, dx && \text{if } n = p \\
&= \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} \varepsilon^{1-n} \, dx && \text{if } n = p \\
&= \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \varepsilon \, |B(0,1)| \to 0 \text{ as } \varepsilon \to 0, && \text{if } n = p.
\end{aligned}
$$

By Gauss-Green's Theorem, we have

$$
\begin{aligned}
I_\varepsilon &= \int_{B(0,R)\setminus B(0,\varepsilon)} |D\Phi|^{p-2} D\Phi \cdot D\varphi \, dx \\
&= -\int_{B(0,R)\setminus B(0,\varepsilon)} \operatorname{div}(|D\Phi|^{p-2} D\Phi)\varphi \, dx \\
&+ \int_{\partial B(0,R)\cup \partial B(0,\varepsilon)} \frac{\partial}{\partial \nu}(|D\Phi|^{p-2} D\Phi) \, \varphi \, dS(x)
\end{aligned}
$$

(where $\nu$ is unit normal pointing outward from $\partial B(0,R)$ and $\partial B(0,\varepsilon)$)

$$
\begin{aligned}
&= \int_{\partial B(0,R)\cup \partial B(0,\varepsilon)} \frac{\partial}{\partial \nu}(|D\Phi|^{p-2} D\Phi) \, \varphi \, dS(x) \\
&= \int_{\partial B(0,\varepsilon)} \Big( \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(|D\Phi|^{p-2} D\Phi)\nu_j \Big) \varphi \, dS(x) \quad \Big( \varphi = 0 \text{ on } \partial B(0,R) \Big)
\end{aligned}
$$

hence,

$$
\begin{aligned}
|I_\varepsilon| &= \left| \int_{\partial B(0,\varepsilon)} \Big( \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (|D\Phi|^{p-2} D\Phi) \nu_j \Big) \varphi \, dS(x) \right| \\
&\leq \int_{\partial B(0,\varepsilon)} \left| \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (|D\Phi|^{p-2} D\Phi) \right| |\nu_j| \, |\varphi| \; dS(x) \\
&\leq \int_{\partial B(0,\varepsilon)} \left| D \, |D\Phi|^{p-2} D\Phi + |D\Phi|^{p-2} DD\Phi \right| \; |\varphi| \; dS(x) \\
&\leq c(n,p) \int_{\partial B(0,\varepsilon)} f(|x|) \; |\varphi| \; dS(x) \\
&= c(n,p) \int_{\partial B(0,1)} f(|x|) \, |\varphi| \; dS(x) \to c(n,p)\varphi(0) \quad \text{as } \varepsilon \to 0
\end{aligned}
$$